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Description	

New Concrete Relation between Trace, Definition Field, and Embedding Degree*

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SUMMARY A pairing over an elliptic curve E/\mathbb{F}_{p^m} to an extension field of \mathbb{F}_{p^m} has begun to be attractive in cryptosystems, from the practical and theoretical point of view. From the practical point of view, many cryptosystems using a pairing, called the pairing-based cryptosystems, have been proposed and, thus, a pairing is a necessary tool for cryptosystems. From the theoretical point of view, the so-called embedding degree k is an indicator of a relationship between the elliptic curve Discrete Logarithm Problem (ECDLP) and the Discrete Logarithm Problem (DLP), where ECDLP over $E(\mathbb{F}_{p^m})$ is reduced to DLP over \mathbb{F}_{p^m} by using the pairing. An elliptic curve is determined by mathematical parameters such as the j -invariant or order of an elliptic curve, however, explicit conditions between these mathematical parameters and an embedding degree have been described only in a few degrees. In this paper, we focus on the theoretical view of a pairing and investigate a new condition of the existence of elliptic curves with pre-determined embedding degrees. We also present some examples of elliptic curves over 160-bit, 192-bit and 224-bit \mathbb{F}_{p^m} with embedding degrees $k < (\log p)^2$ such as $k = 10, 12, 14, 20, 22, 24, 28$.

key words: elliptic curve, embedding degree

1. Introduction

The elliptic curve cryptosystem (ECC) chosen appropriately can offer efficient public key cryptosystems [1]. This is why elliptic curve cryptosystems have been desired in various application such as a smart card, whose memory storage and CPU power are very limited. Recently, a pairing over an elliptic curve E/\mathbb{F}_{p^m} has begun to be attractive in cryptosystems from the practical and theoretical point of view. For example, it can achieve an ID-based cryptosystem [5], short signature [2], etc. The cryptosystems using a pairing are called the pairing-based cryptosystems. On the other hand, a pairing is originally used to solve the Elliptic Curve Discrete Logarithm Problem (ECDLP) by reducing ECDLP on $E(\mathbb{F}_{p^m})$ to Discrete Logarithm Problem (DLP) on \mathbb{F}_{p^m} [8], [12], where k is called the embedding degree. The embedding degree k can be considered to be an indicator of the security of ECDLP. Then, the security level of ECDLP over $E(\mathbb{F}_{p^m})$ is the same as that of DLP over \mathbb{F}_{p^m} by using a pairing from $E(\mathbb{F}_{p^m})$ to \mathbb{F}_{p^m} .

An elliptic curve E/\mathbb{F}_{p^m} can be determined by parameters of j -invariant or order $\#E(\mathbb{F}_{p^m})$. This is one of math-

ematical facts on elliptic curves. On the other hand, from the point of view of cryptology, we are curious about the relation between an embedding degree and these mathematical parameters of j -invariant or $\#E(\mathbb{F}_{p^m})$. The relationship, however, between them has been described only in a few degrees such as $k = 3, 4, 6, 10$, or 12 . It is an open problem to describe an embedding degree of an elliptic curve E/\mathbb{F}_{p^m} explicitly by its j -invariant or $\#E(\mathbb{F}_{p^m})$. Generally, the embedding degree k for a prime-order elliptic curve is $k \approx n$ where $n = \#E(\mathbb{F}_{p^m})$ [3].

A lot of studies to construct elliptic curves having small embedding degrees, such as $k = 2, 3, 4, 5, 6, 10$ and 12 , have been investigated. Miyaji, Nakabayashi and Takano [13] have proposed ordinary elliptic curves with embedding degrees $k = 3, 4$ and 6 . Galbraith, Valenca, Mackee [9] have presented the factorization of cyclotomic polynomials with degrees $5, 10$ and 12 , and applied the results [13] to a hyperelliptic curve. Freeman [6] and Barretto and Naehrig [4] have constructed ordinary elliptic curves with embedding degree $k = 10$ and $k = 12$ using the factorization of cyclotomic polynomials in [9], respectively.

Hitt [11] has investigated a Jacobian of hyperelliptic curve J_C/\mathbb{F}_{2^m} and discussed the way to decide an embedding degree k of $J_C(\mathbb{F}_{2^m})$ from order of p in \mathbb{Z}_n , where n is the largest prime divisor of $\#J_C(\mathbb{F}_{2^m})$. Hitt also gave some examples of $\#J_C(\mathbb{F}_{2^m})$ with embedding degrees $k < (\log p^m)^2$. Unfortunately, by her approach, it is not easy to give concrete J_C/\mathbb{F}_{2^m} themselves. However, her approach sheds additional light on the relations between $\#J_C(\mathbb{F}_{2^m})$ and the embedding degree of J_C by the simple discussion based on the elementary number theory. There might be still room for further improvement in the following points:

1. Her result cannot construct $\#J_C(\mathbb{F}_{2^m})$ with $\rho = \frac{\#J_C(\mathbb{F}_{2^m})}{n} \approx 1$. Her results restrict the relation between trace, definition field, and the largest prime divisor. As a result, $\rho > 1$.
2. Her results cannot decide an embedding degree of k of $\#J_C(\mathbb{F}_p)$ in the case of a prime field \mathbb{F}_p . This means that they suffer from reduction to the actual minimum embedding degree. In fact, the actual security level of her examples is reduced to $\frac{1}{19}$ of their original security level at the minimum and its $\frac{1}{3}$ at the maximum.

In this paper, we investigate the case of elliptic curves by improving the approach to a Jacobian of hyperelliptic curve

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[11]. We prove the new concrete relations between \mathbb{F}_{p^m} , $\#E(\mathbb{F}_{p^m})$ and the embedding degree of E , which resolves the above her drawbacks. In fact, we improve Hitt's results from the point of view of ρ -value and the actual minimum security. As for ρ -value, we do not place any restriction on the relation between trace and definition field. As a result, we can construct elliptic curves with $\rho = 1$. Furthermore, we can enjoy the case of prime field \mathbb{F}_p , and, thus, our results do not suffer from reduction to the minimum embedding degree. We also present some examples of prime orders of elliptic curves over 160-bit, 192-bit and 224-bit \mathbb{F}_p with embedding degrees $k < (\log p)^2$ such as $k = 10, 12, 14, 20, 22, 24, 28$.

This paper is organized as follows. Section 2 summarizes known facts on elliptic curves. In Sect. 3, we review the previous results. Our main contribution appears in Sect. 4, where we show how to decide order of elliptic curves with pre-determined embedding degree. In Sect. 5, we present some experimental results based on Sect. 4. Section 6 compares our results with Hitt's results. Conclusion follows in Sect. 7.

2. Preparation

This section summarizes the known facts on elliptic curves. Let p be a prime, m be a positive integer, and E be an elliptic curve defined over a finite field \mathbb{F}_{p^m} . The trace t is defined as $t = p^m + 1 - \#E(\mathbb{F}_{p^m})$. The embedding degree is defined as follows.

Definition 1: Let E be an elliptic curve over a finite field \mathbb{F}_{p^m} with $\#E(\mathbb{F}_{p^m})$, and n be the largest prime divisor of $\#E(\mathbb{F}_{p^m})$. The embedding degree of E is defined as the smallest positive integer k such that $n \mid p^{mk} - 1$.

The k -th cyclotomic polynomial, $\Phi_k(X)$, is defined as the minimum polynomial of k -th root of unity, which satisfies the following features:

$$X^k - 1 = \prod_{d \mid k} \Phi_d(X).$$

By using the k -th cyclotomic polynomial, we can say that k is the minimal integer such that $n \mid \Phi_k(p^m)$. Further, the following 4 conditions on the embedding degree k of E are equivalent to each other:

1. ECDLP over a subgroup with order n of $E(\mathbb{F}_{p^m})$ is reduced to DLP over that of $\mathbb{F}_{p^{mk}}$.
2. k is the smallest positive integer such that

$$n \mid p^{mk} - 1.$$
3. $\Phi_k(p^m) \equiv 0 \pmod{n}$.
4. $\Phi_k(t - 1) \equiv 0 \pmod{n}$.

In this paper, we need the condition that, for a given order n , an elliptic curve E/\mathbb{F}_{p^m} with order n exists. The following Waterhouse's theorem [14] gives conditions that

Table 1 Elliptic curves with small embedding degrees.

	k	p^m	t
MNT-Curve[13]	3	$12x^2 - 1$	$-1 \pm 6x$
	4	$x^2 + x + 1$	$-x$ or $x + 1$
	6	$4x^2 + 1$	$1 \pm 2x$
Freeman[6]	10	$25x^4 + 25x^3 + 25x^2 + 10x + 3$	$10x^2 + 5x + 3$
BN-Curve[4]	12	$36x^4 + 36x^3 + 24x^2 + 6x + 1$	$6x^2 + 1$

an elliptic curve defined over \mathbb{F}_{p^m} of order $p^m + 1 - t$ exists.

Theorem 1 ([14]): An elliptic curve defined over \mathbb{F}_{p^m} of order $p^m + 1 - t$ exists if and only if one of the following conditions holds:

1. $t \not\equiv 0 \pmod{p}$ and $t^2 \leq 4p^m$.
2. m is odd and one of the following holds:
 - $t = 0$,
 - $t^2 = 2p^m$ and $p = 2$,
 - $t^2 = 3p^m$ and $p = 3$.
3. m is even and one of the following holds:
 - $t^2 = 4p^m$,
 - $t^2 = p^m$ and $p \not\equiv 1 \pmod{3}$,
 - $t = 0$ and $p \not\equiv 1 \pmod{4}$.

3. Previous Results

We summarize previous results that determine the embedding degree explicitly by trace [4], [6], [13] and Hitt's results [11] in detail. Table 1 presents relations between traces t of elliptic curves over \mathbb{F}_{p^m} and their embedding degrees k , where x are integers.

Hitt [11] investigates Jacobians of genus 2 curves J_C over \mathbb{F}_{2^m} and showed that the embedding degree k of $J_C(\mathbb{F}_{2^m})$ is decided from the order of 2 in \mathbb{Z}_n (where n is the largest prime divisor of $\#J_C(\mathbb{F}_{2^m})$), and that $k < (\log 2^m)^2$. The order of $a \in \mathbb{Z}$ in \mathbb{Z}_n is denoted by $\text{ord}_n(a)$. Here we present Hitt's results.

Theorem 2 ([11]): Let $n = \frac{2^{2^L} + 1}{2^{2^r} + 1}$ be a prime for $\exists r \geq 0$, let $L \geq 5$ be odd, and let k be the embedding degree of $J_C(\mathbb{F}_{2^m})$ with respect to the largest prime divisor n of $\#J_C(\mathbb{F}_{2^m})$, where $1 \leq m \leq 2^r(L - 1) - 1$ or $(m, r) = (\frac{L+1}{2}, 0)$. Then, $k = 2^{r+1-i}$ when $\text{gcd}(\text{ord}_n(2), m) = 2^iL$ ($0 \leq i \leq r - 1$), and $k = 2^{r+1-i}L$ when $\text{gcd}(\text{ord}_n(2), m) = 2^i$ ($0 \leq i \leq r + 1$).

Let us present Lemma shown in [11] that we will use later.

Lemma 1 ([11]): Let m be a positive integer, let p and $n \neq p$ be primes, and let k be the smallest positive integer such that $p^{mk} \equiv 1 \pmod{n}$. Then $k = \frac{\text{ord}_n(p)}{\text{gcd}(\text{ord}_n(p), m)}$.

4. The New Relations

We present the new concrete relations between \mathbb{F}_{p^m} , $\#E(\mathbb{F}_{p^m})$

and the embedding degree of E . The embedding degree of $E(\mathbb{F}_{p^m})$ is determined by order of p in \mathbb{Z}_n , where $n = \#E(\mathbb{F}_{p^m})$ is a prime and m is a positive integer. Our approach is to find such a condition that order of an element in \mathbb{Z}_n can be determined. In fact, we will show that order of an element a in \mathbb{Z}_n can be determined when $n = \frac{a^{2^r L} + 1}{\lambda(a^{2^r} + 1)}$ holds for $r, a, \lambda \in \mathbb{Z}$ ($r, \lambda \geq 0$) and an odd prime L . We will, then, set $a = p$ or $a = t - 1$ for a definition field \mathbb{F}_{p^m} of an elliptic curve and the trace t of $E(\mathbb{F}_{p^m})$ when we apply the following lemmas to decide order of an elliptic curve.

The following lemma determines order of an element a over \mathbb{Z}_n .

Lemma 2: Let $r, a, \lambda \in \mathbb{Z}$ ($r, \lambda \geq 0$) and L be an odd prime. If $n = \frac{a^{2^r L} + 1}{\lambda(a^{2^r} + 1)}$ and $a^{2^r} \not\equiv -1 \pmod{n}$, then $\text{ord}_n(a) = 2^{r+1}L$.

proof: From $n = \frac{a^{2^r L} + 1}{\lambda(a^{2^r} + 1)}$, we have $\lambda(a^{2^r} + 1)n = a^{2^r L} + 1$.

Thus, we get $a^{2^r L} \equiv -1 \pmod{n}$. This implies $a^{2^{r+1}L} \equiv 1 \pmod{n}$, and, thus, $\text{ord}_n(a) \mid 2^{r+1}L$. Since L is prime, we get that either $\text{ord}_n(a) = 2^j$ or $\text{ord}_n(a) = 2^j L$ ($0 \leq j \leq r + 1$).

Suppose that $\text{ord}_n(a) = 2^j L$ ($0 \leq j \leq r$). Then, $a^{2^j L} \equiv 1 \pmod{n}$, so $a^{(2^j L)2^{r-j}} \equiv 1 \pmod{n}$ and, thus, $a^{2^r L} \equiv 1 \pmod{n}$. However, this contradicts the above fact that $a^{2^r L} \equiv -1 \pmod{n}$. Therefore, $\text{ord}_n(a) \neq 2^j L$ ($0 \leq j \leq r$). Similarly, we easily get $\text{ord}_n(a) \neq 2^j$ ($0 \leq j \leq r$).

Suppose that $\text{ord}_n(a) = 2^{r+1}$. From the above fact that $a^{2^r L} \equiv -1 \pmod{n}$, we get the following sequences: $-1 \equiv a^{2^r L} \equiv a^{2^{r+1}} a^{2^r(L-2)} \equiv a^{2^r(L-2)} \equiv a^{2^{r+1}} a^{2^r(L-4)} \equiv \dots \equiv a^{2^r} \pmod{n}$ since L is an odd prime. However this contradicts $a^{2^r} \not\equiv -1 \pmod{n}$.

Therefore, we have proved $\text{ord}_n(a) = 2^{r+1}L$.
From Lemmas 1 and 2, we get the following Lemma.

Lemma 3: Let r, m, λ , and $a \in \mathbb{Z}$ ($r, m, \lambda \geq 0$), let L and n be odd primes, and let k be the smallest positive integer such that $a^{mk} \equiv 1 \pmod{n}$. If $n = \frac{a^{2^r L} + 1}{\lambda(a^{2^r} + 1)}$ and $a^{2^r} \not\equiv -1 \pmod{n}$, then $k = 2^{r+1-i}$ when $\text{gcd}(\text{ord}_n(a), m) = 2^i L$ ($0 \leq i \leq r + 1$), and $k = 2^{r+1-i} L$ when $\text{gcd}(\text{ord}_n(a), m) = 2^i$ ($0 \leq i \leq r + 1$).

proof: From the assumption of $n = \frac{a^{2^r L} + 1}{\lambda(a^{2^r} + 1)}$ and $a^{2^r} \not\equiv -1 \pmod{n}$, we get $\text{ord}_n(a) = 2^{r+1}L$ by Lemma 2. Thus, $\text{gcd}(\text{ord}_n(a), m) \mid \text{ord}_n(a)$ and, therefore, $\text{gcd}(\text{ord}_n(a), m) = 2^i L$ or 2^i ($0 \leq i \leq r + 1$). Lemma 1 says that $k = \frac{\text{ord}_n(a)}{\text{gcd}(\text{ord}_n(a), m)}$. Therefore, we get $k = 2^{r+1-i}$ if $\text{gcd}(\text{ord}_n(a), m) = 2^i L$ ($0 \leq i \leq r + 1$); and $k = 2^{r+1-i} L$ else if $\text{gcd}(\text{ord}_n(a), m) = 2^i$ ($0 \leq i \leq r + 1$).

By applying Lemmas 2 and 3 to a prime p of a definition field \mathbb{F}_{p^m} and trace t of an elliptic curve $E(\mathbb{F}_{p^m})$, we prove the following theorem that describes the relation between embedding degree and order.

Theorem 3: Let $r, m, \lambda, a \in \mathbb{Z}$ ($r, m, \lambda \geq 0$), let L be an odd prime, $D = \text{gcd}(\text{ord}_n(p), m)$, and let k be the embedding degree of $E(\mathbb{F}_{p^m})$. Then, the following two results hold:

1. if $\#E(\mathbb{F}_{p^m}) = \frac{p^{2^r L} + 1}{\lambda(p^{2^r} + 1)} = n$ is a prime and $p^{2^r} \not\equiv -1 \pmod{n}$, then
 - $k = 2^{r+1-i} L$ when $D = 2^i$ ($0 \leq i \leq r + 1$); and
 - $k = 2^{r+1-i}$ when $D = 2^i L$ ($0 \leq i \leq r + 1$);
2. if $\#E(\mathbb{F}_{p^m}) = \frac{(t-1)^{2^r L} + 1}{\lambda((t-1)^{2^r} + 1)} = n$ is a prime and $(t-1)^{2^r} \not\equiv -1 \pmod{n}$, then $k = 2^{r+1}L$.

proof: (1). Apply $a = p$ and $\#E(\mathbb{F}_{p^m}) = n = \frac{p^{2^r L} + 1}{\lambda(p^{2^r} + 1)}$ to Lemma 3. Then k in Lemma 3 is the smallest positive integer such that $p^{mk} \equiv 1 \pmod{n}$. Therefore, the embedding degree k of $E(\mathbb{F}_{p^m})$ is $k = 2^{r+1-i} L$ when $D = 2^i$, and $k = 2^{r+1-i}$ when $D = 2^i L$.

(2). Apply $a = t - 1$, and $\#E(\mathbb{F}_{p^m}) = n = \frac{(t-1)^{2^r L} + 1}{\lambda((t-1)^{2^r} + 1)}$ to Lemma 2. In this case, $t = p^m + 1 - n$, and, thus $t - 1 \equiv p^m \pmod{n}$, which implies that $(t-1)^k \equiv p^{mk} \equiv 1 \pmod{n}$. Thus, we get the embedding degree $k = \text{ord}_n(p^m) = \text{ord}_n(t-1) = 2^{r+1}L$. ■

In the next section, we give two algorithms to find elliptic curve parameters such as a definition field \mathbb{F}_p , order of $\#E(\mathbb{F}_p) = n$, and trace t , which have a pre-determined embedding degree by using Theorem 3 and satisfy Waterhouse's theorem.

5. Searching Algorithm

We give two algorithms that search elliptic-curve parameters satisfying Theorem 3. We also present our experimental results. All experiments were done by using MATHEMATICA.

5.1 The Basic Algorithm

Let $n = \#E(\mathbb{F}_{p^m}) = \frac{p^{2^r L} + 1}{\lambda(p^{2^r} + 1)}$ be a prime, where p and L are odd primes, and r, m and λ are non negative integers. This means that n is a factor of $\Gamma = \frac{p^{2^r L} + 1}{p^{2^r} + 1}$. Before showing the concrete algorithm, we will prove that a non-negative integer m can be restricted by the following Proposition.

Proposition 1: Let $n = \#E(\mathbb{F}_{p^m})$ for an odd prime p and a positive integer m . If $|t|$ satisfies the condition of Waterhouse, then $m = \lfloor \log_p n \rfloor$ or $\lfloor \log_p n \rfloor + 1$.

proof: Let $m' = \lfloor \log_p n \rfloor$. Then $m' + 1 > \log_p n \geq m'$, and, thus, $p^{m'+1} > n \geq p^{m'}$ holds.

First we assume that $m > m' + 1$. Then, $m - 1 > m'$ and, thus, $m - 1 \geq m' + 1$ holds since both m and m' are integers. Then, we get the following since $p \geq 3$ and $m \geq 1$:

$$|t| = |p^m + 1 - n| \geq |p^m - p^{m-1} + 1|$$

$$= (p - 1)p^{m-1} + 1 \geq 2\sqrt{p^m} + 1 > 2\sqrt{p^m}.$$

Next we assume that $m' > m$. Then, in the same way as the above, $m' - 1 \geq m$. Then, the following holds:

$$\begin{aligned} |t| &= |n - p^m - 1| \geq |n - p^{m'-1} - 1| \\ &\geq p^{m'} - p^{m'-1} - 1 = (p - 1)p^{m'-1} - 1 \\ &\geq p^m(p - 1) - 1 > 2\sqrt{p^m} \end{aligned}$$

since $p \geq 3$ and $m \geq 1$.

This contradicts the condition of Waterhouse. From this, we get $m = m'$ or $m' + 1$. ■

Here we present Algorithm 1, which applies $a = p$ to Theorem 3.

Algorithm 1:

Input: An odd prime L and a non negative integer r .

Output: The embedding degree k , order n , definition field p^m , and the trace t .

1. Set an odd prime p .
2. Set $\Gamma = \frac{p^{2^r L} + 1}{p^{2^r} + 1}$.
3. Set a large prime factor of Γ to n .
4. If $p^{2^r} \equiv -1 \pmod{n}$, then return to Step 1.
5. Set $m' = \lfloor \log_p n \rfloor$.
6. If $m' + 1/2 < \log_p n$, then $m = m' + 1$. Else $m = m'$.
7. Set $t = p^m + 1 - n$.
8. If (p^m, n) does not satisfy the condition of Waterhouse, then return to Step 1.
9. Set $k = \frac{2^{r+1}L}{\gcd(2^{r+1}L, m)}$.
10. Output $\{k, n, p^m, t\}$.

Remark 1: Step 3 of Algorithm 1 excludes all small prime factors[†] from Γ and checks whether the rest is prime or not.

Table 2 presents elliptic-curve parameters of p^m, n, t which satisfy Theorem 3 and the condition of Waterhouse. They are constructed by Algorithm 1 for $0 \leq r \leq 1, 3 \leq L \leq 7$ and all 16-bit primes p . The total number of 16-bit primes are to 1649.

Algorithm 1 does not work well, since it often fails in Step 8 for the following reason: In order to execute Step 3,

$$\Gamma = \frac{p^{2^r L} + 1}{p^{2^r} + 1} = p^{2^r(L-1)} - p^{2^r(L-2)} + \dots + 1$$

has to be almost prime with a prime factor n , which implies that $n \approx \Gamma$. On the other hand, $p^m \approx n$ due to the condition of Waterhouse. Therefore, $2^r(L - 1) \approx m$. By combining these facts that $n \approx \Gamma$ and $p^m \approx p^{2^r(L-1)}$, we get that

$$|t| = |p^m + 1 - n| \approx p^{m \frac{L-2}{L-1}} + O(p^{m \frac{L-3}{L-1}}),$$

which induces $t^2 > 4p^m$ if L is large. Therefore, it fails in Step 8. As a result, only limited small L can be used, as we

Table 2 The elliptic-curve parameters and k (Algorithm 1).

p	r	L	k	n	m	t	ρ
71993	0	5	10	72341	1	-347	1
74167	0	5	10	74531	1	-363	1
76207	0	5	10	76441	1	-233	1
81023	0	5	10	81401	1	-377	1
65963	1	3	12	66373	1	-409	1
81119	0	7	14	81677	1	-557	1
81847	0	7	14	82223	1	-375	1
75223	1	5	20	75721	1	-497	1
78031	1	5	20	78121	1	-89	1
83579	1	5	20	83621	1	-41	1

have shown in Table 2.

5.2 The Improved Algorithm

In order to resolve the problem of Algorithm 1, we apply $a = t - 1$ to Theorem 3. Then, the condition of Waterhouse usually holds for the following reason. In this case, $\log n \approx \log t^{2^r(L-1)}$, thus any n, p^m and t satisfy $n \approx t^{2^r(L-1)} \geq t^2$, which roughly implies that $2\sqrt{n} > \sqrt{n} \geq t$. The equality in $t^{2^r(L-1)} \geq t^2$ holds if and only if $(r, L) = (0, 3)$. In this case of $(r, L) = (0, 3)$, we get $\Gamma = t^2 - 3t + 3$, and, thus, $p^m \approx \Gamma + t - 1 = t^2 - 2t + 2 > (\frac{t}{2})^2$ always holds.

Algorithm 2:

Input: An odd prime L and a non negative integer r . (i.e. an embedding degree $k = 2^{r+1}L$.)

Output: Order n , a power of prime p^m , and trace t .

1. Set an odd integer^{††} t as a candidate of trace, where the range of t is defined by the size of elliptic curves that we will need. The range is described below in detail.
2. Set $\Gamma = \frac{(t-1)^{2^r L} + 1}{(t-1)^{2^r} + 1}$.
3. If Γ is not almost prime, then return to Step 1.
4. Set the largest prime factor of Γ to n .
5. If $n + t - 1$ is a power of prime^{†††}, then set $n + t - 1 = p^m$. Else, then return to Step 1.
6. If $(t - 1)^{2^r} \equiv -1 \pmod{n}$, then return to Step 1.
7. Output $\{k, n, p^m, t\}$.

Let us investigate the range of t . Algorithm 2 sets $\#E(\mathbb{F}_{p^m}) = n = \frac{(t-1)^{2^r L} + 1}{\lambda((t-1)^{2^r} + 1)}$, and, thus, $\log p^m \approx 2^r(L -$

[†]The size of small prime factors depend on a system. In our experiment, we set the size of small prime factors to be 16 bits.

^{††}Here we consider only a prime-order elliptic curve and a power of an odd prime p . This is why only an odd integer t is dealt.

^{†††}It is easy to check whether $n + t - 1$ is prime or not by using primality tests [1]. In our experimental results in Section 5.3, we deal with only this case.

Table 3 The number of parameters satisfying with (r, L, k) (Algorithm 2).

160-bit prime p			
r	L	k	$\#\{n, p^m, t\}$
1	3	12	136
2	3	24	84
0	5	10	225
1	5	20	180
0	7	14	135
192-bit prime p			
r	L	k	$\#\{n, p^m, t\}$
1	3	12	91
2	3	24	57
0	5	10	154
1	5	20	119
0	7	14	90
0	11	22	91
224-bit prime p			
r	L	k	$\#\{n, p^m, t\}$
1	3	12	73
2	3	24	41
0	5	10	112
1	5	20	85
0	7	14	75
1	7	28	62
0	11	22	70

1) $\log t$. Therefore, t is set to $\frac{160}{2^r(L-1)}$ bits when we construct 160-bit elliptic curves.

5.3 Experimental Results

Experiments are executed for $m = 1$ and (r, L) as shown in Table 3. Here we set $m = 1$ since an elliptic curve over a prime field \mathbb{F}_p do not suffer from reduction to the minimum embedding degree.

Then, the range of t is determined by the above discussion in Sect. 5.2, where t runs over 100,000 kinds of random $\frac{\log p}{2^r(L-1)}$ -bit integers. We constructed examples with 160, 192, 224-bit primes p .

Table 3 shows the total number of elliptic-curve parameters searched by Algorithm 2 under given values of (m, r, L, k) and each 100,000 kinds of t .

The following are some examples.

160-bit elliptic curves

$k = 10$			
$t = 1285693206491$			
(1)	$p = 273243221$	2182434088	1531032711
		6177600129	4482681101
	$n = 273243221$	2182434088	1531032711
		6177600000	8789474611
$\rho = 1$			

$k = 20$			
$t = 1712607$			
(2)	$p = 180499429$	2421198963	8284539265
		0495241704	2967749987
	$n = 180499429$	2421198963	8284539265
		0495241704	2966037381
$\rho = 1$			

192-bit elliptic curves

$k = 14$			
$t = 7223820963$			
(1)	$p = 1912551$	0159002005	2219431877
	1995972452	4321251430	5346292063
	$n = 1912551$	0159002005	2219431877
	1995972452	4321251429	8122471101
$\rho = 1$			

$k = 28$			
$t = -66881$			
(2)	$p = 24447$	5068599953	5868940457
	2401483697	7022909801	1541952959
	$n = 24447$	5068599953	5868940457
	2401483697	7022909801	1542019841
$\rho = 1$			

224-bit elliptic curves

$k = 12$			
$t = -99243129329168669$			
(1)	$p = 1328859$	1151679357	6485698850
	6600860828	7715683196	7144571636
	$n = 1328859$	1151679357	6485698850
	6600860828	7715683196	7154495949
			9048460837

$\rho = 1$			
$k = 20$			
$t = -419108453$			
(2)	$p = 15605638$	2326675970	2773190814
	4467543753	5277124105	2130428296
	$n = 15605638$	2326675970	2773190814
	4467543753	5277124105	2130428296
			5948659121
$\rho = 1$			

6. Comparison between Our Results and Previous Results

Let us compare our results with previous results. First we discuss our advantage over Hitt's results [11]. Table 4 shows the comparison between our results and [11]. Hitt has investigated Jacobians of genus 2 curves J_C over \mathbb{F}_{2^m} and some examples of the parameters for such curves over \mathbb{F}_{2^m} with embedding degrees $k = 8, 13, 16, 23, 26, 37, 46,$ and 52 . However, Hitt's results cannot construct $\rho = 1$ because the results restrict the relation between trace, definition field, and the largest prime divisor. Furthermore, the results suffer from reduction to the actual minimum embedding degree

Table 4 Comparison of [11] and our results.

	[11]								Ours							
genus	2								1							
definition field \mathbb{F}_q	$q = 2^m$								$q = p^m$ (p : a prime)							
largest prime divisor of $\#J_C(\mathbb{F}_{2^m})$ or $\#E(\mathbb{F}_{p^m})$	$\frac{2^{2^L} + 1}{2^{2^L} + 1}$								$\frac{(t-1)^{2^L} + 1}{\lambda((t-1)^{2^L} + 1)}$							
trace	$(-1, 2^m + 2^{2^m+2^{2^m-2^L}})$								$\forall t \leq q^{1/2^L(L-1)}$							
ρ -value	$\frac{4L}{3(L-1)} \leq \rho \leq 2 - \frac{2}{2^L(L-1)}$								1							
constructed embedding degree k	8	13	16	23	26	37	46	52	10	12	14	20	22	24	28	
actual embedding degree (highest case)	$\frac{8}{3}$	$\frac{13}{5}$	$\frac{16}{7}$	$\frac{23}{8}$	$\frac{26}{9}$	$\frac{37}{13}$	$\frac{46}{17}$	$\frac{52}{19}$	10	12	14	20	22	24	28	

since they work only in the case of \mathbb{F}_{2^m} . As a result, the actual security level of all results is reduced to $\frac{1}{19}$ of the original security level at the minimum and to its $\frac{1}{3}$ at the maximum.

On the other hand, we improve on Hitt's ideas from the point of view of ρ -value, the actual minimum security, and elliptic curves. As for ρ -value, we do not place any restrictions on the relation between trace and definition field. As a result, we can construct elliptic curves with $\rho = 1$. Furthermore, we can enjoy the case of prime field \mathbb{F}_p , and, thus, our results do not suffer from reduction to the minimum embedding degree.

Let us compare our results with Cocks-Pinch and Brezing-Weng curves, which are summarized in [7]. In [7], previous results to construct pairing-friendly elliptic curves are investigated from the point of view of embedding degrees k and ρ -value. Embedding degrees of results shown in Section 3 are currently limited to $k \leq 12$ while they achieve $\rho = 1$. On the other hand, Cocks-Pinch and Brezing-Weng curves exist for arbitrary embedding degree k but usually lead to $\rho > 1$. Actually, Cocks-Pinch curve tends to have ρ -value around 2. Brezing-Weng curve improves Cocks-Pinch curve to have ρ -value less than 2 but it still needs $\rho > 1$. Compared with these previous results, our results can deal with both arbitrary embedding degree k and $\rho = 1$.

7. Conclusion

We have investigated the new concrete relation between trace, definition field \mathbb{F}_{p^m} , embedding degree k , and the largest prime divisor n of $\#E(\mathbb{F}_{p^m})$ of elliptic curves E/\mathbb{F}_{p^m} . Compared with the previous Hitt's result, the new relation discovered by us do not place any restrictions on the relation between trace and definition field. As a result, our result can work even when $\rho = 1$. The case of $\rho = 1$ is the most efficient since the size of definition field \mathbb{F}_{p^m} is equal to that of the largest prime divisor n of $\#E(\mathbb{F}_{p^m})$. We also gave some examples of the new relations in the cases of 160-bit, 192-bit and 224-bit elliptic curves.

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