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Petri Nets as Models of Linear Logic

By Keiko Ishihara

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Professor Kunihiko Hiraishi

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Chapter 1

Introduction

In the years 1960-1962, Carl Adam Petri defined Petri nets which is a general purpose mathematical model for describing relations existing between conditions and events. Petri nets consist of two types of elements, places and transitions. Each place models a process in terms of types of resources, and can hold arbitrary nonnegative multiplicity. Each transition represents a state transition rule, i.e., how those resources are consumed or produced by actions. They are described using the notion of multisets. A multiset over a set P is a function, $m : P \rightarrow \mathcal{N}$ [7], [10].

Linear Logic was discovered by J. Y. Girard in 1987 [3], [4], [13], [14]. Linear logic (intuitionistic, classical and predicate) are obtained by deleting the contraction and the weakening rules from standard sequent calculus formulations of corresponding logics. In the Gentzen sequent calculus for intuitionistic logic, a sequent $A_1, \cdots, A_n \rightarrow A$ is written to mean that the formula A is deducible from the assumption formulas $A_1, \cdots, A_n$ (we shall use capital Greek letters as an abbreviation for a sequence of formulas). The calculus has the two structural rules for adding a vacant assumption and removing of a duplicate of assumption.

$$\frac{\Gamma \rightarrow B}{\Gamma, A \rightarrow B} \text{(weakening)},$$

$$\frac{\Gamma, A, A \rightarrow B}{\Gamma, A \rightarrow B} \text{(contraction)}.$$

In the presence of these rules the following two right introduction rules for conjunction

$$\frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \wedge B} (1),$$

$$\frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B} (2)$$

become interderivable in the sense that the first rule can be derived from the second by weakening, and the second from the first by contraction. In linear intuitionistic logic these rules (weakening and contraction) are deleted and the rule of (1) and (2) are no longer interderivable. Without them, propositions cannot be introduced arbitrarily into a list of assumption and a duplication in the list cannot be removed. It is in this sense that linear logic is a *resource conscious* logic.

The connection between linear logic and Petri nets has recently been the subject of great interest [2], [5], [6]. Girard's linear logic has a great deal of interest in how might be useful in the theory of parallelism. The places are like atomic propositions in linear logic and transitions like provability relation. Girard's phase semantics for linear logic in [3] uses quantales [1], [9], [11], [12], [15], and Engberg and Winskel [2] showed a straightforward way in which a Petri net induces a quantale and so becomes a model for intuitionistic linear logic. But they did not prove a completeness theorem for models induced by Petri nets.

In this thesis, we prove completeness for quantales generated by Petri nets. To prove completeness the quantales used in [2] do not work. Although the following proof shows that

$$(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$$

is derivable in intuitionistic linear logic,

$$\dfrac{\dfrac{\dfrac{A \to A}{A \wedge C \to A}(\wedge \to) \quad \dfrac{\dfrac{B \to B}{B \to B \vee C}(\to \vee)}{A \wedge B \to B \vee C}(\wedge \to)}{A \wedge B \to A \wedge (B \vee C)}(\to \wedge) \quad \dfrac{\dfrac{A \to A}{A \wedge C \to A}(\wedge \to) \quad \dfrac{\dfrac{C \to C}{C \to B \vee C}(\to \vee)}{A \wedge C \to B \vee C}(\wedge \to)}{A \wedge C \to A \wedge (B \vee C)}(\to \wedge)}{(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)}(\vee \to),$$

we cannot prove the sequent

$$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C).$$

That is, the *distributivity* of $\wedge$ and $\vee$ does not hold in intuitionistic linear logic. In the quantale given in [2], the distributivity is always valid. Therefore, if we want to prove completeness using the quantales of [2], then we have to add the distributivity to intuitionistic linear logic. However this is not what we intend to do. We construct quantales in which the distributivity is not always valid, and prove completeness. We can also prove completeness of intuitionistic linear logic with a modal operator for quantales by using similar construction.

In Chapter 2, we overview Petri nets and algebraic structures [8]. We introduce algebras including quantales, and closure operations on the algebras which play a crucial role in the proof of completeness.

In Chapter 3, we discuss intuitionistic linear logic (its syntax and semantics) and then prove soundness theorem for quantales generated by Petri nets. Next we show why we cannot prove completeness in the quantales used in [2], and then prove completeness using the quantale based on our construction.

In Chapter 4, we discuss a modal operator of course !. The absence of the rules for weakening and contraction is compensated, to some extent, by the addition of the modal operator !. We consider a semantics with the modal operator using similar construction, and then prove completeness of intuitionistic linear logic with the modal operator for quantales generated by Petri nets.

In Chapter 5, we consider classical quantales for classical linear logic generated by Petri nets.

Chapter 2

Petri Nets and Algebraic Structures

In this chapter, we will discuss Petri nets and algebras including quantales, and introduce closure operations on the algebras.

2.1 Basic Structures

2.1.1 Multisets

Definition 2.1.1 (multiset)

1. *Multiset* over a set P is a mapping $f : P \to \mathcal{N}$, where $f(a) = n$ means that a occurs with multiplicity n,

2. operation $+$ on multisets is defined by $(m + m')(a) = m(a) + m'(a)$ for all $a \in P$,

3. $\underline{\emptyset}$ is the empty multiset.

Example 2.1.2 We shall denote the set of all finite multisets by $\mathcal{M}$. We shall use $\{\cdots\}$ for a set and $\underline{\cdots}$ for a multiset. Let a and b be elements of P, then

- $\{a\}$, $\{b\}$, $\{a, b\}$, $\cdots$ are sets and

$$\{a\} = \{a, a\} \text{ and } \{a\} \cup \{a, b\} = \{a, b\},$$

- $\underline{a}$, $\underline{b}$, $\underline{a, b}$, $\cdots$ are multisets and

$$\underline{a} \neq \underline{a, a} \text{ and } \underline{a} + \underline{a, b} = \underline{a, a, b}.$$

2.1.2 Ordered Structures and Monoids

Definition 2.1.3 (monoid) A structure $\mathbf{M} = \langle X, \cdot, e \rangle$ is a *monoid* with the identity e if $\cdot$ is a binary operation on X and e is an element of X such that for every $a, b, c \in X$,

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,

2. $a \cdot e = e \cdot a = a$.

Remark 2.1.4 When the structure satisfies only Definition 2.1.3 1., it is called a *semigroup.*

Definition 2.1.5 (commutative monoid) A structure $\mathbf{M} = \langle X, \cdot, e \rangle$ is a *commutative monoid* with the identity e if

1. $\langle X, \cdot, e \rangle$ is a monoid,
2. $a \cdot b = b \cdot a$ for every $a, b \in X$.

Proposition 2.1.6 *A structure* $\mathbf{M} = \langle \mathcal{M}, +, \underline{\emptyset} \rangle$ *is a commutative monoid.*

Definition 2.1.7 (preordered set) A structure $\mathbf{X} = \langle X, \leq \rangle$ is *a preordered set* if, $\leq$ is a binary relation on X such that for every $x, y, z \in X$,

1. $x \leq x$,
2. $x \leq y, \ \ y \leq z \rightarrow x \leq z$.

Definition 2.1.8 (preordered commutative monoid) A structure $\mathbf{X} = \langle X, \leq, \cdot, e \rangle$ is a *preordered commutative monoid* if,

1. $\langle X, \cdot, e \rangle$ is a commutative monoid,
2. $\langle X, \leq \rangle$ is a preordered set,
3. $x \leq x', y \leq y' \Rightarrow x \cdot y \leq x' \cdot y'$ for every $x, x', y, y' \in X$.

2.1.3 Lattices

Definition 2.1.9 (infimum & supremum) Let X be a partially ordered set. Then an element a of X is said to be a *lower bound* (*upper bound*) of a subset E of X if $a \leq x$ ($x \leq a$) for every x in E (respectively). Let E_* (E^*) be the set of all lower bounds (upper bounds) of E in X (respectively).

1. If it happens that the set E_* contains a greatest element a (necessarily unique), then a is called the *infimum* of E;denoted by $\inf E$.
2. If it happens that the set E^* contains a least element a (necessarily unique), then a is called the *supremum* of E;denoted by $\sup E$.

$\top$ and $\bot$ are defined as follows:

- $\top$ is the grestest element,
- $\bot$ is the least element.

Definition 2.1.10 (lattice) A structure $\mathbf{L} = \langle L, \cup, \cap \rangle$ is a *lattice* if, $\cup, \cap$ are binary operations on L such that for every $a, b, c \in L$,

1. $a \cup a = a$, $a \cap a = a$,

2. $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$,

3. $a \cup b = b \cup a$, $a \cap b = b \cap a$,

4. $a \cup (a \cap b) = a$, $a \cap (a \cup b) = a$.

Definition 2.1.11 (complete lattice) A structure $\mathbf{L} = \langle L, \cup \rangle$ is *a complete lattice* if $\cup$ is a mapping of $\mathcal{P}(X)$ into X such that

1. $y \leq \cup Y$ for all $y \in Y$,

2. if $y \leq z$ for all $y \in Y$, then $\cup Y \leq z$.

That is, $\cup Y$ is the supremum of Y.

Remark 2.1.12 We can define a mapping $\cap$ the infimum of $\mathcal{P}(X)$ into X by

$$\cap Y := \cup \{z | z \leq y \ \ for \ \ all \ \ y \leq Y\}.$$

In a complete lattice $\langle X, \cup \rangle$, the maxumum and the minimum element exist: in fact

$$\top := \cup X, \ \ \bot := \cap X.$$

$a \cup b$ and $a \cap b$ denote $\cup\{a, b\}$ and $\cap\{a, b\}$, respectively.

Remark 2.1.13 A lattice does not satisfy the distributivity of $\cup$ and $\cap$ in general, i.e.

- $(a \cap b) \cup (a \cap c) \leq a \cap (b \cup c)$,
- $(a \cup b) \cap (a \cup c) \leq a \cup (b \cap c)$.

Example 2.1.14 For a given set U, a structure $\mathbf{P(U)} = \langle \mathcal{P}(U), \cup, \cap \rangle$ forms a complete lattice, where $\cup$ and $\cap$ are usual set union and intersection.

2.2 Petri nets

Definition 2.2.1 (Petri net) *A Petri net* N is a quadruple $\langle P, T, ^\bullet(-), (-)^\bullet \rangle$ such that

1. P is a set (of places),

2. T is a set (of transitions),

3. $^\bullet(-), (-)^\bullet$ are mapping of T into $\mathcal{M}$, for $t \in T$, $^\bullet(t)$ and $(t)^\bullet$ are called pre multiset and post multiset of t respectively.

Definition 2.2.2 (reachability relation) Let $N = \langle P, T, ^\bullet(-), (-)^\bullet \rangle$ be a Petri net. Then we define a relation$\rhd$ on $\mathcal{M}$ called *the reachability relation* of N as follows:

1. For $t \in T$, let $[t\rangle$ be a relation on $\mathcal{M}$ defined by

$$m\ [t\rangle\ m' \ \Leftrightarrow \ \exists m'' \in \mathcal{M}.\ m = m'' +^{\bullet} t \ \textit{and}\ t^{\bullet} + m'' = m'.$$

2. Then we define $\rhd$ by

$$m \rhd m' \Leftrightarrow \exists t_1, t_2, \cdots, t_n \in T, \; m_1, m_2, \cdots, m_n \in \mathcal{M}, n \geq 0. \; m\ [t_1\rangle\ m_1\ [t_2\rangle\ m_2\ [t_3\rangle\ \cdots\ [t_n\rangle\ m_n = m'.$$

Proposition 2.2.3 *A structure* $M_N = \langle \mathcal{M}, \rhd, +, \underline{\emptyset} \rangle$ *is a preordered commutative monoid.*

Proof. In fact, we can show that a structure defined as above satisfies the conditions of Definition 2.1.8,for every $m, m', m'' \in \mathcal{M}$,

1. $m + (m' + m'') = (m + m') + m''$,
2. $m + \underline{\emptyset} = \underline{\emptyset} + m = m$,
3. $m + m' = m' + m$,
4. a structure $\langle \mathcal{M}, \rhd \rangle$ holds Definition 2.1.7 since
   (a) $m \rhd m$,
   (b) if $m \rhd m'$ and $m' \rhd m''$, then $m \rhd m''$,
5. if $x \rhd x', y \rhd y'$ then $x + y \rhd x' + y'$.

2.3 IL-algebras and Quantales

2.3.1 IL-algebras

Definition 2.3.1 (IL-algebra) [8] A structure $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1, \top, \bot \rangle$ is an IL-*algebra* if

1. $\langle A, \cup, \cap, \top, \bot \rangle$ is a *lattice* with the greatest element $\top$ and the least element $\bot$ for which $\top = \bot \rightarrow \bot$ holds ,
2. $\langle A, \bullet, 1 \rangle$ is a *commutative monoid*,
3. $z \bullet (x \cup y) \bullet w = (z \bullet x \bullet w) \cup (z \bullet y \bullet w)$ for every $x, y, z, w \in A$,
4. $x \bullet y \leq z \Leftrightarrow x \leq y \Rightarrow z$ for every $x, y, z \in A$.

Definition 2.3.2 (complete IL-algebra) A structure $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1\rangle$ is a *complete* IL-*algebra* if

1. $\langle A, \cup, \cap\rangle$ is a complete lattice,
2. $\langle A, \bullet, 1\rangle$ is a commutative monoid,
3. $(\bigcup x_i) \bullet y = \bigcup(x_i \bullet y)$ for every $x_i, y \in A$,
4. $x \bullet y \leq z \Leftrightarrow x \leq y \Rightarrow z$ for every $x, y, z \in A$.

Proposition 2.3.3 *Let* $\mathbf{M} = \langle M, \cdot, e\rangle$ *be a commutative monoid with the identity* e*, and for each* $X, Y \subseteq M$*, define sets* $X \bullet Y$ *and* $Y \Rightarrow Z$ *of* M *by*

1. $X \bullet Y := \{x \cdot y | x \in X, y \in Y\}$,
2. $Y \Rightarrow Z := \{x \in M | \forall y \in Y(x \cdot y \in Z)\}$.

Then the structure

$$\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \Rightarrow, \cup, \cap, \bullet, \{e\}\rangle$$

is a complete IL-algebra.

Proof. 1. Definition 2.3.1 1 and 2 are trivial.

2. We show that $(\bigcup X_i) \bullet Y = \bigcup(X_i \bullet Y)$.

   ($\Rightarrow$). Let $x \in (\bigcup X_i) \bullet Y$. Then $\exists y \in \bigcup X_i$, $\exists z \in Y$ $(x = y \bullet z)$. Therefore $\exists i$ $y \in X_i$, and so $x \in X_i \bullet Y \subseteq \bigcup(X_i \bullet Y)$.
   ($\Leftarrow$). Let $X_j \subseteq \cup X_i$. Then $X_j \bullet Y \subseteq (\bigcup X_i) \bullet Y$. Therefore $\bigcup_j(X_j \bullet Y) \subseteq (\cup X_i) \bullet Y$.

3. We show that $X \bullet Y \subseteq Z \Leftrightarrow X \subseteq Y \Rightarrow Z$.

   ($\Rightarrow$). If $x \in X$. Then $\forall y \in Y$ $(x \cdot y \in X \bullet Y)$. Therefore $x \cdot y \in Z$, and so $x \in Y \Rightarrow z$.
   ($\Leftarrow$). For every $x \in X$ and $y \in Y$, $X \subseteq Y \Rightarrow Z$. Therefore $x \cdot y \in Z$, and hence $X \bullet Y \subseteq Z$.

2.3.2 Quantales

Definition 2.3.4 (commutative quantale) [11] [12] [15] A structure $\mathbf{Q} = \langle Q, \cup, \bullet, 1\rangle$ is *a commutative quantale* if

1. $\langle Q, \cup\rangle$ is a complete lattice,
2. $\langle Q, \bullet, 1\rangle$ is a commutative monoid,
3. $(\bigcup x_i) \bullet y = \bigcup(x_i \bullet y)$ for every $x_i, y \in Q$.

Remark 2.3.5 Define a binary operation on $\mathbf{Q}$ by

$$y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}.$$

Then $x \leq y \Rightarrow z$ if and only if $x \bullet y \leq z$.

Proposition 2.3.6 *Let* $\mathbf{M} = \langle M, \cdot, e\rangle$ *be a commutative monoid with the identity* e*, and for each* $X, Y \subseteq M$*, define a subset* $X \bullet Y$ *of* M *by*

$$X \bullet Y := \{x \cdot y | x \in X, y \in Y\}.$$

Then the structure

$$\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\}\rangle$$

is a commutative quantale.

Proof.

1. Definition 2.3.4 1. and 2. are trivial.
2. We show that $(\bigcup X_i) \bullet Y = \bigcup(X_i \bullet Y)$.

   $(\Rightarrow)$. Let $x \in (\bigcup X_i) \bullet Y$. Then $\exists y \in \bigcup X_i$, $\exists z \in Y$ $(x = y \bullet z)$. Therefore $\exists i$ $y \in X_i$, and so $x \in X_i \bullet Y \subseteq \bigcup(X_i \bullet Y)$.
   $(\Leftarrow)$. Let $X_j \subseteq \bigcup X_i$. Then $X_j \bullet Y \subseteq (\bigcup X_i) \bullet Y$. Therefore $\bigcup_j(X_j \bullet Y) \subseteq (\bigcup X_i) \bullet Y$.

Remark 2.3.7

$$Y \Rightarrow Z := \bigcup\{X | X \bullet Y \subseteq Z\} = \{x \in M | \forall y \in Y (x \cdot y \in Z)\}.$$

Remark 2.3.8 It is easy to show that a complete IL-algebra is just a commutative quantale, in which $y \Rightarrow z$ is defined by

$$y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}.$$

Proposition 2.3.9 *A structure is a complete IL-algebra if and only if it is a commutative quantale.*

Proof. $(\Rightarrow)$. is trivial.
$(\Leftarrow)$. Define $y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}$. Then we show that commutative quantale is always complete IL-algebra.

1. Definition 2.3.1 1., 2. and 3. are trivial.
2. We show that $x \bullet y \leq z \Leftrightarrow x \leq y \Rightarrow z$.

   Let $y \Rightarrow z := \bigcup\{u | u \bullet y \leq z\}$.

($\Rightarrow$). If $x \bullet y \leq z$, then $x \in \{u | u \bullet y \leq z\}$, and hence $x \leq y \Rightarrow z$.

($\Leftarrow$). If $x \leq y \Rightarrow z$, then

$$\begin{aligned} x \bullet y \leq (y \Rightarrow z) \bullet y &= (\bigcup\{u | u \bullet y \leq z\}) \bullet y \\ &= \bigcup\{u \bullet y | u \bullet y \leq z\} \leq z, \end{aligned}$$

and hence $x \bullet y \leq z$.

Corollary 2.3.10 *A structure* $\mathbf{P}(M_N) = \langle \mathcal{P}(\mathcal{M}), \cup, \bullet, \{\underline{\emptyset}\} \rangle$ *is a commutative quantale, where* $X \bullet Y = \{m + m' | m \in X, m' \in Y\}$.

2.3.3 Closure Operations on Quantales

Definition 2.3.11 (closure operation) An operation C on a commutative quantale $\mathbf{Q} = \langle Q, \cup, \bullet, 1 \rangle$ is a closure operation on Q if

1. $x \leq Cx$,
2. $x \leq y \Rightarrow Cx \leq Cy$,
3. $CCx \leq Cx$,
4. $Cx \bullet Cy \leq C(x \bullet y)$.

An element x of Q is *C-closed* if $x = Cx$ holds. $C(Q)$ denotes the set of all *C-closed* elements of Q.

Proposition 2.3.12 *If C is a closure operation on a commutative quantale* $\mathbf{Q} = \langle Q, \cup, \bullet, 1 \rangle$, *then*

$$\mathbf{C}(\mathbf{Q}) = \langle C(Q), \cup_C, \bullet_C, C1 \rangle$$

is also a commutative quantale, where $\cup_C$ and $\bullet_C$ are defined by

1. $\cup_C x_i := C(\cup x_i)$,
2. $x \bullet_C y := C(x \bullet y)$.

Proof. We show that a structure $\mathbf{C}(\mathbf{Q}) = \langle C(Q), \cup_C, \bullet_C, C1 \rangle$ defined as above holds Definition 2.3.4.

1. $\langle C(Q), \cup_C \rangle$ is a complete lattice.
   First we show that if $a, b \in C(Q)$, then $a \cap b \in C(Q), a \cup_C b \in C(Q), a \Rightarrow b \in C(Q), \top \in C(Q)$ and $C\bot \in C(Q)$.
   (a) If $a \cap b \leq a$, then $C(a \cap b) \leq Ca = a$. If $a \cap b \leq b$, then $C(a \cap b) \leq Cb = b$. Therefore $C(a \cap b) \leq a \cap b \leq C(a \cap b)$, and so $a \cap b \in C(Q)$.

(b) $C(a \cup_C b) = C(C(a \cup b)) = C(a \cup b) = a \cup_C b$. Therefore $a \cup_C b \in C(Q)$.

(c) $C(a \Rightarrow b) \bullet a = C(a \Rightarrow b) \bullet Ca \leq C((a \Rightarrow b) \bullet a) = b$. Therefore $C(a \Rightarrow b) \leq a \Rightarrow b \leq C(a \Rightarrow b)$, and so $a \Rightarrow b \in C(Q)$.

(d) $\top \in C(Q)$ since $C\top \leq \top \leq C\top$.

(e) Since $\forall a \in Q \ (\bot \leq a)$, by definition $C\bot \leq Ca = a$.

Next we show that a structure $\langle C(Q), \cup_C \rangle$ is a lattice and holds Definition 2.1.11. It is enough to prove for $\cup_C$.

(a) For $a \cup_C a = a$, $a \cup_C a = C(a \cup a) = Ca = a$.

(b) For $a \cup_C (b \cup_C c) = (a \cup_C b) \cup_C c$, $a \cup_C (b \cup_C c) = C(a \cup C(b \cup c)) = C(Ca \cup C(b \cup c))$
$\leq C(C(a \cup (b \cup c))) = C(a \cup (b \cup c))$
$\leq C(a \cup C(b \cup c)) = a \cup_C (b \cup_C c)$. Therefore $a \cup_C (b \cup_C c) = C(a \cup (b \cup c)) = C((a \cup b) \cup c) = (a \cup_C b) \cup_C c$.

(c) For $a \cup_C b = b \cup_C a$, $a \cup_C b = C(a \cup b) = C(b \cup a) = b \cup_C a$.

(d) For $a \cap (a \cup_C b) = a$, $a \geq a \cap (a \cup_C b) = a \cap C(a \cup b) \geq a \cap (a \cup b) = a$. For $a \cup_C (a \cap b) = a$, $a \cup_C (a \cap b) = C(a \cup (a \cap b)) = C(a) = a$.

2. $\langle C(Q), \bullet_C, C1 \rangle$ is a commutative monoid.
First we show that $C(Cx \bullet Cy) = C(x \bullet y)$.
By definition $x \bullet y \leq Cx \bullet Cy$, and hence $C(x \bullet y) \leq C(Cx \bullet Cy)$. Therefore $C(Cx \bullet Cy) \leq C(C(x \bullet y)) = C(x \bullet y)$, and so $C(Cx \bullet Cy) = C(x \bullet y)$. We show that a structure $\langle C(Q), \bullet_C, C1 \rangle$ holds Definition 2.1.3.
For every $a, b, c \in C(Q)$,

(a) For $a \bullet_C (b \bullet_C c) = (a \bullet_C b) \bullet_C c$, $a \bullet_C (b \bullet_C c) = C(a \bullet C(b \bullet c)) = C(Ca \bullet C(b \bullet c))$
$= C(a \bullet (b \bullet c)) = C((a \bullet b) \bullet c) = C(C(a \bullet b) \bullet Cc)$
$= C(C(a \bullet b) \bullet c) = (a \bullet_C b) \bullet_C c$.

(b) For $a \bullet_C b = b \bullet_C a$, $a \bullet_C b = C(a \bullet b) = C(b \bullet a) = b \bullet_C a$.

(c) For $a \bullet_C C1 = C1 \bullet_C a = Ca$, $C1 \bullet_C a = C(C1 \bullet a) = C(C1 \bullet Ca)$
$= C(1 \bullet a) = C(a) = a$.

3. $(\bigcup_C S) \bullet_C b = C(C(\bigcup S) \bullet b) = C(C(\bigcup S) \bullet Cb) = C(\bigcup S \bullet b) = C(\bigcup_{a \in S} (a \bullet b)) = C(\bigcup_{a \in S} C(a \bullet b)) = \bigcup_{a \in S} {}_C(a \bullet_C b)$. Therefore for every $S \subseteq C(Q)$ and $a \in S$, $(\bigcup_C S) \bullet_C b = \bigcup_C (a \bullet_C b)$.

4. $(\Rightarrow)$. $a \bullet b = Ca \bullet Cb \leq C(a \bullet b) = a \bullet_C b \leq c$, and hence $a \leq b \Rightarrow c$.
$(\Leftarrow)$. If $a \leq b \Rightarrow c$, then $a \bullet b \leq c$. Therefore $a \bullet_C b = C(a \bullet b) \leq C(c) = c$, and hence $a \bullet_C b \leq c$. Therefore for every $a, b, c \in C(A)$, $a \bullet_C b \leq c$ if and only if $a \leq b \Rightarrow c$.

Proposition 2.3.13 *Let* $\mathbf{M} = \langle M, \leq, \cdot, e \rangle$ *be a preordered commutative monoid and define an operation* $\downarrow$ *on* $\mathbf{P}(\mathbf{M})$ *by*

$$\downarrow X := \{y \in M | \exists x \in X \ \ (y \leq x)\}.$$

Then $\downarrow$ *is a closure operation on the commutative quantale* $\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\} \rangle$.

Definition 2.3.14 Let $\mathbf{M} = \langle M, \leq, \cdot, e \rangle$ be a preordered commutative monoid and define an operation C_1 on $\mathbf{P}(\mathbf{M})$ by

$$C_1 X :=\downarrow X.$$

Definition 2.3.15 Let $\mathbf{M} = \langle M, \leq, \cdot, e \rangle$ be an preordered commutative monoid and define two operations $\rightarrow$ and $\leftarrow$ on $\mathcal{P}(M)$ by

$$X^{\rightarrow} := \{y \in M | \forall x \in X (x \leq y)\},$$

$$X^{\leftarrow} := \{y \in M | \forall x \in X (y \leq x)\},$$

and let

$$C_2 X := (X^{\rightarrow})^{\leftarrow}.$$

Proposition 2.3.16 C_1 *is a closure operation on the commutative quantale* $\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\} \rangle$.

Proof. We show that a function C defined as above holds Definition 2.3.11.

1. If $x \in X$, then $x \leq x$, and hence $\exists y \in X \ (x \leq y)$. Therefore $x \in C_1 X$, and so $X \subseteq C_1 X$.

2. If $z \in C_1 X$, then $\exists x \in X \ (z \leq x)$, and since $X \subseteq Y$, so $\exists x \in Y \ (z \leq x)$. Therefore $z \in C_1 Y$, and hence $C_1 X \subseteq C_1 Y$.

3. If $x \in C_1 C_1 X$, then $\exists y \in C_1 X \ (x \leq y)$ and $\exists z \in X \ (y \leq z)$. Therefore $x \leq z$, so $x \leq C_1 X$, and hence $C_1 C_1 X \subseteq C_1 X$.

4. We show that if $x \in C_1 X, y \in C_1 Y$, then $x * y \in C_1(X \bullet Y)$. Since $x \in C_1 X$, $\exists x' \in X \ (x \leq x')$. Since $y \in C_1 Y$, $\exists y' \in Y \ (y \leq y')$. By definition $x * y \leq x' * y'$, so $x * y \in X \bullet Y$, $\exists x' * \exists y' \in X \bullet Y \ (x * y \leq x' * y')$. Therefore $x * y \in C_1(X \bullet Y)$, and hence $C_1 X \bullet C_1 Y \subseteq C_1(X \bullet Y)$.

Proposition 2.3.17 C_2 *is a closure operation on the commutative quantale* $\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\} \rangle$.

Proof. We show that a function C defined as above holds Definition 2.3.11.

1. If $x \in X$, then $\forall y \in X^{\rightarrow} \ (x \leq y)$. Therefore $x \in (X^{\rightarrow})^{\leftarrow} = C_2 X$, and hence $X \subseteq C_2 X$.

2. For $X \subseteq Y \Rightarrow Y^{\rightarrow} \subseteq X^{\rightarrow} \cdots (1)$, if $z \in Y^{\rightarrow}$, then $\forall y \in Y \ (y \leq z)$. Therefore $\forall x \in X \ (x \leq z)$, and hence $z \in X^{\rightarrow}$.
   For $X \subseteq Y \Rightarrow Y^{\leftarrow} \subseteq X^{\leftarrow} \cdots (2)$, if $z \in Y^{\leftarrow}$, then $\forall y \in Y \ (z \leq y)$. Therefore $\forall x \in X \ (z \leq x)$, and hence $z \in X^{\leftarrow}$.
   By (1) and (2), if $X \subseteq Y$, then $Y^{\rightarrow} \subseteq X^{\rightarrow}$. Therefore $(X^{\rightarrow})^{\leftarrow} \subseteq (Y^{\rightarrow})^{\leftarrow}$, and hence $C_2X \subseteq C_2Y$.

3. By 1. $X \subseteq (X^{\rightarrow})^{\leftarrow}$, and by 2. $((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} \subseteq X^{\rightarrow} \cdots (1)$.
   If $x \in X^{\rightarrow}$, then by definition $\forall y \in (X^{\rightarrow})^{\leftarrow} \ (y \leq x)$. Therefore $x \in ((X^{\rightarrow})^{\leftarrow})^{\rightarrow}$, and hence $X^{\rightarrow} \subseteq ((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} \cdots (2)$.
   By (1) and (2) $((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} = X^{\rightarrow}$. Therefore $C_2C_2X = ((C_2X)^{\rightarrow})^{\leftarrow} \subseteq (X^{\rightarrow})^{\leftarrow} = C_2X$, and since $(C_2X)^{\rightarrow} = X^{\rightarrow}$,then $C_2C_2X = C_2X$.

4. We show that if $x \in C_2X, y \in C_2Y$, then $x * y \in C_2(X \bullet Y)$.
   Suppose $z \in (X \bullet Y)^{\rightarrow}$. Then $\forall u \in X, \ \forall v \in Y \ (u * v \leq z)$, and hence $u \leq v \rightarrow z \ (\forall u \in X)$.
   Since $u \in X$ is arbitrary, so $v \rightarrow z \in X^{\rightarrow} \ (\forall v \in Y)$. Therefore $x \leq v \rightarrow z \ (\forall v \in Y)$, so $v * x = x * v \leq z \ (\forall v \in Y)$, and hence $v \leq x \rightarrow z \ (\forall v \in Y)$.
   Since $v \in Y$ is arbitrary, so $x \rightarrow z \in Y^{\rightarrow}$, and hence $y \leq x \rightarrow z$, so $x*y = y*x \leq z$. Therefore $x * y \in ((X \bullet Y)^{\rightarrow})^{\leftarrow}$, and hence $C_2X \bullet C_2Y \subseteq C_2(X \bullet Y)$.

Remark 2.3.18 C_1 is used in [2]. In the commutative quantales constructed from Petri nets using C_1, since C_1 closed sets are downwards closed,

$$\downarrow m \subseteq \downarrow m' \Leftrightarrow m \rhd m'.$$

Therefore C_1 adequates for the reachability relation of Petri nets. Also C_2 closed sets are downwards closed, hence it adequates for the reachability relation.

Lemma 2.3.19

$$C_2(\{x\}) = \{y \mid y \leq x\}.$$

Proof.

- $(\Rightarrow)$. If $m \in C_2(\{\underline{A}\})$, and since $\leq$ is reflexive, then $\underline{A} \in \{\underline{A}\}^{\rightarrow}$, and hence $m \leq \underline{A}$.
- $(\Leftarrow)$. Suppose $m \leq \underline{A}$. If $m' \in \underline{A}^{\rightarrow}$, then $\underline{A} \leq m'$. Since $\leq$ is transitive, then $m \leq m'$, and hence $m \in (\{\underline{A}\}^{\rightarrow})^{\leftarrow} = C_2(\{\underline{A}\})$.

2.3.4 Modal-Operators

Definition 2.3.20 A structure $\mathbf{Q} = \langle Q, \cup, \bullet, !, 1 \rangle$ is a modal quantale if

- $\langle Q, \cup, \bullet, 1 \rangle$ is a quantale,
- $!$ is a unary operation on Q such that

1. $!a \leq a$,
2. $!1 = 1$,
3. $!a \leq !!a$,
4. $!(a \cap b) = !a \bullet !b$.

Lemma 2.3.21 In every modal quantale, the following holds:

1. $!a \leq 1$,
2. $!a \leq !a \bullet !a$,
3. $!a \bullet !b \leq !(!a \bullet !b)$,
4. $a \leq b \Rightarrow !a \leq !b$.

We remark here that, in every quantale, condition 4. in Definition 2.3.20 is equivalent to the conjunction of 1., 2., 3. and 4., under the assumptions 1. to 3. of Definition 2.3.20.

Definition 2.3.22 A modal classical quantale is a classical quantale with additional unary operation ? satisfying:

1. $!(a \rightarrow b) \leq ?a \rightarrow ?b$,
2. $a \leq ?a$,
3. $?0 = 0$,
4. $??a \leq ?a$,
5. $0 \leq ?a$.

Lemma 2.3.23 In every modal classical quantale, the following holds:

1. $a \leq b \Rightarrow ?a \leq ?b$,
2. $!a * b \leq ?c \Rightarrow !a * ?b \leq ?c$.

2.4 CL-algebras and Classical Quantales

2.4.1 CL-algebras

Definition 2.4.1 (complete CL-algebra) A structure $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1, 0 \rangle$ is a complete CL-algebra if

1. $\langle A, \cup, \cap \rangle$ is a complete lattice,
2. $\langle A, \bullet, 1 \rangle$ is a commutative monoid,

3. $(\bigcup x_i) \bullet y = \bigcup (x_i \bullet y)$ for every $x_i, y \in A$,

4. $x \bullet y \leq z \Leftrightarrow x \leq y \Rightarrow z$ for every $x, y, z \in A$,

5. $\sim\sim x = x (\sim x := x \Rightarrow 0)$.

Definition 2.4.2 Let $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1, \rangle$ be a complete IL-algebra. For an element $0 \in A$, we will define $\mathcal{K}$ below:

$$\mathcal{K}x := (x \Rightarrow 0) \Rightarrow 0.$$

Proposition 2.4.3 *A given complete IL-algebra* $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1 \rangle$*, a mapping* $\mathcal{K}$ *from* A *to* A *is a closure operation on* $\mathbf{A}$*.*

Proof. For a given IL-algebra $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1 \rangle$, a mapping $\mathcal{K}$ from A to A is a closure operation on $\mathbf{A}$ if

1. $x \leq \mathcal{K}x$,

2. $x \leq y \Rightarrow \mathcal{K}x \leq Ky$,

3. $\mathcal{K}\mathcal{K}x \leq \mathcal{K}x$,

4. $\mathcal{K}x \bullet \mathcal{K}y \leq \mathcal{K}(x \bullet y)$.

Let $f(x) := \mathcal{K}x \ (= \ (x \Rightarrow 0) \Rightarrow 0)$.

1. We show $x \leq f(x)$.

   Since $x \Rightarrow 0 \leq x \Rightarrow 0$ then $(x \Rightarrow 0) \bullet x \leq 0$.

   Terefore $x \bullet (x \Rightarrow 0) \leq 0$ and hence $x \leq (x \Rightarrow 0) \Rightarrow 0$.

2. We show $x \leq y \Rightarrow f(x) \leq f(y)$.

   Since $y \Rightarrow 0 \leq y \Rightarrow 0$ then $(y \Rightarrow 0) \bullet y \leq 0$.

   Therefore $(y \Rightarrow 0) \bullet x \leq 0$ and hence

   $$y \Rightarrow 0 \leq x \Rightarrow 0 \cdots (1).$$

   Since $(x \Rightarrow 0) \Rightarrow 0 \leq (x \Rightarrow 0) \Rightarrow 0$ then

   $$((x \Rightarrow 0) \Rightarrow 0) \bullet (x \Rightarrow 0) \leq 0,$$

   and hence $((x \Rightarrow 0) \Rightarrow 0) \bullet (y \Rightarrow 0) \leq 0$ by (1).

3. We show $f(f(x)) = f(x)$.

Since $f(x) \Rightarrow 0 \leq f(x) \Rightarrow 0$ then

$$(f(x) \Rightarrow 0) \bullet f(x) \leq 0.$$

Therefore $(f(x) \Rightarrow 0) \bullet x \leq 0$ and hence

$$f(x) \Rightarrow 0 \leq x \Rightarrow 0 \cdots (1).$$

Since $x \Rightarrow 0 \leq f(x \Rightarrow 0)$ by 1. and

$$f(x \Rightarrow 0) = ((x \Rightarrow 0) \Rightarrow 0) \Rightarrow 0 = f(x) \Rightarrow 0$$

then

$$x \Rightarrow 0 \leq f(x) \Rightarrow 0 \cdots (2).$$

By (1) and (2), $x \Rightarrow 0 = f(x) \Rightarrow 0$, and hence

$$f(f(x)) = (f(x) \Rightarrow 0) \Rightarrow 0 = (x \Rightarrow 0) \Rightarrow 0 = f(x).$$

4. We show $f(x) \bullet f(y) \leq f(x \bullet y)$.

   Since $(a \Rightarrow b) \bullet (b \Rightarrow c) \bullet a \leq c$ then $a \Rightarrow b \leq (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$.

$$\begin{aligned} x &\leq y \rightarrow x * y \\ &\leq ((x * y) \rightarrow 0) \rightarrow (y \rightarrow 0) \\ &\leq ((y \rightarrow 0) \rightarrow 0) \rightarrow ((x * y) \rightarrow 0) \rightarrow 0 \\ &= f(y) \rightarrow f(x * y). \end{aligned}$$

   Thus $x \bullet f(y) \leq f(x \bullet y)$ then $f(x * f(y)) \leq f(x * y)$.

   Therefore $f(x) \bullet f(y) \leq f(x \bullet f(y))$ and hence

$$f(x) \bullet f(y) \leq f(x \bullet y).$$

Proposition 2.4.4 *Let* $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1 \rangle$ *be a complete IL-algebra. Then*

$$\mathbf{K}(\mathbf{A}) = \langle \mathcal{K}(A), \Rightarrow, \cup_K, \cap, \bullet_K, K1 \rangle$$

is a complete CL-algebra, where $\cup_K$ *and* $\bullet_K$ *are defined by*

1. $x \cup_K y := K(x \cup y)$ $(\cup_K x_i := K(\cup x_i))$,
2. $x \bullet_K y := K(x \bullet y)$.

Proof. We have that for a given complete IL-algebra $\mathbf{A} = \langle A, \Rightarrow, \cup, \cap, \bullet, 1 \rangle$, a mapping $\mathcal{K}$ from A to A is a closure operation on $\mathbf{A}$. Therefor $\mathbf{K}(\mathbf{A}) = \langle \mathcal{K}(A), \Rightarrow, \cup_K, \cap, \bullet_K, K1 \rangle$ is a complete IL-algebra, and so Definition 2.4.1 from 1. to 4. are trivial. For 5. $\sim\sim x = x$ $(\sim x := x \Rightarrow 0)$, we show $\mathcal{K}x = x$ $(i.e.\ (x \Rightarrow 0) \Rightarrow 0 = x)$, hence $\mathcal{K}0 = 0$ $(i.e.\ (0 \Rightarrow 0) \Rightarrow 0 = 0)$.

1. ($\Leftarrow$). $0 \leq (0 \Rightarrow 0) \Rightarrow 0$ is trivial.

2. ($\Rightarrow$). Since $1 \bullet 1 \leq 0 \Rightarrow 0$ then

$$1 \leq 1 \Rightarrow (0 \Rightarrow 0) \leq ((0 \Rightarrow 0) \Rightarrow 0) \Rightarrow (1 \Rightarrow 0),$$

and hence

$$(0 \Rightarrow 0) \Rightarrow 0 \leq 1 \Rightarrow 0 = (1 \Rightarrow 0) \bullet 1 \leq 0.$$

Therefore $(0 \Rightarrow 0) \Rightarrow 0 \leq 0$.

2.4.2 Classical Quantales

Definition 2.4.5 A structure $\mathbf{Q} = \langle Q, \cup, \bullet, 1\rangle$ is a *commutative classical quantale* if

1. $\langle Q, \cup\rangle$ is a *complete lattice*,

2. $\langle Q, \bullet, 1\rangle$ is a *commutative monoid*,

3. $(\bigcup x_i) \bullet y = \bigcup (x_i \bullet y)$,

4. $\sim\sim x = x$ $(\sim x := x \Rightarrow 0)$.

Remark 2.4.6 Define a binary operation on $\mathbf{Q}$ by

$$y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}.$$

Then $x \leq y \Rightarrow z$ if and only if $x \bullet y \leq z$.

Remark 2.4.7 It is easy to show that a complete CL-algebra is just a commutative classical quantale, in which $y \Rightarrow z$ is defined by

$$y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}.$$

Proposition 2.4.8 *A structure is a complete CL-algebra if and only if it is a commutative classical quantale.*

Proof. ($\Rightarrow$). is trivial.
($\Leftarrow$). Define $y \Rightarrow z := \bigcup\{x | x \bullet y \leq z\}$. Then we show that commutative quantale is always complete IL-algebra.

1. Definition 2.3.1 1., 2. and 3. are trivial.

2. We show that $x \bullet y \leq z \Leftrightarrow x \leq y \Rightarrow z$.

   Let $y \Rightarrow z := \bigcup\{u | u \bullet y \leq z\}$.

$(\Rightarrow)$. If $x \bullet y \le z$, then $x \in \{u | u \bullet y \le z\}$, and hence $x \le y \Rightarrow z$.

$(\Leftarrow)$. If $x \le y \Rightarrow z$, then

$$\begin{aligned} x \bullet y \le (y \Rightarrow z) \bullet y &= (\bigcup\{u | u \bullet y \le z\}) \bullet y \\ &= \bigcup\{u \bullet y | u \bullet y \le z\} \le z, \end{aligned}$$

and hence $x \bullet y \le z$.

Chapter 3

Intuitionistic Linear Logic

In this chapter, we will discuss intuitionistic linear logic (its syntax and semantics) and then will prove soundness theorem for quantales generated by Petri nets. And then we will show how to prove completeness using the quantale based on our construction.

3.1 Syntax

3.1.1 Formulas

The language of intuitionistic linear logic (ILL) has an alphabet consisting of

1. propositional variables : $a, b, c, \ldots,$
2. propositional constants : $1, \top, \bot,$
3. connectives : $*, \vee, \wedge, \supset,$
4. auxiliary symbols : (,).

The connectives carry traditional names:

- $*$: conjunction (times),
- $\vee$: disjunction (or),
- $\wedge$: conjunction (and).

Formulas are inductively defined by

1. The propositional variables and constants are formulas,
2. if A and B are formulas, then $A * B, A \vee B, A \wedge B$ and $A \supset B$ are formulas.

We shall use $A \equiv B$ as an abbreviation, for $(A \supset B) \wedge (B \supset A)$, and denote the set of all formulas by Φ.

3.1.2 Sequents

A sequent of ILL is an expression of the form

$$\Gamma \to \theta,$$

where Γ is a finite sequence of formulas and θ is a formula. Both Γ and θ may be empty. In the sequel, capital Greek letters will denote finite (possibly empty) sequences of formulas.

3.1.3 Axioms (initial sequents) and Rules

Definition 3.1.1 (axioms and rules of inference of ILL) The axioms of ILL are the instances of the four axiom-schemes:

$$A \to A$$

$$\to 1$$

$$\Gamma \to \top$$

$$\Gamma, \bot, \Delta \to A$$

The rules of inference of ILL are the following structural rules:

$$\frac{\Gamma, \Delta \to A}{\Gamma, 1, \Delta \to A} \text{ (1 - weakening)}$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C} \text{ (exchange)}$$

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta, \Gamma, \Sigma \to C} \text{ (cut)}$$

and the following logical rules:

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} (\vee \to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B} (\to \vee 1) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B} (\to \vee 2)$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} (\wedge 1 \to) \qquad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} (\wedge 2 \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B} (\to \wedge)$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, A * B, \Delta \to C}\ (* \to) \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B}\ (\to *)$$

$$\frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, A \supset B, \Gamma, \Sigma \to C}\ (\supset\to) \qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\to\supset)$$

Remark 3.1.2 With contraction and weakening,

$$\frac{\Gamma \to B}{\Gamma, A \to B}\ (\text{weakening}), \qquad \frac{\Gamma, A, A \to B}{\Gamma, A \to B}\ (\text{contraction}),$$

we can prove $A * B \equiv A \wedge B$ as follows.

- For $A * B \to A \wedge B$,

$$\dfrac{\dfrac{\dfrac{A \to A}{A, B \to A}\ (\text{weakening}) \quad \dfrac{B \to B}{A, B \to B}\ (\text{weakening})}{A, B \to A \wedge B}\ (\to \wedge)}{A * B \to A \wedge B}\ (\to *) \quad .$$

- For $A \wedge B \to A * B$,

$$\dfrac{\dfrac{\dfrac{A \to A}{A \wedge B \to A}\ (\wedge \to) \quad \dfrac{B \to B}{A \wedge B \to B}\ (\wedge \to)}{A \wedge B, A \wedge B \to A * B}\ (\to *)}{A \wedge B \to A * B}\ (\text{contraction}) \quad .$$

3.1.4 Examples of Proofs

Example 3.1.3 As an example of a proof, we can derive the rule

$$\frac{\Gamma \to A \supset B}{\Gamma, A \to B}$$

from $(\supset\to)$ by:

$$\dfrac{\Gamma \to A \supset B \quad \dfrac{A \to A \quad B \to B}{A, A \supset B \to B}\ (\to\supset)}{\Gamma, A \to B}\ (\text{cut})$$

Example 3.1.4 We can derive that $(A \vee B) * C \equiv (A * C) \vee (B * C)$.

- For $(A \vee B) * C \to (A * C) \vee (B * C)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A \quad C \to C}{A, C \to A * C}\ (\to *)}{A, C \to A * C \vee B * C}\ (\to \vee 1) \quad \dfrac{\dfrac{B \to B \quad C \to C}{B, C \to B * C}\ (\to *)}{B, C \to A * C \vee B * C}\ (\to \vee 2)}{A \vee B, C \to A * C \vee B * C}\ (\vee \to)}{(A \vee B) * C \to A * C \vee B * C}\ (* \to)$$

- For $(A * C) \vee (B * C) \to (A \vee B) * C$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A \vee B}(\to \vee 1) \quad C \to C}{A, C \to (A \vee B) * C}(\to *)}{A * C \to (A \vee B) * C}(* \to) \quad \dfrac{\dfrac{\dfrac{B \to B}{B \to A \vee B}(\to \vee 2) \quad C \to C}{B, C \to (A \vee B) * C}(\to *)}{B * C \to (A \vee B) * C}(* \to)}{A * C \vee B * C \to (A \vee B) * C}(\vee \to)$$

Example 3.1.5 We can derive that $(A * B) \to C \equiv A \to B \supset C$,

- For $(A * B) \to C \to A \to B \supset C$,

$$\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}(\to *) \quad A * B \to C}{A, B \to C}(\text{cut})}{A \to B \supset C}(\to\supset)$$

- For $A \to B \supset C \to (A * B) \to C$,

$$\dfrac{\dfrac{A \to B \supset C \quad \dfrac{B \to B \quad C \to C}{B \supset C, B \to C}(\supset\to)}{A, B \to C}(\text{cut})}{A * B \to C}(* \to)$$

3.2 Semantics

3.2.1 Valuation on Quantale

Definition 3.2.1 (valuation) A valuation v on a commutative quantale $\mathbf{Q} = \langle Q, \cup, \bullet, 1 \rangle$ is a mapping of Φ into Q satisfying the following conditions for every $A, B \in \Phi$

1. $v(A \wedge B) = v(A) \cap v(B)$,
2. $v(A \vee B) = v(A) \cup v(B)$,
3. $v(A * B) = v(A) \bullet v(B)$,
4. $v(A \supset B) = v(A) \Rightarrow v(B)$,
5. $v(\top) = \top$,
6. $v(\bot) = \bot$,
7. $v(1) = 1$.

3.2.2 Validity

Definition 3.2.2 (valid) A formula A is said to be

1. *true* in a valuation v on a commutative quantale Q if
$$1 \leq v(A),$$
which will be denoted by $Q, v \models A$;

2. *valid* with respect to a class $\mathcal{Q}$ of commutative quantales if for each commutative quantale $Q \in \mathcal{Q}$ and each valuation v on Q,
$$Q, v \models A,$$
holds, which will be denoted as $\mathcal{Q} \models A$;

3. A sequent $\Gamma \to A$ is said to be *valid* with $\mathcal{Q}$ if and only if
$$\mathcal{Q} \models \mathit{\Gamma}^* \supset A,$$
where Γ^* is defined by $< >^* := 1$ and $(\Gamma, A)^* := \Gamma^* * A$.

3.2.3 Soundness

Theorem 3.2.3 (soundness) *If a sequent* $\Gamma \to A$ *is provable in* ILL*, then it is valid with respect to the class of all commutative quantales.*

Proof. Soundness is proved by a straightforward induction on hight of proof.

- Initial sequents are valid,
- for the rules of inference (structural rules and logical rules), if upper sequent(s) is valid, then lower sequent is valid.

We show that initial sequents, structural rules and logical rules are valid:

1. initial sequents

   (a) $\models A \to A \Leftrightarrow \models A \supset A$
$$\begin{aligned} &\Leftrightarrow \quad 1 \leq v(A \supset A) \\ &\Leftrightarrow \quad 1 \leq v(A) \Rightarrow v(A) \\ &\Leftrightarrow \quad 1 \bullet v(A) \leq v(A) \\ &\Leftrightarrow \quad v(A) \leq v(A). \end{aligned}$$

   (b) $\models \to 1 \Leftrightarrow \models 1$
$$\begin{aligned} &\Leftrightarrow \quad 1 \leq v(1) \\ &\Leftrightarrow \quad 1 \leq 1. \end{aligned}$$

(c) $\models \Gamma \to \top \Leftrightarrow \models \Gamma^* \supset \top$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* \supset \top) \\ &\Leftrightarrow \ 1 \leq v(\Gamma^*) \Rightarrow v(\top) \\ &\Leftrightarrow \ 1 \bullet v(\Gamma^*) \leq v(\top) \\ &\Leftrightarrow \ v(\Gamma^*) \leq \top \\ &\quad\ \text{(since } \top \text{ is the greatest element).} \end{aligned}$$

(d) $\models \Gamma, \bot, \Delta \to A \Leftrightarrow \models \Delta^* * \bot * \Delta^* \supset A$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * \bot * \Delta^*) \Rightarrow v(A) \\ &\Leftrightarrow \ v(\Gamma^* * \bot * \Delta^*) \leq v(A) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(\bot) \bullet v(\Delta^*) \leq v(A) \\ &\Leftrightarrow \ v(\bot) \bullet v(\Gamma^*) \bullet v(\Delta^*) \leq v(A) \\ &\Leftrightarrow \ v(\bot) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(A) \\ &\quad\ \text{(since } \bot \text{ is the least element).} \end{aligned}$$

2. structural rules

(a) For $1 - weakening$,

- $\models \Gamma, \Delta \to A \Leftrightarrow \models \Gamma^* * \Delta^* \supset A$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * \Delta^*) \Rightarrow v(A) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(\Delta^*) \leq v(A), \end{aligned}$$

- $\models \Gamma, 1, \Delta \to A \Leftrightarrow \models \Gamma^* * 1 * \Delta^* \supset A$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * 1 * \Delta^*) \Rightarrow v(A) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(1) \bullet v(\Delta^*) \leq v(A) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(\Delta^*) \leq v(A). \end{aligned}$$

(b) For exchange,

- $\models \Gamma, A, B, \Delta \to C \Leftrightarrow \models \Gamma^* * A * B * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * A * B * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \leq v(C), \end{aligned}$$

- $\models \Gamma, B, A, \Delta \to C \Leftrightarrow \models \Gamma^* * B * A * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * B * A * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(B) \bullet v(A) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \leq v(C). \end{aligned}$$

(c) For cut,

- $\models \Gamma \to A \Leftrightarrow \models \Gamma^* \supset A$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Gamma^*) \Rightarrow v(A) \\ &\Leftrightarrow v(\Gamma^*) \leq v(A) \cdots (1), \end{aligned}$$

- $\models \Delta, A, \Sigma \to C \Leftrightarrow \models \Delta^* * A * \Sigma^* \supset C$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Delta^* * A * \Sigma^*) \Rightarrow v(C) \\ &\Leftrightarrow v(\Delta^*) \bullet v(A) \bullet v(\Sigma^*) \leq v(C) \\ &\Leftrightarrow v(A) \leq v(\Delta^*) \bullet v(\Sigma^*) \Rightarrow v(C) \cdots (2), \end{aligned}$$

- $\models \Delta, \Gamma, \Sigma \to C \Leftrightarrow \models \Delta^* * \Gamma^* * \Sigma^* \supset C$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Delta^* * \Gamma^* * \Sigma^*) \Rightarrow v(C) \\ &\Leftrightarrow v(\Delta^*) \bullet v(\Gamma^*) \bullet v(\Sigma^*) \leq v(C) \\ &\Leftrightarrow v(\Gamma^*) \leq v(\Delta^*) \bullet v(\Sigma^*) \Rightarrow v(C) \\ &\quad \text{(since (1) and (2)).} \end{aligned}$$

3. logical rules

(a) For $(\vee \to)$,

- $\models \Gamma, A, \Delta \to C \Leftrightarrow \models \Gamma^* * A * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Gamma^* * A * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow v(\Gamma^*) \bullet v(A) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow v(A) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(C) \cdots (1), \end{aligned}$$

- $\models \Gamma, B, \Delta \to C \Leftrightarrow \models \Gamma^* * B * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Gamma^* * B * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow v(B) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(C) \cdots (2), \end{aligned}$$

- $\models \Gamma, A \vee B, \Delta \to C \Leftrightarrow \models \Gamma^* * (A \vee B) * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow 1 \leq v(\Gamma^* * (A \vee B) * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow v(\Gamma^*) \bullet v(A \vee B) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow v(A) \cup v(B) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(C) \\ &\quad \text{(since (1) and (2)).} \end{aligned}$$

(b) For $(\to \vee 1)$,

- By cut $\cdots (1) \models \Gamma \to A \Leftrightarrow v(\Gamma^*) \leq v(A)$,

- $\models \Gamma \rightarrow A \vee B \Leftrightarrow\ \models \Gamma^* \supset (A \vee B)$

$$\begin{aligned} &\Leftrightarrow\ 1 \leq v(\Gamma^*) \Rightarrow v(A \vee B) \\ &\Leftrightarrow\ v(\Gamma^*) \leq v(A \vee B) \\ &\Leftrightarrow\ v(\Gamma^*) \leq v(A) \cup v(B). \end{aligned}$$

(c) The proof is similar for $(\rightarrow \vee 2)$.

(d) For $(\wedge 1 \rightarrow)$,

- By cut $\cdots(2) \models \Gamma, A, \Delta \rightarrow C \Leftrightarrow v(A) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(C)$,
- $\models \Gamma, A \wedge B, \Delta \rightarrow C \Leftrightarrow\ \models \Gamma^* * (A \wedge B) * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow\ 1 \leq v(\Gamma^* * (A \wedge B) * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(A \wedge B) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet (v(A) \cap v(B)) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow\ v(A) \cap v(B) \leq v(\Gamma^*) \bullet v(\Delta^*) \Rightarrow v(C). \end{aligned}$$

(e) The proof is similar for $(\wedge 2 \rightarrow)$.

(f) For $(\rightarrow \wedge)$,

- By cut $\cdots(1) \models \Gamma \rightarrow A \Leftrightarrow v(\Gamma^*) \leq v(A) \cdots (1)$ and,
- $\models \Gamma \rightarrow B \Leftrightarrow v(\Gamma^*) \leq v(B) \cdots (2)$,
- $\models \Gamma \rightarrow A \wedge B \Leftrightarrow\ \models \Gamma^* \supset (A \wedge B)$

$$\begin{aligned} &\Leftrightarrow\ 1 \leq v(\Gamma^*) \Rightarrow v(A \wedge B) \\ &\Leftrightarrow\ v(\Gamma^*) \leq v(A \wedge B) \\ &\Leftrightarrow\ v(\Gamma^*) \leq v(A) \cap v(B) \\ &\quad\ \text{(since (1) and (2))}. \end{aligned}$$

(g) For $(* \rightarrow)$,

- $\models \Gamma, A, B, \Delta \rightarrow C \Leftrightarrow\ \models \Gamma^* * A * B * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow\ 1 \leq v(\Gamma^* * A * B * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \leq v(C), \end{aligned}$$

- $\models \Gamma, A * B, \Delta \rightarrow C \Leftrightarrow\ \models \Gamma^* * (A * B) * \Delta^* \supset C$

$$\begin{aligned} &\Leftrightarrow\ 1 \leq v(\Gamma^* * (A * B) * \Delta^*) \Rightarrow v(C) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(A * B) \bullet v(\Delta^*) \leq v(C) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \leq v(C). \end{aligned}$$

(h) For $(\rightarrow *)$,

- by cut $\cdots(1) \models \Gamma \rightarrow A \Leftrightarrow\ v(\Gamma^*) \leq v(A)$ and,

- $\models \Delta \to B \Leftrightarrow \ v(\Delta^*) \leq v(B)$,
- $\models \Gamma, \Delta \to A * B \Leftrightarrow \ \models \Gamma^* * \Delta^* \supset A * B$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Gamma^* * \Delta^*) \Rightarrow v(A * B) \\ &\Leftrightarrow \ v(\Gamma^*) \bullet v(\Delta^*) \leq v(A) \bullet v(B). \end{aligned}$$

(i) For $(\supset\to)$,
first we show that if $a \leq a'$ and $b \leq b'$, then $a' \Rightarrow b \leq a \Rightarrow b'$. Since $a \leq a'$ and $b \leq b'$, $a \bullet (a' \Rightarrow b) \leq a' \bullet (a' \Rightarrow b) \leq b \leq b'$, and hence $a' \Rightarrow b \leq a \Rightarrow b'$.

- By cut $\cdots (1) \models \Gamma \to A \Leftrightarrow v(\Gamma^*) \leq v(A)$ and by cut $\cdots (2) \models \Delta, B, \Sigma \to C \Leftrightarrow v(B) \leq v(\Delta^*) \bullet v(\Sigma^*) \Rightarrow v(C)$,
- $\models \Delta, A \supset B, \Gamma, \Sigma \to C \Leftrightarrow \ \models \Delta^* * (A \supset B) * \Gamma^* * \Sigma^* \supset C$

$$\begin{aligned} &\Leftrightarrow \ 1 \leq v(\Delta^* * (A \supset B) * \Gamma^* * \Sigma^*) \Rightarrow v(C) \\ &\Leftrightarrow \ v(\Delta^* * (A \supset B) * \Gamma^* * \Sigma^*) \leq v(C) \\ &\Leftrightarrow \ v(\Delta^*) \bullet (v(A) \Rightarrow v(B)) \bullet v(\Gamma^*) \\ &\qquad \bullet v(\Sigma^*) \leq v(C) \\ &\Leftrightarrow \ (v(A) \Rightarrow v(B)) \leq v(\Delta^*) \bullet v(\Gamma^*) \\ &\qquad \bullet v(\Sigma^*) \Rightarrow v(C) \\ &\qquad \text{(by the result above).} \end{aligned}$$

(j) The proof is similar for $(\to\supset)$.

3.3 Completeness

We have constructed quantales from a Petri net:

$$\begin{array}{ll} \text{Petri net } N, & \\ \Downarrow & \text{Proposition 2.2.3} \\ \textit{preordered commutative monoid } M_N, & \\ \Downarrow & \text{Proposition 2.3.6} \\ \textit{quantale } \mathcal{P}(M), & \\ \Downarrow & \text{Proposition 2.3.16 (or Proposition 2.3.17)} \\ \textit{quantale } C_1(\mathcal{P}(M_N)), (C_2(\mathcal{P}(M_N))). & \end{array}$$

Let $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be classes of commutative quantales defined by

$$\mathcal{Q}_0 := \{\ \mathcal{P}(M_N) : N \text{ is a Petri net } \},$$

$$\mathcal{Q}_1 := \{\ C_1(\mathcal{P}(M_N)) : N \text{ is a Petri net } \},$$

$$\mathcal{Q}_2 := \{\ C_2(\mathcal{P}(M_N)) : N \text{ is a Petri net } \},$$

where $\mathcal{P}(M_N)$, $C_1(\mathcal{P}(M_N)$ and $C_2(\mathcal{P}(M_N))$ are commutative quantale defined from a pre-ordered commutative monoid M_N as in proposition 2.3.6, and definition 2.3.14 and 2.3.15 respectively.

Definition 3.3.1 We say that ILL is *complete* for a class $\mathcal{Q}$ of commutative quantales, if $\Gamma \rightarrow A$ is provable in ILL whenever $\Gamma \rightarrow A$ is valid with respect to $\mathcal{Q}$.

$\mathcal{Q}_1$ is the class of commutative quantales used in Engberg and Winskel [2]. To prove completeness of ILL for a class of commutative quantales generated by Petri nets, $\mathcal{Q}_0$ and $\mathcal{Q}_1$ do not work. Although the following proof shows that

$$(A \wedge B) \vee (A \wedge C) \rightarrow A \wedge (B \vee C)$$

is derivable in ILL

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{A \wedge C \rightarrow A}(\wedge\rightarrow) \quad \dfrac{\dfrac{B \rightarrow B}{B \rightarrow B \vee C}(\rightarrow\vee)}{A \wedge B \rightarrow B \vee C}(\wedge\rightarrow)}{A \wedge B \rightarrow A \wedge (B \vee C)}(\rightarrow\wedge) \quad \dfrac{\dfrac{A \rightarrow A}{A \wedge C \rightarrow A}(\wedge\rightarrow) \quad \dfrac{\dfrac{C \rightarrow C}{C \rightarrow B \vee C}(\rightarrow\vee)}{A \wedge C \rightarrow B \vee C}(\wedge\rightarrow)}{A \wedge C \rightarrow A \wedge (B \vee C)}(\rightarrow\wedge)}{(A \wedge B) \vee (A \wedge C) \rightarrow A \wedge (B \vee C)}(\vee\rightarrow),$$

we cannot prove the sequent

$$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C).$$

That is, the *distributivity* of $\wedge$ and $\vee$ does not hold in ILL. In any quantale in $\mathcal{Q}_0$ or $\mathcal{Q}_1$, the distributivity is always valid. Therefore, if we want to prove completeness for $\mathcal{Q}_0$ or $\mathcal{Q}_1$, then we have to add the distributivity to ILL. Then we will consider the class $\mathcal{Q}_2$, and prove completeness for $\mathcal{Q}_2$, the distributivity is not always valid in a quantale in $\mathcal{Q}_2$.

In the sequel, we shall prove completeness for $\mathcal{Q}_2$.

Theorem 3.3.2 (completeness) *If a sequent $\Gamma \rightarrow A$ is valid in the $\mathcal{Q}_2$, then it is provable in ILL.*

Proof. First we construct a Petri net $N = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ as follows:

1. $P := \Phi$,

2. $T := \{(\Gamma, \Delta) | \Gamma \rightarrow \Delta^* \text{ is provable in ILL}\}$,

3. for each $t = (\Gamma, \Delta) \in T$,

   (a) ${}^{\bullet}t := \underline{\Gamma}$,

   (b) $t^{\bullet} := \underline{\Delta}$.

Note that in the preordered commutative monoid $M_N = \langle \mathcal{M}, \rhd, +, \underline{\emptyset} \rangle$, if $\underline{A} \rhd \underline{B}$, then $A \to B$ is provable in ILL. Next we define a mapping v of Φ into the quantale $C_2(\mathcal{P}(M_N))$ by

$$v(\alpha) := C_2(\{\underline{\alpha}\}).$$

We can show by induction on the complexity of α that v is a valuation on $C_2(\mathcal{P}(M_N))$.

We show that $\mathbf{Q}_2 = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_C, \bullet_C, C_2(\{\underline{\emptyset}\}) \rangle$ satisfies Definition 4.2.1,

1. $v(A \wedge B) = v(A) \cap v(B)$,
2. $v(A \vee B) = v(A) \cup_C v(B)$,
3. $v(A * B) = v(A) \bullet_C v(B)$,
4. $v(A \supset B) = v(A) \Rightarrow v(B)$,
5. $v(\top) = \mathcal{M}$,
6. $v(\bot) = C_2\emptyset$,
7. $v(1) = C_2(\{\underline{\emptyset}\})$.

*Case*1. $\alpha \equiv A \wedge B$.
We show $v(A \wedge B) = v(A) \cap v(B)$. Since

$$v(A \wedge B) = C(\{\underline{A \wedge B}\}),$$
$$v(A) \cap v(B) = C(\{\underline{A}\}) \cap C(\{\underline{B}\}),$$

then we show $C(\{\underline{A \wedge B}\}) = C(\{\underline{A}\}) \cap C(\{\underline{B}\})$, by Lemma 2.3.19.

- $(\leq)$. If $A \wedge B \to A$, then $A \wedge B \leq A$, and if $A \wedge B \to B$, then $A \wedge B \leq B$. Therefore $C(\{\underline{A \wedge B}\}) \subseteq C(\{\underline{A}\}) \cap C(\{\underline{B}\})$.
- $(\geq)$. If $C \to A, C \to B$, then $C \to A \wedge B$. Therefore if $C \leq A$ and $C \leq B$, then $C \leq A \wedge B$, and hence $C(\{\underline{A}\}) \cap C(\{\underline{B}\}) \subseteq C(\{\underline{A \wedge B}\})$.

*Case*2. $\alpha \equiv A \vee B$.
We show $v(A \vee B) = v(A) \cup_C v(B)$. Since

$$v(A \vee B) = C(\{\underline{A \vee B}\}),$$

and since $x \cup_C y := C(x \cup y)$,

$$\begin{aligned} v(A) \cup_C v(B) &= C(\{\underline{A}\}) \cup_C C(\{\underline{B}\}) \\ &= C(C(\{\underline{A}\}) \cup C(\{\underline{B}\})) \\ &= C(\{\underline{A}\} \cup \{\underline{B}\}) \end{aligned}$$

(since $C(Cx \cup Cy) = C(x \cup y)$),

then we show $\{\underline{A \vee B}\} = \{\underline{A}\} \cup \{\underline{B}\}$, by Lemma 2.3.19.

- ($\leq$). If $C \to A \vee B$, then $C \to A$ or $C \to B$. Therefore if $C \leq A \vee B$, then $C \leq A$ or $C \leq B$, and hence $\{\underline{A \vee B}\} \leq \{\underline{A}\} \cup \{\underline{B}\}$.
- ($\geq$). If $A \to A \vee B$, then $A \leq A \vee B$, and if $B \to A \vee B$, then $B \leq A \vee B$. Therefore $\{\underline{A}\} \cup \{\underline{B}\} \leq \{\underline{A \vee B}\}$.

*Case*3. $\alpha \equiv A * B$.
We show $v(A * B) = v(A) \bullet_C v(B)$. Since

$v(A * B) = C(\{\underline{A * B}\})$,

and since $x *_C y := C(x * y)$,

$$\begin{aligned} v(A) *_C v(B) &= C(\{\underline{A}\}) *_C C(\{\underline{B}\}) \\ &= C(C(\{\underline{A}\}) * C(\{\underline{B}\})) \\ &= C(\{\underline{A}\} * \{\underline{B}\}) \end{aligned}$$

(since $C(Cx * Cy) = C(x * y)$)

$$= C(\{\underline{A} + \underline{B}\}),$$

then we show $\{\underline{A}\} + \{\underline{B}\} = \{\underline{A * B}\}$.

- ($\leq$). We show $\{\underline{A}\} + \{\underline{B}\} \leq \{\underline{A * B}\}$.

  $\underline{A} + \underline{B} =^{\bullet} (t) + \underline{\emptyset} = \underline{A, B} \cdots \Gamma \equiv A, B$,
  $\underline{A * B} = (t)^{\bullet} + \underline{\emptyset} = \underline{A * B} \cdots \Gamma \equiv A * B$.

  If $\Gamma \to \Delta^*$ is provable, then $A, B \to A * B * 1$ is provable.

$$\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}(\to *) \qquad \to 1}{A, B \to A * B * 1}(\to *)$$

- ($\geq$). We show $\{\underline{A * B}\} \leq \{\underline{A}\} + \{\underline{B}\}$.

  $\underline{A * B} =^{\bullet} (t) + \underline{\emptyset} = \underline{A * B}$ ...... $\Gamma \equiv A * B$,
  $\underline{A} + \underline{B} = (t)^{\bullet} + \underline{\emptyset} = \underline{A, B}$ ...... $\Gamma \equiv A, B$.

If $\Gamma \to \Delta^*$ is provable, then $A * B \to A * B * 1$ is provable.

$$\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}(\to *)}{A * B \to A * B}(* \to) \qquad \to 1}{A * B \to A * B * 1}(\to *)$$

*Case*4. $\alpha \equiv A \supset B$.
We show $v(A \supset B) = v(A) \to v(B)$. Since

$$v(A \supset B) = C(\{\underline{A \supset B}\}),$$
$$v(A) \to v(B) = C(\{\underline{A}\}) \to C(\{\underline{B}\}),$$

then we show $C(\{\underline{A \supset B}\}) = C(\{\underline{A}\}) \to C(\{\underline{B}\})$.

- $(\leq)$. For $C(\{\underline{A \supset B}\}) \leq C(\{\underline{A}\}) \to C(\{\underline{B}\})$,

  (1) if $\underline{\Gamma} \in C(\{\underline{A \supset B}\})$, then $\Gamma \to A \supset B$,

  (2) if $\forall \underline{\Delta} \in C(\{\underline{A}\})$, then $\Delta \to A$.

  By (1) and (2),

  $$\dfrac{\Delta \to A \quad \Gamma, A \to B}{\Gamma, \Delta \to B}(\text{cut}).$$

  Then $\Gamma, \Delta \to B$, and hence $\underline{\Gamma, \Delta} \in C(\{\underline{B}\})$. Since $\underline{\Gamma, \Delta} = \underline{\Gamma} + \underline{\Delta}$, then $\Gamma \leq C(\{\underline{A}\}) \to C(\{\underline{B}\})$.
- $(\geq)$. For $C(\{\underline{A}\}) \to C(\{\underline{B}\}) \leq C(\{\underline{A \supset B}\})$,

  (1) $\underline{\Gamma} \in C(\{\underline{A}\}) \to C(\{\underline{B}\})$,

  (2) $\forall \underline{\Delta} \in C(\{\underline{A}\})$.

  By (1) and (2), $\underline{\Gamma} + \underline{\Delta} \in C(\{\underline{B}\})$, and hence $\underline{\Gamma} + \underline{\Delta} \to B$. Therefore $\underline{\Gamma} \to A \supset B$, and so $C(\{\underline{A}\}) \to C(\{\underline{B}\}) \leq C(\{\underline{A \supset B}\})$.

*Case*5. $\alpha \equiv \top$.
We show $v(\top) = \mathcal{M}$. By definition, $C_2(\{\underline{\top}\}) = \{y \mid y \leq \underline{\top}\}$, then

$$v(\top) := C_2(\{\underline{\top}\}) = \mathcal{M}.$$

*Case*6. $\alpha \equiv \bot$.
We show $v(\bot) = C\emptyset$.

$$\emptyset^{\to} := \{y \mid \forall x \in \emptyset \ (x \leq y)\} = \mathcal{M},$$
$$\mathcal{M}^{\leftarrow} := \{y \mid \forall x \in \mathcal{M} \ (y \leq x)\} = \{\underline{\bot}\}.$$

*Case*7. $\alpha \equiv 1$.
We can show $v(1) = C_2(\{\underline{\emptyset}\})$.

Finally we prove that $\rightarrow A$ is provable in ILL whenever $1 \leq v(A)$. If $1 \leq v(A)$, then $C_2(\{\underline{\emptyset}\}) \subseteq C_2(\{\underline{A}\})$, and hence $\underline{\emptyset} \leq \underline{A}$ in the original preordered monoid M_N. Thus $\rightarrow A$ is provable in ILL. If $\rightarrow \Gamma^* \supset A$ is provable in ILL, then so is $\Gamma \rightarrow A$: in fact

$$\dfrac{\dfrac{\Gamma \rightarrow \Gamma^* \quad \dfrac{\Gamma^* \rightarrow \Gamma^* \quad A \rightarrow A}{\Gamma^* \supset A, \Gamma^* \rightarrow A}(\supset\rightarrow)}{\rightarrow \Gamma^* \supset A \quad \Gamma^* \supset A, \Gamma \rightarrow A}(\text{cut})}{\Gamma \rightarrow A}(\text{cut}) \quad .$$

Chapter 4

The "Of Course" Operator

In this chapter, we will discuss intuitionistic linear logic with a modal operator "of course" (its syntax and semantics) and then will prove soundness theorem for quantales generated by Petri nets. And then we will show how to prove completeness for quantales by using similar construction in chapter 3.

The absence of the rules for weakening and contraction is compensated,to some extent,by the addition of the modal operator "of cource".

4.1 Syntax

In this section we show only formulas, axioms, rules and examples of proofs we have to add to ILL in chapter 3.

4.1.1 Formulas

The language of ILL with a modal operator has an alphabet consisting of

unary connectives : !.

Formulas are inductively defined by

1. The propositional variables and constants are formulas,
2. if A is a formula, then $!A$ is a formula.

4.1.2 Sequents

A sequent of ILL is an expression of the form

$$\Gamma \rightarrow \theta,$$

where Γ is a finite sequence of formulas and θ is a formula. Both Γ and θ may be empty. In the sequel, capital Greek letters will denote finite (possibly empty) sequences of formulas.

4.1.3 Axioms (initial sequents) and Rules

Definition 4.1.1 The basic calculus is obtained from the calculus ILL by adding the following rules for the modal operator !:

$$!A \to A \ \ (1)$$

$$!A \to 1 \ \ (2)$$

$$!A \to !A * !A \ \ (3)$$

$$\frac{B \to A \quad B \to 1 \quad B \to B * B}{B \to !A} \ (4)$$

Given a proposition A, the assertion of !A has the possibility of being instantiated by the proposition A, the unit 1 or !A∗!A, and thus of arbitrarily many assertions of !A.

Remark 4.1.2 How this operator compensates for the absence of the two structural rules can be seen from the derived rules Girard originally presented:

$$\frac{\Gamma, A \to B}{\Gamma, !A \to B} \ (! \ \to)$$

$$\frac{\Gamma \to B}{\Gamma, !A \to B} \ (! \text{ - weakening})$$

$$\frac{\Gamma, !A, !A \to B}{\Gamma, !A \to B} \ (! \text{ - contraction})$$

$$\frac{!\Gamma \to B}{!\Gamma \to !B} \ (\to \ !)$$

where $!\Gamma$ is a shorthand for $!A_1, ..., !A_n$ where $\Gamma = A_1, ..., A_n$.

Remark 4.1.3 The rules of (1), (2) and (3) of Definition 4.1.1 are derivable from the above original rules.

$$\frac{A \to A}{!A \to A} \ (! \ \to)$$

$$\frac{\to 1}{!A \to 1} \ (! \text{ - weakening})$$

$$\frac{\dfrac{!A \to !A \quad !A \to !A}{!A, !A \to !A * !A} \ (\to *)}{!A \to !A * !A} \ (! \text{ - contraction})$$

Proposition 4.1.4 *The rule of (1), (2) and (3) of Definition 4.1.1 for* $!A$ *are interderivable with the following single rule:*

$$!A \to 1 \wedge A \wedge (!A * !A)$$

Proof.

$$\dfrac{\dfrac{!A \to A \quad !A \to 1}{!A \to 1 \wedge A}\,(\to \wedge) \qquad !A \to !A * !A}{!A \to 1 \wedge A \wedge (!A * !A)}\,(\to \wedge)$$

Proposition 4.1.5

$$1 \wedge A \wedge (!A * !A) \to !A$$

Proof. Define that $B := 1 \wedge A \wedge (!A * !A)$.

$$\dfrac{B \to A \quad B \to 1 \quad \dfrac{B \to !A * !A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A \to A \quad !A \to 1}{!A \to 1 \wedge A}\,(\to \wedge) \qquad !A \to !A * !A}{!A \to B}\,(\to \wedge) \qquad !A \to B}{!A, !A \to B * B}\,(\to *)}{!A * !A \to B * B}\,(* \to)}{B \to B * B}\,(\text{cut})}{B \to !A}\,(2)$$

4.1.4 Examples of Proofs

Example 4.1.6 We derive that $!A \to A * !A$.

$$\dfrac{\dfrac{\dfrac{A \to A \quad !A \to !A}{A, !A \to A * !A}\,(\to *)}{!A, !A \to A * !A}\,(! \to)}{!A \to A * !A}\,(! \text{ - contruction})$$

Example 4.1.7 We derive that $!A \to !(A * A)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A \quad A \to A}{A, A \to A * A}\,(\to *)}{!A, !A \to A * A}\,(! \to)}{!A \to A * A}\,(! \text{ - weakening})}{!A \to !(A * A)}\,(\to !)$$

Example 4.1.8 We derive that $!(A \wedge B) \equiv !A * !B$.

- $(\Rightarrow)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \wedge B \to A}{!(A \wedge B) \to A}\,(* \to)}{!(A \wedge B) \to !A}\,(\to *) \qquad \dfrac{\dfrac{A \wedge B \to B}{!(A \wedge B) \to B}\,(\to *)}{!(A \wedge B) \to !B}\,(* \to)}{!(A \wedge B), !(A \wedge B) \to !A * !B}\,(\to *)}{!(A \wedge B) \to !A * !B}\,(! \text{ - weakening})$$

- $(\Leftarrow)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{!A \to A}{!A, !B \to A}\ (!\text{ - weakening}) \quad \dfrac{!B \to B}{!A, !B \to B}\ (!\text{ - weakening})}{!A, !B \to A \wedge B}\ (\to \wedge)}{!A, !B \to !(A \wedge B)}\ (!\text{ - weakening})}{!A * !B \to !(A \wedge B)}\ (* \to)$$

4.2 Semantics

4.2.1 Valuation on Quantale

Definition 4.2.1 (valuation) A valuation v on a commutative quantale $\mathbf{Q} = \langle Q, \cup, \bullet, 1 \rangle$ is a mapping of Φ into Q satisfying the following conditions for every $A, B \in \Phi$

1. $v(A \wedge B) = v(A) \cap v(B)$,
2. $v(A \vee B) = v(A) \cup v(B)$,
3. $v(A * B) = v(A) \bullet v(B)$,
4. $v(A \supset B) = v(A) \Rightarrow v(B)$,
5. $v(\top) = \top$,
6. $v(\bot) = \bot$,
7. $v(1) = 1$.
8. $v(!A) = !v(A)$.

4.2.2 Validity

Definition 4.2.2 (valid) A formula A is said to be

1. *true* in a valuation v on a commutative quantale Q if
$$1 \leq v(A),$$
which will be denoted by $Q, v \models A$;

2. *valid* with respect to a class $\mathcal{Q}$ of commutative quantales if for each commutative quantale $Q \in \mathcal{Q}$ and each valuation v on Q,
$$Q, v \models A,$$
holds, which will be denoted as $\mathcal{Q} \models A$;

3. A sequent $\Gamma \to A$ is said to be *valid* with $\mathcal{Q}$ if and only if
$$\mathcal{Q} \models \Gamma^* \supset A,$$
where Γ^* is defined by $< >^* := 1$ and $(\Gamma, A)^* := \Gamma^* * A$.

4.2.3 Soundness

Theorem 4.2.3 (soundness) *If a sequent* $\Gamma \to A$ *is provable in* ILL*, then it is valid with respect to the class of all commutative quantales.*

Proof. Soundness is proved by a straightforward induction on hight of proof.

- Initial sequents are valid,
- for the rules of inference (structural rules and logical rules), if upper sequent(s) is valid, then lower sequent is valid.

We show that initial sequents, structural rules and logical rules are valid: (In this section we show only four structural rules for of course operator we have to add to ILL in chapter 3.)

structural rules

(a) For $! \to$,

- $\models \Gamma, A \to B \Leftrightarrow\ \models \Gamma^* * A \supset B$

$$\begin{aligned} &\Leftrightarrow\ 1 \le v(\Gamma^* * A) \Rightarrow v(B) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(A) \le v(B) \\ &\Leftrightarrow\ v(A) \le v(\Gamma^*) \Rightarrow v(B), \end{aligned}$$

- $\models \Gamma, !A \to B \Leftrightarrow\ \models \Gamma^* *!A \supset B$

$$\begin{aligned} &\Leftrightarrow\ 1 \le v(\Gamma^* *!A) \Rightarrow v(B) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(!A) \le v(B) \\ &\Leftrightarrow\ v(!A) \le v(\Gamma^*) \Rightarrow v(B) \\ &\Leftrightarrow\ !v(A) \le v(\Gamma^*) \Rightarrow v(B) \\ &\qquad \text{(since } !a \le a). \end{aligned}$$

(b) For $! - \mathit{weakening}$,

- $\models \Gamma \to B \Leftrightarrow\ \models \Gamma^* \supset B$

$$\Leftrightarrow\ 1 \le v(\Gamma^*) \Rightarrow v(B),$$

- $\models \Gamma, !A \to B \Leftrightarrow\ \models \Gamma^* *!A \supset B$

$$\begin{aligned} &\Leftrightarrow\ 1 \le v(\Gamma^* *!A) \Rightarrow v(B) \\ &\Leftrightarrow\ v(\Gamma^*) \bullet v(!A) \le v(B) \\ &\Leftrightarrow\ v(!A) \le v(\Gamma^*) \Rightarrow v(B) \\ &\Leftrightarrow\ !v(A) \le v(\Gamma^*) \Rightarrow v(B) \\ &\qquad \text{(since } !a \le \mathit{1}). \end{aligned}$$

(c) For $! - contraction$,

- $\models \Gamma, !A, !A \rightarrow B \Leftrightarrow \models \Gamma^{*} {*} !A {*} !A \supset B$

$$\begin{aligned}
&\Leftrightarrow 1 \leq v(\Gamma^{*}) \bullet v(!A) \bullet v(!A) \Rightarrow v(B) \\
&\Leftrightarrow v(\Gamma^{*}) \bullet v(!A) \bullet v(!A) \leq v(B) \\
&\Leftrightarrow v(!A) \bullet v(!A) \leq v(\Gamma^{*}) \Rightarrow v(B) \\
&\Leftrightarrow !v(A) \bullet !v(A) \leq v(\Gamma^{*}) \Rightarrow v(B),
\end{aligned}$$

- $\models \Gamma, !A \rightarrow B \Leftrightarrow \models \Gamma^{*} {*} !A \supset B$

$$\begin{aligned}
&\Leftrightarrow 1 \leq v(\Gamma^{*}) \bullet v(!A) \Rightarrow v(B) \\
&\Leftrightarrow v(\Gamma^{*}) \bullet v(!A) \leq v(B) \\
&\Leftrightarrow v(!A) \leq v(\Gamma^{*}) \Rightarrow v(B) \\
&\Leftrightarrow !v(A) \leq v(\Gamma^{*}) \Rightarrow v(B) \\
&\quad \text{(since } !a \leq !a {*} !a).
\end{aligned}$$

(d) For $\rightarrow !$,

- $\models !\Gamma \rightarrow B \Leftrightarrow \models \Gamma^{*} \supset B$

$$\begin{aligned}
&\Leftrightarrow 1 \leq v((!\Gamma)^{*}) \Rightarrow v(B) \\
&\Leftrightarrow v(!(\Gamma^{*})) \leq v(B),
\end{aligned}$$

- $\models !\Gamma \rightarrow !B \Leftrightarrow \models (!\Gamma)^{*} \supset !B$

$$\begin{aligned}
&\Leftrightarrow 1 \leq v((!\Gamma)^{*}) \Rightarrow v(!B) \\
&\Leftrightarrow v((!\Gamma)^{*}) \leq v(!B).
\end{aligned}$$

By $v(!(\Gamma^{*})) \leq v(B)!$,

$$\begin{aligned}
&\Leftrightarrow !v((!\Gamma)^{*}) \leq !v(B) \\
&\quad \text{(since } a \leq b \rightarrow !a \leq !b) \\
&\Leftrightarrow v(!(!\Gamma)^{*}) \leq v(!B) \\
&\Leftrightarrow v((!!\Gamma)^{*}) \leq v(!B) \\
&\Leftrightarrow v((!\Gamma)^{*}) \leq v(!B) \\
&\quad \text{(since } !a \leq !!a).
\end{aligned}$$

4.3 Completeness

We have constructed quantales from a Petri net:

Petri net N,

$\Downarrow$ Proposition 2.2.3

preordered commutative monoid M_N,

$\Downarrow$ Proposition 2.3.6

quantale $\mathcal{P}(M)$,

$\Downarrow$ Proposition 2.3.16 (or Proposition 2.3.17)

quantale $C_1(\mathcal{P}(M_N)), (C_2(\mathcal{P}(M_N)))$.

Let $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be classes of commutative quantales defined by

$\mathcal{Q}_0 := \{ \ \mathcal{P}(M_N) : N$ is a Petri net $\}$,

$\mathcal{Q}_1 := \{ \ C_1(\mathcal{P}(M_N)) : N$ is a Petri net $\}$,

$\mathcal{Q}_2 := \{ \ C_2(\mathcal{P}(M_N)) : N$ is a Petri net $\}$,

where $\mathcal{P}(M_N)$, $C_1(\mathcal{P}(M_N)$ and $C_2(\mathcal{P}(M_N))$ are commutative quantale defined from a preordered commutative monoid M_N as in proposition 2.3.6, and definition 2.3.14 and 2.3.15 respectively.

Definition 4.3.1 We say that ILL is *complete* for a class $\mathcal{Q}$ of commutative quantales, if $\Gamma \to A$ is provable in ILL whenever $\Gamma \to A$ is valid with respect to $\mathcal{Q}$.

$\mathcal{Q}_1$ is the class of commutative quantales used in Engberg and Winskel [2]. To prove completeness of ILL for a class of commutative quantales generated by Petri nets, $\mathcal{Q}_0$ and $\mathcal{Q}_1$ do not work. Although the following proof shows that

$$(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)$$

is derivable in ILL

$$\dfrac{\dfrac{\dfrac{A \to A}{A \wedge C \to A}(\wedge \to) \quad \dfrac{\dfrac{B \to B}{B \to B \vee C}(\to \vee)}{A \wedge B \to B \vee C}(\wedge \to)}{A \wedge B \to A \wedge (B \vee C)}(\to \wedge) \quad \dfrac{\dfrac{A \to A}{A \wedge C \to A}(\wedge \to) \quad \dfrac{\dfrac{C \to C}{C \to B \vee C}(\to \vee)}{A \wedge C \to B \vee C}(\wedge \to)}{A \wedge C \to A \wedge (B \vee C)}(\to \wedge)}{(A \wedge B) \vee (A \wedge C) \to A \wedge (B \vee C)}(\vee \to) \quad ,$$

we cannot prove the sequent

$$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C).$$

That is, the *distributivity* of $\wedge$ and $\vee$ does not hold in ILL. In any quantale in $\mathcal{Q}_0$ or $\mathcal{Q}_1$, the distributivity is always valid. Therefore, if we want to prove completeness for $\mathcal{Q}_0$ or $\mathcal{Q}_1$, then we have to add the distributivity to ILL. Then we will consider the class $\mathcal{Q}_2$, and prove completeness for $\mathcal{Q}_2$, the distributivity is not always valid in a quantale in $\mathcal{Q}_2$.

In the sequel, we shall prove completeness for $\mathcal{Q}_2$.

Theorem 4.3.2 (completeness) *If a sequent $\Gamma \rightarrow A$ is valid in the $\mathcal{Q}_2$, then it is provable in ILL.*

Proof. First we construct a Petri net $N = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ as follows:

1. $P := \Phi$,
2. $T := \{(\Gamma, \Delta) | \Gamma \rightarrow \Delta^* \text{ is provable in ILL}\}$,
3. for each $t = (\Gamma, \Delta) \in T$,
   (a) ${}^\bullet t := \underline{\Gamma}$,
   (b) $t^\bullet := \underline{\Delta}$.

Note that in the preordered commutative monoid $M_N = \langle \mathcal{M}, \rhd, +, \underline{\emptyset} \rangle$, if $\underline{A} \rhd \underline{B}$, then $A \rightarrow B$ is provable in ILL. Next we define a mapping v of Φ into the quantale $C_2(\mathcal{P}(M_N))$ by

$$v(\alpha) := C_2(\{\underline{\alpha}\}).$$

We can show by induction on the complexity of α that v is a valuation on $C_2(\mathcal{P}(M_N))$. (In this section we show only the valuation for of course operator we have to add to ILL in chapter 3.)

Let

$$!X := \{!m \ : \ m \in X\},$$

and let

$$!_C X := C_2 !X.$$

Then we show that

$$\mathbf{Q} = \langle C_2(M), \cup_C, \bullet_C, !_C, C_2(\{\underline{\emptyset}\}) \rangle$$

is a modal classical quantale, i.e. it satisfies the condition in Definition 2.3.20.

If $C_2 X = X$ and $C_2 Y = Y$, then

1. $!_C X \subseteq X$,

2. $!_C 1_C = 1_C$,

3. $!_C X \subseteq !_C !_C X$,

4. $!_C (X \cap Y) = !_C X *_C !_C Y$,

and $C_2\{\underline{!A}\} = !_C C_2\{\underline{A}\}$.

First we show 1. to 4.

1. $Case 1.\ !_C X \subseteq X$.

   For $!_C X \subseteq X$, suppose that $m \in !_C X$ and $m' \in X^{\rightarrow}$, ($i.e.\ \forall m'' \in X\ \ (m'' \leq m')$). Therefore

   $$\forall m'' \in X\ \ (!m'' \leq !m'),$$

   and hence $!m' \in (!X)^{\rightarrow}$, and so

   $$m \leq !m'.$$

   Since $!m' \leq m'$, then $m \leq m'$, and hence

   $$m \in (X^{\rightarrow})^{\leftarrow} = C_2 X = X.$$

2. $Case 2.\ !_C 1_C = 1_C$.

   - $(\Rightarrow)$. $!_C 1_C \subseteq 1_C$ is trivial.
   - $(\Leftarrow)$. For $!_C 1_C \supseteq 1_C$, suppose that $m \in 1_C$, then

     $$\forall m' \in (!1_c)^{\rightarrow} \text{and} \forall m'' \in !1_C\ \ (m'' \leq m').$$

     Since $\underline{1} \in 1_C$, then $!\underline{1} = \underline{!1} \in !1_C$, and hence

     $$\underline{!1} \leq m'.$$

     And since $\underline{1} \leq \underline{!1}$, then

     $$\underline{1} \leq m'.$$

     Therefore $m' \in \{\underline{1}\}^{\rightarrow}$, and hence $m \leq m'$, and so

     $$m \in C_2 !1_C = !_C 1_C.$$

3. $Case 3.\ !_C X \subseteq !_C !_C X$.

   For $!_C X \subseteq !_C !_C X$, we show $!X \subseteq C_2 !X$.

   Suppose $!m \in !X$, then $\forall m' \in (!_c !X)^{\rightarrow}$ and $!m \in !X \subseteq C_2 !X\ \ (!!m \leq m')$.

   And $!m \leq !!m$, then $!m \leq m'$.

   Therefore $!m \in C_2 !C_2 !X$, then $!X \subseteq C_2 !C_2 !X$, and hence

   $$!C_2 !X \subseteq C_2 !C_2 !X.$$

4. $Case4.$ $!_C(X \cap Y) =!_C X *_C !_C Y$.

   - $(\Rightarrow)$. For $!_C(X \cap Y) \subseteq !_C X *_C !_C Y$,

     $$!_C X *_C !_C Y = C_2(C_2!X * C_2!Y) = C_2(!X*!Y).$$

     Suppose that $m \in C_2!(X \cap Y)$ and $m' \in (!X*!Y)^{\rightarrow}$. We show $m' \in (!(X \cap Y))^{\rightarrow}$.

     Let $m'' \in X \cap Y$, then $!m'' \in !X$ and $!m'' \in !Y$, and hence

     $$!m''+!m'' \leq m'.$$

     Since $!m'' \leq !m''+!m''$, then $!m'' \leq m'$, and hence

     $$m' \in (!(X \cap Y))^{\rightarrow},$$

     and so

     $$m \leq m'.$$

   - $(\Leftarrow)$. Similar to the proof of $!_C(X \cap Y) \subseteq !_C X *_C !_C Y$.

Next we show $C_2\{\underline{!A}\} =!_C C_2\{\underline{A}\}$.

- $(\Rightarrow)$. For $C_2\{\underline{!A}\} \subseteq !_C C_2\{\underline{A}\}$, suppose $m \in C_2\{\underline{!A}\}$ (*i.e.* $m \leq \underline{!A}$).

  Let $m' \in (!C_2\{\underline{A}\})^{\rightarrow}$, since $\underline{A} \in C_2\{\underline{A}\}$, then

  $$!\underline{A} = \underline{!A} \leq m'.$$

  Therefore $m \leq m'$ and hence

  $$C_2\{\underline{A}\} \subseteq !_C C_2\{\underline{A}\}.$$

- $(\Leftarrow)$. For $C_2\{\underline{!A}\} \supseteq !_C C_2\{\underline{A}\}$,

  $$!_C C_2\{\underline{A}\} = C_2!C_2\{\underline{A}\} \subseteq C_2\{!A\}.$$

  Suppose $m \in !_C C_2\{\underline{A}\}$ and $m' \in \{\underline{!A}\}^{\rightarrow}$ (*i.e.* $!A \leq m'$).

  $\forall !m'' \in !C_2\{\underline{A}\}$ $(m'' \leq \underline{A})$, then

  $$!m'' \leq !\underline{A}.$$

  Since $!m'' \leq m'$, then

  $$m' \in !C_2\{\underline{A}\}^{\rightarrow},$$

  and hence

  $$m \leq m',$$

  and so

  $$m \in C_2\{\underline{!A}\}^{\rightarrow}.$$

Finally we prove that $\rightarrow A$ is provable in ILL whenever $1 \leq v(A)$. If $1 \leq v(A)$, then $C_2(\{\underline{\emptyset}\}) \subseteq C_2(\{\underline{A}\})$, and hence $\underline{\emptyset} \leq \underline{A}$ in the original preordered monoid M_N. Thus $\rightarrow A$ is provable in ILL. If $\rightarrow \Gamma^* \supset A$ is provable in ILL, then so is $\Gamma \rightarrow A$: in fact

$$\dfrac{\rightarrow \Gamma^* \supset A \quad \dfrac{\Gamma \rightarrow \Gamma^* \quad \dfrac{\Gamma^* \rightarrow \Gamma^* \quad A \rightarrow A}{\Gamma^* \supset A, \Gamma^* \rightarrow A}(\supset\rightarrow)}{\Gamma^* \supset A, \Gamma \rightarrow A}(\text{cut})}{\Gamma \rightarrow A}(\text{cut}) \quad .$$

Chapter 5

Concluding Remarks

We have seen how to construct quantales from Petri net, and we have proved completeness of ILL for the quantale generated by Petri net. In this concluding chapter, moreover we consider the connection between classical linear logic (CLL) and Petri nets.

Definition Syntax (formulas, axioms and rules) of CLL are follows:
(In this section we show only formulas, axioms, rules and examples of proofs we have to add to ILL in chapter 3 and 4.)

1. Formulas

   The language of CLL has an alphabet consisting of

   - a propositional constant : 0,
   - an unary connective : ?,
   - a binary connective : $\oplus$.

   The connective carry traditional names:

   $\oplus$: disjunction (par).

   Formulas are inductively defined by

   - The propositional variables and constants are formulas,
   - if A is a formula, then $?A$ is a formula.

2. Axioms and Rules

   The basic calculus is obtained from the calculus ILL by adding the following rules.

   (a) The adding axiom of CLL is the instance of one axiom-schemes:

$$\Gamma, 0 \rightarrow \Delta$$

(b) The adding rule of inference of CLL is the following structural rule:

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, 0} \text{ (0 - weakening)}$$

$$\frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda} \text{ (exchange)}$$

The adding rules of inference of CLL are the following logical rules:

$$\frac{A, \Gamma \to \Delta \quad B, \Pi \to \Lambda}{A \oplus B, \Gamma, \Pi \to \Delta, \Lambda} (\oplus \to) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \oplus B} (\to \oplus)$$

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, ?A} \text{ (? - weakening)}$$

$$\frac{\Gamma \to \Delta, ?A, ?A}{\Gamma \to \Delta, ?A} \text{ (? - contraction)}$$

$$\frac{A \to ?\Sigma}{?A \to ?\Sigma} (\to ?) \qquad \frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, ?A} (? \to)$$

where $?\Gamma$ is a shorthand for $?A_1, ..., ?A_n$ where $\Gamma = A_1, ..., A_n$.

For examples of proofs we can derive as follows:

1. We can derive that $(A * B)^\perp \equiv A^\perp \oplus B^\perp$.

   - $(\Rightarrow)$.

$$\cfrac{\cfrac{\cfrac{A \to A}{A, B \to A * B} (\to *) \quad B \to B}{(A * B)^\perp \to A^\perp, B^\perp} (\perp \to, \to \perp)}{(A * B)^\perp \to A^\perp \oplus B^\perp} (\to \oplus)$$

   - $(\Leftarrow)$.

$$\cfrac{\cfrac{\cfrac{\cfrac{A \to A}{A^\perp, A \to} (\perp \to) \quad \cfrac{B \to B}{B^\perp, B \to} (\perp \to)}{A, B, A^\perp \oplus B^\perp \to} (\oplus \to)}{A * B, A^\perp \oplus B^\perp \to} (* \to)}{A^\perp \oplus B^\perp \to (A * B)^\perp} (\to \perp)$$

2. We derive that $A \supset B \equiv A^\perp \oplus B$.

- $(\Rightarrow)$.

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{\rightarrow A, A^{\perp}}(\rightarrow \perp) \qquad B \rightarrow B}{A \supset B) \rightarrow A^{\perp}, B}(\supset\rightarrow)}{A \supset B \rightarrow A^{\perp} \oplus B}(\rightarrow \oplus)$$

- $(\Leftarrow)$.

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{A, A^{\perp} \rightarrow}(\perp \rightarrow) \qquad B \rightarrow B}{A, A^{\perp} \oplus B \rightarrow B}(\oplus \rightarrow)}{A^{\perp} \oplus B \rightarrow A \supset B}(\rightarrow\supset)$$

3. We derive that $(!A)^{\perp} \equiv ?A^{\perp}$.

- $(\Rightarrow)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\rightarrow A, A^{\perp}}(\rightarrow \perp)}{\rightarrow A, ?A^{\perp}}(\rightarrow ?)}{\rightarrow !A, ?A^{\perp}}(\rightarrow !)}{(!A)^{\perp} \rightarrow ?A^{\perp}}(\perp \rightarrow)$$

- $(\Leftarrow)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A^{\perp}, A \rightarrow}(\perp \rightarrow)}{A^{\perp}, !A \rightarrow}(! \rightarrow)}{?A^{\perp}, !A \rightarrow}(? \rightarrow)}{?A^{\perp} \rightarrow (!A)^{\perp}}(\rightarrow \perp)$$

4. We derive that $A^{\perp} \equiv Arightarrow0$. We use this later.

- $(\Rightarrow)$.

$$\dfrac{\dfrac{0 \rightarrow \qquad A \rightarrow A}{A, A \supset 0 \rightarrow}(\supset\rightarrow)}{A \supset 0 \rightarrow A^{\perp}}(\rightarrow \perp)$$

- $(\Leftarrow)$.

$$\dfrac{\dfrac{\dfrac{A \rightarrow A}{A, A^{\perp} \rightarrow}(\perp \rightarrow)}{A, A^{\perp} \rightarrow 0}(0 \text{ - weakening})}{A^{\perp} \rightarrow A \supset 0}(\rightarrow\supset)$$

Suppose we want to see how to express the negative propaty for Petri net (for example, two processes cannot be in their critical regions at the same time). Then we consider the operation of linear negation.

Definition

$$A^{\perp} := A \to 0.$$

Its semantics with respect to a quantale is determined by a choice for the denotation 0. In a net we define the interpretation of the logical constant 0 as follows:

Definition

$0 := \{m \mid m$ is the set of all markings which can not be reached from the empty markings$\}$.

The consequence of this choice for 0 is that whatever property we could state before in terms of validity of a formula A can now be stated negativery as $\models A^{\perp}$.

Now we can prove soundness for this problem, but have not proved completeness of CLL for classical quantale by Petri net. It is one of subjects of further research.

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Bibliography

[1] V.M.Abrusci, *Sequent Calculus for Intuitionistic Linear Propositional Logic*, Mathematical Logic, Edited by P.P.Petkov, Plenum Press, New York, (1990), pp.223-242.

[2] U.Engberg and G.Winskel, *Petri Nets as Models of Linear Logic*, LNCS, 431, Springer, (1990), pp.147-161.

[3] J.Y.Girard, *Linear Logic : its syntax and semantics*, Advances in Linear Logic, Edited by J.Y.Girard, Y.Lafont and L.Regnier, (1995), pp.1-42.

[4] J.Y.Girard and Y.Lafont, *Linear Logic and Lazy Computation*, In proc. TAPSOFT 87 (Pisa), vol. 2, Springer-Verlag (LNCS 250), (1987), pp.52-66.

[5] J.Lilius, *High-level Nets and Linear Logic*, LNCS, 612, Springer, (1992), pp.310-327.

[6] Narciso Marti-Oliet and Jose Meseguer. *From Petri Nets to Linear Logic*, In Category Theory and Computer Science, Manchester, UK. Spring-Verlag (LNCS 389), (1989).

[7] 村田忠夫：ペトリネットの解析と応用，近代科学社，(1992).

[8] H Ono, *Algebraic aspect of logics without structural rules*, in: Contemporary Mathematics, Proceedings of the International Conference on Algebra Honoring A. Mal'cev, Novosibirsk,(1989), L.A. Bokut, Yu.L.Ershov, O.H.Kegel and A.I.Kostrikin eds., American Mathematical Society.

[9] H Ono, *Semantics for Substructural Logics*, in: Substructural logics, Oxford Univ. Press, (1993), pp.259-291.

[10] Wolfgang Reisig, *Petri Nets*, An Introduction, Volume 4 of EATCS Monogaphs on Theoretical Computer Science. Spring-Verlag, (1985).

[11] K.I. Rosenthal, *A note on Girard quantales*, Cahiers de Topologie et Geometrie Differentielle Categoriques 31, 3-11, (1990).

[12] K.I. Rosenthal, *Quantales and their Applications*, Longman Scientific & Technical, (1990).

[13] 竹内外史：線形論理入門，日本評論社，(1995).

[14] A.S.Troelstra *Lectures on Linear Logic*, CSLI, (1991).

[15] D.N.Yetter, *Quantales and (noncommutative) linear logic*, The Journal of Symbolic Logic 55, 41-64, (1990).