Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 15, no. 4, pp. 485–501 (2011)

The Voronoi game on graphs and its complexity

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Abstract

The Voronoi game is a two-person game which is a model for a competitive facility location. The game is played on a continuous domain, and only two special cases (one-dimensional case and one-round case) are well investigated. We introduce the *discrete* Voronoi game in which the game arena is given as a graph. We first analyze the game when the arena is a large complete k-ary tree, and give an optimal strategy. When both players play optimally, the first player wins when k is odd, and the game ends in a tie for even k. Next we show that the discrete Voronoi game is intractable in general. Even for the one-round case in which the strategy adopted by the first player consist of a fixed single node, deciding whether the second player can win is NP-complete. We also show that deciding whether the second player can win is PSPACE-complete in general.

Submitted:	Reviewed:	Revised:	Accepted:
May 2009	October 2009	July 2011	August 2011
	Final:	Published:	
	August 2011	August 2011	

Article type:	Communicated by:
Regular Paper	H. Meijer

A preliminary version was presented at the 2nd IEEE Symposium on Computational Intelligence and Games (CIG 2006). A part of this work was done while the third author was visiting MIT, USA.

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1 Introduction

The Voronoi game is an idealized model of competitive facility location, first proposed by Ahn, Cheng, Cheong, Golin, and Oostrum [1]. The Voronoi game is played on a bounded continuous arena by two players. Two players $\mathcal{W}$ (white) and $\mathcal{B}$ (black) place n points alternately, and the continuous arena is subdivided according to the *nearest neighbor rule* (Voronoi diagram). At the final step, the player who dominates the larger area wins.

The Voronoi game is a natural game, but the general case seems to be very hard to analyze from the theoretical point of view. Hence, Ahn et al. [1] investigated the case in which the game arena is a bounded one-dimensional continuous domain. On the other hand, Cheong, Har-Peled, Linial, and Matoušek [2], and Fekete and Meijer [3] deal with a two-dimensional case, but they restrict themselves to one-round games; first, $\mathcal{W}$ places all n points, and next $\mathcal{B}$ places all n points.

In this paper, we introduce *discrete* Voronoi game. Two players alternately occupy n vertices on a graph, which is a bounded discrete arena. (Hence the graph contains at least $2n$ vertices.) This restriction seems to be appropriate since real estates are already bounded in general, and we have to build shops in the bounded area. More precisely, the discrete Voronoi game is played on a given finite graph G, instead of a bounded continuous arena. Each vertex of G can be assigned to the nearest vertices occupied by $\mathcal{W}$ or $\mathcal{B}$, according to the *nearest neighbor rule*. (Hence some vertex can be a "tie" when it has the same distance from a vertex occupied by $\mathcal{W}$ and another vertex occupied by $\mathcal{B}$.) Finally, the player who dominates the larger area (or the larger number of vertices) wins. We note that two players can tie in some cases.

First we consider the case in which the graph G is a complete k-ary tree. A complete k-ary tree is a natural generalization of a path which is the analogue of the one-dimensional continuous domain. We also mention that complete k-ary trees form a very natural and nontrivial graph class. Ahn et al. [1] showed that the second player $\mathcal{B}$ has an advantage on a one-dimensional continuous domain. In contrast, we first show that the first player $\mathcal{W}$ has an advantage for the discrete Voronoi game on a complete k-ary tree, when the tree is sufficiently large (comparing to n and k). More precisely, we show that $\mathcal{W}$ has a winning strategy if (1) $2n \le k$, or (2) k is odd and the complete k-ary tree contains at least $(k^3n^2 - 1)/(k-1)$ vertices. On the other hand, when k is even, $2n > k$, and the tree contains at least $(k^3n^2 - 1)/(k-1)$ vertices, two players tie if they play optimally. (We leave open the case when the complete k-ary tree contains at most $(k^3n^2 - 1)/(k-1)$ vertices with $2n > k$.)

Next we show the hardness results for the discrete Voronoi game. When we admit a general graph as a game arena, the discrete Voronoi game becomes NP-hard even in a strongly restricted case. We consider the following strongly restricted case; the game arena is an arbitrary graph, the first player $\mathcal{W}$ occupies just one vertex which is predetermined, and the second player $\mathcal{B}$ occupies n vertices in any way. The decision problem for the strongly restricted discrete Voronoi game is to determine whether $\mathcal{B}$ has a winning strategy. This restricted case seems to be advantageous to $\mathcal{B}$. However, this decision problem is NP-complete. This result is also quite different from the previously known results for the two-dimensional case (i.e., $\mathcal{B}$ can always dominate a $\frac{1}{2} + \varepsilon$ fraction of the two- or higher-dimensional domain) by Cheong et al. [2] and Fekete and Meijer [3]. However, Fekete and Meijer [3] showed that maximizing the area that $\mathcal{B}$ can claim is NP-hard in the one-round game when the given arena is a polygon with holes.

We also show that the discrete Voronoi game is PSPACE-complete in the general case. That is, for a given graph G and the number n of turns, determining whether $\mathcal{W}$ has a winning strategy on G is PSPACE-complete. Fekete and Meijer [3] conjectured that the two-player multiple-round (continuous) Voronoi game is PSPACE-hard. Thus, although we make the game discrete, our result gives a kind of positive answer to their conjecture.

2 Problem definitions

In this section, we formulate the discrete Voronoi game on a graph. Let us denote a Voronoi game $VG(G, n)$, where G is the game arena, and the players play n rounds. Hereafter, the game arena will be an undirected and unweighted simple graph $G = (V, E)$ with $N = |V|$ vertices.

For each round, the two players, $\mathcal{W}$ (white) and $\mathcal{B}$ (black), alternately occupy an *empty* vertex on the graph G ($\mathcal{W}$ always starts the game, as in Chess). The empty vertex is defined as a vertex which has not been occupied so far. This implies that $\mathcal{W}$ and $\mathcal{B}$ cannot occupy the same vertex simultaneously. Hence it is implicitly assumed that the game arena G contains at least $2n$ vertices.

Let W_i (resp. B_i) be the set of vertices occupied by player $\mathcal{W}$ (resp. $\mathcal{B}$) at the end of the i-th round. We define the distance $d(v, w)$ between two vertices v and w as the number of edges along the shortest path between them if such a path exists, otherwise $d(v, w) = \infty$. Each vertex of G can be assigned to the nearest among the vertices occupied by $\mathcal{W}$ and $\mathcal{B}$, according to the *nearest neighbor rule*. So, we define a *dominance set* $\mathcal{V}(A, B)$ (or *Voronoi regions*) of a subset $A \subset V$ against a subset $B \subset V$, where $A \cap B = \emptyset$ as

$$\mathcal{V}(A, B) = \{u \in V \mid \min_{v \in A} d(u, v) < \min_{w \in B} d(u, w)\}.$$

The dominance sets $\mathcal{V}(W_i, B_i)$ and $\mathcal{V}(B_i, W_i)$ represent the sets of vertices dominated at the end of the i-th round by $\mathcal{W}$ and $\mathcal{B}$, respectively. Let $\mathcal{V}_{\mathcal{W}}$ and $\mathcal{V}_{\mathcal{B}}$ denote $\mathcal{V}(W_n, B_n)$ and $\mathcal{V}(B_n, W_n)$, respectively. Since some vertex can be a "tie" when it has the same distance from a vertex occupied by $\mathcal{W}$ and another vertex occupied by $\mathcal{B}$, there may exist the set N_i of *neutral* vertices, $N_i := \{u \in V \mid \min_{v \in W_i} d(u, v) = \min_{w \in B_i} d(u, w)\}$, which do not belong to any of $\mathcal{V}(W_i, B_i)$ and $\mathcal{V}(B_i, W_i)$.

Finally, the player who dominates the larger number of vertices wins, in the discrete Voronoi game. More precisely, $\mathcal{W}$ *wins* if $|\mathcal{V}_{\mathcal{W}}| > |\mathcal{V}_{\mathcal{B}}|$, $\mathcal{B}$ wins (or $\mathcal{W}$ *loses*) if $|\mathcal{V}_{\mathcal{W}}| < |\mathcal{V}_{\mathcal{B}}|$, and $\mathcal{W}$ and $\mathcal{B}$ *tie* otherwise, since the *outcome* for each player, $\mathcal{W}$ or $\mathcal{B}$, is the size of the dominance set $|\mathcal{V}_{\mathcal{W}}|$ or $|\mathcal{V}_{\mathcal{B}}|$. In our model, note that any vertex in N_n does not contribute to the outcomes $\mathcal{V}_{\mathcal{W}}$ and $\mathcal{V}_{\mathcal{B}}$ of both players (see Figure 1).

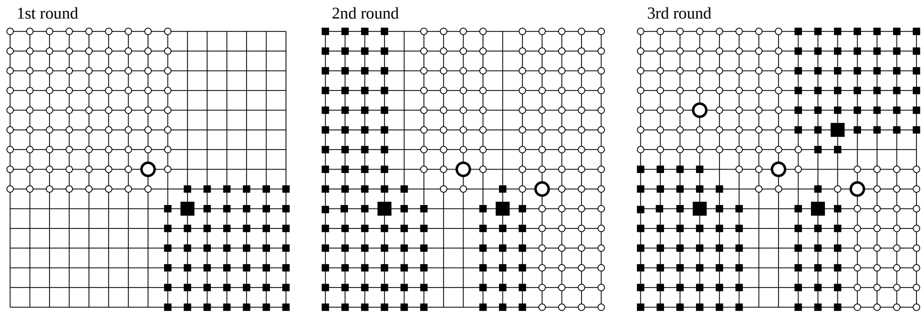

Figure 1: Example for a discrete Voronoi game $VG(G, 3)$, where G is the 15×15 grid graph; each bigger circle is a vertex occupied by $\mathcal{W}$, each smaller circle is an empty vertex dominated by $\mathcal{W}$, each bigger black square is a vertex occupied by $\mathcal{B}$, each smaller black square is an empty vertex dominated by $\mathcal{B}$, and the other are neutral vertices. In this example, the 2nd player $\mathcal{B}$ won by 108–96.

3 Discrete Voronoi games on a complete k-ary tree

In this section, we consider the case in which the game arena G is a complete k-ary tree with $k > 1$, which is a rooted tree whose inner vertices have exactly k children, and all leaves are at the same (highest) level. We will say that a (sub)tree T is said to be *unoccupied* if no vertex in T is occupied at all.

Firstly, we show a simple observation for Voronoi games $VG(T, n)$ in which satisfy $2n \leq k$. In this game of a few rounds, $\mathcal{W}$ occupies the root of T with his first move, and then $\mathcal{W}$ can dominate at least $\frac{N-1}{k}n + 1$ vertices, where $\frac{N-1}{k}$ is the number of vertices of a subtree rooted at level 1. Since $\mathcal{B}$ can dominate at most $\frac{N-1}{k}n$ vertices, $\mathcal{W}$ wins. More precisely, $\mathcal{W}$ plays the game using the following algorithm.

Algorithm 1: Simple strategy

if *the root of T is not occupied* **then**
  occupy the root of T;
else
  occupy a child v of the root such that v is the root of an unoccupied subtree;
end

Since $2n \leq k$, $\mathcal{W}$ can always occupy a vertex of level 1 that is the root of an unoccupied subtree. On the other hand, we can assume that $\mathcal{B}$ also occupies a child of the root in his optimal play. That is, $\mathcal{W}$ and $\mathcal{B}$ alternately occupy one of the unoccupied children of the root in their play. This strategy is obviously well-defined and a winning strategy for $\mathcal{W}$, whenever the game arena T satisfies $2n \leq k$.

Proposition 1 *Let $VG(G, n)$ be the discrete Voronoi game such that G is a complete k-ary tree with $2n \leq k$. Then the first player $\mathcal{W}$ always wins.*

We next turn to the more general cases. We call a k-ary tree *odd* (respectively, *even*) if k is odd (resp. even). Let T be a complete k-ary tree as a game arena, and H be the height of T. Note that the number N of vertices of T is given by $N = \frac{k^{H+1}-1}{k-1}$. For this game, we show the following theorem.

Theorem 1 *Let G be a complete k-ary tree ($k > 1$) with $N = \frac{k^{H+1}-1}{k-1}$ vertices. We assume that $N \geq \frac{k^3n^2-1}{k-1}$. Then, if k is even, the discrete Voronoi game $VG(G, n)$ ends in tie when the players play optimally. On the other hand, if k is odd, the first player $\mathcal{W}$ can always win.*

Hereafter, we assume that the tree is sufficiently large and contains $N \geq \frac{k^3n^2-1}{k-1}$ vertices throughout this section.

In subsection 3.1, we first show a winning strategy for the first player $\mathcal{W}$ when k is odd and the complete k-ary tree. Since our game arena is discrete, it is necessary to consider the relation between the number of children k and the number of rounds n. Indeed, $\mathcal{W}$ chooses one of two strategies according to the relation between k and n. We next consider the even k-ary tree in subsection 3.2, which completes the proof of Theorem 1.

3.1 Discrete Voronoi game on a large complete odd k-ary tree

We generalize the simple strategy to Voronoi games $VG(T, n)$ on a large complete k-ary tree, where $2n > k$ and k is odd ($k \geq 3$). Let T_i denote the number of vertices in a subtree rooted at a vertex in level i (i.e., $T_H = 1$, $T_i = kT_{i+1} + 1$, and $T_0 = N$). We say that a level h is a *key level* if the number k^h of vertices in the level satisfies $k^{h-1} < n \leq k^h$. A vertex in the key level is called *key vertex*.

We first show a simple proposition that gives a relationship between h and H.

Proposition 2 *If $N \geq \frac{k^3n^2-1}{k-1}$, the key level h always exists, and the size of a subtree rooted at a vertex in level $h+1$ is larger than a subtree induced by the vertices in levels from 0 to $h-1$.*

Proof: By the definition of h and $N = \frac{k^{H+1}-1}{k-1} \geq \frac{k^3n^2-1}{k-1}$, we have $k^{H+1} \geq k^3n^2 > k^3(k^{h-1})^2 = k^{2h+1}$. Hence $H > 2h$, which implies $H \geq 2h + 1$ since both H and h are integers. Thus the height of a subtree rooted at a vertex in level $h+1$ is at least h, and the subtree induced by the vertices in levels from 0 to $h-1$ has at most height $h-1$, which imply the claim. $\square$

Proposition 2 implies that the game arena T is so large that the subtrees rooted at level at most $h+1$ contain sufficient vertices compared the number of vertices between levels 0 and $h-1$. Thus, essentially, the player who takes more subtrees at levels around h than the other will win. (In other words, the vertices at level up to around h have little influence.)

Let $\{V_1^h, V_2^h, \ldots, V_{k^{h-1}}^h\}$ be a partition of vertices in the key level h such that for each i set V_i^h consists of k vertices which have a common parent. As mentioned above, a winning strategy is sensitive to the relation between k, h, and n. So, we here introduce a magic number $\alpha = \frac{2n}{k^h}$. By definition of the key level, we have $\frac{2}{k} < \alpha \leq 2$ (see Figure 2; we note that $\alpha = \frac{2n}{k^h} \neq 1$ since k is odd).

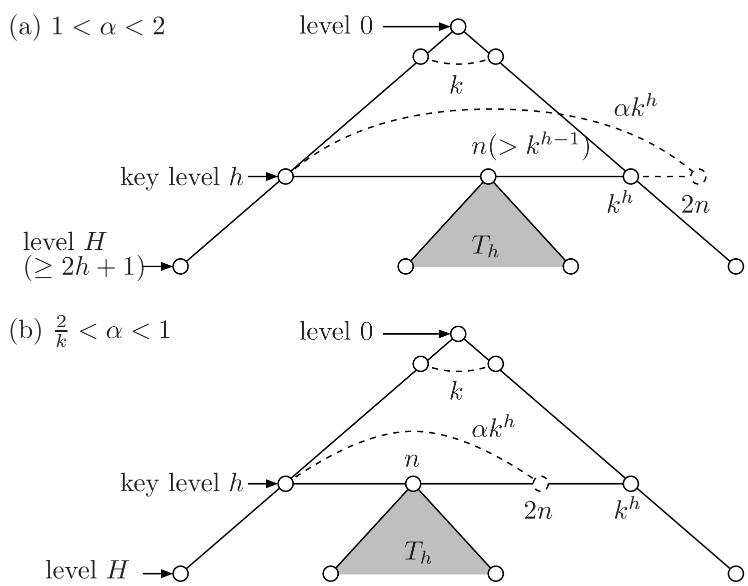

Figure 2: Notations on the game arena T.

Now we show a winning strategy for $\mathcal{W}$. For given N, n, and odd $k \geq 3$, $\mathcal{W}$ first computes the key level h with $k^{h-1} < n \leq k^h$ and the magic number $\alpha = \frac{2n}{k^h}$. Then $\mathcal{W}$ chooses one of two strategies according to the value of α. The precise strategy is shown in Algorithm 2; if $\alpha > \frac{k^2-2}{k(k-1)}$, $\mathcal{W}$ performs the strategy in case (A), and otherwise $\mathcal{W}$ performs the strategy in case (B).

Algorithm 2: Key-level strategy for $\mathcal{W}$

```
if α > (k²−2)/(k(k−1)) then /* We will refer to this case by (A) */
    if there is a key-vertex set V_i^h such that no vertex in V_i^h is occupied
    by W then /* Step (A)-1 */
        let V_i^h be a key-vertex set such that no vertex in V_i^h is occupied
        by W and the number of occupied vertices by B is maximum
        among V_j^h with 1 ≤ j ≤ k^{h−1};
        occupy an unoccupied key vertex in V_i^h;
    else if there is an unoccupied key vertex v then /* Step (A)-2 */
        occupy the key vertex v;
    else
        /* Step (A)-3                                              */
        let B be a set of unoccupied vertices v such that v is dominated
        by B in the sense that v is a child of u occupied by B;
        occupy a vertex v in B such that v has the minimum level but not
        less than h;
    end
else
    /* We will refer to this case by (B)                           */
    if there is an unoccupied vertex v in level h − 1 then /* Step (B)-1
    */
        occupy the vertex v in level h − 1;
    else if there is an unoccupied key vertex v whose parent is occupied
    by B then /* Step (B)-2 */
        occupy the unoccupied key vertex v dominated by B;
    else
        /* Step (B)-3                                              */
        if there is an unoccupied vertex v in level h + 1 whose parent is
        occupied by B then
            occupy the vertex v in level h + 1;
        else
            occupy any unoccupied vertex in level h + 1 whose parent is
            occupied by W;
        end
    end
end
```

Lemma 1 *The key-level strategy in Algorithm 2 is well-defined in a discrete Voronoi game $VG(T, n)$.*

Proof: We first observe that there exists the key level h by Proposition 2. According to the value of $\alpha = \frac{2n}{k^h}$ we have two cases.

First, we assume that $\mathcal{W}$ is in the case (A), that is, $\alpha > \frac{k^2-2}{k(k-1)}$. Then $\mathcal{W}$ tries to occupy as many vertices in level h as possible in steps (A)-1 and (A)-2. When $\mathcal{W}$ can occupy n vertices in level h, we are done. Hence assume that $\mathcal{W}$ occupies $x_h < n$ vertices in level h and $y_h = k^h - x_h$ vertices are occupied by $\mathcal{B}$. Then, $\mathcal{W}$ next tries to occupy the children of the vertices in level at least h occupied by $\mathcal{B}$. Now $ky_h - (n - x_h) = k^{h+1} - (n + (k-1)x_h) \geq k^{h+1} - (n + (k-1)(n-1)) = k^{h+1} - (kn - (k-1)) \geq k - 1 > 0$. Hence y_h vertices occupied by $\mathcal{B}$ have enough children to be occupied by $\mathcal{W}$. Even if one of these children is occupied by $\mathcal{B}$, it produces k more children. Since H is big enough, $\mathcal{B}$ cannot occupy all descendants of them. Hence $\mathcal{W}$ can always occupy a child of a vertex occupied by $\mathcal{B}$; in other words, step (A)-3 always can be performed.

We next assume that $\alpha \leq \frac{k^2-2}{k(k-1)}$, or $\mathcal{W}$ is in case (B). In this case, we can use the same argument above again, and show that $\mathcal{B}$ cannot occupy all the vertices in levels $h-1$, h, and $h+1$ under the condition $n \leq k^h$. Hence $\mathcal{W}$ always can find an unoccupied child of a vertex occupied by $\mathcal{B}$ in steps (B)-2 and (B)-3 after step (B)-1.

Therefore, the key-level strategy is well-defined. □

Lemma 2 *The strategy shown in Algorithm 2 is a winning strategy for $\mathcal{W}$ in a discrete Voronoi game $VG(T, n)$, where T is a sufficient large complete odd k-ary tree so that consists of at least $\frac{k^3n^2-1}{k-1}$ vertices.*

Proof: We first consider the case in which $\mathcal{W}$ uses the strategy in case (A); that is, $\alpha = \frac{2n}{k^h} > \frac{k^2-2}{k(k-1)} = 1 + \frac{k-2}{k(k-1)} > 1$. Then, we also have $2n = \alpha \times k^h > k^h$. According to the strategy of $\mathcal{B}$, we have the following two subcases; (A)-(a) $\mathcal{W}$ occupies n key vertices in level h, or (A)-(b) $\mathcal{W}$ cannot occupy n key vertices in level h.

Case (A)-(a): The first subcase of case (A) is when $\mathcal{W}$ occupies n key vertices in level h. We first specify the optimal strategy for $\mathcal{B}$. Since $k^{h-1} < n$, at least one key vertex in V_i^h will be occupied by $\mathcal{W}$ for every i with $1 \leq i \leq k^{h-1}$. Thus, to dominate other key vertices, $\mathcal{B}$ occupies each vertex in level $h-1$. (Then $\mathcal{B}$ also dominates all vertices in levels from 0 to $h-2$.) Moreover, using $n - k^{h-1}$ vertices, $\mathcal{B}$ also occupies the children of the key vertices occupied by $\mathcal{W}$. It is not difficult to see that this is an optimal strategy for $\mathcal{B}$ in this subcase. Thus we have the following equations.

$$|\mathcal{V}_{\mathcal{B}}| = \frac{k^h - 1}{k - 1} + (k^h - n)T_h + (n - k^{h-1})T_{h+1}$$
$$|\mathcal{V}_{\mathcal{W}}| = n + (kn - (n - k^{h-1}))T_{h+1}$$

Hence we have

$$\begin{aligned}|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| &= n - \frac{k^h - 1}{k - 1} + ((k-2)n + 2k^{h-1})T_{h+1} - (k^h - n)T_h \\ &= 2n - k^h - \frac{k^h - 1}{k - 1} + (2(k-1)n + 2k^{h-1} - k^{h+1})T_{h+1} \\ &\qquad\qquad\qquad\qquad (\text{by } T_h = kT_{h+1} + 1)\end{aligned}$$

In case (A), we have $\frac{2n}{k^h} = \alpha > \frac{k^2-2}{k(k-1)} = 1 + \frac{k-2}{k(k-1)} > 1$. Hence $2n - k^h > 0$. On the other hand, by $\frac{2n}{k^h} = \alpha$, $(2(k-1)n+2k^{h-1}-k^{h+1}) = (k(k-1)\alpha+2-k^2)k^{h-1}$. Here, by $\alpha > \frac{k^2-2}{k(k-1)}$, we obtain $(k(k-1)\alpha+2-k^2)k^{h-1} > (k^2-2+2-k^2)k^{h-1} = 0$. We remind that $(2(k-1)n + 2k^{h-1} - k^{h+1})$ gives the number of dominated subtrees rooted at level $h+1$. Hence $(2(k-1)n + 2k^{h-1} - k^{h+1}) > 0$ implies $(2(k-1)n + 2k^{h-1} - k^{h+1}) \geq 1$. Thus $|\mathcal{V}_\mathcal{W}| - |\mathcal{V}_\mathcal{B}| > 0 - \frac{k^h-1}{k-1} + T_{h+1} > 0$ by Proposition 2. Therefore $\mathcal{W}$ wins in this case.

Case (A)-(b): The second subcase of case (A) is when $\mathcal{W}$ cannot occupy n vertices in level h. Let x_h be the number of key vertices occupied by $\mathcal{W}$. By the algorithm, $\mathcal{B}$ occupies $k^h - x_h$ key vertices in this case. Let $y_h := k^h - x_h$. Then, an optimal strategy for $\mathcal{B}$ is to occupy the children of the key vertices occupied by $\mathcal{W}$ as possible as $\mathcal{B}$ can after occupying y_h key vertices. Then, on the other hand, $\mathcal{W}$ also occupies the children of the key vertices occupied by $\mathcal{B}$ after occupying x_h key vertices. We here observe that $y_h + kx_h = k^h + (k-1)x_h > n$ and $x_h + ky_h = k^h + (k-1)y_h > n$. Thus, in their (optimal) playing, both players occupy the vertices in levels only h and $h+1$. We note that a subtree rooted at an unoccupied vertex v of level $h+1$ is dominated by the parent of v. Moreover, by step (A)-1, $\mathcal{B}$ cannot dominate all vertices in V_i^h for some i. This implies that we can ignore all vertices in levels from 0 to $h-1$ since they are tie and not dominated in their optimal playing. Thus, we have

$$\begin{aligned} |\mathcal{V}_\mathcal{W}| &= x_h + (n - x_h)T_{h+1} + (kx_h - (n - y_h))T_{h+1} \\ |\mathcal{V}_\mathcal{B}| &= y_h + (n - y_h)T_{h+1} + (ky_h - (n - x_h))T_{h+1} \end{aligned}$$

and hence

$$|\mathcal{V}_\mathcal{W}| - |\mathcal{V}_\mathcal{B}| = (x_h - y_h) + ((k-2)(x_h - y_h))T_{h+1}.$$

Here, we have $x_h > y_h$ since $\mathcal{W}$ is the first player and $k \geq 3$ is odd. Thus $|\mathcal{V}_\mathcal{W}| - |\mathcal{V}_\mathcal{B}| > 0$ and $\mathcal{W}$ wins.

Now we turn to the other case; $\mathcal{W}$ uses the strategy in case (B). Then we have $\alpha \leq \frac{k^2-2}{k(k-1)} = 1 + \frac{k-2}{k(k-1)}$. We also have $\frac{2}{k} < \alpha = \frac{2n}{k^h}$ by the definition of the key level ($k^{h-1} < n$). Then, according to the strategy of $\mathcal{B}$, we have the following two subcases; (B)-(a) $\mathcal{W}$ occupies all k^{h-1} vertices in level $h-1$, or (B)-(b) $\mathcal{W}$ cannot occupy all k^{h-1} vertices in level $h-1$.

Case (B)-(a): In this case, $\mathcal{W}$ first occupies all k^{h-1} vertices. During this process, an optimal playing of $\mathcal{B}$ is to occupy as much key vertices as $\mathcal{B}$ can (since T_h is greater than the vertices in levels 0 to $h-2$). Then, after occupying all vertices in level $h-1$, $\mathcal{W}$ occupies the children of the key vertices occupied by $\mathcal{B}$. Thus, after their optimal playing, $k^h - n$ key vertices are not occupied and hence dominated by $\mathcal{W}$. Hence, in this case, we have

$$\begin{aligned}|\mathcal{V}_\mathcal{W}| - |\mathcal{V}_\mathcal{B}| &= \left(\frac{k^h-1}{k-1} + (k^h-n)T_h + (n-k^{h-1})T_{h+1}\right)\\ &\quad - (n + (kn - (n-k^{h-1}))T_{h+1})\\ &= \frac{k^h-1}{k-1} - n + (k^h-n)T_h + (2(n-k^{h-1}) - kn)T_{h+1}\\ &= \frac{k^h-1}{k-1} - 2n + k^h + (k^{h+1} - 2kn + 2n - 2k^{h-1})T_{h+1}\\ &\qquad\qquad \text{(by } T_h = kT_{h+1}+1.)\end{aligned}$$

Here $k^{h+1} - 2kn + 2n - 2k^{h-1} = (k^2 - 2 - (k-1)\alpha k)k^{h-1}$ by $2n = \alpha k^h$, and $k^2 - 2 - (k-1)\alpha k \geq k^2 - 2 + (1-k)k\frac{k^2-2}{k(k-1)} = 0$ by $\alpha \leq \frac{k^2-2}{k(k-1)}$. Hence it is sufficient to show that $\frac{k^h-1}{k-1} - 2n + k^h > 0$. Here we again use $2n = \alpha k^h$ and $\alpha \leq \frac{k^2-2}{k(k-1)} = 1 + \frac{2}{k} - \frac{1}{k-1}$, we have $\frac{k^h-1}{k-1} - 2n + k^h = \frac{k^h-1}{k-1} + k^h - \alpha k^h \geq \frac{k^h-1}{k-1} + k^h - k^h - 2k^{h-1} + \frac{k^h}{k-1} = \frac{2k^{h-1}-1}{k-1} > 0$. Thus $\mathcal{W}$ wins in this subcase.

Case (B)-(b): In this case, $\mathcal{W}$ first tries to occupy all k^{h-1} vertices, but it is obstructed by $\mathcal{B}$. Let x_{h-1} and y_{h-1} be the numbers of vertices in level $h-1$ occupied by $\mathcal{W}$ and $\mathcal{B}$, respectively. Since $\mathcal{W}$ is the first player, $x_{h-1} > \frac{k^{h-1}}{2} > y_{h-1}$ and $x_{h-1} + y_{h-1} = k^{h-1}$. After occupying the level $h-1$, optimal playing for $\mathcal{B}$ is to occupy the key vertices dominated by $\mathcal{W}$. On the other hand, simultaneously, $\mathcal{W}$ also occupies the key vertices dominated by $\mathcal{B}$. According to the values of y_{h-1} (or strategy of $\mathcal{B}$) and α, we again have two more subcases;

(B)-(b)-(i) $\mathcal{W}$ can occupy $n - x_{h-1}$ key vertices and the game is over, or

(B)-(b)-(ii) $\mathcal{W}$ occupies all ky_{h-1} key vertices which are children of the vertices occupied by $\mathcal{B}$ in level $h-1$ and the game still continues.

Case (B)-(b)-(i): In this case, $\mathcal{W}$ occupies x_{h-1} vertices in level $h-1$ and $(n - x_{h-1})$ key vertices under the vertices occupied by $\mathcal{B}$. We first observe that $kx_{h-1} + y_{h-1} = kx_{h-1} + (n - x_{h-1}) = n + (k-1)x_{h-1} > n$. Hence, since there exist enough unoccupied key vertices, $\mathcal{B}$ occupies $y_{h-1}(= k^{h-1} - x_{h-1})$ vertices in level $h-1$ and $n - y_{h-1}$ key vertices after his optimal playing under this assumption. Here note that $x_{h-1} > y_{h-1}$ and there are no occupied vertices in levels from 0 to $h-2$. We ignore this positive benefit for $\mathcal{W}$ in this area since they are not essential. Thus, in this case, we have

$$\begin{aligned}|\mathcal{V}_\mathcal{W}| - |\mathcal{V}_\mathcal{B}| &= (x_{h-1} + (n - x_{h-1})T_h + (kx_{h-1} - n + y_{h-1})T_h)\\ &\quad - (y_{h-1} + (n - y_{h-1})T_h + (ky_{h-1} - n + x_{h-1})T_h)\\ &= (x_{h-1} - y_{h-1}) + k(x_{h-1} - y_{h-1})T_h > 0\end{aligned}$$

since $\mathcal{W}$ is the first player which implies that $x_{h-1} > y_{h-1}$.

Case (B)-(b)-(ii): In this case, $\mathcal{W}$ occupies all ky_{h-1} key vertices which are children of the vertices occupied by $\mathcal{B}$ in level $h-1$, and $x_{h-1}+ky_{h-1} < n$. As in the case (B)-(b)-(i), since $kx_{h-1}+y_{h-1} = kx_{h-1}+(n-x_{h-1}) = n+(k-1)x_{h-1} > n$, $\mathcal{B}$ occupies y_{h-1} vertices in level $h-1$ and $n-y_{h-1}$ key vertices which are children of the occupied vertices by $\mathcal{W}$ in level $h-1$ in his optimal playing. Hence, in this case, $\mathcal{W}$ first occupies x_{h-1} vertices in level $h-1$, and next occupies ky_{h-1} key vertices which were dominated by $\mathcal{B}$. Now, we observe that $x_{h-1}+ky_{h-1}+k(n-y_{h-1}) = k^{h-1}+(k-2)y_{h-1}+kn > n$. Thus, finally, $\mathcal{W}$ occupies $n-x_{h-1}-ky_{h-1}$ vertices in level $h+1$ which are the children of the key vertices occupied by $\mathcal{B}$, and the game is over. To simplify, we ignore the vertices up to level $h-2$ which are not essential. Then, we have

$$\begin{aligned}
|\mathcal{V}_{\mathcal{W}}| &= x_{h-1} + (ky_{h-1} + kx_{h-1} - n + y_{h-1})T_h + (n - x_{h-1} - ky_{h-1})T_{h+1} \\
&= (k+1)(x_{h-1} + y_{h-1}) - n + (k^2(x_{h-1} + y_{h-1}) - nk + n - x_{h-1})T_{h+1} \\
&= (k+1)k^{h-1} - n + (k^{h+1} - (k-1)n - x_{h-1})T_{h+1} \\
|\mathcal{V}_{\mathcal{B}}| &= y_{h-1} + (n - y_{h-1}) + (k(n - y_{h-1}) - n + x_{h-1} + ky_{h-1})T_{h+1} \\
&= n + (kn - n + x_{h-1})T_{h+1}.
\end{aligned}$$

Thus,

$$|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| = (k+1)k^{h-1} - \alpha k^h + (k^{h+1} - (k-1)\alpha k^h - 2x_{h-1})T_{h+1},$$

and letting $\alpha \geq \frac{k^2-2}{k(k-1)}$, we have

$$|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| \geq \frac{k^{h-1}}{k-1} + 2(k^{h-1} - x_{h-1})T_{h+1} > 0.$$

Hence $\mathcal{W}$ wins.

Therefore, the first player $\mathcal{W}$ always wins when he follows Algorithm 2. This completes the proof of Lemma 2. □

3.2 Discrete Voronoi game on a large complete even k-ary tree

We consider the case in which the game arena T is a large complete even k-ary tree. We first define a *symmetric strategy* for $\mathcal{B}$ as follows. We assume that the game arena T is drawn in a natural way; the root is the top, and k^i vertices in each level i are ordered and numbered from 1 to k^i from left to right. Then a symmetric strategy for $\mathcal{B}$ is to occupy the vertex j in level i if $\mathcal{W}$ occupies the vertex $k^i - j + 1$ in the previous turn.

Theorem 2 *We assume that the game $VG(T,n)$ satisfies $k > 2n$, k is even, and $N \geq \frac{k^3n^2-1}{k-1}$. Then the game always ends in tie if both players play optimally.*

Proof: We first observe that if $\mathcal{W}$ employs the key-level strategy in Algorithm 2, $\mathcal{B}$ can employ the symmetric strategy of the key-level strategy. This is possible since $\mathcal{W}$ never occupies the root of the tree, and hence $\mathcal{B}$ can always occupy the symmetric vertex against $\mathcal{W}$. Then the game ends in tie. More precisely, when $\alpha = \frac{2n}{k^h} > \frac{k^2-2}{k(k-1)}$, we have $2n > k^h$ and hence obtain case (A)-(b). Then, we have $|\mathcal{V}_{\mathcal{W}}| - |\mathcal{V}_{\mathcal{B}}| = (x_h - y_h) + ((k-2)(x_h - y_h))T_{h+1}$ with $x_h = y_h$. On the other hand, when $\alpha = \frac{2n}{k^h} \leq \frac{k^2-2}{k(k-1)}$, we have case (B)-(b). When $\mathcal{W}$ plays according to Algorithm 2 and $\mathcal{B}$ plays according to the symmetric key-level strategy, (B)-(b)-(i) occurs if $\frac{2}{k} \leq \alpha \leq \frac{k-1}{k} = 1 - \frac{1}{k}$, and (B)-(b)-(ii) occurs if $1 - \frac{1}{k} < \alpha \leq \frac{k^2-2}{k(k-1)} = 1 + \frac{1}{k} + \frac{1}{k(k-1)}$. In both cases, the symmetric strategy works and the game ends in tie.

We now consider the game from the viewpoint of $\mathcal{W}$. It is easy to see that $\mathcal{W}$ can use the key-level strategy; if $\mathcal{B}$ does not employ the symmetric strategy, then $\mathcal{W}$ wins or the game ends in tie. In other words, $\mathcal{W}$ never loses if $\mathcal{W}$ employs the key-level strategy.

We next consider the game from the viewpoint of $\mathcal{B}$. Imagine that $\mathcal{B}$ employs the following strategy. As long as $\mathcal{W}$ uses the key-level strategy, $\mathcal{B}$ employs the symmetric strategy. Once $\mathcal{W}$ is out of the key-level strategy, $\mathcal{B}$ proceeds to his own key-level strategy according to Algorithm 2. Then we can show that $\mathcal{B}$ wins or the game ends in tie. Intuitively, even if $\mathcal{W}$ occupies the vertices in levels from 0 to $h-1$, they have little influences and $\mathcal{B}$ obtains more vertices from only one subtree rooted at h. On the other hand, even if $\mathcal{W}$ occupies the vertices v in levels $h+1$ or more, $\mathcal{B}$ also obtains more vertices from the subtrees rooted at h that contains v. The details are obtained straightforwardly by the same analyses in the proof of Lemma 2, and omitted here. Thus $\mathcal{W}$ cannot win as long as $\mathcal{B}$ employs the (symmetric) key-level strategy.

To complete the proof, we here note that taking the root gives no advantage to the players. If one player A takes the root, the other player B can proceed the key-level strategy. Since the dominated subtree containing the root is so small, this may produce good results for B, but never for A. Therefore, without loss of generality, we can assume that both players never occupy the root if they play optimally.

Hence, if both players play optimally, they never lose, and the game always ends in tie. $\square$

4 NP-hardness for general graphs

In this section, we show that the discrete Voronoi game is intractable on general graphs even if we restrict ourselves to the one-round case. To show this, we consider the following special case:

Problem 1:
Input: A graph $G = (V, E)$, a vertex $u \in V$, and n. **Output:** Determine whether $\mathcal{B}$ has a winning strategy on G by n occupations after just one occupation of u by $\mathcal{W}$.

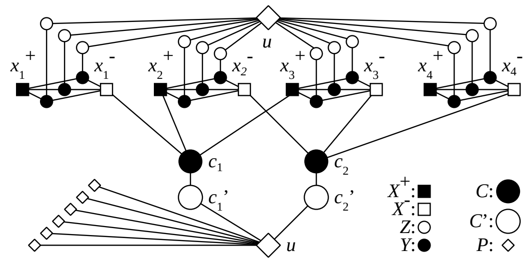

Figure 3: Reduction from $F = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$

That is, $\mathcal{W}$ first occupies u, and never occupy any more, and $\mathcal{B}$ can occupy n vertices in any way. Then we have the following Theorem:

Theorem 3 *Problem 1 is* NP*-complete.*

Proof: It is clear that Problem 1 is in NP. Hence we prove the completeness by showing the polynomial time reduction from a restricted 3SAT such that each variable appears at most three times in a given formula [5, Proposition 9.3]. Let F be a given formula with the set X of variables $\{x_1, x_2, \ldots, x_n\}$ and the set C of clauses $\{c_1, c_2, \ldots, c_m\}$, where $n = |X|$ and $m = |C|$.

Now we show a construction of G. Let $X^+ := \{x_i^+ \mid x_i \in X\}$, $X^- := \{x_i^- \mid x_i \in X\}$, $Y := \{y_i^j \mid i \in \{1, 2, \ldots, n\}, j \in \{1, 2, 3\}\}$, $Z := \{z_i^j \mid i \in \{1, 2, \ldots, n\}, j \in \{1, 2, 3\}\}$, $C' := \{c'_1, c'_2, \ldots, c'_m\}$, $P := \{p_1, p_2, \ldots, p_{2n-2}\}$. Then the set of vertices of G is defined by $V := \{u\} \cup X^+ \cup X^- \cup Y \cup Z \cup C \cup C' \cup P$. The set of edges E is defined by the union of the following edges; $\{\{u, z\} \mid z \in Z\}$, $\{\{y_i^j, z_i^j\} \mid y_i^j \in Y, z_i^j \in Z\}$, $\{\{x_i^+, y_i^j\} \mid x_i^+ \in X^+, y_i^j \in Y\}$, $\{\{x_i^-, y_i^j\} \mid x_i^- \in X^-, y_i^j \in Y\}$, $\{\{x_i^+, c_j\} \mid x_i^+ \in X^+, c_j \in C$ if c_j contains literal $x_i\}$, $\{\{x_i^-, c_j\} \mid x_i^- \in X^-, c_j \in C$ if c_j contains literal $\bar{x}_i\}$, $\{\{c_j, c'_j\} \mid c_j \in C, c'_j \in C'\}$, $\{\{c'_j, u\} \mid c'_j \in C'\}$, and $\{\{u, p_i\} \mid p_i \in P\}$.

An example of the reduction for the formula $F = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$ is depicted in Figure 3. Small white and black circles are the vertices in Z and Y, respectively; small black and white boxes are the vertices in $X^+ \cup X^-$; large black and white circles are the vertices in C and C', respectively; two white large diamonds are the same vertex u; and small diamonds are the vertices in P. It is easy to see that G contains $10n + 2m - 1$ vertices, and hence the reduction can be done in polynomial time.

Now we show that F is satisfiable if and only if $\mathcal{B}$ has a winning strategy. We first observe that for $\mathcal{B}$, occupying the vertices in $X^+ \cup X^-$ gives more outcome than occupying the vertices in $Y \cup Z \cup C \cup C'$: Each vertex in $Z \cup C' \cup P$ has distance 1 from u (aside from u, described in white in Figure 3), each vertex in $Y \cup C$ has distance 2 from u, and each vertex in $X^+ \cup X^-$ has distance 3 from u. Moreover, each vertex in $Y \cup C$ is adjacent to two or three vertices in $X^+ \cup X^-$. Furthermore, $\mathcal{B}$ has n chances to occupy which is equal to $|X^+| = |X^-|$. Thus, occupying either x_i^+ or x_i^- for each i with $1 \le i \le n$, $\mathcal{B}$ dominates all vertices in $X^+ \cup X^- \cup Y$, and any other way archives less outcome. Therefore, we can assume that $\mathcal{B}$ occupies one of x_i^+ and x_i^- for each i with $1 \le i \le n$.

When there is an assignment $(a_1, a_2, \ldots, a_n)$ that satisfies F, $\mathcal{B}$ can also dominate all vertices in C by occupying x_i^+ if $a_i = 1$, and occupying x_i^- if $a_i = 0$. Hence, $\mathcal{B}$ dominates $5n + m$ vertices in the case, and then $\mathcal{W}$ dominates all vertices in Z, C' and P, that is, $\mathcal{W}$ dominates $1+3n+m+2n-2 = 5n+m-1$ vertices. Therefore, $\mathcal{B}$ wins if F is satisfiable.

On the other hand, if F is unsatisfiable, $\mathcal{B}$ can dominate at most $5n+m-1$ vertices. In this case, the vertex in C corresponding to the unsatisfied clause is dominated by u. Thus $\mathcal{W}$ dominates at least $5n + m$ vertices, and hence $\mathcal{W}$ wins if F is unsatisfiable.

Therefore, Problem 1 is NP-complete. □

Next we show that the discrete Voronoi game is NP-hard even in the one-round case. More precisely, we show the NP-completeness of the following problem:

Problem 2:
Input: A graph $G = (V, E)$, a vertex set $S \subseteq V$ with $n := \lvert S \rvert$. **Output:** Determine whether $\mathcal{B}$ has a winning strategy on G by n occupations after n occupations of the vertices in S by $\mathcal{W}$.

Corollary 1 *Problem 2 is* NP*-complete.*

Proof: We use the same reduction in Theorem 3. We call each vertex of degree 1 in P *pendant vertex*. Let S be a set that contains u and $(n-1)$ pendant vertices in P. Then we immediately have NP-completeness of Problem 2. □

5 PSPACE-completeness for general graphs

In this section, we show that the discrete Voronoi game is intractable on general graphs. More precisely, we consider the following general case:

Problem 3:
Input: A graph $G = (V, E)$ and n. **Output:** Determine whether $\mathcal{W}$ has a winning strategy on G after n occupations.

Then we have the following Theorem:

Theorem 4 *The discrete Voronoi game is* PSPACE*-complete in general.*

Proof: We show that Problem 3 is PSPACE-complete. It is clear that Problem 3 is in PSPACE. Hence we prove the completeness by showing the polynomial time reduction from the following two-person game:

G_{pos}(Pos Dnf):
Input: A positive DNF formula A (that is, a DNF formula containing no negative literal). **Rule:** Two players alternately choose some variable of A which has not been chosen yet. The game ends after all variables of A have been chosen. The first player wins if and only if A is true when all variables chosen by the first player are set to 1 and all variables chosen by the second player are set to 0. (In other words, the first player wins if and only if he takes every variable of some disjunct.) **Output:** Determine whether the first player has the winning strategy for A.

The game G_{pos}(Pos Dnf) is PSPACE-complete even with inputs restricted to DNF formulas having at most 11 variables in each disjunct (see [6, Game 5(b)]).

Let A be a positive DNF formula with n variables $\{x_1, \ldots, x_n\}$ and m disjuncts $\{d_1, \ldots, d_m\}$. Without loss of generality, we assume that n is even. Now we show a construction of $G = (V, E)$. Let $X = \{x_1, \ldots, x_n\}$, $D = \{d_1, \ldots, d_m\}$, $U = \{u_1, u_2\}$, and $P = \{p_1, \ldots, p_{2n^2+6n}\}$. Then the set of vertices of G is defined by $V := X \cup D \cup U \cup P$.

In this reduction, each pendant vertex in P is attached to some vertex in $X \cup U$ to make it "heavy."

The set of edges E consists of the following edges; (1) make X a clique with edges $\{x_i, x_j\}$ for each $1 \le i < j \le n$, (2) join a vertex x_i in X with a vertex d_j in D if A has a disjunct d_j that contains x_i, (3) join each d_j with u_2 by $\{d_j, u_2\}$ for each $1 \le j \le m$, (4) join u_1 and u_2 by $\{u_1, u_2\}$, (5) attach $2n$ pendant vertices to each x_i with $1 \le i \le n$, and (6) attach $3n$ pendant vertices to each u_i with $i = 1, 2$.

An example of the reduction for the formula $A = (x_1 \wedge x_2 \wedge x_4 \wedge x_5) \vee (x_3 \wedge x_5 \wedge x_7 \wedge x_8) \vee (x_6 \wedge x_8)$ is depicted in Figure 4: Large white numbered circles are pendant vertices, and the number indicates the number of pendant vertices attached to the vertex.

Each player will occupy $(n/2) + 1$ vertices in G. It is easy to see that G contains $n + m + 2 + 6n + 2n^2 = 2n^2 + 7n + m + 2$ vertices, and hence the reduction can be done in polynomial time.

Now we show that the first player of G_{pos}(Pos Dnf) for A wins if and only if $\mathcal{W}$ of the discrete Voronoi game for G wins.

Since the vertices in X and U are heavy enough, $\mathcal{W}$ and $\mathcal{B}$ always occupy the vertices in X and U. In fact, occupying a vertex d_j in D does not bring any advantage; since X induces a clique, the pendant vertices attached to some x_i in $N(d_j)$ will be canceled by occupying any $x_{i'}$ by the other player.

Since the vertices in U are heavier than the vertices in X, $\mathcal{W}$ and $\mathcal{B}$ first occupy one of u_1 and u_2, and occupy the vertices in X, and the game will end when all vertices in X are occupied.

The player $\mathcal{W}$ has two choices.

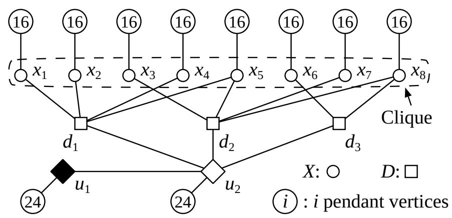

Figure 4: Reduction from $A = (x_1 \wedge x_2 \wedge x_4 \wedge x_5) \vee (x_3 \wedge x_5 \wedge x_7 \wedge x_8) \vee (x_6 \wedge x_8)$

We first consider the case $\mathcal{W}$ occupies u_2. Then $\mathcal{B}$ has to occupy u_1, and $\mathcal{W}$ and $\mathcal{B}$ occupy $n/2$ vertices in X. It is easy to see that in the case, they are in tie on the graph induced by $U \cup X \cup P$. Hence the game depends on the domination for D. In G_{pos}(Pos Dnf), if the first player has the winning strategy for A, the first player can take every variable of a disjunct d_j. Hence, following the strategy, $\mathcal{W}$ can occupy every variable in $N(d_j)$ on G. Then, since $\mathcal{W}$ also occupies u_2, d_j is dominated by $\mathcal{W}$. On the other hand, $\mathcal{B}$ cannot dominate any vertex in D since $\mathcal{W}$ occupies u_2. Hence, if the first player of G_{pos}(Pos Dnf) has a winning strategy, so does $\mathcal{W}$. (Otherwise, the game ends in tie.)

Next, we consider the case in which $\mathcal{W}$ occupies u_1. Then $\mathcal{B}$ can occupy u_2. The game again depends on the occupation for D. However, in this case, $\mathcal{W}$ cannot dominate any vertex in D since $\mathcal{B}$ has already occupied u_2. Hence $\mathcal{W}$ will lose or they will be in tie at best.

Thus $\mathcal{W}$ has to occupy u_2 at first, and then $\mathcal{W}$ has winning strategy if and only if the first player of G_{pos}(Pos Dnf) has it.

Therefore, Problem 3 is PSPACE-complete. □

6 Conclusion

We gave winning strategies for the first player $\mathcal{W}$ on the discrete Voronoi game $VG(T, n)$ where T is a large complete k-ary tree with odd k. It seems that $\mathcal{W}$ has an advantage even if the complete k-ary tree is not large, which is future work. As a special case, it remains open when T is just a path. By computer experiments, we obtain that all games on a path of length at most 30 are in tie. So we conjecture that $VG(T, n)$ is always in tie for a path T.

In our strategy, it is essential that each subtree of the same depth has the same size. Considering general trees is the natural next problem. The simplest case, $n = 1$, can be solved as follows. When $n = 1$, the discrete Voronoi game on a tree is essentially equivalent to finding a *median* vertex of a tree. The deletion of a median vertex partitions the tree so that no component contains more than $n/2$ of the original n vertices. It is well known that a tree has either one or two median vertices, which can be found in linear time; see, e.g., [4]. In the former case, $\mathcal{W}$ wins by occupying the median vertex. In the latter case, two players tie. This algorithm corresponds to our Algorithm 1.

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