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On Basic Structures of General Topology in Constructive Mathematics

By Tatsuji Kawai

A project paper submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
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Written under the supervision of
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March, 2012

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Associate Professor Kazuhiro Ogata
Professor Hajime Ishihara
Professor Satoshi Tojo

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Chapter 1

Introduction

1.1 Constructivism

There are several approaches to constructive mathematics. This thesis follows the approach set out by Errett Bishop, Bishop's Constructive Mathematics, in his book *Foundation of Constructive Mathematics* [10]. The distinguishing feature of his approach is that it is consistent with classical mathematics as well as the other two main schools of constructive mathematics, Brouwer's intuitionism and Markov's Constructive Recursive Mathematics.

All three schools accept the principles of intuitionistic logic. However, Brouwer and Markov accepted principles which are inconsistent with classical mathematics. Brouwer, for example, accepted the continuity principle which states that all functions from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$ are continuous. The principle expresses a view that the value of such functions at each sequence is completely determined by some initial segment of its argument. The continuity principle, together with a certain form of axiom of choice, implies that all functions on the real numbers are continuous, which clearly contradicts classical mathematics. On the other hand, Markov's view that all mathematical objects are algorithms led him to accept Church's Thesis which states that all functions on the natural numbers are recursive. He also accepted Markov's principle which states that for any (recursive) function $\alpha \in \mathbf{N}^{\mathbf{N}}$

$$\mathrm{MP}: \neg\neg\exists n(\alpha(n) \neq 0) \rightarrow \exists n\,(\alpha(n) \neq 0)$$

which expresses a view that if it is impossible that an algorithm does not halt then it must halt. Church's Thesis, together with Markov's principle, also implies that all functions on the real line are continuous. It is also known that Brouwer's intuitionism and Markov's Constructive Recursive Mathematics contradict each other. Thus we can say that Bishop's Constructive Mathematics is the most general style of constructive mathematics.

Although Bishop did not make his foundational background explicit, it is generally accepted that Bishop's Constructive Mathematics is also predicative: it is not permissible to define a set d in term of a collection in which d is to be an element. For example, to define the transitive closure of a relation $r \subseteq X \times X$ by the intersection of all the transitive relations which include r is not permissible.

Since the publication of Bishop's book, two possible foundational approaches to Bishop's Constructive Mathematics had emerged. One of them is Constructive Set Theory by Myhill [19] and the other is Intuitionistic Type Theory by Martin-Löf [17] (henceforth, simply the type theory). Constructive Set Theory was introduced for formalization of Bishop's mathematics in a style that is close to that of classical set theories. On the other hand, the type theory makes the intuitionistic reading of the logical connectives explicit, and for this reason, it is regarded as the most fundamental framework for constructive mathematics. However, the presentation of mathematics in the type theory is very different from widespread set theoretical presentations. Aczel, in [1, 2, 3], introduced a version of Constructive Set Theory, the constructive Zermelo-Frankel set theory (CZF). The distinguishing feature of CZF compared to that of Myhill's is that it has an interpretation in the type theory, which means the theorems in CZF are also valid in the type theory, and hence they are given constructive justification. CZF uses intuitionistic logic and some modifications of the axioms of the classical set theory ZF to avoid impredicativity. Hence, CZF does not have Powerset axiom nor Full Separation scheme; separations are restricted to bounded formulae. Instead, CZF introduced the axioms of Strong Collection and Subset Collection. The full set of axioms of CZF are described in Section 2.1.

This thesis takes CZF as a foundational framework. We also accept two axioms of CZF: *Relativized Dependent Choice* (RDC) and *Regular Extension Axiom* (REA). Section 2.1 describes these axioms as well as how to carry out some of the basic mathematical constructions in CZF.

For the history of constructive mathematics and constructive mathematics in general, see [27]. For the practice of Bishop's Constructivism Mathematics, see, for example, [10, 11, 12, 13, 18].

1.2 Basic Pairs and Concrete Spaces

The topic of basic pair and concrete space was initiated by Sambin in [24]. The notion of basic pair and concrete space arose from the analysis of the usual notion of topological space in the light of intuitionistic and predicative foundations, such as the type theory and CZF.

The standard definition of a topological space $(X, \mathcal{O}(X))$ where X is a set and $\mathcal{O}(X)$ is a set of open subsets of X which satisfies the following axioms is unacceptable constructively.

O1 $\emptyset, X \in \mathcal{O}(X)$,

O2 if $U, V \in \mathcal{O}(X)$, then $U \cap V \in \mathcal{O}(X)$,

O3 for any family $(U_i)_{i\in I}$ of $\mathcal{O}(X)$, $\bigcup_{i\in I} U_i \in \mathcal{O}(X)$.

The problem is that the open sets of an inhabited space cannot form a set in CZF. This can be seen by the following argument which is due to Fox [15, Chapter 2].

Proposition 1.2.1. *If the open subsets of an inhabited space forms a set, then Powerset axiom is derivable in* CZF*.*

Proof. Suppose that $(X, \mathcal{O}(X))$ is an inhabited topological space, i.e. there is an element $a \in X$. Note that $U_p = \{x \in X \mid 0 \in p\}$ is an open set for each $p \in \mathrm{Pow}(\{0\})$: we have $(\forall x \in U_p)\, x \in X \subseteq U_p$ and X is an open set. Then the mapping $p \mapsto U_p$ is a bijection between $\mathrm{Pow}(\{0\})$ and $\mathcal{U} = \{U_p \mid p \in \mathrm{Pow}(\{0\})\}$. It is clearly surjective. Moreover, suppose that $U_p = U_q$ for $p, q \in \mathrm{Pow}(\{0\})$. Let $n \in p$. Then $n = 0$, so $a \in U_p$, and hence $a \in U_q$, i.e. $0 \in q$. Therefore $p \subseteq q$. Similarly we have $q \subseteq p$, and thus $p = q$. So the mapping is injective. Now, suppose that $\mathcal{O}(X)$ forms a set. Then we have $\mathcal{U} = \mathcal{V} = \{U \in \mathcal{O}(X) \mid (\forall x, y \in X)\, x \in U \leftrightarrow y \in U\}$. The inclusion $\mathcal{U} \subseteq \mathcal{V}$ is clear. For the other direction, let $U \in \mathcal{V}$, and put $p = \{n \in \{0\} \mid (\forall x \in X)\, x \in U\}$. Then, it is easy to see $U = U_p$, and hence $\mathcal{V} \subseteq \mathcal{U}$. Therefore $\mathcal{U}$ is a set by Bounded Separation scheme, so $\mathrm{Pow}(\{0\})$ is a set. Given any set A, $\mathrm{Pow}(\{0\})^A$ is a set by Exponentiation (cf. Definition 2.1.19). Then, we can show that $\mathrm{Pow}(\{0\})^A \cong \mathrm{Pow}(A)$ (cf. [1, Proposition 2.3]). Thus $\mathrm{Pow}(A)$ is a set for every set A. ☐

A constructively acceptable definition of a topological space is obtained by specifying a certain set of subsets of X as a base for a topology, that is, a topological space is a pair $(X, \mathcal{B})$ of set X and set $\mathcal{B}$ of subsets of X such that

1. $(x \in X)\,(A, B \in \mathcal{B})\,[x \in A \cap B \to (\exists C \in \mathcal{B})\, x \in C \subseteq A \cap B]$,
2. $X = \bigcup_{B \in \mathcal{B}} B$.

A subset $U \in \mathrm{Pow}(X)$ is defined to be open if $U = \bigcup_{B \in \mathcal{C}} B$ for some subset $\mathcal{C}$ of $\mathcal{B}$. This is essentially the notion of *neighbourhood space* by Bishop [10].

We can generalize the above definition of topological space as follows: first, we consider that the base for a topology is given by a family $(\mathsf{ext}\, a)_{a \in S}$ of subsets $\mathsf{ext}\, a$ of X indexed by some set S. Since the family $(\mathsf{ext}\, a)_{a \in S}$ corresponds bijectively to a binary relation $\Vdash \subseteq X \times S$ such that $\mathsf{ext}\, a = \{x \in X \mid x \Vdash a\}$, we consider a topological space to be a triple $(X, \Vdash, S)$ where X and S are sets and $\Vdash \subseteq X \times S$ is a relation such that the family $(\mathsf{ext}\{a\})_{a \in S}$ satisfies the above two conditions for a base. Thus we have arrived at the definition of concrete space: a *concrete space* is a triple $(X, \Vdash, S)$ of sets X and S and a relation $\Vdash \subseteq X \times S$ such that

(B1) $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}(a \downarrow b)$,

(B2) $X = \mathsf{ext}\, S$,

where $a \downarrow b = \{c \in S \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}$. The notion of basic pair is obtained by dropping the conditions (B1) and (B2). Thus, a basic pair is just a pair of sets together with a binary relation between them. However, this simple structure is enough to define a notion of open and closed subsets of S as well as of X. Moreover, the notion of map between basic pairs $(X_1, \Vdash_1, S_1)$ and $(X_2, \Vdash_2, S_2)$ is defined to be a pair (r, s) of relations $r \subseteq X_1 \times X_2$ and $s \subseteq S_1 \times S_2$, called a relation pair, which makes the square

$$\begin{array}{ccc} X_1 & \xrightarrow{\Vdash_1} & S_1 \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X_2 & \xrightarrow{\Vdash_2} & S_2 \end{array}$$

commute rather than a function between the sets X_1 and X_2 as in the case of a continuous function between topological spaces. The notion of map between concrete spaces is then obtained by adding certain conditions to a relation pair so that it preserves the structure of the open sets on a concrete space. The usual notion of continuous function becomes a special case of that of relation pair. Thus, one can say that the notion of basic pair and concrete space are generalizations of that of topological space.

Since the notion of basic pair and concrete space appear to be new, the literature on these subjects is scarce [25, 26, 23, 15], although the publication of the forthcoming monograph [21] may change the current situation. Hence, it is still to be seen whether those notions are fruitful compared to the other existing notions of general topology in constructive mathematics, in particular that of formal topology [22].

1.3 Contributions

The primary contribution of this thesis consists of the results in [16] where we showed that the category of basic pairs (**BP**) and that of concrete spaces (**CSpa**) are both complete and cocomplete, and moreover that **CSpa** is a coreflective subcategory of **BP**. The categorical structure of **BP** and **CSpa** has been given less attention in the study of basic pairs and concrete spaces, and the only known result is [14], where the existence of binary products of **BP** is mentioned. Also no adjunctions between **BP** and **CSpa** has been constructed so far. Our results fill in the missing pieces. Working in the extension of CZF, we showed that both **BP** and **CSpa** have arbitrary (co)products and (co)equalisers. The distinctive feature of our construction of (co)products and (co)equalisers is a uniform application of the notion of a generalized geometric theory [6] to deal with predicativity problems: the difficulties in showing that a certain collection of objects forms a set. Our construction of (co)equalisers of **BP** and **CSpa** and products of **CSpa** are good demonstrations of how the notion of generalized geometric theory is useful for dealing with such kind problems.

1.4 Organization

Our original results consist of Section 4.6 and Chapter 5. The other chapters and sections contain background materials and basic facts about basic pairs and concrete spaces which are well described in [21].

In Chapter 2, we describe background materials: CZF, generalized geometric theory, category theory, and some basic mathematical notions which are frequently used in the later chapters. In Chapter 3, we introduce the notion of basic pair and relation pair between them, and define the category of basic pairs **BP**. We show that **BP** is isomorphic to its own dual $\mathbf{BP}^{\mathrm{op}}$, from which it follows that **BP** is complete if and only if it is cocomplete. In Chapter 4, we describe concrete spaces and convergent relation pairs, which are shown to form a coreflective subcategory **CSpa** of **BP**. We also define the notion of convergent subset and ideal point of a basic pair. Then we introduce a weak separation axiom T_0 and sobriety of basic pairs, and consider relations between category of T_0 basic pairs and **BP** and between the category of sober concrete spaces and that of the sober topological spaces. In Chapter 5, we describe the categorical constructions, (co)products and (co)equalisers, of **BP** and **CSpa** and show that both categories are complete and cocomplete.

1.5 Note to the reader

There is no prerequisite for reading this thesis. All the necessary backgrounds of CZF, category theory, generalized geometric theory, basic pairs and concrete spaces are included with proofs. The reader who has basic understanding of CZF and category theory can start from Chapter 3. However, since the notion of generalized geometric theory plays a crucial role in Chapter 5, she should at least look at Definition 2.2.2 and Theorem 2.2.13. The notion of four operators given in Section 2.4 are frequently used in Chapter 3 and Chapter 4, but not in Section 4.6 and Chapter 5, where all of our contributions are presented. The reader who is only interested in our own results can jump to Section 4.6 and Chapter 5, consulting Section 2.2 and Chapter 3 and Chapter 4 for basic definitions and notations.

We use "iff" for the phrase "if and only if" throughout this thesis.

Chapter 2

Preliminary

2.1 Constructive Zermero-Frankel Set Theory

In this section, we describe the axiom system CZF, *Constructive Zermelo-Frankel Set Theory*, and show how some of basic mathematical constructions, e.g. ordered pairs, relations and functions, products and exponentiations, can be carried out in CZF. We also describe two axioms for CZF, the Relativized Dependent Choice (RDC) and the Regular Extension Axiom (REA). This chapter is a summary of Chapter 3, 4, 10 and 11 of [7]; see [7] for further details of CZF.

2.1.1 The axiom system CZF

CZF is a first order theory using intuitionistic predicate logic with equality with a binary predicate symbol $\in$ as its only non-logical constant symbol. CZF is based on the following axioms and axiom schemes:

Extensionality

$$\forall a \forall b \, [\forall \, (x \in a \leftrightarrow x \in b) \rightarrow a = b]$$

Paring

$$\forall a \forall b \exists y \forall u \, [u \in y \leftrightarrow u = a \vee u = b]$$

Union

$$\forall a \exists y \forall x \, [x \in y \leftrightarrow \exists u \in a \, (x \in u)]$$

Bounded Separation

$$\forall a \exists b \forall x \, [x \in b \leftrightarrow x \in a \wedge \varphi(x)]$$

for all *bounded* formulae $\varphi(x)$, where y is not free in $\varphi(x)$. A formula is bounded if all quantifiers are bounded, i.e. occur only in one of the forms $\forall x \in a$ or $\exists x \in a$.

Strong Collection

$$\forall a \, [\forall x \in a \exists y \, \varphi(x, y) \rightarrow \exists b \, [\forall x \in a \exists y \in b \, \varphi(x, y) \wedge \forall y \in b \exists x \in a \, \varphi(x, y)]]$$

for all formulae $\varphi(x, y)$.

Subset Collection

$$\forall a \forall b \exists c \forall u [\forall x \in \exists y \in b\, \varphi(x,y,u) \to \\ \exists d \in c\, [\forall x \in a \exists y \in d\, \varphi(x,y,u) \wedge \forall y \in d \exists x \in a\, \varphi(x,y,u)]\,]$$

for all formulae $\varphi(x, y, u)$.

Strong Infinity

$$\exists a\, [Ind(a) \wedge \forall b\ Ind(b) \to \forall x \in a\, (x \in b)]$$

where we use the following abbreviations.

- $Empty(y)$ for $(\forall z \in y)\bot$,
- $Succ(x, y)$ for $\forall z\, [z \in y \leftrightarrow z \in x \vee z = x]$,
- $Ind(a)$ for $\exists y \in a\, Empty(y) \wedge (\forall x \in a)\, (\exists y \in a)\, Succ(x, y)$.

Set induction

$$\forall a\, [\forall x \in a\, \varphi(x) \to \varphi(a)] \to \forall a\, \varphi(a)$$

for all formula $\varphi(x)$.

Remark 2.1.1.

- The set y asserted to exist by Pairing is unique by Extensionality and is denoted by $\{a, b\}$. We also write $\{a\} = \{a, a\}$. In the following, we shall not explicitly say that a certain set is uniquely determined by Extensionality.
- The set y asserted to exist by Union is denoted by $\bigcup b$.
- The set b asserted to exist by Bounded Separation is denoted by $\{x \in a \mid \varphi(x)\}$.

Class notation

When working in CZF, we freely exploit the notion of class and notations for classes.

Definition 2.1.2. For any formula $\varphi(x)$, we call the collection of sets of the form $\{x \mid \varphi(x)\}$ a *class*. A set x is an *element* of a class $A = \{x \mid \varphi(x)\}$, denoted by $x \in A$, if $\varphi(x)$. A class A is a *subclass* of a class B, denoted by $A \subseteq B$, if $\forall x\, [x \in A \to y \in B]$. Classes A and B are equal if $A \subseteq B$ and $B \subseteq A$. A set a is identified as a class $\{x \mid x \in a\}$.

We introduce the following class notations:

- $\emptyset = \{x \mid \bot\}$,
- $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \cdots \vee x = a_n\}$. When $n = 0$, it is $\emptyset$,
- $\bigcup A = \{x \mid (\exists y \in A)\, x \in y\}$,
- $A \cup B = \{x \mid x \in A \vee x \in B\}$,

- $a^+ = a \cup \{a\}$,
- $\mathrm{Pow}(A) = \{x \mid x \subseteq A\}$,
- $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$,
- $A \times B = \{z \mid (\exists x \in A)\, (\exists y \in B)\, z = \langle x, y\rangle\}$,
- $\{x \in A \mid \varphi(x)\} = \{x \mid x \in A \wedge \varphi(x)\}$,

where A, B range over the classes and $a, a_1, \dots, a_n, b$ range over the sets. We also use the following notation

$$\{a \mid \varphi\} \quad \text{for} \quad \{z \mid (\exists x_1, \dots, x_n)\, z = a \wedge \varphi\}$$

where a denotes a set which depends on variables $x_1, \dots, x_n$. For example, we often write

$$\{\langle x, y\rangle \mid x \in A \wedge y \in B\} \quad \text{for} \quad \{z \mid (\exists x \in A)\, (\exists y \in B)\, z = \langle x, y\rangle\}.$$

Remark 2.1.3.

- $\emptyset$ is a set, for we can write $\emptyset = \{x \in b \mid \bot\}$ where b is an arbitrary set; take b to be the set whose existence is guaranteed by Strong Infinity.
- $\{a_1, \dots, a_n\}$ is a set by Pairing.
- $\bigcup A$ is a set when A is a set by Union. If A and B are sets, then $A \cup B$ is a set by Union and Pairing, and so is $A \times B$; see Proposition 2.1.14.
- $\langle a, b\rangle$ is also denoted by (a, b).

2.1.2 Elementary mathematical construction in CZF

In this section, we show that some of the basic mathematical constructions can be performed in CZF.

Definition 2.1.4. For any sets a and b, $\langle a, b\rangle$ is the *ordered pair* of a and b.

Proposition 2.1.5. $\langle a, b\rangle = \langle c, d\rangle$ *iff* $a = c$ *and* $b = d$.

Proof. The implication form right to left is trivial. Conversely, suppose that $\langle a, b\rangle = \langle c, d\rangle$. Since $\{a\} \in \langle c, d\rangle$, either $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. In either case, we have $a = c$. Similarly, we have $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. In either case, we have $b = c$ or $b = d$. If $b = c$, then $a = c = b$, and hence $b = d$. Therefore, in either case, we have $a = c$ and $b = d$. □

For any natural number n, we define an ordered n tuple by induction $\langle\rangle = \emptyset$, $\langle a\rangle = a$, $\langle a_1, \dots, a_n, a_{n+1}\rangle = \langle\langle a_1, \dots, a_n\rangle, a_{n+1}\rangle$.

We introduce an auxiliary axiom which is a theorem of CZF.

Definition 2.1.6. *Replacement* is a statement

$$\forall x \in a \exists ! y \, \varphi(x,y) \to \exists b \forall y \, [y \in b \leftrightarrow \exists x \in a \, \varphi(x,y)]$$

for any formula $\varphi(x,y)$, where b is not free in $\varphi(x,y)$.

Proposition 2.1.7. *Strong Collection implies Replacement.*

Proof. Suppose that $\forall x \in a \exists ! y \varphi(x,y)$. Then, there is a set b such that

$$\forall x \in a \exists y \in b \varphi(x,y) \wedge \forall y \in b \exists x \in a \varphi(x,y)$$

by Strong Collection. Then, $\forall y \, [y \in b \leftrightarrow \exists x \in a \varphi(x,y)]$. □

A relation is defined as usual.

Definition 2.1.8. A relation is a set R of ordered pairs. The *domain* and *range* of a relation R are defined by

$$\begin{aligned} \mathbf{dom}(R) &= \{x \mid \exists y \, (\langle x,y \rangle \in R)\}, \\ \mathbf{ran}(R) &= \{y \mid \exists x \, (\langle x,y \rangle \in R)\}. \end{aligned}$$

Remark 2.1.9. $\mathbf{dom}(R)$ and $\mathbf{cod}(R)$ are both sets, for let $A = \bigcup \bigcup R$, then we can write

$$\mathbf{dom}(R) = \{x \in A \mid (\exists y \in A) \, (\exists z \in R) \, z = \langle x,y \rangle\}$$

and similarly for $\mathbf{ran}(R)$.

A relation f is a *function* if

$$\forall x \in \mathbf{dom}(f) \exists ! y \in \mathbf{ran}(f) \, \langle x,y \rangle \in f.$$

We write $f : A \to B$ to assert that f is a function with $\mathbf{dom}(f) = A$ and $\mathbf{ran}(f) \subseteq B$. For any $x \in \mathbf{dom}(f)$, we write $f(x)$ for the unique $y \in \mathbf{ran}(f)$ such that $\langle x,y \rangle \in f$.

Lemma 2.1.10. *If* $\forall x \in a \exists ! y \varphi(x,y)$, *then there exists a unique function* f *with* $\mathbf{dom}(f) = a$ *such that* $\forall x \in a \, \varphi(x, f(x))$.

Proof. Suppose that $\forall x \in a \exists ! y \, \varphi(x,y)$. Let $\theta(x,z)$ be a formula such that $\theta(x,z) = \exists y \, (z = \langle x,y \rangle \wedge \varphi(x,y))$. Then $\forall x \in a \exists ! z \, \theta(x,z)$. Hence, there is a set f such that

$$\forall z \, [z \in f \leftrightarrow \exists x \in a \, \theta(x,z)].$$

by Replacement. It is straightforward to verify that f is a function with domain a and that $\forall x \in a \, \varphi(x, f(x))$. The uniqueness of f is obvious. □

We introduce a few more class notations.

Definition 2.1.11. Let A be a class and $\theta(x, y)$ be a formula. A family of classes $(B_a)_{a \in A}$ over A is a collection

$$B_a = \{y \mid \theta(a, y)\}$$

for each $a \in A$. A family of classes over A is also called a family of classes indexed by A or an A-indexed family of classes.

Let $(B_a)_{a \in A}$ be a family of classes. We define classes:

$$\bigcup_{a \in A} B_a = \{y \mid (\exists a \in A)\, y \in B_a\},$$

$$\bigcap_{a \in A} B_a = \{y \mid (\forall a \in A)\, y \in B_a\}.$$

If R is a class of ordered pairs, then we write aRb for $\langle a, b\rangle \in R$. If A and B are classes, then a *class function* from A to B, denoted by $F : A \to B$, is a class $F \subseteq A \times B$ such that

$$\forall x \in A \exists! y \in B\, [xFy].$$

Lemma 2.1.12. *For any class function $F : A \to B$, if A is a set, then so is F.*

Proof. Let $F : A \to B$ be a class function where A is a set. Then $\forall x \in A \exists! y \langle x, y\rangle \in F$. By Lemma 2.1.10, there exists a unique function f with $\mathbf{dom}(f) = A$ such that $\forall x \in A \langle x, f(x)\rangle \in F$. Clearly, $f = F$, so F is a set. ☐

Lemma 2.1.13. *Let A be a set and $(B_a)_{a \in A}$ be a family of sets over A. Then*

1. *$\bigcup_{a \in A} B_a$ is a set,*
2. *if A is inhabited, i.e. $\exists a_0 \in A$, then $\bigcap_{a \in A} B_a$ is a set.*

Proof. 1. Since $\forall a \in A \exists! y\, y = B_a$, there exists a set

$$b = \{y \mid (\exists a \in A)\, y = B_a\}$$

by Replacement. Then $\bigcup_{a \in A} B_a = \bigcup b$ is a set by Union.

2 Let $a_0 \in A$. Since $\forall x \in A \exists! y\, y = B_a$, there is a unique function with domain A such that $(\forall x \in A)\, f(x) = B_a$. Hence, we have

$$\bigcap_{a \in A} B_a = \{y \in f(a_0) \mid (\forall a \in A)\, y \in f(a)\},$$

so $\bigcap_{a \in A} B_a$ is a set by Bounded Separation. ☐

Proposition 2.1.14. *$A \times B$ is a set whenever A and B are sets.*

Proof. Let $a \in A$. Since $\forall y \in B \exists! z\, z = \langle a, y\rangle$, we have a set

$$\{a\} \times B = \{z \mid (\exists y \in B)\, z = \langle a, y\rangle\}$$

by Replacement. Therefore, $A \times B = \bigcup_{a \in A} \{a\} \times B$ is a set by Lemma 2.1.13. ☐

Definition 2.1.15. Let I be a class and $(A_i)_{i \in I}$ be a family over I. The *sum* of $(A_i)_{i \in I}$ is the class

$$\sum_{i \in I} A_i = \{\langle i, a \rangle \mid i \in I \wedge a \in A_i\}.$$

Proposition 2.1.16. *Let $(A_i)_{i \in I}$ be a family of sets over a set I. Then, $\sum_{i \in I} A_i$ is a set.*

Proof. Since $\sum_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$, $\sum_{i \in I} A_i$ is a set by Proposition 2.1.14 and Lemma 2.1.13 (1). □

Definition 2.1.17. Let A and $R \subseteq A \times A$ be classes. The class R is an *equivalence relation* on A if for all $a, b, c \in A$

1. aRb,
2. $aRb \Rightarrow bRa$,
3. aRb and $bRc \Rightarrow aRc$.

For each $a \in A$, the class

$$[a]_R = \{x \in A \mid xRa\}$$

is called the *equivalence class* of a with respect to R.

Proposition 2.1.18. *Let A be a set and R be an equivalence relation on A. If R is a set, $[a]_R$ is a set for each $a \in A$, and moreover, the* quotient of A *with respect to R*

$$A/R = \{[a]_R \mid a \in A\}$$

is a set.

Proof. If R is a set, then $[a]_R$ is a set for each $a \in A$ by Bounded Separation. Then, A/R is a set by Replacement. □

Definition 2.1.19. Let a and b be sets and let $b^a = \{f \mid f : a \to b\}$. We also write $a \to b$ for b^a. Let $\mathbf{mv}(a, b)$ be the class of relations $r \subseteq A \times B$ such that $\forall u \in a \exists v \in b\, u\, r\, v$. A set c is *full in* $\mathbf{mv}(a, b)$ if $c \subseteq \mathbf{mv}(a, b)$ and

$$\forall r \in \mathbf{mv}(a, b) \exists s \in c\, [s \subseteq r].$$

Fullness is a statement

$$\forall a \forall b \exists c\, [c \text{ is full in } \mathbf{mv}(a, b)].$$

Exponentiation is a statement

$$\forall a \forall b \exists c \forall f\, [f \in c \leftrightarrow f \in b^a].$$

Proposition 2.1.20. *Fullness is derivable in* CZF.

Proof. Let a, b be sets. Let $\varphi(x,z,u)$ be a formula such that

$$\varphi(x,z,u) = \exists y \in b\,[\langle x,y\rangle = z \wedge z \in u]\,.$$

By Subset Collection, there is a set d such that

$$\forall u[\forall x \in a \exists z \in a \times b\, \varphi(x,z,u) \to \exists r \in d\,[\forall x \in a \exists z \in r\, \varphi(x,z,u) \wedge \forall z \in r \exists x \in a\, \varphi(x,z,u)]].$$

Then

$$c = \{r \in d \mid r \in \mathbf{mv}(a,b)\}$$

is a set by Bounded Separation. Let $r \in \mathbf{mv}(a,b)$. Then $\forall x \in a \exists z \in a \times b\, \varphi(x,z,r)$, and thus there exists a set $s \in d$ such that

$$\forall x \in a \exists z \in s\, \varphi(x,z,r) \wedge \forall z \in s \exists x \in a\, \varphi(x,z,r).$$

Then $s \subseteq r$ and $s \in \mathbf{mv}(a,b)$, and thus $s \in c$. Therefore, c is full in $\mathbf{mv}(a,b)$. $\square$

Corollary 2.1.21. *Fullness implies Exponentiation.*

Proof. Let a,b be sets. By Fullness, there is a set c which is full in $\mathbf{mv}(a,b)$. Then

$$X = \{f \in c \mid f \in b^a\}$$

is a set by Bounded Separation. Let $f \in b^a$. Since $f \in \mathbf{mv}(a,b)$, there is $g \in c$ such that $g \subseteq f$. As f is a function, we must have $g = f$, and hence $f \in X$. Therefore, $X = b^a$ so b^a is a set. $\square$

Definition 2.1.22. Let I be a set and $(A_i)_{i\in I}$ be a family of classes over I. The *dependent product* (or Cartesian product) of $(A_i)_{i\in I}$ is the class

$$\prod_{i\in I} A_i = \Big\{f \mid f : I \to \bigcup_{i\in I} A_i \wedge (\forall i \in I)\, f(i) \in A_i\Big\}.$$

Proposition 2.1.23. *Let $(A_i)_{i\in I}$ be a family of sets over a set I. Then, $\prod_{i\in I} A_i$ is a set.*

Proof. $\bigcup_{i\in I} A_i$ is a set by Lemma 2.1.13, and hence $I \to \bigcup_{i\in I} A_i$ is a set by Exponentiation. Therefore $\prod_{i\in I} A_i$ is a set by Bounded Separation. $\square$

2.1.3 The Natural Numbers

We show that the set which is asserted to exist by the Strong Infinity satisfies the usual property of natural numbers including the well-know axiom of Dedekind and Peano. We also introduce the notion of finitely enumerable set.

Lemma 2.1.24. *Let $Ind(a)$ and $\theta(a)$ be formulae*

$$Ind(a) \equiv 0 \in a \wedge (\forall x \in a)\,(\exists y \in a)\, y = x^+,$$
$$\theta(a) \equiv Ind(a) \wedge \forall y\,[Ind(y) \to a \subseteq y]$$

where $0 \equiv \emptyset$. If $\theta(a)$ and $\theta(b)$, then $a = b$.

Proof. If $\theta(a)$ and $\theta(b)$, then $a \subseteq b$ and $b \subseteq a$, i.e. $a = b$. $\square$

Definition 2.1.25. The unique set a such that $\theta(a)$ is denoted by ω or $\mathbf{N}$.

Theorem 2.1.26.

1. $\varphi(0) \wedge \forall n \in \omega\,[\varphi(n) \to \varphi(n^+)] \to (\forall n \in \omega)\,\varphi(n)$ *for every bounded formula* $\varphi(n)$.
2. $\forall n \in \omega\,[n = 0 \vee (\exists m \in \omega)\, n = m^+]$.
3. $\forall n \in \omega\,(0 \neq n^+)$.
4. $\forall n \in \omega\,(\forall x \in n\, x \subseteq n)$.
5. $\forall n \in \omega\,(n \notin n)$.
6. $\forall n, m \in \omega\,[n \in m \to n^+ \in m \vee n^+ = m]$.
7. $\forall n, m \in \omega\,[n^+ = m^+ \to n = m]$.
8. $\forall n \in \omega\,(0 \in n^+)$.
9. $\forall n, m \in \omega\,[n \in m \vee n = m \vee m \in n]$.
10. $\forall n, m \in \omega\,[m \in n \vee m \notin n] \wedge \forall n, m \in \omega\,[m = n \vee m \neq n]$.

Proof. 1. Let φ be a bounded formula. Suppose that $\varphi(0) \wedge \forall n \in \omega\,[\varphi(n) \to \varphi(n^+)]$. Let $a = \{n \in \omega \mid \varphi(n)\}$ be a set by Bounded Separation. Clearly, $Ind(a)$ so $\omega \subseteq a$, and thus $(\forall n \in \omega)\varphi(n)$.

2. Let $\varphi(n)$ be a formula such that $\varphi(n) \equiv n = 0 \vee (\exists m \in \omega)\, n = m^+$. Then $\varphi(0)$, and if $\varphi(n)$, then $\varphi(n^+)$ for all $n \in \omega$. Therefore, 2 follows from 1.

3. We have $n \notin 0$, and $n \in n^+$ for any $n \in \omega$. Thus $0 \neq n^+$ by Extensionality.

4. Let $\varphi(n) \equiv \forall n \in \omega m \subseteq n$. Then $\varphi(0)$. Let $n \in \omega$, and suppose that $\varphi(n)$. Let $m \in n^+$. Either $m = n$ or $m \in n$. If $m \in n$, since $\varphi(n)$, we have $m \subseteq n$. So in either case, we have $m \subseteq n \subseteq n^+$; the conclusion follows from 1.

5. Trivially $0 \notin 0$. Suppose that $n \notin n$ and $n^+ \in n^+$. Then either $n^+ = n$ or $n^+ \in n$. In either case, we have $n^+ \subseteq n$ by 4, and hence $n \in n$, a contradiction.

6. Let $\varphi(m) \equiv \forall n \in \omega\,[n \in m \to n^+ \in m \vee n^+ = m]$. Then $\varphi(0)$. Suppose that $\varphi(m)$, and let $n \in m^+$. Either $n \in m$ or $n = m$. If $n \in m$, then $n^+ \in m$ or $n^+ = m$, so $n^+ \in m^+$. Hence, in either case, we have $n^+ \in m^+ \vee n^+ = m^+$.

7. Suppose that $n^+ = m^+$. Then, either $n = m$ or $n \in m$. In either case, we have $n \subseteq m$ by 4. Similarly, we have $m \subseteq n$. Thus $n = m$.

8. Let $\varphi(n) \equiv 0 \in n^+$. The conclusion follows from 1.

9. Let $\varphi(n) \equiv \forall m \in \omega\,[n \in m \vee n = m \vee m \in n]$. Since $0 = m \vee \exists n \in \omega n = m^+$ for all $m \in \omega$ by 2, we have $\varphi(0)$ by 8. Assume that $\varphi(n)$, and let $m \in \omega$. Then, we have $n \in m \vee n = m \vee m \in n$. If $n \in m$, then $n^+ \in m \vee n^+ = m$ by 6. If $n = m$ or $m \in n$, then $m \in n^+$. Hence $\varphi(n^+)$.

10. The first part follows from 4, 5 and 9. The second part follows from 5 and 9. $\square$

Definition 2.1.27. The *Dedekind-Peano axioms* for the structure $(\mathbb{N}, 0, S)$ is the following list of axioms.

1. $0 \in \mathbb{N}$.

2. $S : \mathbb{N} \to \mathbb{N}$.

3. $0 \neq S(n)$ for all $n \in \mathbb{N}$.

4. $S(n) = S(m) \to n = m$ for all $n, m \in \mathbb{N}$.

5. For each subset $X \subseteq \mathbb{N}$, if $0 \in X$ and $S(n) \in X$ for all $n \in X$, then $n \in X$ for all $n \in \mathbb{N}$.

Proposition 2.1.28. $(\omega, 0, S)$ *satisfies the Dedekind-Peano axioms, where* $0 = \emptyset$ *and for all* $n \in \omega$*,* $S(n) = n^+$*.*

Proof. Since $Ind(\omega)$, the first two and the last axioms are immediate. The others are 3 and 7 of Theorem 2.1.26. □

Structure $(\mathbb{N}, 0, S)$ satisfying the Dedekind-Peano axioms can be shown to be unique up to isomorphism (See [7, Chapter 6]).

To close this section, we given a notion of finiteness of a set.

Definition 2.1.29. A set A is *finitely enumerable* if there exist $n \in \omega$ and a surjective function $f : n \to A$. Note that $n = \{m \in \omega \mid m \in n\}$. Let $\mathrm{Fin}(S)$ denote the class of all finitely enumerable subsets of a set S.

Lemma 2.1.30. $\mathrm{Fin}(S)$ *is a set whenever* S *is a set.*

Proof. Since S^n is a set for each $n \in \omega$ by Exponentiation, $F_n = \{\mathbf{ran}(f) \mid f \in S^n\}$ is a set by Replacement. Therefore, $\mathrm{Fin}(S) = \bigcup_{n \in \omega} F_n$ is a set by Replacement and Union. □

2.1.4 Choice Principles in CZF

We introduced three constructive choice principles which are available in CZF, the last one was shown to be valid in the interpretation in the type theory [2], from which the others follow.

Definition 2.1.31.

The Axiom of Countable Choice (AC_ω) states that

> For any function F with domain ω, if $\forall n \in \omega \exists y \in F(n)$, then there exists a function $f : \omega \to \bigcup_{n \in \omega} F(n)$ such that $(\forall n \in \omega)\, f(n) \in F(n)$.

The *Axiom of Dependent Choice* (DC) states that

> For any set a and formula φ, if $(\forall x \in a)\,(\exists y \in a)\,\varphi(x, y)$, then for any $a_0 \in a$ there exists a function $f : \omega \to a$ such that $f(0) = a_0 \wedge (\forall n \in \omega)\,\varphi(f(n), f(n+1))$.

The *Axiom of Relativized Dependent Choice* (RDC) states that

> For any formulae φ and ψ, if $\forall x\, \varphi(x) \to \exists y\, [\varphi(y) \land \psi(x,y)]$, then for any set a_0 such that $\varphi(a_0)$ there exists a function f with domain ω such that $f(0) = a_0 \land (\forall n \in \omega)\, \varphi(f(n)) \land \psi(f(n), f(n+1))$.

Proposition 2.1.32.

1. DC *implies* AC_ω.

2. RDC *implies* DC.

Proof. 1. Let F be a function with domain ω such that $\forall n \in \omega \exists y \in F(n)$. Let $A = \sum_{n\in\omega} F(n)$. Then $\forall (i,x) \in A \exists (j,y) \in A\, [j = i+1]$. Pick $a_0 \in F(0)$. Applying DC, there is a function $f : \omega \to A$ with $f : n \mapsto (i_n, x_n)$ such that $(i_0, x_0) = (0, a_0)$ and $(\forall n \in \omega)\, i_{n+1} = i_n + 1$. Then, $i_n = n$ for all $n \in w$, and thus we have a function $g : \omega \to \bigcup_{n\in\omega} F(n)$ defined by $g(n) = x_n$ such that $\forall n \in \omega\, g(n) \in F(n)$.

2. Given a set a and a formula ψ such that $(\forall x \in a)\, (\exists y \in a)\, \psi(x,y)$, put $\varphi(x) \equiv x \in a$ and apply RDC. □

2.1.5 The Regular Extension Axiom

The *Regular Extension Axiom* was introduced in CZF to accommodate inductive definitions [3].

Definition 2.1.33. A set A is *transitive* if $(\forall a \in A)\, a \subseteq A$, and it is *regular* if it is transitive and for any $a \in A$ and $R \in \mathbf{mv}(a, A)$, there exists $b \in A$ such that

$$(\forall x \in a)\, (\exists y \in b)\, (x,y) \in R \land (\forall y \in b)\, (\exists x \in a)\, (x,y) \in R.$$

The axiom REA asserts that

REA Every set is a subset of a regular set.

A set A is *union-closed* if $\bigcup a \in A$ for each $a \in A$. The axiom uREA asserts that

uREA Every set is a subset of a union-closed regular set.

A regular set A is RRS_2-*regular* if for each $A' \in \mathrm{Pow}(A)$, $R \in \mathbf{mv}(A' \times A', A')$ and $a_0 \in A'$, there exists $A_0 \in A$ such that $a_0 \in A_0 \subseteq A'$ and

$$\forall (x,y) \in A_0 \times A_0 \exists z \in A_0\, ((x,y), z) \in R.$$

The axiom RRS_2-uREA asserts that

$\mathbf{RRS_2}$**-uREA** Every set is a subset of a union-closed RRS_2-regular set.

Lemma 2.1.34. *Let A be a regular set. Then,* $\mathbf{ran}(f) \in A$ *for any $a \in A$ and $f : a \to A$.*

Proof. Let $f : a \to A$. Since $f \in \mathbf{mv}(a, A)$ and A is regular, there exists a set $b \in A$ such that

$$\forall x \in a \exists y \in b f(x) = y \wedge \forall y \in b \exists x \in a f(x) = y.$$

Then, obviously $\mathbf{ran}(f) = b \in A$. □

Lemma 2.1.35. *Let A be a regular set such that $\mathbf{2} = \{0, 1\} \in A$. Then,*

1. $\forall x, y \in A \{x, y\} \in A$, *and*

2. $f \in A$ *for all* $a \in A$ *and* $f : a \to A$.

Proof. 1. Let $x, y \in A$. Define a function $g : \mathbf{2} \to A$ by $g(0) = x$ and $g(1) = y$. Then, $\{x, y\}$ by Lemma 2.1.34.

2. Let $a \in A$ and $f : a \to A$. Then, for each $x \in a$, we have $\langle x, f(x)\rangle \in A$ by 1. Hence, we have a function $x \mapsto \langle x, f(x)\rangle$ from a to A, and thus $f = \{\langle x, f(x)\rangle \mid x \in a\} \in A$ by Lemma 2.1.34. □

Lemma 2.1.36. *Let A be a union-closed regular set such that $\mathbf{2} \in A$. Then, for any $I \in A$ and a family $(a_i)_{i\in I}$ of elements of A, $\sum_{i\in I} a_i \in A$.*

Proof. Let $I \in A$ and let $(a_i)_{i\in I}$ be a family of elements of A. For each $i \in I$, we have a function $f_i : a_i \to A$ defined by $f_i(x) = \langle i, x\rangle$ by Lemma 2.1.35. Thus, we have a function $g : I \to A$ defined by $g(i) = \mathbf{ran}(f_i) = \{i\} \times a_i$ by Lemma 2.1.34. Since A is union-closed regular, we have $\bigcup \mathbf{ran}(g) = \bigcup_{i\in I}\{i\} \times a_i = \sum_{i\in I} a_i \in A$ by Lemma 2.1.34. □

Corollary 2.1.37. *For a union-closed regular set A such that $\mathbf{2} \in A$, $a \times b \in A$ for each $a, b \in A$.*

Lemma 2.1.38. *Let A be a union-closed regular set such that $\mathbf{N} \in A$. Then,* $\mathrm{Fin}(a) \in A$ *for each $a \in A$.*

Proof. Let $a \in A$. Since $\mathrm{Fin}(a) = \bigcup_{n\in\mathbf{N}} \{\mathbf{ran}(f) \mid f \in a^n\}$, A is union-closed regular and $\mathbf{N} \in A$, it suffices to show that $a^n \in A$ for each $n \in \mathbf{N}$. If $n = 0$, then since $\mathbf{N} \subseteq A$, trivially $a^0 = \mathbf{1} \in A$. For the induction step, assume that $a^n \in A$. Let $f \in a^n$. Since A is union-closed regular and $\mathbf{N} \in A$, we have $f \cup \{\langle n + 1, x\rangle\} \in A$ for each $x \in a$, and therefore $G(f) = \{f \cup \{\langle n + 1, x\rangle\} \mid x \in a\} \in A$. Hence, $a^{n+1} = \bigcup_{f\in a^n} G(f) \in A$. □

Proposition 2.1.39. uREA *and* DC *imply* RRS_2-uREA.

Proof. Assume uREA and DC. Let S be a set. Then, there exists a union-closed regular set A such that $\{\mathbf{N}\} \cup S \subseteq A$. We claim that A is RRS_2-regular. Let $A' \in \mathrm{Pow}(A)$, $a_0 \in A'$ and $R \in \mathbf{mv}(A' \times A', A')$. Let $A_{A'} = \{a \in A \mid a \subseteq A'\}$ and $a \in A_{A'}$. Then

$$\forall (x, y) \in a \times a \exists z \in A [(x, y) Rz \wedge z \in A'].$$

Since $a \times a \in A$ by Corollary 2.1.37, there is a set $b \in A$ such that

$$\forall (x, y) \in a \times a \exists z \in b [(x, y) Rz \wedge z \in A'] \wedge \forall z \in b \exists (x, y) \in a \times a [(x, y) Rz \wedge z \in A'].$$

by regularity of A. Hence $b \subseteq A'$, and thus $b \in A_{A'}$. Since $\mathbf{2} \in A$ and A is union-closed regular, we have $a \cup b \in A$, and so $a \cup b \in A_{A'}$. Thus, we have

$$\forall a \in A_{A'} \exists b \in A_{A'} \left[a \subseteq b \wedge (\forall (x, y) \in a \times a) (\exists z \in b) (x, y) R z \right].$$

Since $a_0 \in A'$, $\mathbf{1} \in A$ and A is regular, we have $\{a_0\} \in A'$, and hence $\{a_0\} \in A_{A'}$. Applying DC, there exists a function $f : \mathbf{N} \to A_{A'}$ such that $f(0) = a_0$ and

$$(\forall n \in \mathbf{N}) \, f(n) \subseteq f(n+1) \wedge (\forall (x, y) \in f(n) \times f(n)) (\exists z \in f(n+1)) (x, y) R z.$$

Since $\mathbf{N} \in A$ and A is union-closed regular, we have $A_0 = \bigcup \mathbf{ran}(f) \in A_{A'}$ by Lemma 2.1.34. Then, $a_0 \in A_0$, and for any $(x, y) \in A_0 \times A_0$ then there exists $n \in \mathbf{N}$ such that $x, y \in f(n)$, so there exists $z \in f(n+1)$ such that $(x, y) R z$. Therefore, $R \in \mathbf{mv}(A_0 \times A_0, A_0)$. □

2.2 Set-generated classes and generalized geometric theories

We introduce the notion of set-generated class and generalized geometric theory by Ishihara et al. [6]. In the practice of constructive mathematics, we often face difficulties in showing that a certain collection forms a set. However, in the case where the object X in question is a collection of subsets of a set S, we can often construct a subset G of X which generates X in the following sense:

Definition 2.2.1. A class C of subsets of a set S is *set-generated* if there is a subset G of C such that

$$(\forall U \in C)\,(\forall \sigma \in \mathrm{Fin}\,(U))\,(\exists V \in G)\,\sigma \subseteq V \subseteq U,$$

We call G a generating subset (shortly, generating set) of C and say that G generates C.

It often turns out that a generating subset G suffices for a particular purpose. Ishihara et al. showed that the class of models of a generalized geometric theory over a give set is set generated. In practice, if we want to show that a collection X of subsets of a set S is set-generated, we formulate a generalized geometric theory over the set S in such a way that a subset α of S is a model of the theory iff $\alpha \in X$. The notion of generalized geometric theory plays a crucial role in Chapter 5 where we construct (co)equalisers and products of the categories of basic pairs and concrete spaces.

2.2.1 Generalized geometric theories

First, we review the notion of generalized geometric theory [6].

Definition 2.2.2. A *generalized geometric implication* (shortly, implication) and a *generalized geometric theory* (shortly, theory) over a set S, and their ranks, are defined simultaneously by

1. s is a generalized geometric implication of rank 0 for each $s \in S$;
2. if σ is a finitely enumerable subset of S and Γ is a set of generalized geometric theories of rank n, then $\bigwedge \sigma \to \bigvee_{U \in \Gamma} \bigwedge U$ is a generalized geometric implication of rank $n+1$;
3. a set T of generalized geometric implications of rank $\leq n$ is a generalized geometric theory of rank n.

Remark 2.2.3. More precisely, the classes $\mathcal{I}_n$ and $\mathcal{T}_n$ of implications of rank $\leq n$ and theories of rank n, respectively, over a set S are defined by simultaneous induction as follows:

$$\begin{aligned} \mathcal{I}_0 &= S, \\ \mathcal{I}_{n+1} &= \mathcal{I}_n \cup \{(n+1, (\sigma, \Gamma)) \mid \sigma \in \mathrm{Fin}(S) \wedge \Gamma \in \mathrm{Pow}(\mathcal{T}_n)\}\,, \\ \mathcal{T}_n &= \mathrm{Pow}(\mathcal{I}_n). \end{aligned}$$

A class $\mathcal{C}$ is *predicative* if it can be is presented by a bounded formula π, i.e.

$$\forall x\, (x \in \mathcal{C} \leftrightarrow x \in \pi(x))\,.$$

If a class $\mathcal{C}$ is predicative and presented by a formula π, then $\mathrm{Pow}(\mathcal{C})$ is also predicative, for we have $\forall y\, (y \in \mathrm{Pow}(\mathcal{C}) \leftrightarrow \forall x \in y \pi(x))$. Therefore, for an externally given natural number n, the classes $\mathcal{I}_n$ and $\mathcal{T}_n$ are predicative.

We introduce the following abbreviations for geometric implications.

$$\begin{aligned} s &\equiv \textstyle\bigwedge \sigma && \text{if } \sigma = \{s\}, \\ \theta &\equiv \textstyle\bigwedge U && \text{if } U = \{\theta\}, \\ \textstyle\bigwedge U &\equiv \textstyle\bigvee_{U \in \Gamma} \bigwedge U && \text{if } \Gamma = \{U\}, \\ \textstyle\bigvee_{U \in \Gamma} \bigwedge U &\equiv \textstyle\bigwedge \emptyset \to \bigvee_{U \in \Gamma} \bigwedge U. \end{aligned}$$

where $\sigma \in \mathrm{Fin}(S)$, and U and Γ are a theory and a set of theories respectively. For an implication $\varphi \equiv \bigwedge \sigma \to \bigvee_{U \in \Gamma} \bigvee U$ of positive rank, we write $\sigma_\varphi = \sigma$ and $\Gamma_\varphi = \Gamma$.

Definition 2.2.4. The relation $\models$ between a subset α of S, and implication s (of rank 0), φ (of positive rank) and a theory T over S is defined by

1. $\alpha \models s$ if $s \in \alpha$,
2. $\alpha \models \varphi$ if $\sigma_\varphi \subseteq \alpha$ implies $\alpha \models U$ for some $U \in \Gamma_\varphi$,
3. $\alpha \models T$ if $\alpha \models \theta$ for all $\theta \in T$.

A subset α of S is a *model* of a theory T if $\alpha \models T$. The class of models of a theory T will be denoted by $\mathfrak{M}(T)$.

Remark 2.2.5. More precisely, the class relations $\models_{\mathcal{I}_n}$ and $\models_{\mathcal{T}_n}$ between $\mathrm{Pow}(S)$ and $\mathcal{I}_n$ and $\mathcal{T}_n$ are defined by simultaneous induction:

1. $\alpha \models_{\mathcal{I}_0} s \iff s \in S$,
2. $\alpha \models_{\mathcal{I}_{n+1}} \theta \iff \alpha \models_{\mathcal{I}_n} \theta$ if $\theta \in \mathcal{I}_n$,
3. $\alpha \models_{\mathcal{I}_{n+1}} (n+1, (\sigma, \Gamma)) \iff \sigma \subseteq \alpha$ implies $\alpha \models_{\mathcal{T}_n} U$ for some $U \in \Gamma$,
4. $\alpha \models_{\mathcal{T}_n} T \iff \alpha \models_{\mathcal{I}_n} \theta$ for each $\theta \in T$.

Note that, for an externally given n, the class relations $\models_{\mathcal{I}_n}$ and $\models_{\mathcal{T}_n}$ are predicative.

An *extension* S' of a set S is a set with an inclusion (i.e. an injection) $\iota : S \to S'$. Let S' be an extension of a set S with an inclusion ι. Then we can naturally extend the inclusion ι to an inclusion $\hat{\iota}$ from the implications and the theories over S into the implications and the theories over S' of the same rank by

$$\begin{aligned}\hat{\iota}(s) &= \iota(s), \\ \hat{\iota}(\varphi) &= \bigwedge \iota(\sigma_\varphi) \to \bigvee_{U \in \Gamma_\varphi} \bigwedge \hat{\iota}(U), \\ \hat{\iota}(T) &= \{\hat{\iota}(\theta) \mid \theta \in T\},\end{aligned}$$

where s and φ are implications of rank 0 and of positive rank respectively, and T is a theory.

Remark 2.2.6. More precisely, we define the class functions $\hat{\iota}_{\mathcal{I}_n} : \mathcal{I}_n \to \mathcal{I}'_n$ and $\hat{\iota}_{\mathcal{T}_n} : \mathcal{T}_n \to \mathcal{T}'_n$, where $\mathcal{I}'_n$ and $\mathcal{T}'_n$ are the classes of implications of rank $\leq n$ and theories of rank n over S' respectively, by simultaneous induction:

1. $\hat{\iota}_{\mathcal{I}_0}(s) = \iota(s)$,
2. $\hat{\iota}_{\mathcal{I}_{n+1}}(\theta) = \hat{\iota}_{\mathcal{I}_n}(\theta)$ if $\theta \in \mathcal{I}_n$,
3. $\hat{\iota}_{\mathcal{I}_{n+1}}(n+1, (\sigma, \Gamma)) = (n+1, (\iota(\sigma), \{\hat{\iota}_{\mathcal{T}_n}(U) \mid U \in \Gamma\}))$,
4. $\hat{\iota}_{\mathcal{T}_n}(T) = \{\hat{\iota}_{\mathcal{I}_n}(\theta) \mid \theta \in T\}$.

Note that the image $\hat{\iota}_{\mathcal{T}_n}(T)$ of a set T is a set by Strong Collection.

Lemma 2.2.7. *For an externally given n, $\hat{\iota}_{\mathcal{I}_n}$ and $\hat{\iota}_{\mathcal{T}_n}$ are injective.*

Proof. The proof is by induction on n. Trivially, $\hat{\iota}_{\mathcal{I}_0}$ is injective, and so is $\hat{\iota}_{\mathcal{T}_0}$. Assume that $\hat{\iota}_{\mathcal{I}_n}$ and $\hat{\iota}_{\mathcal{T}_n}$ are injective. Let $\theta, \theta' \in \mathcal{I}_n$, and suppose that $\hat{\iota}_{\mathcal{I}_{n+1}}(\theta) = \hat{\iota}_{\mathcal{I}_{n+1}}(\theta')$. Then $\hat{\iota}_{\mathcal{I}_n}(\theta) = \hat{\iota}_{\mathcal{I}_n}(\theta')$, and so $\theta = \theta'$ because $\hat{\iota}_{\mathcal{I}_n}$ is injective. Let $\sigma, \sigma' \in \mathrm{Fin}(S)$ and $\Gamma, \Gamma' \in \mathrm{Pow}(\mathcal{T}_n)$, and suppose that $\hat{\iota}_{\mathcal{I}_{n+1}}(n+1, (\sigma, \Gamma)) = \hat{\iota}_{\mathcal{I}_{n+1}}(n+1, (\sigma', \Gamma'))$. Then, $(\iota(\sigma), \{\hat{\iota}_{\mathcal{T}_n}(T) \mid U \in \Gamma\}) = (\iota(\sigma'), \{\hat{\iota}_{\mathcal{T}_n}(T) \mid U \in \Gamma'\})$, so $\sigma = \sigma'$ and $\Gamma = \Gamma'$ by injectivity of ι and $\hat{\iota}_{\mathcal{T}_n}$. Hence, $(n+1, (\sigma, \Gamma)) = (n+1, (\sigma', \Gamma'))$. Finally, let $T, T' \in \mathcal{T}_{n+1}$ and suppose that $\hat{\iota}_{\mathcal{T}_{n+1}}(T) = \hat{\iota}_{\mathcal{T}_{n+1}}(T')$. Then $\{\hat{\iota}_{\mathcal{I}_n}(\theta) \mid \theta \in T\} = \{\hat{\iota}_{\mathcal{I}_n}(\theta) \mid \theta \in T'\}$. Since $\hat{\iota}_{\mathcal{I}_n}$ is injective, we have $T = T'$. □

Lemma 2.2.8. *For an externally given n,*

$$\forall \theta \in \mathcal{I}_n \forall \alpha' \in \mathrm{Pow}(S') \left[\iota^{-1}(\alpha') \models_{\mathcal{I}_n} \theta \iff \alpha' \models_{\mathcal{I}'_n} \hat{\iota}_{\mathcal{I}_n}(\theta)\right],$$

and

$$\forall T \in \mathcal{T}_n \forall \alpha' \in \mathrm{Pow}(S') \left[\iota^{-1}(\alpha') \models_{\mathcal{T}_n} T \iff \alpha' \models_{\mathcal{T}'_n} \hat{\iota}_{\mathcal{T}_n}(T)\right].$$

Proof. The proof is by induction on n. Let $s \in \mathcal{I}_0 = S$ and $\alpha \in \mathrm{Pow}(S')$. Then

$$\iota^{-1}(\alpha') \models_{\mathcal{I}_0} s \iff s \in \iota^{-1}(\alpha') \iff \iota(s) \in \alpha' \iff \alpha' \models_{\mathcal{I}_0} \hat{\iota}_{\mathcal{I}_0}(s).$$

Now, let $T \in \mathcal{T}_0 \in \mathrm{Pow}(\mathrm{Pow}(S))$ and $\alpha' \in \mathrm{Pow}(S')$. Then

$$\begin{aligned}\iota^{-1}(\alpha') \models_{\mathcal{T}_0} T &\iff (\forall s \in T)\, \iota^{-1}(\alpha') \models_{\mathcal{I}_0} s \\ &\iff (\forall s \in T)\, \alpha' \models_{\mathcal{I}_0} \hat{\iota}_{\mathcal{I}_0}(s) \\ &\iff \alpha' \models_{\mathcal{T}_0} \hat{\iota}_{\mathcal{T}_0}(T).\end{aligned}$$

Assume that the assertion holds for $\models_{\mathcal{I}_n}$ and $\models_{\mathcal{T}_n}$. Let $\alpha' \in \mathrm{Pow}(S')$. If $\theta \in \mathcal{I}_n$, then we have

$$\iota^{-1}(\alpha') \models_{\mathcal{I}_{n+1}} \theta \iff \alpha' \models_{\mathcal{I}'_{n+1}} \hat{\iota}_{\mathcal{I}_{n+1}}(\theta)$$

by the definitions of $\models_{\mathcal{I}_{n+1}}$ and $\hat{\iota}_{\mathcal{I}_{n+1}}$. Let $\sigma \in \mathrm{Fin}(S)$ and $\Gamma \in \mathrm{Pow}(\mathcal{T}_n)$. Then

$$\begin{aligned}\iota^{-1}(\alpha') \models_{\mathcal{I}_{n+1}} (n+1, (\sigma, \Gamma)) &\iff \sigma \subseteq \iota^{-1}(\alpha') \Rightarrow (\exists U \in \Gamma)\, \iota^{-1}(\alpha') \models_{\mathcal{T}_n} U \\ &\iff \iota(\sigma) \subseteq \alpha' \Rightarrow (\exists U \in \Gamma)\, \alpha' \models_{\mathcal{T}_n} \hat{\iota}_{\mathcal{T}_n}(U) \\ &\iff \alpha' \models_{\mathcal{I}'_{n+1}} (n+1, (\iota(\sigma), \{\hat{\iota}_{\mathcal{T}_n}(U) \mid U \in \Gamma\})) \\ &\iff \alpha' \models_{\mathcal{I}'_{n+1}} \hat{\iota}_{\mathcal{I}_{n+1}}(n+1, (\sigma, \Gamma)).\end{aligned}$$

Finally, if $T \in \mathcal{T}_{n+1}$, then

$$\begin{aligned}\iota^{-1}(\alpha') \models_{\mathcal{T}_{n+1}} T &\iff (\forall \theta \in T)\, \iota^{-1}(\alpha') \models_{\mathcal{I}_{n+1}} \theta \\ &\iff (\forall \theta \in T)\, \alpha' \models_{\mathcal{I}_{n+1}} \hat{\iota}_{\mathcal{I}_{n+1}}(\theta) \\ &\iff \alpha' \models_{\mathcal{T}'_{n+1}} \hat{\iota}_{\mathcal{I}_{n+1}}(T).\end{aligned}$$

□

Hence, we have the following lemma.

Lemma 2.2.9. *Let T be a theory over S, and let S' be an extension of S with inclusion ι. Then $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$ if and only if $\alpha' \in \mathfrak{M}(\hat{\iota}(T))$ for each $\alpha' \in \mathrm{Pow}(S')$.*

Let S' be an extension of a set S with an inclusion ι. A theory T' over S' is an *extension* of a theory T over S if $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$ for each $\alpha' \in \mathfrak{M}(T')$, and the extension is *conservative* if for each $\alpha \in \mathfrak{M}(T)$, there exists $\alpha' \in \mathfrak{M}(T')$ such that $\alpha = \iota^{-1}(\alpha')$. Note that $\hat{\iota}(T)$ is a conservative extension of T by Lemma 2.2.9.

Proposition 2.2.10. *Each theory of rank $n+1$ $(n \geq 1)$ has a conservative extension of rank n.*

Proof. Let T be a theory of rank $n+1$ $(n \geq 1)$ over a set S. Divide T into the set P of implications of rank $\leq n$ and the set Q of implications rank $n+1$. Define an extension S' of S by $S' = S + \sum_{\varphi \in Q} \sum_{U \in \Gamma_\varphi} U$, and let $\iota_S : S \to S'$ and $\iota_U : U \to S'$ $(U \in \Gamma_\varphi, \varphi \in Q)$ be the canonical inclusions. Let $\tilde{Q}$ be a theory over S' of rank 1 defined by

$$\tilde{Q} = \left\{ \bigwedge \iota_S(\sigma_\varphi) \to \bigvee_{U \in \Gamma_\varphi} \bigwedge \iota_U(U) \mid \varphi \in Q \right\},$$

and, for an implication $\theta \in U$ $(U \in \Gamma_\varphi, \varphi \in Q)$, define an implication $\tilde{\theta}$ of rank 1 or the same rank by

$$\tilde{\theta} \equiv \begin{cases} \iota_U(\theta) \to \iota_S(\theta) & \text{for } \theta \text{ of rank } 0, \\ \bigwedge (\iota_S(\sigma_\theta) \cup \{\iota_U(\theta)\}) \to \bigvee_{V \in \Gamma_\theta} \bigwedge \hat{\iota}_S(V) & \text{for } \theta \text{ of positive rank.} \end{cases}$$

Finally, define a theory T' over S' of rank n by

$$T' = \hat{\iota_S}(P) \cup \tilde{Q} \cup \left\{ \tilde{\theta} \mid \theta \in U, U \in \Gamma_\varphi, \varphi \in Q \right\}.$$

We show that T' is a conservative extension of T. First, we show that T' is an extension of T. Let $\alpha' \in \mathfrak{M}(T')$. We must show that ${\iota_S}^{-1}(\alpha') \in \mathfrak{M}(T)$, i.e. ${\iota_S}^{-1}(\alpha') \models P$ and ${\iota_S}^{-1}(\alpha') \models Q$. Since $\alpha' \models \hat{\iota_S}(P)$ and P is a theory over S, ${\iota_S}^{-1}(\alpha') \models P$ by Lemma 2.2.9. Let $\varphi \in Q$, and suppose that $\sigma_\varphi \subseteq {\iota_S}^{-1}(\alpha')$. Then $\iota_S(\sigma_\varphi) \subseteq \alpha'$. Since $\alpha' \models \tilde{Q}$, there exists $U_0 \in \Gamma_\varphi$ such that $\alpha' \models \iota_{U_0}(U_0)$, so $\iota_{U_0}(U_0) \subseteq \alpha'$. Let $\theta \in U_0$. If θ is of rank 0, then, since $\alpha' \models \tilde{\theta}$ and $\{\iota_{U_0}(\theta)\} \subseteq \alpha'$, we have $\alpha' \models \iota_S(\theta)$, and hence ${\iota_S}^{-1}(\alpha') \models \theta$. If θ is of positive rank, suppose that $\sigma_\theta \subseteq {\iota_S}^{-1}(\alpha')$. Then $\iota_S(\sigma_\theta) \cup \{\iota_{U_0}(\theta)\} \subseteq \alpha'$, and since $\alpha' \models \tilde{\theta}$, there exists $V \in \Gamma_\theta$ such that $\alpha' \models \hat{\iota}_S(V)$. Hence, ${\iota_S}^{-1}(\alpha') \models V$ by Lemma 2.2.9, and therefore ${\iota_S}^{-1}(\alpha') \models \theta$. Thus, ${\iota_S}^{-1}(\alpha') \models U_0$, so ${\iota_S}^{-1}(\alpha') \models Q$.

Next, we show that T' is a conservative extension of T. Let $\alpha \in \mathfrak{M}(T)$, and let α' be a subset of S' such that

$$\alpha' = \iota_S(\alpha) \cup \bigcup \{\iota_U(U) \mid \alpha \models U, U \in \Gamma_\varphi, \varphi \in Q\}.$$

Note that α' is a set since $\models$ is a predicative class as mentioned in Remark 2.2.5. Clearly, $\alpha = {\iota_S}^{-1}(\alpha')$. It remains to be shown that $\alpha' \models T'$. First, since ${\iota_S}^{-1}(\alpha') = \alpha \models P$, we have $\alpha' \models \hat{\iota_S}(P)$ by Lemma 2.2.9. Next, let $\varphi \in Q$, and suppose that $\iota_S(\sigma_\varphi) \subseteq \alpha'$. Then, $\sigma_\varphi \subseteq {\iota_S}^{-1}(\alpha') = \alpha$. Since $\alpha \models Q$, there exists $U \in \Gamma_\varphi$ such that $\alpha \models U$. Then, $\iota_U(U) \subseteq \alpha'$ by the definition of α', i.e. $\alpha' \models \iota_U(U)$, and hence $\alpha' \models \tilde{Q}$. Finally, let $\varphi \in Q, U \in \Gamma_\varphi$, and $\theta \in U$. If θ is of rank 0, and if $\iota_U(\theta) \in \alpha'$, then $\theta \in U$ and ${\iota_S}^{-1}(\alpha') = \alpha \models U$. Therefore $\alpha' \models \iota_S(\theta)$ by Lemma 2.2.9. Thus $\alpha' \models \tilde{\theta}$. If θ is of positive rank, suppose that $\iota_S(\sigma_\theta) \cup \{\iota_U(\theta)\} \subseteq \alpha'$. Since $\iota_U(\theta) \in \alpha'$ we have $\alpha \models \theta$, and since $\sigma_\theta \subseteq {\iota_S}^{-1}(\alpha') = \alpha$, there exists $V \in \Gamma_\theta$ such that ${\iota_S}^{-1}(\alpha') = \alpha \models V$, so $\alpha' \models \hat{\iota}_S(V)$ by Lemma 2.2.9, and thus, $\alpha \models \tilde{\theta}$. Therefore, $\alpha' \models \left\{ \tilde{\theta} \mid \theta \in U, U \in \Gamma_\varphi, \varphi \in Q \right\}$. □

Proposition 2.2.11. *Let T' be a conservative extension of a theory T. If the class $\mathfrak{M}(T')$ is set-generated, so is the class $\mathfrak{M}(T)$.*

Proof. Let $\iota : S \to S'$ be an extension. Let T be a theory over S and T' be a theory over S', where T' is a conservative extension of T. Suppose that $\mathfrak{M}(T')$ is set-generated, and let G' be a generating subset of $\mathfrak{M}(T')$. We show that $G = \{\iota^{-1}(\alpha') \mid \alpha' \in G'\}$ is a generating subset of $\mathfrak{M}(T)$. To this end, let $\alpha \in \mathfrak{M}(T)$ and $\sigma \subseteq \mathrm{Fin}(\alpha)$. Since T' is a conservative extension of T, there exists $\beta \in \mathfrak{M}(T')$ such that $\iota^{-1}(\beta) = \alpha$, and so $\iota(\sigma) \subseteq \beta$. Then, since G' generates $\mathfrak{M}(T')$, there exists $\alpha' \in G'$ such that $\iota(\sigma) \subseteq \alpha' \subseteq \beta$. Therefore $\sigma \subseteq \iota^{-1}(\alpha') \subseteq \iota^{-1}(\beta) = \alpha$. □

In the following sections, we give two different proofs for the following theorem in an extension of CZF.

Theorem 2.2.12. *The class $\mathfrak{M}(T)$ of models of a theory T of rank 1 is set-generated.*

Combining this with the above two propositions, we have the following theorem.

Theorem 2.2.13. *Assume* RRS_2-uREA *or* RDC. *Then the class $\mathfrak{M}(T)$ of models of a theory T of rank n is set-generated.*

2.2.2 A regular extension axiom

In this section, we give proof of Theorem 2.2.12 in CZF extended with the axiom RRS_2-uREA [6, Theorem 5.2].

Theorem 2.2.14. *Assume* RRS_2-uREA. *Then the class $\mathfrak{M}(T)$ of models of a theory T of rank 1 is set-generated.*

Proof. Let T be a theory over a set S of rank 1. Let $P = \mathcal{I}_0 \cap T$ and $Q = T - P$. Then $P \subseteq S$, and for each $\varphi \in Q$, we have $\Gamma_\varphi \subseteq \mathrm{Pow}(S)$. For each $\alpha \in \mathrm{Pow}(S)$, let $Q_\alpha = \{\varphi \in Q \mid \sigma_\varphi \subseteq \alpha\}$. Since $\sigma_\varphi \in \mathrm{Fin}(S)$ for each $\varphi \in Q$, we have $Q_\alpha = \bigcup_{\tau \in \mathrm{Fin}(\alpha)} Q_\tau$. Let A be a union-closed RRS_2-regular set containing $\{\mathbf{N}, S, P\} \cup \{Q_\tau \mid \tau \in \mathrm{Fin}(S)\} \cup \{\Gamma_\varphi \mid \varphi \in Q\}$ and let

$$G = \{\alpha \in A \mid \alpha \models \mathfrak{M}(T)\}.$$

Note that G is a set by Bounded Separation. We show that G is a generating subset of $\mathfrak{M}(T)$. To this end, let $\gamma \in \mathfrak{M}(T)$ and $A_\gamma = \{\alpha \in A \mid P \subseteq \alpha \subseteq \gamma\}$. Let R be a relation on A_γ such that

$$R = \{(\alpha, \beta) \mid (\forall \varphi \in Q_\alpha)\,(\exists U \in \Gamma_\varphi)\, U \subseteq \beta \wedge \alpha \subseteq \beta\}$$

Let $\alpha \in A_\gamma$. Then $\mathrm{Fin}(\alpha) \in A$ by Lemma 2.1.38. Since $(\forall \tau \in \mathrm{Fin}(\alpha))\,(\exists y \in A)\, y = Q_\tau$ and A is union-closed regular, we have $Q_\alpha = \bigcup_{\tau \in \mathrm{Fin}(\alpha)} Q_\tau \in A$. Since $Q_\alpha \subseteq Q_\gamma$ and $\gamma \models Q$, we have

$$(\forall \varphi \in Q_\alpha)\,(\exists U \in A)\, U \in \Gamma_\varphi \wedge U \subseteq \gamma.$$

Hence, since A is regular, there exists $D \in A$ such that

$$(\forall \varphi \in Q_\alpha)\,(\exists U \in D)\,[U \in \Gamma_\varphi \wedge U \subseteq \gamma] \wedge (\forall U \in D)\,(\exists \varphi \in Q_\alpha)\,[U \in \Gamma_\varphi \wedge U \subseteq \gamma],$$

and since A is union-closed, we have $\delta = \bigcup D \in A_\gamma$. Thus, since $\{\delta, \alpha, P\} \subseteq A$ and A is union-closed regular, we have $\beta = \delta \cup \alpha \cup P \in A$. Hence, $\beta \in A_\gamma$ because $P \subseteq \gamma$. Therefore, $(\alpha, \beta) \in R$, and thus $R \in \mathbf{mv}(A_\gamma, A_\gamma)$. Define a relation $R' \subseteq (A_\gamma \times A_\gamma) \times A_\gamma$ by

$$R' = \{((\alpha, \beta), \eta) \mid (\alpha \cup \beta)\, R \eta\}.$$

Then, since A is union-closed regular, $\alpha \cup \beta \in A_\gamma$ for each $\alpha, \beta \in A_\gamma$. Thus, since R is total, so is R'. Now, let $\tau \in \mathrm{Fin}(\gamma)$. Since $\mathbf{N} \in A$ and A is regular, we have $\tau \in A$ and

$\tau \subseteq \gamma$, and so $\tau' = P \cup \tau \in A_\gamma$. Since A is RRS$_2$-regular, there exists a set $A_0 \in A$ such that $\tau' \in A_0 \subseteq A_\gamma$ and

$$(\forall \alpha, \beta \in A_0)\,(\exists \eta \in A_0)\,(\alpha \cup \beta)\, R\eta.$$

Let $\alpha' = \bigcup A_0$. Then $\tau \subseteq \tau' \subseteq \alpha' \subseteq \gamma$. Also, since A is union-closed, $\alpha' \in A$. It remains to be shown that $\alpha' \models T$, i.e. $\alpha' \models P$ and $\alpha' \models Q$. The former is trivial. For the latter, let $\varphi \in Q$ and suppose that $\sigma_\varphi \subseteq \alpha'$. Since $\sigma_\varphi \in \mathrm{Fin}(\alpha')$, there exists $\beta \in A_0$ such that $\sigma_\varphi \subseteq \beta$. Since R' is total, there exists $\eta \in A_0$ such that $\beta R \eta$, i.e.

$$(\forall \varphi \in Q_\beta)\,(\exists U \in \Gamma_\varphi)\,[U \subseteq \eta \wedge \beta \subseteq \eta]$$

Since $\varphi \in Q_\beta$, we have $U \in \Gamma_\varphi$ such that $U \subseteq \alpha'$, i.e. $\alpha' \models U$. Therefore $\alpha' \models Q$, and thus $\alpha' \models T$. ☐

2.2.3 The relativized dependent choice

In this section, we given a proof of Theorem 2.2.12 in CZF extended with the axiom RDC [6, Theorem 4.1]. First we prove the following lemma for the main theorem.

Lemma 2.2.15. *Let a, b and R be sets, and let*

$$\begin{aligned} r \in \mathbf{mv}_R(a,b) &\iff r \in \mathbf{mv}(a,b) \wedge r \subseteq R, \\ \mathrm{Full}_R(a,b,c) &\iff c \subseteq \mathbf{mv}_R(a,b) \wedge (\forall r \in \mathbf{mv}_R(a,b))\,(\exists s \in c)\, s \subseteq r. \end{aligned}$$

Then, there exists a set c such that $\mathrm{Full}_R(a,b,c)$.

Proof. By Fullness, there exists a set d which is full in $\mathbf{mv}(a,b)$. Let $c = \{r \in d \mid r \subseteq R\}$ be a set by Bounded Separation. Then, $c \subseteq \mathbf{mv}_R(a,b)$. Let $r \in \mathbf{mv}_R(a,b)$. Since $r \in \mathbf{mv}(a,b)$, there exists $s \in d$ such that $s \subseteq r$. Then $s \subseteq R$, and so $s \in c$. ☐

Theorem 2.2.16. *Assume* RDC. *Then the class $\mathfrak{M}(T)$ of models of a theory T of rank 1 is set-generated.*

The claim follows from the series of propositions. Let T be a theory over a set S of rank 1. Let $P = \mathcal{I}_0 \cap T$ and $Q = T - P$, and note that $P \subseteq S$ and $\Gamma_\varphi \subseteq \mathrm{Pow}(S)$ for each $\varphi \in Q$. For $\alpha \in \mathrm{Pow}(S)$, let $Q_\alpha = \{\varphi \in Q \mid \sigma_\varphi \subseteq \alpha\}$.

Let $b = \bigcup_{\varphi \in Q} \Gamma_\varphi$ and $R = \sum_{\varphi \in Q} \Gamma_\varphi$, and define a class $\mathcal{V}$ by

$$\mathcal{V} = \{(\alpha, c) \mid \alpha \in \mathrm{Pow}(S) \wedge \mathrm{Full}_R(Q_\alpha, b, c)\}.$$

Proposition 2.2.17. *There exists a set $V \subseteq \mathcal{V}$ such that*

1. $\forall \tau \in \mathrm{Fin}(S) \exists c\, ((\tau, c) \in V)$,
2. $\forall\, (\alpha, c) \in V \forall r \in c \exists\, (\alpha', c') \in V\, (P \cup \alpha \cup \bigcup \mathrm{ran}(r) = \alpha')$.

Proof. Let ψ be a formula defined by

$$\psi(X,Y) \equiv \forall(\alpha,c) \in X \forall r \in c \exists\, (\alpha',c') \in Y \left(P \cup \alpha \cup \bigcup \mathrm{ran}(r) = \alpha'\right).$$

We show that $\forall X \in \mathrm{Pow}(\mathcal{V}) \exists Y \in \mathrm{Pow}(\mathcal{V}) \psi(X,Y)$. To this end, let X be a set such that $X \subseteq \mathcal{V}$. For each $(\alpha,c) \in X$ and $r \in c$, let $\alpha' = P \cup \alpha \cup \bigcup \mathrm{ran}(r)$, and let c' be a set such that $\mathrm{Full}_R(Q_{\alpha'}, b, c')$ by Lemma 2.2.15. Then $(\alpha',c') \in \mathcal{V}$. Hence, we have

$$\forall\, ((\alpha,c),r) \in \textstyle\sum_{(\alpha,c)\in X} c \exists (\alpha',c') \in \mathcal{V}\, (P \cup \alpha \cup \bigcup \mathrm{ran}(r) = \alpha'),$$

and therefore, there is a set $Y \subseteq \mathcal{V}$ such that

$$\forall\, ((\alpha,c),r) \in \textstyle\sum_{(\alpha,c)\in X} c \exists (\alpha',c') \in Y\, (P \cup \alpha \cup \bigcup \mathrm{ran}(r) = \alpha').$$

by Strong Collection. Thus $\psi(X,Y)$. Since $\forall \tau \in \mathrm{Fin}(S) \exists c\, \mathrm{Full}_R(Q_\tau, b, c)$ by Lemma 2.2.15, there exists a set $X_0 \subseteq \mathcal{V}$ such that $\forall \tau \in \mathrm{Fin}(S) \exists c\, ((\tau,c) \in X_0)$ by Strong Collection. Applying RDC to $\forall X \in \mathrm{Pow}(\mathcal{V}) \exists Y \in \mathrm{Pow}(\mathcal{V}) \psi(X,Y)$ and X_0, we have a function f with domain $\mathbf{N}$ such that $f(0) = X_0$ and

$$\forall n \in \mathbf{N}\, [f(n) \subseteq \mathcal{V} \wedge \psi\,(f(n), f(n+1))].$$

Let $V = \bigcup_{n\in\mathbf{N}} f(n)$. Then it is straightforward to see (1) and (2). □

Using Exponentiation, Bounded Separation and Strong Collection, define sets B and G by

$$B = \left\{\langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in V^{\mathbf{N}} \mid \forall n \in \mathbf{N}\, \exists r \in c_n\, (P \cup \alpha_n \cup \bigcup \mathrm{ran}(r) = \alpha_{n+1})\right\},$$
$$G = \left\{\textstyle\bigcup_{n\in\mathbf{N}} \alpha_n \mid \langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in B\right\}.$$

Proposition 2.2.18. *Each α in G is a model of T.*

Proof. Let $\alpha \in G$. Then there exists $\langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in B$ such that $\alpha = \bigcup_{n\in\mathbf{N}} \alpha_n$. Note that $P \subseteq \alpha_1 \subseteq \alpha$, so $\alpha \models P$. Now, let $\varphi \in Q$, and suppose that $\sigma_\varphi \subseteq \alpha$. Then, since $\sigma_\varphi \in \mathrm{Fin}(S)$ and $\alpha_n \subseteq \alpha_{n+1}$ for each $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ and $r \in c_m$ such that $\sigma_\varphi \subseteq \alpha_m$ and $P \cup \alpha_m \cup \bigcup \mathrm{ran}(r) = \alpha_{m+1}$. Therefore, since $r \in \mathbf{mv}_R(Q_{\alpha_m}, b)$ and $\varphi \in Q_{\alpha_m}$, there exists $U \in b$ such that $U \in \Gamma_\varphi$ and $\varphi\, r\, U$, and hence $U \subseteq \bigcup \mathrm{ran}(r) \subseteq \alpha_{m+1} \subseteq \alpha$. Hence $\alpha \models Q$, and thus $\alpha \models T$. □

Proposition 2.2.19. *Let γ be a model of T, and let $\tau \in \mathrm{Fin}(\gamma)$. Then there exists $\beta \in G$ such that $\tau \subseteq \beta \subseteq \gamma$.*

Proof. Let $V_\gamma = \{(\alpha,c) \in V \mid \alpha \subseteq \gamma\}$. We show that

$$\forall\, (\alpha,c) \in V_\gamma \exists\, (\alpha',c') \in V_\gamma \exists s \in c \left(P \cup \alpha \cup \bigcup \mathrm{ran}(s) = \alpha'\right).$$

Let $(\alpha,c) \in V_\gamma$. Define a set

$$r = \{(\varphi, U) \in R \mid U \in \Gamma_\varphi \wedge U \subseteq \gamma\},$$

by Bounded Separation. Then, since $Q_\alpha \subseteq Q_\gamma$ and γ is a model of T, for each $\varphi \in Q_\alpha$ there exists $U \in \Gamma_\varphi$ such that $U \subseteq \gamma$. Therefore $r \in \mathbf{mv}_R(Q_\alpha, b)$, so there exists $s \in c$ such that $s \subseteq r$. Note that $\bigcup \mathrm{ran}(s) \subseteq \bigcup \mathrm{ran}(r) \subseteq \gamma$. Then, there exists $(\alpha', c') \in V$ such that $\alpha' = P \cup \alpha \cup \bigcup \mathrm{ran}(s) \subseteq \gamma$ by Proposition 2.2.17 (2). Thus $(\alpha', c') \in V_\gamma$.

By Proposition 2.2.17 (1), there exists c such that $(\tau, c) \in V_\gamma$. Applying DC, we have a function $h : \mathbf{N} \to V_\gamma$ with $h(n) = (\alpha_n, c_n)$ such that $(\alpha_0, c_0) = (\tau, c)$ and $\forall n \in \mathbf{N}\, \exists s \in c_n\, (P \cup \alpha_n \cup \bigcup \mathrm{ran}(s) = \alpha_{n+1})$. Therefore, since $h \in B$, we have $\beta = \bigcup_{n \in \mathbf{N}} \alpha_n \in G$ and $\tau \subseteq \beta \subseteq \gamma$. □

2.3 Category Theory

In this section, we introduce all the concepts of category theory that we use in this thesis. The topics are largely drawn from the first chapter of [9]. They can be found in any other introductory textbooks on category theory; see, e.g. [8]. The reader who is new to category theory should be warned that some of the important concepts of category theory, e.g. pullbacks, are deliberately omitted since they are not treated in this thesis.

2.3.1 Categories

Definition 2.3.1. A *category* $\mathbf{C}$ consists of a class $\mathrm{Ob}(\mathbf{C})$ of *objects* of $\mathbf{C}$ (called $\mathbf{C}$-objects), and a class $\mathrm{Arr}(\mathbf{C})$ of *arrows* of $\mathbf{C}$ (called $\mathbf{C}$-arrows) such that

1. there are assignments $\mathrm{dom}, \mathrm{cod} : \mathrm{Arr}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$ which assigns to each $f \in \mathrm{Arr}(\mathbf{C})$, objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$, called the *domain* and the *codomain* of f respectively. If $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$, we write

$$f : X \to Y \text{ or } X \xrightarrow{f} Y$$

   for the statement

$$f \in \mathrm{Arr}(\mathbf{C}) \wedge \mathrm{dom}(f) = X \wedge \mathrm{cod}(f) = Y.$$

2. there is an assignment $\circ : \mathrm{Arr}(\mathbf{C}) \times \mathrm{Arr}(\mathbf{C}) \to \mathrm{Arr}(\mathbf{C})$ which assigns to each arrows $f : X \to Y$ and $g : Y \to Z$ with $\mathrm{dom}(g) = \mathrm{cod}(f)$, an arrow $g \circ f : X \to Z$ called the *composite* (or *composition*) of f and g.

3. there is an assignment $1 : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Arr}(\mathbf{C})$ which assigns to each $X \in \mathrm{Ob}(\mathbf{C})$, an arrow $1_X : X \to X$ called the *identity arrow* on X.

These data are required to satisfy the following axioms:

Associativity law: For any arrows $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Identity law: For any $Y \in \mathrm{Ob}(\mathbf{C})$, and for any $f : X \to Y$ and $g : Y \to Z$

$$1_Y \circ f = f, \qquad\qquad g \circ 1_Y = g.$$

Remark 2.3.2.

- Arrows are also called *morphisms*. We use arrow and morphism synonymously.
- The composite $g \circ f$ is defined iff $\mathrm{dom}(g) = \mathrm{cod}(f)$.
- By Associativity law, we can unambiguously write $h \circ g \circ f$ for $h \circ (g \circ f)$ or $(h \circ g) \circ f$.

Notations. Throughout this section, we follow the following conventions.

- $\mathbf{C}, \mathbf{D}, \mathbf{E} \ldots$ denote categories
- $A, B, C, \ldots, X, Y, Z, \ldots$ denote objects.
- $f, g, h, \ldots$ denote arrows.

Definition 2.3.3. Let $\mathbf{C}$ be a category. For each pair of $\mathbf{C}$-objects X and Y, the class

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathrm{Arr}(\mathbf{C}) \mid f : X \to Y\}$$

is called the *hom-set* of X and Y. We also write $\mathbf{C}(X, Y)$ for $\mathrm{Hom}_{\mathbf{C}}(X, Y)$, or just $\mathrm{Hom}(X, Y)$ if $\mathbf{C}$ is clear from the context.

Definition 2.3.4. A category $\mathbf{C}$ is *small* if $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Arr}(\mathbf{C})$ are sets, otherwise it is *large*. It is locally small if $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a set for each $X, Y \in \mathrm{Ob}(\mathbf{C})$.

Examples 2.3.5.

1. **Set** is a category whose objects are the class of all sets and whose arrows are the class of functions between sets. The composition is the usual composition of functions, and the identity arrow on a set X is the identity function id_X. **Set** is a large category but it is locally small.

2. Similarly, **Top** is a category whose objects are the class of topological spaces and whose arrows are the class of continuous functions between topological spaces. The composition and the identity are defined as in **Set**.

3. **Rel** is a category whose objects are the class of all sets and whose arrows are binary relations between sets. The composition of relations $r : X \to Y$ and $s : Y \to Z$, namely $r \subseteq X \times Y$ and $s \subseteq Y \times Z$, is a relation $s \circ r \subseteq X \times Z$ defined by

$$s \circ r = \{(x, z) \in X \times Z \mid (\exists y \in Y)\, x\, r\, y \wedge y\, s\, z\}\,.$$

   The identity arrow on a set X is the *diagonal relation* on X, namely $\Delta_X = \{(x, x) \mid x \in X\}$. Note that **Rel** is neither small nor locally small in CZF.

4. A *preordered class* is a class P with a relation $\leqslant$ on P such that

   - (reflexive) $p \leqslant p$,
   - (transitive) $p \leqslant q \wedge q \leqslant r \to p \leqslant r$.

   If, moreover, $\leqslant$ satisfies

   - (anti-symmetry) $p \leqslant q \wedge q \leqslant p \to p = q$,

   then P is called a *partially ordered class.* Every preordered class determines a category as follows:

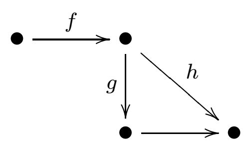
- Objects — The elements of P.
- Arrows — For any $p, q \in P$, $\mathrm{Hom}(p, q) = \{(p, q) \mid p \leqslant q\}$.
- Compositions — For any arrows (p, q) and (q, r), their composite is $(q, r) \circ (p, q) = (p, r)$.
- Identities — For any $p \in P$, $1_p = (p, p)$.

Note that $\mathrm{Hom}(p, q)$ has at most one arrow for any objects $p, q \in P$. Also, the composition and the identity are well-defined by transitivity and reflexivity of $\leqslant$. We write $\mathbf{C}(P)$ for the category associated with a preordered class P.

5. The *empty* category, denoted by $\mathbf{0}$, has the empty set of objects and the empty set of arrows. Similarly, the *degenerate* category, denoted by $\mathbf{1}$, has just one object and one arrow, namely, the identity arrow on the object.

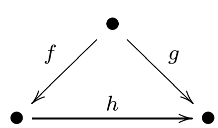
Since category theory makes heavy use of diagrams, we define an informal notion of diagram in a category. Much more formal definition of a diagram in a category is given in 2.3.54.

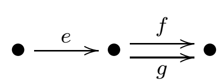
Definition 2.3.6. A *diagram* D in a category $\mathbf{C}$ consists of a set V of objects of $\mathbf{C}$ and a set of arrows of $\mathbf{C}$ whose domain and codomain are in V. A diagram can be depicted as



A *path* in a diagram D is a finite sequence $\langle f_1, \ldots, f_n \rangle$ of arrows of D with $\mathrm{dom}(f_{i+1}) = \mathrm{cod}(f_i)$ for each $i < n$. A diagram D is said to *commute* if for any two paths $\langle f_1, \ldots, f_n \rangle$ and $\langle g_1, \ldots, g_m \rangle$ with $n \geq 2$ or $m \geq 2$ and $\mathrm{dom}(f_1) = \mathrm{dom}(g_1)$ and $\mathrm{cod}(f_n) = \mathrm{cod}(g_m)$, we have $f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1$.

Examples 2.3.7. Here are two diagrams.

(i) 

(ii) 

The diagram (i) commutes iff $g = h \circ f$. The diagram (ii) commutes iff $f \circ e = g \circ e$.

Definition 2.3.8. A category $\mathbf{D}$ is a *subcategory* of a category $\mathbf{C}$ if

- $\mathrm{Ob}(\mathbf{D}) \subseteq \mathrm{Ob}(\mathbf{C})$,
- For each $X, Y \in \mathrm{Ob}(\mathbf{D})$, $\mathbf{D}(X, Y) \subseteq \mathbf{C}(X, Y)$,
- The composition $\circ_{\mathbf{D}}$ of $\mathbf{D}$ is the restriction of the composition $\circ_{\mathbf{C}}$ of $\mathbf{C}$ to the arrows of $\mathbf{D}$.

- For any $X \in \mathrm{Ob}(\mathbf{D})$, the identity arrow on X in $\mathbf{D}$ is $1_X \in \mathbf{C}(X, X)$ in $\mathbf{C}$.

A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *full* if for every $X, Y \in \mathrm{Ob}(\mathbf{D})$, $\mathbf{D}(X, Y) = \mathbf{C}(X, Y)$.

Remark 2.3.9. A full subcategory is completely determined by specifying its objects.

Example 2.3.10.

- **Set** is a subcategory of **Rel**. It is not full.
- The category **Finset** of the category of finitely enumerable sets and functions between them is a full subcategory of **Set**.

Definition 2.3.11. For any category $\mathbf{C}$, its *opposite category* $\mathbf{C}^{\mathrm{op}}$ is a category defined by

- $\mathrm{Ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathbf{C})$,
- For each $X, Y \in \mathrm{Ob}(\mathbf{C})$, $\mathbf{C}^{\mathrm{op}}(X, Y) = \mathbf{C}(Y, X)$,
- For any $\mathbf{C}^{\mathrm{op}}$-arrows $f : X \to Y$ and $g : Y \to Z$, their composite $g \circ_{op} f : X \to Z$ in $\mathbf{C}^{\mathrm{op}}$ is the composite $f \circ g : Z \to X$ in $\mathbf{C}$,
- For each $X \in \mathrm{Ob}(\mathbf{C}^{\mathrm{op}})$, the identity 1_X in $\mathbf{C}^{\mathrm{op}}$ is the identity arrow on X in $\mathbf{C}$.

Note that $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$. Informally, $\mathbf{C}^{\mathrm{op}}$ is obtained from $\mathbf{C}$ by reversing all arrows, i.e. interchanging the domain and the codomain of all arrows.

Formally a category can be defined as a two-sorted first order theory with equality with variables $X, Y, Z, \ldots.$ for objects and $f, g, h, \ldots$ for arrows and four function symbols dom, cod, $\circ$ and 1 with axioms

$$
\begin{aligned}
\mathrm{dom}(1_X) &= X, & \mathrm{cod}(1_X) &= X, \\
f \circ 1_{\mathrm{dom}(f)} &= f, & 1_{\mathrm{cod}(f)} \circ f &= f, \\
\mathrm{dom}(g \circ f) &= \mathrm{dom}(f), & \mathrm{cod}(g \circ f) &= \mathrm{cod}(g), \\
h \circ (g \circ f) &= (h \circ g) \circ f.
\end{aligned}
$$

More precisely, since $g \circ f$ is defined only when $\mathrm{dom}(g) = \mathrm{cod}(f)$, we must append this condition to each equation containing $\circ$.

Definition 2.3.12. Let S be a statement of a category. The *dual* of S, denoted by S^{op}, is a statement obtained from S by recursively substituting

- $f \circ g$ for $g \circ f$,
- dom for cod,
- cod for dom.

Then for any category $\mathbf{C}$, S^{op} holds in $\mathbf{C}$ iff S holds in $\mathbf{C}^{\mathrm{op}}$. Thus we have the following principle.

Proposition 2.3.13 (Duality principle)**.** *Let S be any statement of categories. If S holds for all categories, then S^{op} holds for all categories.*

Proof. Suppose that S holds for all categories. Let $\mathbf{C}$ be a category. Then S holds in $\mathbf{C}^{\mathrm{op}}$, and thus S^{op} holds in $\mathbf{C}$. □

Similarly, for any concept or construct W defined in terms of the language of categories, its *dual*, denoted by W^{op}, is a concept or construct of categories obtained by applying the above substitution to its definition.

A construct or concept is *self dual* iff $W = W^{op}$.

2.3.2 Arrows

Unless otherwise noted, we fixed the base category $\mathbf{C}$ in the following arguments.

Definition 2.3.14. An arrow $f : X \to Y$ is an *isomorphism* if there is an arrow $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Such arrow g is called an *inverse* of f. We often write $f : X \cong Y$ for an isomorphism $f : X \to Y$.

Proposition 2.3.15. *For any $f : X \to Y, g : Y \to X$ and $h : Y \to X$, if $g \circ f = 1_X$ and $f \circ h = 1_Y$, then $g = h$.*

Proof. Suppose that $g \circ f = 1_X$ and $f \circ h = 1_Y$. Then $g = g \circ 1_Y = g \circ f \circ h = 1_X \circ h = h$. □

Corollary 2.3.16. *If g and h are inverses of an isomorphism f, then $g = h$.*

Remark 2.3.17.

- By the corollary, an inverse of any isomorphism f is unique, and we write f^{-1} for *the* inverse of f.
- The statement "f is and isomorphism" is self dual.

The following are obvious.

Proposition 2.3.18.

1. *If f is an isomorphism, so is f^{-1}.*
2. *If $f : X \to Y$ and $g : Y \to Z$ are isomorphisms, so is $g \circ f$.*

Definition 2.3.19. Objects A and B are *isomorphic*, denoted by $A \cong B$, if there is an isomorphism $f : A \to B$ between them.

Definition 2.3.20. An arrow $f : X \to Y$ is a *monomorphism* if for any $g_1, g_2 : Z \to X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

The dual concept of a monomorphism is that of an epimorphism.

Definition 2.3.21. An arrow $f : X \to Y$ is an *epimorphism* if for any $g_1, g_2 : Y \to Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Examples 2.3.22. In **Set**, an arrow $f : X \to Y$ is

- an isomorphism iff f is bijective,
- a monomorphism iff f is injective,
- an epimorphism iff f is surjective.

We give a proof for the last fact since most textbooks on category theory give non-constructive proofs. The proof is due to Hajime Ishihara.

Proposition 2.3.23. *In* **Set***, a function $f : X \to Y$ is an epimorphism iff f is surjective.*

Proof. First, suppose that f is surjective. Let $g_1, g_2 : Y \to Z$ be functions such that $g_1 \circ f = g_2 \circ f$. Let $y \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = y$. Thus, $g_1(y) = g_1(f(x)) = g_2(f(x)) = g(y)$, and therefore $g_1 = g_2$. Conversely, suppose that f is an epimorphism. Define a set Z and functions $g_1, g_2 : Y \to Z$ by

$$\begin{aligned} Z &= \{\{y\} \mid y \in Y\} \cup \left\{ff^{-1}\{y\} \mid y \in Y\right\}, \\ g_1(y) &= ff^{-1}\{y\}, \\ g_2(y) &= \{y\} \end{aligned}$$

for any $y \in Y$. Then, we have $g_1(f(x)) = ff^{-1}\{f(x)\} = \{f(x)\} = g_2(f(x))$ for all $x \in X$. Since f is an epimorphism, we have $g_1 = g_2$, and hence $\{y\} = ff^{-1}\{y\}$ for all $y \in Y$. Thus f is surjective. □

2.3.3 Construction in categories

Definition 2.3.24. An object $A \in \mathrm{Ob}(\mathbf{C})$ is an *initial object* of $\mathbf{C}$ if for any $B \in \mathrm{Ob}(\mathbf{C})$, there is a unique arrow $f : A \to B$.

Proposition 2.3.25. *An initial object of a category is* unique up to isomorphism*, i.e. if A and B are initial objects in $\mathbf{C}$, then $A \cong B$.*

Proof. Suppose that $A, B \in \mathrm{Ob}(\mathbf{C})$ are initial objects. Then, there are arrows $f : A \to B$ and $g : B \to A$, so we have $g \circ f : A \to A$ and $f \circ g : B \to B$. Since $1_A : A \to A$ and $1_B : B \to B$, we must have $g \circ f = 1_A$ and $f \circ g = 1_B$ by initiality. Therefore f is an isomorphism, and thus $A \cong B$. □

Notation. An initial object in a category, if it exists, is usually denoted by 0.

The dual notion of initial object is that of terminal object.

Definition 2.3.26. An object $A \in \mathrm{Ob}(\mathbf{C})$ is a *terminal object* of $\mathbf{C}$ if for any $B \in \mathrm{Ob}(\mathbf{C})$, there is a unique arrow $f : B \to A$.

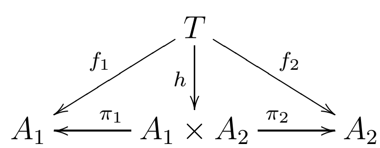
By the duality principle, a terminal object, if it exists, is unique up to isomorphism.

Notation. A terminal object in a category, if it exists, is usually denoted by 1.

Examples 2.3.27.

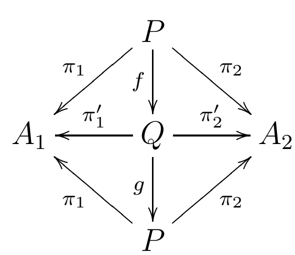
- In **Set**, the empty set $\emptyset$ is an initial object, and any one point set $\{*\}$ is a terminal object.
- In the category $\mathbf{C}(P)$ associated with a partially ordered class P, an initial object is the *least* element and a terminal object is the *largest* element.
- In **Rel**, $\emptyset$ is both an initial and terminal object.

Definition 2.3.28. A *product* of objects A_1, A_2 is an object $A_1 \times A_2$ together with arrows $A_1 \xleftarrow{\pi_1} A_1 \times A_2 \xrightarrow{\pi_2} A_2$, called *projections*, such that for any pair of arrows $f_1 : T \to A_1$ and $f_2 : T \to A_2$ with a common domain, there is a unique arrow $h : T \to A_1 \times A_2$ such that the following diagram commutes.



Proposition 2.3.29. *A product of two objects is unique up to isomorphism, i.e. if* $A_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} A_2$ *and* $A_1 \xleftarrow{\pi_1'} Q \xrightarrow{\pi_2'} A_2$ *are products of* A_1 *and* A_2*, then* $P \cong Q$*.*

Proof. Let $A_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} A_2$ and $A_1 \xleftarrow{\pi_1'} Q \xrightarrow{\pi_2'} A_2$ be products of A_1 and A_2. Then there are arrows $f : P \to Q$ and $g : Q \to P$ such that the following diagram commutes.

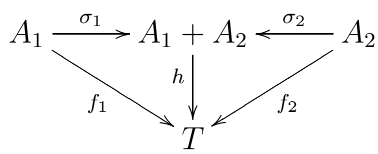


Thus, we have $\pi_1 \circ g \circ f = \pi_1$ and $\pi_2 \circ g \circ f = \pi_2$, but we also have $\pi_1 \circ 1_P = \pi_1$ and $\pi_2 \circ 1_P = \pi_2$. Therefore $g \circ f = 1_P$ by uniqueness of such arrow. Similarly, we have $f \circ g = 1_Q$. Thus $f : P \cong Q$. $\square$

Notation. We write $A_1 \times A_2$ for the product of A_1 and A_2 and $\langle f_1, f_2 \rangle$ for the unique arrow $h : T \to A_1 \times A_2$ which corresponds to arrows $f_1 : T \to A_1$ and $f_2 : T \to A_2$. Note that $\langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle$ iff $f_i = g_i$ $(i = 1, 2)$; for $\langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle$ implies $f = \pi_1 \circ \langle f_1, f_2 \rangle = \pi_1 \circ \langle g_1, g_2 \rangle = g_1$ and similarly $f_2 = g_2$. The converse is trivial.

The dual notion of product of two objects is that of coproduct of two objects.

Definition 2.3.30. A *coproduct* of objects A_1, A_2 is an object $A_1 + A_2$ together with arrows $A_1 \xrightarrow{\sigma_1} A_1 + A_2 \xleftarrow{\sigma_2} A_2$, called *injections*, such that for any pair of arrows $f_1 : A_1 \to T$ and $f_2 : A_2 \to T$ with common codomain, there is a unique arrow $h : A_1 + A_2 \to T$ such that the following diagram commutes.



By the duality principle, coproducts of two objects are unique up to isomorphism.

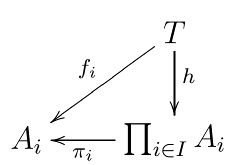
Definition 2.3.31. A category **C** *has binary products* (or coproducts) if the product $A \times B$ (respectively coproduct $A + B$) exists for every pair of objects A and B of **C**.

Examples 2.3.32.

- In **Set**, the product of sets A and B is the Cartesian product $A \times B$ together with projection functions. Their coproduct is the disjoint union $A + B = (\{1\} \times A) \cup (\{2\} \times B)$ together with injection functions $\sigma_A : a \mapsto (1, a)$ and $\sigma_B : b \mapsto (2, b)$.
- In **Top**, the product of topological spaces (X_1, τ_1) and (X_2, τ_2), where τ_i $(i = 1, 2)$ are topologies of X_1 and X_2, is the topological product $(X_1 \times X_2, \tau_1 \times \tau_2)$ together with projection functions.
- In the category $\mathbf{C}(P)$ associated with partially ordered class P, the product of $p, q \in P$ is the *meet* (or *infimum*) $p \wedge q$, and their coproduct is the *join* (or *supremum*) $p \vee q$.

The notion of a (binary) product and coproduct can be extended to a product of an arbitrary set-indexed family of objects.

Definition 2.3.33. A *products* of a set-indexed family $(A_i)_{i \in I}$ of objects is an object, denoted by $\prod_{i \in I} A_i$, together with a family of arrows $\left(\pi_i : \prod_{i \in I} A_i \to A_i\right)_{i \in I}$ such that for any family of arrows $(f_i : T \to A_i)_{i \in I}$, there is a unique arrow $h : T \to \prod_{i \in I} A_i$ such that the diagram commutes



for each $i \in I$. Note that the empty product, i.e. a product in which $I = \emptyset$, is a terminal object, and the product of a singleton family $\{A\}$ is A itself.

The dual concept of product of a family is that of *coproduct* of a family. Its definition can be obtained by reversing all arrows in the definition of a product of a family. Note that the empty coproduct is an initial object, and the coproduct of a singleton family $\{A\}$ is A itself.

Examples 2.3.34. In **Set**, the product of a family of sets $(A_i)_{i\in I}$ is the Cartesian product $\prod_{i\in I} A_i$ of the family together with a family of projections $\left(\pi_i : \prod_{i\in I} A_i \to A_i\right)_{i\in I}$ defined by $\pi_i(f) = f(i)$ for all $f \in \prod_{i\in I} A_i$ and $i \in I$. The coproduct of a family $(A_i)_{i\in I}$ is the disjoint sum $\sum_{i\in I} A_i$ together with a family of injections $\left(\sigma_i : A_i \to \sum_{i\in I} A_i\right)_{i\in I}$ defined by $\sigma_i(a) = (i, a)$ for all $i \in I$ and $a \in A_i$.

In the category $\mathbf{C}(P)$ associated with a partially ordered class P, the product (coproduct) of a family of elements $(p_i)_{i\in I}$ is the *meet* $\bigwedge_{i\in I} p_i$ (respectively join $\bigvee_{i\in I} p_i$).

Definition 2.3.35. A category $\mathbf{C}$ *has (finite) products (coproducts)* if a product (respectively coproduct) of any family of objects indexed by any (finite) set exists in $\mathbf{C}$.

Proposition 2.3.36. *A category* $\mathbf{C}$ *has finite products if it has a terminal object and binary products.*

Proof. Suppose that $\mathbf{C}$ has a terminal object and binary products. It suffices to show that $\mathbf{C}$ has a product of any finite sequence of objects. Let $(A_i)_{i<n}$ be a family of objects in $\mathbf{C}$ where $n \in \mathbf{N}$. The proof is by induction on n.
Basis: If $n = 0$, then the product of the family is a terminal object.
Induction step: Given a family of objects $(A_i)_{i<n+1}$, by induction hypothesis, we have a product $\prod_{i<n} A_i$ of the family $(A_i)_{i<n}$ with projections $(q_j : \prod_{i<n} A_i \to A_i)_{i<n}$. Let $\prod_{i<n} A_i \xleftarrow{p_1} P \xrightarrow{p_2} A_n$ be the product of $\prod_{i<n} A_i$ and A_n. We show that P together with a family of projections $(\pi_i : P \to A_i)_{i<n+1}$ defined by

$$\pi_i = \begin{cases} q_i \circ p_1 & \text{if } i < n, \\ p_2 & \text{if } i = n \end{cases}$$

is a product of $(A_i)_{i<n+1}$. Let $(f_i : T \to A_i)_{i<n+1}$ be a family of arrows in $\mathbf{C}$. Then, there is a unique arrow $h : T \to \prod_{i<n} A_i$ such that $f_i = q_i \circ h$ for all $i < n$, so there is a unique arrow $u : T \to P$ such that $h = p_1 \circ u$ and $f_n = p_2 \circ u$. Then we have $f_i = q_i \circ h = q_i \circ p_1 \circ u$ for each $i < n$, and hence the diagram

$$\begin{array}{ccc} & & T \\ & \swarrow^{f_i} & \downarrow u \\ A_i & \xleftarrow[\pi_i]{} & P \end{array}$$

commutes for each $i < n + 1$. Suppose that $u' : T \to P$ is an arrow which makes the above diagram commute for each $i < n + 1$. Then, we have $f_i = \pi_i \circ u' = q_i \circ p_2 \circ u'$ for each $i < n$, so we must have $p_2 \circ u' = h$ by the uniqueness of h. Therefore $u' = u$ by the uniqueness of u. Thus P is a product of $(A_i)_{i<n+1}$. ☐

Corollary 2.3.37. *A category* $\mathbf{C}$ *has finite coproducts if it has an initial object and binary coproducts.*

Definition 2.3.38. An *equaliser* for a parallel pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$, i.e. two arrows with common domain and codomain, is an object E together with an arrow $e : E \to A$ such that

1. the diagram $E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$ commutes,

2. for any arrow $h : T \to A$ which makes the diagram $T \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$ commute, there is a unique arrow $u : T \to E$ such that $h = e \circ u$.

Dually, a *coequaliser* for a parallel pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an object Q together with an arrow $q : B \to Q$ such that

1. the diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} Q$ commutes,

2. for any arrow $h : B \to T$ which makes the diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} T$ commute, there is a unique arrow $u : Q \to T$ such that $h = u \circ q$.

Definition 2.3.39. A category $\mathbf{C}$ *has equalisers (coequalisers)* if an equaliser (respectively coequaliser) for any parallel pair of $\mathbf{C}$-arrows exists in $\mathbf{C}$.

Example 2.3.40. In **Set**, an equaliser for functions $f, g : A \to B$ is an insertion $i : E \hookrightarrow A$ from the set $E = \{a \in A \mid f(a) = g(a)\}$. A coequaliser for functions $f, g : A \to B$ is the quotient $B/\equiv$ of B with respect to the smallest equivalence relation $\equiv$ on B which contains a set $R = \{(f(a), g(a)) \mid a \in A\}$, i.e. $\equiv$ is the reflexive, symmetric and transitive closure of R. The function $q : B \to Q$ is the natural projection which sends each $b \in B$ to its equivalence class $[b]$.

2.3.4 Functors and natural transformations

Definition 2.3.41. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of functions $F_0 : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ and $F_1 : \mathrm{Arr}(\mathbf{C}) \to \mathrm{Arr}(\mathbf{D})$ such that

1. for each $f : A \to B$ in $\mathbf{C}$, $F_1(f) : F_0(A) \to F_0(B)$ in $\mathbf{D}$,

2. for each pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$,

3. for each $A \in \mathrm{Ob}(\mathbf{C})$, $F_1(1_A) = 1_{F_0(A)}$.

Notation. For a functor F, the subscripts of F_0 and F_1 are usually omitted; we simply write $F(A)$ and $F(f)$ for $F_0(A)$ and $F_1(f)$, or even FA and Ff.

Examples 2.3.42.

- For any category $\mathbf{C}$, there is an obvious identity functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ such that $1_{\mathbf{C}}(A) = A$ and $1_{\mathbf{C}}(f) = f$ for all $A \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathrm{Arr}(\mathbf{C})$.

- Similarly, for any subcategory $\mathbf{D}$ of a category $\mathbf{C}$, there is an insertion functor $I : \mathbf{D} \to \mathbf{C}$.

- A functor $F : \mathbf{C}(P) \to \mathbf{C}(Q)$ between categories associated with partially ordered classes P and Q is just an order preserving function.

- Let $\mathbf{C}$ be a locally small category. For each $A \in \text{Ob}(\mathbf{C})$, we have the *hom-functor* $H_A : \mathbf{C} \to \mathbf{Set}$ defined by $H_A(X) = \mathbf{C}(A, X)$ for all $X \in \text{Ob}(\mathbf{C})$ and for each $f : X \to Y \in \text{Arr}(\mathbf{C})$, $H_A(f) : \mathbf{C}(A, X) \to \mathbf{C}(A, Y)$ is the function such that $H_A(f)(g) = f \circ g$ for all $g \in \mathbf{C}(A, X)$. The functor H_A is also denoted by $\text{Hom}(A, -)$ or $\text{hom}(A, -)$ in other literature.

Definition 2.3.43. A functor $F : \mathbf{C}^{\text{op}} \to \mathbf{D}$ whose domain is an opposite category is called a *contravariant* functor from $\mathbf{C}$ to $\mathbf{D}$.

Example 2.3.44. Let $\mathbf{C}$ be a locally small category. For each $A \in \text{Ob}(\mathbf{C})$, we have the contravariant *hom-functor* $H^A : \mathbf{C}^{\text{op}} \to \mathbf{Set}$ defined by $H^A(X) = \mathbf{C}(X, A)$ for all $X \in \text{Ob}(\mathbf{C})$ and for each $f : Y \to X \in \text{Arr}(\mathbf{C})$, $H^A(f) : \mathbf{C}(X, A) \to \mathbf{C}(Y, A)$ is the function such that $H^A(f)(g) = g \circ f$ for all $g \in \mathbf{C}(X, A)$. The functor H^A is also denoted by $\text{Hom}(-, A)$ or $\text{hom}(-, A)$ in other literature.

Definition 2.3.45. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ be functors. The composite of F and G is a functor $G \circ F : \mathbf{C} \to \mathbf{E}$ defined by $G \circ F(A) = GFA$ and $G \circ F(f) = GFf$ for all $A \in \text{Ob}(\mathbf{C})$ and $f \in \text{Arr}(\mathbf{C})$. We often write GF for $G \circ F$.

Definition 2.3.46. A functor $F : \mathbf{C} \to \mathbf{D}$ is called an *isomorphism* if there is a functor $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F = 1_{\mathbf{C}}$ and $F \circ G = 1_{\mathbf{D}}$. The categories $\mathbf{C}$ and $\mathbf{D}$ are said to be *isomorphic*, denoted by $\mathbf{C} \cong \mathbf{D}$, if that there is an isomorphism $F : \mathbf{C} \to \mathbf{D}$.

Definition 2.3.47. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be

- *full* if for each $A, B \in \text{Ob}(\mathbf{C})$, its restriction

$$F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$$

  is surjective.

- *faithful* if the above restriction is injective for each $A, B \in \text{Ob}(\mathbf{C})$.

- *dense* if for any $B \in \text{Ob}(\mathbf{D})$ there exists $A \in \text{Ob}(\mathbf{C})$ such that $FA \cong B$.

- *embedding* if it is faithful and injective on objects.

Example 2.3.48.

- Every insertion functor $I : \mathbf{D} \to \mathbf{C}$ from a subcategory is an embedding. It is full iff $\mathbf{D}$ is a full subcategory of $\mathbf{C}$.

- The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which assigns to each topological space (X, τ_X) its underlying set X and to each continuous function $f : X \to Y$ its underlying function f, is faithful and dense, but not full.

Definition 2.3.49. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* η from F to G is a family $(\eta_A : FA \to GA)_{A \in \mathrm{Ob}(\mathbf{C})}$ of arrows of $\mathbf{D}$ such that for each $f \in \mathbf{C}(A, B)$ the following diagram in $\mathbf{D}$ commutes.

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

The functors F and G are call the domain and codomain of η respectively, and we write $\eta : F \to G$. For each $A \in \mathrm{Ob}(\mathbf{C})$, $\eta_A : FA \to GA$ is called a *component* of η.

Example 2.3.50. For every functor $F : \mathbf{C} \to \mathbf{D}$, there is an obvious identity natural transformation $1_F : F \to F$ such that $(1_F)_A = 1_{FA}$ for each $A \in \mathrm{Ob}(\mathbf{C})$.

Definition 2.3.51. Let $F, G, H : \mathbf{C} \to \mathbf{D}$ be functors and let $\eta : F \to G$ and $\varepsilon : G \to H$ be natural transformations. The composite $\varepsilon \circ \eta$ of η and ε is a natural transformation $\varepsilon \circ \eta : F \to H$ whose component at $A \in \mathrm{Ob}(\mathbf{C})$ is $\varepsilon_A \circ \eta_A : FA \to HA$.

Definition 2.3.52. A natural transformation $\eta : F \to G$ is called a *natural isomorphism* if there is a natural transformation $\varepsilon : G \to F$ such that $\varepsilon \circ \eta = 1_F$ and $\eta \circ \varepsilon = 1_G$, and we write $\eta : F \cong G$ for a natural isomorphism $\eta : F \to G$. Functors $F, G : \mathbf{C} \to \mathbf{D}$ are said to be *naturally isomorphic*, denoted by $F \cong G$, if there is a natural isomorphism $\eta : F \to G$ between them.

Proposition 2.3.53. *A natural transformation $\eta : F \to G$ is a natural isomorphism iff each component $\eta_A : FA \to GA$ is an isomorphism.*

Proof. The implication from left to right is obvious. For the converse, suppose that η_A is an isomorphism for all $A \in \mathrm{Ob}(\mathbf{C})$. Define a family of $\mathbf{D}$-arrows $(\varepsilon_A : GA \to FA)_{A \in \mathrm{Ob}(\mathbf{C})}$ by $\varepsilon_A = {\eta_A}^{-1}$ for each $A \in \mathrm{Ob}(\mathbf{C})$. It is easy to see that ε is a natural transformation from G to F, and by the definition of ε, we have $\varepsilon \circ \eta = 1_F$ and $\eta \circ \varepsilon = 1_G$. □

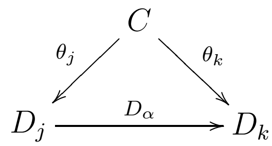
2.3.5 Limits

Definition 2.3.54. Let $\mathbf{C}$ and $\mathbf{J}$ be categories with $\mathbf{J}$ small. A *diagram* of type $\mathbf{J}$ in $\mathbf{C}$ is a functor $D : \mathbf{J} \to \mathbf{C}$. A digram $D : \mathbf{J} \to \mathbf{C}$ is *finite* if $\mathrm{Ob}(\mathbf{J})$ and $\mathrm{Arr}(\mathbf{J})$ are finite sets.

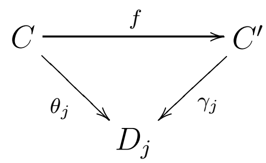
Notations. For a diagram $D : \mathbf{J} \to \mathbf{C}$, we follow the following conventions.

- The objects of $\mathbf{J}$ are denoted by the lowercase letters $i, j, \dots$ and the arrows of $\mathbf{J}$ are denoted by the lowercase Greek letters $\alpha, \beta, \dots$.
- The values of a diagram are denoted by D_i and D_α instead of $D(i)$ and $D(\alpha)$.
- We write $\mathbf{J}_0$ and $\mathbf{J}_1$ instead of $\mathrm{Ob}(\mathbf{J})$ and $\mathrm{Arr}(\mathbf{J})$ respectively.

Definition 2.3.55. A *cone* to a diagram $D : \mathbf{J} \to \mathbf{C}$ is a family $(\theta_j : C \to D_j)_{j\in\mathbf{J}_0}$ of $\mathbf{C}$-arrows with domain $C \in \mathrm{Ob}(\mathbf{C})$ such that for each $\alpha : j \to k$ in $\mathbf{J}$, the diagram



commutes. A morphism $\eta : (\theta_j : C \to D_j)_{j\in\mathbf{J}_0} \to (\gamma_j : C' \to D_j)_{j\in\mathbf{J}_0}$ between cones to D is a $\mathbf{C}$-arrow $f : C \to C'$ such that for each $j \in \mathbf{J}_0$ the diagram

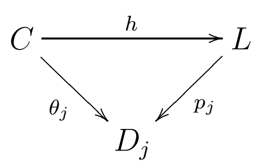


commutes. We have a category

$$\mathbf{Cone}(D)$$

whose objects are all cones to D and whose morphism are all morphisms between such cones with obvious compositions and identities.

Definition 2.3.56. A *limit* for a diagram $D : \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(D)$, namely a cone $(p_j : L \to D_j)_{j\in\mathbf{J}_0}$ such that for any cone $(\theta_j : C \to D_j)_{j\in\mathbf{J}_0}$ to D, there is unique $\mathbf{C}$-arrow $h : C \to L$ such that the diagram



commutes for all $j \in \mathbf{J}_0$.

Examples 2.3.57.

- Let $\mathbf{J}$ be a category such that $\mathrm{Ob}(\mathbf{J}) = \{1, 2\}$ and $\mathrm{Arr}(\mathbf{J}) = \emptyset$. Then, a diagram $D : \mathbf{J} \to \mathbf{C}$ is a pair (D_1, D_2) of $\mathbf{C}$-objects. A cone to D is a pair of $\mathbf{C}$-arrows

  $$D_1 \xleftarrow{f_1} C \xrightarrow{f_2} D_2\ ,$$

  and the limit of D is just a product

  $$D_1 \xleftarrow{\pi_1} D_1 \times D_2 \xrightarrow{\pi_2} D_2$$

  of D_1 and D_2.

  Similarly, a product of a set-indexed family of $\mathbf{C}$-objects $(A_i)_{i\in I}$ is the limit of a diagram $D : \mathbf{J} \to \mathbf{C}$ such that $\mathrm{Ob}(\mathbf{J}) = I$, $\mathrm{Arr}(\mathbf{J}) = \emptyset$ and $D_i = A_i$ for each $i \in I$.

- The limit of the diagram $D : \mathbf{0} \to \mathbf{C}$ from the empty category is a terminal object in $\mathbf{C}$, and the limit of the diagram $D : \mathbf{J} \to \mathbf{C}$, where $\mathbf{J} = \bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet$, is an equaliser for D_α and D_β.

Corollary 2.3.58. *Terminal objects, products and equalisers are all unique up to isomorphisms.*

The dual notion of cone and limit are those of cocone and colimit.

Definition 2.3.59. A *cocone* from a diagram $D : \mathbf{J} \to \mathbf{C}$ is a family $(\theta_j : D_j \to C)_{j \in \mathbf{J}_0}$ of $\mathbf{C}$-arrows with codomain $C \in \mathrm{Ob}(\mathbf{C})$ such that for each $\alpha : j \to k$ in $\mathbf{J}$, the diagram

$$\begin{array}{ccc} D_j & \xrightarrow{D_\alpha} & D_k \\ & {\scriptstyle\theta_j}\searrow \quad \swarrow{\scriptstyle\theta_k} & \\ & C & \end{array}$$

commutes. A morphism of cocones $f : (\eta_j : D_j \to C)_{j \in \mathbf{J}_0} \to (\gamma_j : D_j \to C')_{j \in \mathbf{J}_0}$ is a $\mathbf{C}$-arrow $f : C \to C'$ such that for each $j \in \mathbf{J}_0$ the diagram

$$\begin{array}{ccc} & D_j & \\ & {\scriptstyle\eta_j}\swarrow \quad \searrow{\scriptstyle\gamma_j} & \\ C & \xrightarrow{f} & C' \end{array}$$

commutes. We have a category

$$\mathbf{Cocone}(D)$$

defined similarly as $\mathbf{Cone}(D)$. Then, a *colimit* of D is an initial object of $\mathbf{Cocone}(D)$, namely a cocone $(q_j : D_j \to L)_{j \in \mathbf{J}_0}$ from D such that for any cocone $(\eta_j : D_j \to C)_{j \in \mathbf{J}_0}$ from D, there is a unique $\mathbf{C}$-arrow $h : L \to C$ such that the diagram

$$\begin{array}{ccc} & D_j & \\ & {\scriptstyle q_j}\swarrow \quad \searrow{\scriptstyle\eta_j} & \\ L & \xrightarrow{h} & C \end{array}$$

commutes for all $j \in \mathbf{J}_0$.

The colimits of the digram given in Examples 2.3.57 are a (binary) coproduct, a coproduct of a family of objects, an initial object and a coequaliser respectively. By duality principle, we have the following.

Corollary 2.3.60. *Initial objects, coproducts and coequalisers are all unique up to isomorphisms.*

Definition 2.3.61. A category $\mathbf{C}$ is (finitely) *complete* (or *cocomplete*) if the limit (respectively colimit) of any (finite) diagram in $\mathbf{C}$ exists in $\mathbf{C}$.

Proposition 2.3.62. *A category* $\mathbf{C}$ *is (finitely) complete iff it has (finite) products and equalisers.*

Proof. Since products and equalisers are limits, the implication from left to right is obvious.

Conversely, suppose that $\mathbf{C}$ has products and equalisers, and let $D : \mathbf{J} \to \mathbf{C}$ be a diagram in $\mathbf{C}$. Let $\prod_{j \in \mathbf{J}_0} D_j$ and $\prod_{\alpha \in \mathbf{J}_1} D_{\mathrm{cod}(\alpha)}$ be products of the families $(D_j)_{j \in \mathbf{J}_0}$ and $\big(D_{\mathrm{cod}(\alpha)}\big)_{\alpha \in \mathbf{J}_1}$ with projections

$$\Big(\pi_j : \textstyle\prod_{j \in \mathbf{J}_0} D_j \to D_j\Big)_{j \in \mathbf{J}_0},$$
$$\big(\pi'_\alpha : \textstyle\prod_{\alpha \in \mathbf{J}_1} D_{\mathrm{cod}(\alpha)} \to D_{\mathrm{cod}(\alpha)}\big)_{\alpha \in \mathbf{J}_1}$$

respectively. For the following families of arrows

$$\Big(\pi_{\mathrm{cod}(\alpha)} : \textstyle\prod_{j \in \mathbf{J}_0} D_j \to D_{\mathrm{cod}(\alpha)}\Big)_{\alpha \in \mathbf{J}_1},$$
$$\Big(D_\alpha \circ \pi_{\mathrm{dom}(\alpha)} : \textstyle\prod_{j \in \mathbf{J}_0} D_j \to D_{\mathrm{cod}(\alpha)}\Big)_{\alpha \in \mathbf{J}_1}$$

we have arrows $\varphi, \psi : \prod_{j \in \mathbf{J}_0} D_j \to \prod_{\alpha \in \mathbf{J}_1} D_{\mathrm{cod}(\alpha)}$ such that

$$\pi'_\alpha \circ \varphi = \pi_{\mathrm{cod}(\alpha)},$$
$$\pi'_\alpha \circ \psi = D_\alpha \circ \pi_{\mathrm{dom}(\alpha)}$$

for each $\alpha \in \mathbf{J}_1$. Let $e : E \to \prod_{j \in \mathbf{J}_0} D_j$ be an equaliser for φ and ψ, and let $p = (p_j : E \to D_j)_{j \in \mathbf{J}_0}$ where $p_j = \pi_j \circ e$ for each $j \in \mathbf{J}_0$. We claim that p is the limit of D.

First, we see that p is a cone to D. Since e is an equaliser for φ and ψ, we have

$$\begin{aligned} D_\alpha \circ p_j &= D_\alpha \circ \pi_j \circ e \\ &= \pi'_\alpha \circ \psi \circ e \\ &= \pi'_\alpha \circ \varphi \circ e \\ &= \pi_k \circ e \\ &= p_k \end{aligned}$$

for all $\alpha : j \to k \in \mathbf{J}_1$, and thus p is a cone to D.

Now, given any cone $(\theta_j : T \to D_j)_{j \in \mathbf{J}_0}$ to D, let $h : T \to \prod_{j \in \mathbf{J}_0} D_j$ be the unique arrow such that $\theta_j = \pi_j \circ h$ for each $j \in \mathbf{J}_0$. Since $(\theta_j)_{j \in \mathbf{J}_0}$ is a cone to D, we have

$$\begin{aligned} \pi'_\alpha \circ \varphi \circ h &= \pi_{\mathrm{cod}(\alpha)} \circ h \\ &= \theta_{\mathrm{cod}(\alpha)} \\ &= D_\alpha \circ \theta_{\mathrm{dom}(\alpha)} \\ &= D_\alpha \circ \pi_{\mathrm{dom}(\alpha)} \circ h \\ &= \pi'_\alpha \circ \psi \circ h \end{aligned}$$

for each $\alpha \in \mathbf{J}_1$, and thus $\varphi \circ h = \psi \circ h$. Since e is an equaliser for φ and ψ, there is a unique arrow $u : T \to E$ such that $h = e \circ u$. Note that $h = e \circ u$ iff $\theta_j = \pi_j \circ h = \pi_j \circ e \circ u = p_j \circ u$ for each $j \in \mathbf{J}_0$. So $u : T \to E$ is the unique arrow from cone $(\theta_j)_{j \in \mathbf{J}_0}$ to p. Thus p is the limit of D. □

By the duality principle, we have the following.

Corollary 2.3.63. *A category* $\mathbf{C}$ *is (finitely) cocomplete iff it has (finite) coproducts and coequalisers.*

2.3.6 Adjunctions

Definition 2.3.64. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *adjunction* between $\mathbf{C}$ and $\mathbf{D}$ consists of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and a family of bijections

$$\varphi_{A,B} : \mathbf{C}(A, GB) \cong \mathbf{D}(FA, B)$$

indexed by $A \in \mathrm{Ob}(\mathbf{C})$ and $B \in \mathrm{Ob}(\mathbf{B})$ which is natural in both A and B. Here, $\varphi_{A,B}$ is natural in A if for any $\mathbf{C}$-arrow $f : A' \to A$ and $B \in \mathrm{Ob}(\mathbf{D})$ the diagram

$$\begin{array}{ccc} \mathbf{C}(A, GB) & \xrightarrow{\varphi_{A,B}} & \mathbf{D}(FA, B) \\ \downarrow{\scriptstyle H^{GB} f} & & \downarrow{\scriptstyle H^{B} Ff} \\ \mathbf{C}(A', GB) & \xrightarrow{\varphi_{A',B}} & \mathbf{D}(FA', B) \end{array}$$

commutes, where $H^{GB} f(h) = h \circ f$ and $H^{B} Ff(k) = k \circ Ff$ for each $h \in \mathbf{C}(A, GB)$ and $k \in \mathbf{D}(FA, B)$. This means that for each $f \in \mathbf{C}(A', A)$ and $h \in \mathbf{C}(A, GB)$, the following equation holds.

$$\varphi_{A',B}(h \circ f) = \varphi_{A,B}(h) \circ Ff. \tag{2.3.6.1}$$

Similarly, $\varphi_{A,B}$ is natural in B if for any $\mathbf{D}$-arrow $g : B \to B'$ and $A \in \mathrm{Ob}(\mathbf{C})$, the diagram

$$\begin{array}{ccc} \mathbf{C}(A, GB) & \xrightarrow{\varphi_{A,B}} & \mathbf{D}(FA, B) \\ \downarrow{\scriptstyle H_A Gg} & & \downarrow{\scriptstyle H_{FA}\, g} \\ \mathbf{C}(A, GB') & \xrightarrow{\varphi_{A,B'}} & \mathbf{D}(FA, B') \end{array}$$

commutes, where $H_A Gg(h) = Gg \circ h$ and $H_{FA} g(k) = g \circ k$ for each $h \in \mathbf{C}(A, GB)$ and $k \in \mathbf{D}(FA, B)$. This means that for each $g \in \mathbf{D}(B, B')$ and $h \in \mathbf{C}(A, GB)$, the following equation holds.

$$\varphi_{A,B'}(Gg \circ h) = g \circ \varphi_{A,B}(h). \tag{2.3.6.2}$$

F is called the *left adjoint* and G is called the *right adjoint* of the adjunction. We write $\langle F, G, \varphi \rangle$ for the adjunction which consists of the left adjoint F, the right adjoint G, and a natural bijection φ. We also write $F \dashv G$ to assert that F and G are the left and right adjoint of an adjunction.

Proposition 2.3.65. *Let $\langle F, G, \varphi \rangle$ be an adjunction between* $\mathbf{C}$ *and* $\mathbf{D}$*. Then*

1. *There is a natural transformation*

$$\eta : 1_{\mathbf{C}} \to GF$$

with the following universal property;

for any $A \in \mathrm{Ob}(\mathbf{C})$ and $B \in \mathrm{Ob}(\mathbf{D})$, and for any $f \in \mathbf{C}(A, GB)$, there is a unique arrow $g \in \mathbf{D}(FA, B)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle Gg} \\ & & GB \end{array}$$

commutes.

2. *There is a natural transformation*

$$\varepsilon : FG \to 1_{\mathbf{D}}$$

with the following universal property;

for any $B \in \mathrm{Ob}(\mathbf{D})$ and $A \in \mathrm{Ob}(\mathbf{C})$, and for any $g \in \mathbf{D}(FA, B)$, there is a unique arrow $f \in \mathbf{C}(A, GB)$ such that the diagram

$$\begin{array}{ccc} FA & & \\ {\scriptstyle Ff}\downarrow & \searrow{\scriptstyle g} & \\ FGB & \xrightarrow{\varepsilon_B} & B \end{array}$$

commutes.

Proof. 1. For each $A \in \mathrm{Ob}(\mathbf{C})$, define

$$\eta_A = {\varphi_{A,FA}}^{-1}(1_{FA}).$$

We claim that $(\eta_A : A \to GFA)_{A \in \mathrm{Ob}(\mathbf{C})}$ is a natural transformation with required universal mapping property.

First, note that the universal property of η is equivalent to the assertion that there is a bijection $g \mapsto \hat{g} : \mathbf{D}(FA, B) \to \mathbf{C}(A, GB)$ such that $\hat{g} = Gg \circ \eta_A$. To see this, let $g \in \mathbf{D}(FA, B)$. Then we have

$$\begin{aligned} \hat{g} &= Gg \circ \eta_A \\ &= Gg \circ {\varphi_{A,FA}}^{-1}(1_{FA}) \\ &= {\varphi_{A,B}}^{-1}(g \circ \varphi_{A,FA}({\varphi_{A,FA}}^{-1}(1_{FA}))) && \text{by (2.3.6.2)} \\ &= {\varphi_{A,B}}^{-1}(g \circ 1_{FA}) \\ &= {\varphi_{A,B}}^{-1}(g). \end{aligned}$$

Thus, since $\varphi_{A,B}$ is a bijection, so is $g \mapsto \hat{g}$.

To see that η is a natural transformation from 1_C to GF, let $f \in \mathbf{C}(A, A')$. Then we have

$$\begin{aligned}
\varphi_{A,FA'}(GFf \circ \eta_A) &= Ff \circ \varphi_{A,FA}(\eta_A) && \text{by (2.3.6.2)} \\
&= Ff \circ 1_{FA} \\
&= 1_{FA'} \circ Ff \\
&= \varphi_{A',FA'}(\eta_{A'}) \circ Ff && \text{by (2.3.6.1)} \\
&= \varphi_{A,FA'}(\eta_{A'} \circ f).
\end{aligned}$$

Since $\varphi_{A,FA'}$ is a bijection, we have $GFf \circ \eta_A = \eta_{A'} \circ f$, i.e. the following diagram commutes.

$$\begin{array}{ccc}
A & \xrightarrow{\eta_A} & GFA \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\
A' & \xrightarrow{\eta_{A'}} & GFA'
\end{array}$$

Hence $\eta : 1_{\mathbf{C}} \to GF$ is a natural transformation.

2. For each $B \in \mathrm{Ob}(\mathbf{D})$, define

$$\varepsilon_B = \varphi_{GB,B}(1_{GB}).$$

We claim that $(\varepsilon_B : FGB \to B)_{B \in \mathrm{Ob}(\mathbf{D})}$ is a natural transformation with the required universal mapping property.

First, as in the proof of 1, we show that the mapping $f \mapsto \tilde{f} : \mathbf{C}(A, GB) \to \mathbf{D}(FA, B)$ such that $\tilde{f} = \varepsilon_B \circ Ff$ is a bijection. To see this, let $f \in \mathbf{C}(A, GB)$. Then we have

$$\begin{aligned}
\tilde{f} &= \varepsilon_B \circ Ff \\
&= \varphi_{GB,B}(1_{GB}) \circ Ff \\
&= \varphi_{A,B}(1_{GB} \circ f) && \text{by (2.3.6.1)} \\
&= \varphi_{A,B}(f).
\end{aligned}$$

Thus, the mapping $f \mapsto \tilde{f}$ is a bijection.

To see that ε is a natural transformation, let $g \in \mathbf{D}(B, B')$. Then we have

$$\begin{aligned}
g \circ \varepsilon_B &= g \circ \varphi_{GB,B}(1_{GB}) \\
&= \varphi_{GB,B'}(Gg \circ 1_{GB}) && \text{by (2.3.6.2)} \\
&= \varphi_{GB,B'}(1_{GB'} \circ Gg) \\
&= \varphi_{GB',B'}(1_{GB'}) \circ FGg && \text{by (2.3.6.1)} \\
&= \varepsilon_{B'} \circ FGg.
\end{aligned}$$

Therefore, $\varepsilon : FG \to 1_D$ is a natural transformation. $\square$

Definition 2.3.66. The natural transformations $\eta : 1_C \to GF$ and $\varepsilon : FG \to 1_D$ are call the *unit* and the *counit* of an adjunction respectively. For each $g \in \mathbf{D}(FA, B)$, the arrow $\hat{g} = Gg \circ \eta_A \in \mathbf{C}(A, GB)$ is called the (left) *transpose* of g across $F \dashv G$, and for each $f \in \mathbf{C}(A, GB)$, the arrow $\tilde{f} = \varepsilon_B \circ Ff \in \mathbf{D}(FA, B)$ is called the (right) transpose of f across $F \dashv G$.

Proposition 2.3.67. *Let $F \dashv G$ be an adjunction between categories $\mathbf{C}$ and $\mathbf{D}$ with unit η and counit ε. Then for any $h \in \mathbf{C}(A, GB)$ and $k \in \mathbf{D}(FA, B)$, we have*

$$\tilde{\eta_A} = 1_{FA}, \qquad \hat{\varepsilon_B} = 1_{GB}, \tag{2.3.6.3}$$

$$\widehat{(\tilde{h})} = h, \qquad \widetilde{(\hat{k})} = k, \tag{2.3.6.4}$$

$$\varepsilon_{FA} \circ F\eta_A = 1_{FA}, \qquad G\varepsilon_B \circ \eta_{GB} = 1_{GB}, \tag{2.3.6.5}$$

$$(\widetilde{\eta_{A'} \circ f}) = Ff, \qquad (\widehat{g \circ \varepsilon_B}) = Gg, \tag{2.3.6.6}$$

for all $f \in \mathbf{C}(A, A')$ and $g \in \mathbf{D}(B, B')$, and

$$(\widetilde{h \circ v}) = \tilde{h} \circ Fv, \qquad (\widehat{k \circ Fv}) = \hat{k} \circ v, \tag{2.3.6.7}$$

$$(\widehat{u \circ k}) = Gu \circ \hat{k}, \qquad (\widetilde{Gu \circ h}) = u \circ \tilde{h}, \tag{2.3.6.8}$$

for all $v \in \mathbf{C}(C, A)$ and $u \in \mathbf{D}(B \to E)$.

Proof. The equations (2.3.6.3),(2.3.6.4) and (2.3.6.5) are obvious from the proof of Proposition 2.3.65. For (2.3.6.6), we have

$$\begin{aligned}(\widetilde{\eta_{A'} \circ f}) &= \varepsilon_{FA'} \circ F\eta_{A'} \circ Ff \\ &= 1_{FA'} \circ Ff \\ &= Ff,\end{aligned}$$

and

$$\begin{aligned}(\widehat{g \circ \varepsilon_B}) &= Gg \circ G\varepsilon_B \circ \eta_{GB} \\ &= Gg \circ 1_{BG} \\ &= Gg.\end{aligned}$$

For (2.3.6.7), we have

$$\begin{aligned}(\widetilde{h \circ v}) &= \varepsilon_B \circ Fh \circ Fv \\ &= \tilde{h} \circ Fv,\end{aligned}$$

and

$$\begin{aligned}(\widehat{k \circ Fv}) &= Gk \circ GFv \circ \eta_C \\ &= Gk \circ \eta_A \circ v \\ &= \hat{k} \circ v.\end{aligned}$$

For (2.3.6.8), we have

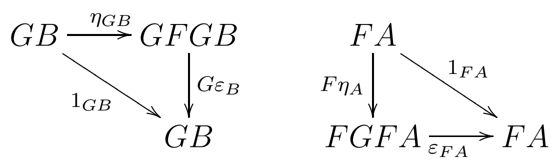
$$\begin{aligned}\widehat{(u \circ k)} &= G(u \circ k) \circ \eta_A \\ &= Gu \circ Gk \circ \eta_A \\ &= Gu \circ \hat{k},\end{aligned}$$

and

$$\begin{aligned}\widetilde{(Gu \circ h)} &= \varepsilon_E \circ F(Gu \circ h) \\ &= \varepsilon_E \circ FGu \circ Fh \\ &= u \circ \varepsilon_B \circ Fh \\ &= u \circ \tilde{h}.\end{aligned}$$

□

Definition 2.3.68. The equations (2.3.6.5) are called the *triangular identities* which assert that the following diagrams commute.



There are several equivalent characterizations of adjunctions.

Proposition 2.3.69. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then the following are equivalent.*

1. *There is an adjunction* $\langle F, G, \varphi\rangle$.

2. *There are functors* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{C}$ *and a natural transformation* $\eta : 1_{\mathbf{C}} \to GF$ *such that for each* $f \in \mathbf{C}(A, GB)$ *there is a unique arrow* $g \in \mathbf{D}(GA, B)$ *such that* $f = Gg \circ \eta_A$.

3. *There exists a functor* $G : \mathbf{D} \to \mathbf{C}$ *and a family of pairs* $(\langle A^*, \eta_A\rangle)_{A\in \mathrm{Ob}(\mathbf{C})}$ *where* $A^* \in \mathrm{Ob}(\mathbf{D})$ *and* $\eta_A \in \mathbf{C}(A, GA^*)$, *such that for each* $f \in \mathbf{C}(A, GB)$ *there is a unique arrow* $g \in \mathbf{D}(A^*, B)$ *such that* $f = Gg \circ \eta_A$.

4. *There are functors* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{C}$ *and a natural transformation* $\varepsilon : FG \to 1_{\mathbf{D}}$ *such that for each* $g \in \mathbf{D}(FA, B)$ *there is a unique arrow* $f \in \mathbf{C}(A, GB)$ *such that* $g = \varepsilon_B \circ Ff$.

5. *There exists a functor* $F : \mathbf{C} \to \mathbf{D}$ *and a family of pairs* $(\langle B^*, \varepsilon_B\rangle)_{B\in \mathrm{Ob}(\mathbf{D})}$, *where* $B^* \in \mathrm{Ob}(\mathbf{C})$ *and* $\varepsilon_B \in \mathbf{D}(FB^*, B)$, *such that for each* $g \in \mathbf{D}(FA, B)$ *there is a unique arrow* $f \in \mathbf{C}(A, B^*)$ *such that* $g = \varepsilon_B \circ Ff$.

Proof. (1) → (2) and (1) → (4) follows from Proposition 2.3.65. (2) → (3) and (4) → (5) are obvious.

(3) → (2): Suppose that we are given a family of pairs $(\langle A^*, \eta_A \rangle)_{A \in \mathrm{Ob}(\mathbf{C})}$ with the specified property. We define a functor $F : \mathbf{C} \to \mathbf{D}$ as follows. On object, we define $F(A) = A^*$ for each $A \in \mathrm{Ob}(\mathbf{C})$. For each $\mathbf{C}$-arrow $f : A \to A'$, let $f^* : FA \to FA'$ be the unique $\mathbf{D}$-arrow such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gf^*} \\ A & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commutes. Define $F(f) = f^*$. Then for each $A \in \mathrm{Ob}(\mathbf{C})$, $F(1_A)$ is the unique arrow such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle GF(1_A)} \\ A & \xrightarrow{\eta_A} & GFA \end{array}$$

commutes. By the uniqueness of such arrow, we must have $F(1_A) = 1_{FA}$. So F preserves the identity. It is easy to see that F also preserves the composition of arrows by the similar argument. Therefore F is a functor from $\mathbf{C}$ to $\mathbf{D}$. By the definition of F, $(\eta_A)_{A \in \mathrm{Ob}(\mathbf{C})}$ is a natural transformation $\eta : 1_{\mathbf{C}} \to GF$.

(2) → (1): Suppose that we are given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and a natural transformation $\eta : 1_{\mathbf{C}} \to GF$ with the specified properties. For each $A \in \mathrm{Ob}(\mathbf{C})$ and $B \in \mathrm{Ob}(\mathbf{D})$, define a function $\varphi_{A,B} : \mathbf{C}(A, GB) \to \mathbf{D}(FA, B)$ by

$$\varphi_{A,B}(h) = \text{unique } k : GA \to B \text{ such that } h = Gk \circ \eta_A$$

for each $h \in \mathbf{C}(A, GB)$. Clearly, $\varphi_{A,B}$ is a bijection between $\mathbf{C}(A, GB)$ and $\mathbf{D}(FA, B)$. It remains to show that $\varphi_{A,B}$ satisfies (2.3.6.1) and (2.3.6.2). To see that (2.3.6.1) holds, let $f \in A' \to A$ and $h \in \mathbf{C}(A, GB)$. Then we have

$$\varphi_{A',B}(h \circ f) = \text{unique } k : GA \to B \text{ such that } h \circ f = Gk \circ \eta_{A'}.$$

However, we have also

$$\begin{aligned} G(\varphi_{A,B}(h) \circ Ff) \circ \eta_{A'} &= G(\varphi_{A,B}(h)) \circ GFf \circ \eta_{A'} \\ &= G(\varphi_{A,B}(h)) \circ \eta_A \circ f \\ &= h \circ f. \end{aligned}$$

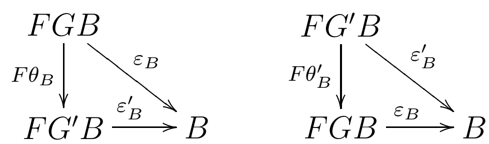
Thus, by the uniqueness, we must have $\varphi_{A',B}(h \circ f) = \varphi_{A,B}(h) \circ Ff$. Therefore, $\varphi_{A,B}$ satisfies (2.3.6.1). Similarly, $\varphi_{A,B}$ satisfies (2.3.6.2). So $\langle F, G, \varphi \rangle$ is an adjunction between $\mathbf{C}$ and $\mathbf{D}$.

(5) → (4) and (4) → (1) can be shown similarly. □

Proposition 2.3.70. *A functor has at most one right adjoint up to natural isomorphism, i.e. for any functors $F : \mathbf{C} \to \mathbf{D}$ and $G, G' : \mathbf{D} \to \mathbf{C}$, if $F \dashv G$ and $F \dashv G'$, then $G \cong G'$.*

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ and $G, G' : \mathbf{D} \to \mathbf{C}$ be functors such that $F \dashv G$ and $F \dashv G'$, and let ε and ε' be the counits of $F \dashv G$ and $F \dashv G'$ respectively.

Then, for each $B \in \mathrm{Ob}(\mathbf{D})$, there exist unique arrows $\theta_B : GB \to G'B$ and $\theta'_B : G'B \to GB$ such that the diagrams



commute. Hence we have

$$\begin{aligned} \varepsilon_B \circ F(\theta'_B \circ \theta_B) &= \varepsilon_B \circ F\theta'_B \circ F\theta_B \\ &= \varepsilon'_B \circ F\theta_B \\ &= \varepsilon_B. \end{aligned}$$

Thus, by the universal property of ε, we have $\theta'_B \circ \theta_B = 1_{GB}$. Similarly, $\theta_B \circ \theta'_B = 1_{G'B}$, so θ_B is a bijection.

We show that θ_B form the components of natural isomorphism $\theta : G \cong G'$. To this end, let $g \in \mathbf{D}(B, B')$. Then by the naturality of ε_B and ε'_B, we have

$$\begin{aligned} \varepsilon'_{B'} \circ F(G'g \circ \theta_B) &= \varepsilon'_{B'} \circ FG'g \circ F\theta_B \\ &= g \circ \varepsilon'_B \circ F\theta_B \\ &= g \circ \varepsilon_B \\ &= \varepsilon_{B'} \circ FGg \\ &= \varepsilon'_{B'} \circ F\theta_{B'} \circ FGg \\ &= \varepsilon'_{B'} \circ F(\theta_{B'} \circ Gg). \end{aligned}$$

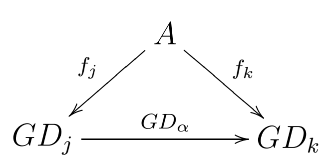
Therefore, by the universal property of ε', we have $G'g \circ \theta_B = \theta_{B'} \circ Gg$. Thus $\theta : G \cong G'$. □

Proposition 2.3.71. *A left adjoint preserves colimit. Dually, a right adjoint preserves limits.*

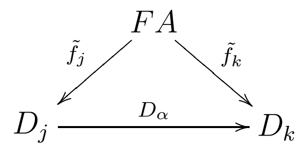
Proof. Let $F \dashv G$ be an adjunction between $\mathbf{C}$ and $\mathbf{D}$. We show that G preserves limits. The fact that F preserves colimits can be shown similarly.

Let $D : \mathbf{J} \to \mathbf{D}$ be a diagram in $\mathbf{D}$ and let $(\theta_j : L \to D_j)_{j \in \mathbf{J}_0}$ be the limit of D. We must show that $(G\theta_j : GL \to GD_j)_{j \in \mathbf{J}_0}$ is a limit of the diagram $GD : \mathbf{J} \to \mathbf{C}$.

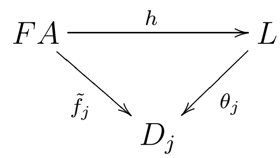
Let $(f_j : A \to GD)_{j \in \mathbf{J}_0}$ be a cone to GD. Then, for each $\alpha : j \to k \in \mathbf{J}_1$, the diagram



commutes. Then, by (2.3.6.8), the diagram



commutes for each $\alpha : j \to k \in \mathbf{J}_1$. So $(\tilde{f}_j : FA \to D_j)_{j \in \mathbf{J}_0}$ is a cone to D. Thus, there is a unique arrow $h : FA \to L$ such that the diagram



commutes for each $j \in \mathbf{J}_0$. Therefore, by (2.3.6.4) and (2.3.6.8), we have

$$f_j = G\theta_j \circ \hat{h}$$

for each $j \in \mathbf{J}_0$. Thus, $\hat{h} : A \to GL$ is a morphism of cones from $(f_j : A \to GD_j)_{j \in \mathbf{J}_0}$ to $(G\theta_j : GL \to GD_j)_{j \in \mathbf{J}_0}$. Given any $\mathbf{C}$-arrow $k : A \to GL$ such that $f_j = G\theta_j \circ k$ for each $j \in \mathbf{J}_0$, we have

$$\tilde{f}_j = \theta_j \circ \tilde{k}$$

for each $j \in \mathbf{J}_0$ by (2.3.6.8). Since θ is a limit of D, we must have $\tilde{k} = h$. Therefore, $k = \hat{h}$, so that $\hat{h}$ is the unique morphism from $(f_j : A \to GD_j)_{j \in \mathbf{J}_0}$ to $(G\theta_j : GL \to GD_j)_{j \in \mathbf{J}_0}$. Hence, $(G\theta_j : GL \to GD_j)_{j \in \mathbf{J}_0}$ is a limit of GD. □

Definition 2.3.72. A functor $F : \mathbf{C} \to \mathbf{D}$ is an *equivalence* if there is a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\eta : 1_{\mathbf{C}} \cong GF$ and $\varepsilon : FG \cong 1_{\mathbf{D}}$. In this case, G is called a *quasi-inverse* of F.

The categories $\mathbf{C}$ and $\mathbf{D}$ are said to be *equivalent*, denoted by $\mathbf{C} \simeq \mathbf{D}$, if there is an equivalence $F : \mathbf{C} \to \mathbf{D}$.

Proposition 2.3.73. *The following are equivalent for a functor $F : \mathbf{C} \to \mathbf{D}$.*

1. *F is an equivalence.*
2. *F is full and faithful, and there is a family of pairs $(\langle B^*, \varepsilon_B \rangle)_{B \in \mathrm{Ob}(\mathbf{D})}$ where $B^* \in \mathrm{Ob}(\mathbf{C})$ and $\varepsilon_B : FB^* \cong B$.*

First, we prove the following lemma.

Lemma 2.3.74. *Let $F : \mathbf{C} \to \mathbf{D}$ be a full and faithful functor. For each $A, A' \in \mathrm{Ob}(\mathbf{C})$ and $\mathbf{D}$-isomorphism $g : FA \cong FA'$, there is a unique $\mathbf{C}$-isomorphism $f : A \cong A'$ such that $Ff = g$.*

Proof. Let $A, A' \in \mathrm{Ob}(\mathbf{C})$ and let $g : FA \cong FA'$ be a $\mathbf{D}$-isomorphism. Since F is full and faithful there are unique arrows $f : A \to A'$ and $f' : A' \to A$ such that $Ff = g$ and $Ff' = g^{-1}$. Then $F(f' \circ f) = Ff' \circ Ff = g^{-1} \circ g = 1_{FA} = F1_A$. Since F is faithful, we have $f' \circ f = 1_A$ and similarly, $f \circ f' = 1_{A'}$. Therefore f is an isomorphism such that $Ff = g$. $\square$

Proof of Proposition 2.3.73. Let $F : \mathbf{C} \to \mathbf{D}$ be an equivalence with the quasi-inverse $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\eta : 1_{\mathbf{C}} \cong GF$ and $\varepsilon : FG \cong 1_{\mathbf{D}}$.

Then $(\langle GB, \varepsilon_B \rangle)_{B \in \mathrm{Ob}(\mathbf{D})}$ is a family such that $GB \in \mathrm{Ob}(\mathbf{C})$ and $\varepsilon_B : FGB \cong B$. To see that F is faithful, let $f, f' \in \mathbf{C}(A, A')$, and suppose that $Ff = Ff'$. Since η is a natural transformation, we have

$$\eta_{A'} \circ f = GFf \circ \eta_A = GFf' \circ \eta_A = \eta_{A'} \circ f'$$

Since $\eta_{A'}$ is an isomorphism, we have $f = f'$. Therefore, F is faithful. Similarly, G is faithful. To see that F is full, let $A, A' \in \mathrm{Ob}(\mathbf{C})$ and $g \in \mathbf{D}(FA, FA')$. Put $f = {\eta_{A'}}^{-1} \circ Gg \circ \eta_A : A \to A'$. Since η is a natural isomorphism, we have

$$GFf = \eta_{A'} \circ f \circ {\eta_A}^{-1} = \eta_{A'} \circ \left({\eta_{A'}}^{-1} \circ Gg \circ \eta_A\right) \circ {\eta_A}^{-1} = Gg,$$

and since G is faithful, we have $Ff = g$. Therefore F is full.

Conversely, suppose that F is full and faithful, and let $(\langle B^*, \varepsilon_B \rangle)_{B \in \mathrm{Ob}(\mathbf{D})}$ be a family with $B^* \in \mathrm{Ob}(\mathbf{C})$ and $\varepsilon_B : FB^* \cong B$. Define a functor $G : \mathbf{D} \to \mathbf{C}$ as follows: On objects, we define $G(B) = B^*$ for each $B \in \mathrm{Ob}(\mathbf{D})$. To define an arrow part of G, let $g : B \to B'$. Since ${\varepsilon_{B'}}^{-1} \circ g \circ \varepsilon_B : FB^* \to FB'^*$ and F is full and faithful, there is a unique arrow $f : B^* \to B'^*$ such that the diagram

$$\begin{array}{ccc} FB^* & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle g} \\ FB'^* & \xrightarrow{\varepsilon_{B'}} & B' \end{array}$$

commutes. We put $Gg = f$. Then, since the diagrams

$$\begin{array}{ccc} FGB & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle F(Gg' \circ Gg)}\downarrow & & \downarrow{\scriptstyle g' \circ g} \\ FGB'' & \xrightarrow{\varepsilon_{B''}} & B' \end{array} \qquad \begin{array}{ccc} FGB & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle FG1_B}\downarrow & & \downarrow{\scriptstyle 1_B} \\ FGB & \xrightarrow{\varepsilon_B} & B \end{array}$$

are commutative for any $g : B \to B$ and $g' : B' \to B''$, we have $G(g' \circ g) = Gg' \circ Gg$ and $G1_B = 1_{GB}$ because F is full and faithful. Therefore G is a functor, and ε_B form the components of natural isomorphism $\varepsilon : FG \cong 1_{\mathbf{D}}$.

For each $A \in \mathrm{Ob}(\mathbf{C})$, since F is full and faithful and ε_{FA} is an isomorphism, there is a unique isomorphism $\eta_A : A \to GFA$ such that $F({\eta_A}^{-1}) = \varepsilon_{FA}$ by the lemma. Then, for any $f : A \to A'$, the diagram

$$\begin{array}{ccc} FGFA & \xrightarrow{F({\eta_A}^{-1})} & FA \\ {\scriptstyle FGFf}\downarrow & & \downarrow{\scriptstyle Ff} \\ FGFA' & \xrightarrow{F({\eta_{A'}}^{-1})} & FA' \end{array}$$

commutes. Since F is faithful, it follows that $GFf \circ \eta_A = \eta_{A'} \circ f$. Thus η_A is a component of a natural isomorphism $\eta : 1_{\mathbf{C}} \cong GF$. □

Definition 2.3.75. A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is *reflective* in $\mathbf{C}$ if the insertion functor $I : \mathbf{D} \to \mathbf{C}$ has a left adjoint $L : \mathbf{C} \to \mathbf{D}$. The adjunction $L \dashv I$ is called a *reflection* of $\mathbf{C}$ in $\mathbf{D}$.

Dually, a subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is *coreflective* in $\mathbf{C}$ if the insertion functor has a right adjoint $L : \mathbf{C} \to \mathbf{D}$. The adjunction $I \dashv L$ is called a *coreflection* of $\mathbf{C}$ in $\mathbf{D}$.

Remark 2.3.76. By Proposition 2.3.69 (3), a subcategory $\mathbf{D}$ is reflective in $\mathbf{C}$ iff there is a family of pairs $(\langle A^*, \eta_A \rangle)_{A \in \mathrm{Ob}(\mathbf{C})}$, where $A^* \in \mathrm{Ob}(\mathbf{D})$ and $\eta_A \in \mathbf{C}(A, A^*)$, such that for any $B \in \mathrm{Ob}(\mathbf{D})$ and $f \in \mathbf{C}(A, B)$, there is a unique arrow $g \in \mathbf{D}(A^*, B)$ such that $f = g \circ \eta_A$. The similar remark applies to coreflections by Proposition 2.3.69 (5).

2.3.7 Galois Connections

Definition 2.3.77. An adjunction $F \dashv G$ between categories $\mathbf{C}(P)$ and $\mathbf{D}(Q)$ associated with partially ordered classes $(P, \leq_P)$ and $(Q, \leq_Q)$, is called a *Galois connection* between P and Q. In this case, $F \dashv G$ iff for any $p \in P$ and $q \in Q$

$$Fp \leq_Q q \iff p \leq_P Gq. \tag{2.3.7.1}$$

Remark 2.3.78. Since a functor between partially ordered classes is just an order preserving function (also called a monotone function), a Galois connection between partially ordered classes P and Q consists of a pair of order preserving functions $F : P \to Q$ and $G : Q \to P$ which satisfy (2.3.7.1).

Definition 2.3.79. Let $(P, \leq)$ be a partially ordered class. A function $cl : P \to P$ is called a *closure operator* on P if

(monotone) $\quad p \leq p' \Rightarrow cl(p) \leq cl(p'),$

(idempotent) $\quad cl(cl(p)) \leq cl(p),$

(expansive) $\quad p \leq cl(p).$

Dually, a function $int : P \to P$ is called an *interior operator* on P if it is monotone and idempotent, and satisfies the following condition.

(contractive) $\quad int(p) \leq p.$

Proposition 2.3.80. *Let $F \dashv G$ be a Galois connection between partially ordered classes P and Q. Then*

1. *The composite $GF : P \to P$ is a closure operator on P, i.e. for all $p, p' \in P$*

$$p \leq p' \Rightarrow GFp \leq GFp, \qquad p \leq GFp, \qquad GFGFp \leq GFp.$$

2. The composite $FG : Q \to Q$ is an interior operator on Q, i.e. for all $p, p' \in Q$

$$q \leq q' \Rightarrow FGq \leq FGq, \qquad FGq \leq q, \qquad FGq \leq FGFGq.$$

Proof. These are direct consequences of (2.3.7.1) and the fact that F and G are monotone operators. $\square$

Proposition 2.3.81. *Let $F \dashv G$ be an adjunction between categories $\mathbf{C}$ and $\mathbf{D}$ and let η and ε be the unit and counit of $F \dashv G$. Let $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{D}}$ be full subcategories of $\mathbf{C}$ and $\mathbf{D}$ respectively determined by*

$$\begin{aligned} \mathrm{Ob}(\tilde{\mathbf{C}}) &= \{A \in \mathrm{Ob}(\mathbf{C}) \mid \eta_A : A \cong GFA\}\,, \\ \mathrm{Ob}(\tilde{\mathbf{D}}) &= \{B \in \mathrm{Ob}(\mathbf{D}) \mid \varepsilon_B : FGB \cong B\}\,. \end{aligned}$$

Then, the functor F restricts to an equivalence between $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{D}}$, and G restricts to an equivalence between $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{C}}$.

Proof. It suffices to show that F restricts to $\tilde{F} : \tilde{\mathbf{C}} \to \tilde{\mathbf{D}}$ and G restricts to $\tilde{G} : \tilde{\mathbf{D}} \to \tilde{\mathbf{C}}$, so that η and ε restrict to natural isomorphisms $\eta : 1_{\tilde{\mathbf{C}}} \to \tilde{G}\tilde{F}$ and $\varepsilon : \tilde{F}\tilde{G} \to 1_{\tilde{\mathbf{D}}}$.

For any $A \in \mathrm{Ob}(\tilde{\mathbf{C}})$, by (2.3.6.5) and since in general functors preserve isomorphisms, we have, $\varepsilon_{FA} = (F\eta_A)^{-1}$. Thus, $FA \in \mathrm{Ob}(\tilde{\mathbf{D}})$ for all $A \in \mathrm{Ob}(\tilde{\mathbf{C}})$. Therefore, $\tilde{F} : \tilde{\mathbf{C}} \to \tilde{\mathbf{D}}$. Dually, G restricts to $\tilde{G} : \tilde{\mathbf{D}} \to \tilde{\mathbf{C}}$. $\square$

Corollary 2.3.82. *If $F \dashv G$ is a Galois connection between partially ordered classes P and Q, then F restricts to an order preserving isomorphism between subclasses*

$$\{p \in P \mid p = GFp\} \quad \textit{and} \quad \{q \in Q \mid FGq = q\}\,,$$

of P and Q, namely the classes of fixed points of the closure operator GF and the interior operator FG respectively.

2.4 Basic mathematical tools

This section introduces basic mathematical concepts which are frequently used in Chapter 3 and Chapter 4.

First, we define some notions associated with sets and binary relations between sets. For any set S, $\mathrm{Pow}(S)$ denotes the class of subsets of S. For any subsets $U, V \in \mathrm{Pow}(S)$, we define

$$U \between V \iff \exists a \in S \, [a \in U \,\&\, a \in V] \,.$$

The diagonal relation id_X on X, the inverse relation $r^- \subseteq S \times X$ of a relation $r \subseteq X \times S$, and the composition $r_2 \circ r_1 \subseteq X \times Z$ of relations $r_1 \subseteq X \times Y$ and $r_2 \subseteq Y \times Z$ are defined as usual:

$$\begin{aligned} id_X &= \{(x, x) \mid x \in X\} \,, \\ r^- &= \{(a, x) \in S \times X \mid x \, r \, a\} \,, \\ r_2 \circ r_1 &= \{(x, z) \in X \times Z \mid (\exists y \in Y) \, x \, r_1 \, y \,\&\, y \, r_2 \, z\}. \end{aligned}$$

2.4.1 Four operators associated with a relation

For any relation $r \subseteq X \times S$ between sets, we can define four operators between $\mathrm{Pow}(X)$ and $\mathrm{Pow}(S)$. Those operators together with their notations are heavily used in the following chapters. See [21, Section 1.3] for further details of this notion.

Definition 2.4.1. Let X and S be sets and $r \subseteq X \times S$ be a relation. Then, there are four monotone operators $r, r^{-*} : \mathrm{Pow}(X) \to \mathrm{Pow}(S)$ and $r^-, r^* : \mathrm{Pow}(S) \to \mathrm{Pow}(X)$ between partially ordered classes $(\mathrm{Pow}(X), \subseteq)$ and $(\mathrm{Pow}(S), \subseteq)$ defined by

$$\begin{aligned} rD &= \{a \in S \mid (\exists x \in D) \, x \, r \, a\} \,, \\ r^{-*}D &= \{a \in S \mid (\forall x \in X) \, x \, r \, a \to x \in D\} \,, \\ r^-U &= \{x \in X \mid (\exists a \in U) \, x \, r \, a\} \,, \\ r^*U &= \{x \in X \mid (\forall a \in S) \, x \, r \, a \to a \in U\} \end{aligned}$$

for all $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$.

Remark 2.4.2. Since $rD = \bigcup_{x \in D} r\{x\}$ and $r^-U = \bigcup_{a \in U} r^-\{a\}$ for all $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$, the operators r and r^- are completely determined by their restrictions to singletons.

Notations.

- We often write $r : X \to S$ to assert that r is a relation $r \subseteq X \times S$. Of course, there is a danger of confusing the relation r with a function with domain X and codomain S, but the context always makes clear whether r is a relation or a function.
- The composite of two *operators* associated with relations are denoted by juxtapositions, for example rr^* denotes the composite of the operators $r : \mathrm{Pow}(X) \to \mathrm{Pow}(S)$ and $r^* : \mathrm{Pow}(S) \to \mathrm{Pow}(X)$ associated with a relation $r \subseteq X \times S$. The composite of two *relations*, e.g. $r \subseteq X \times Y$ and $s \subseteq Y \times Z$, are always denoted by $s \circ r$.

- We often write rx for $r\{x\}$ when the argument is a singleton set. Similar conventions also apply to the other operators.

Proposition 2.4.3. *Let X and S be sets and $r \subseteq X \times S$ be a relation. Then*

$$rD \between U \iff D \between r^{-}V, \tag{2.4.1.1}$$
$$D \subseteq r^{*}U \iff rD \subseteq U, \tag{2.4.1.2}$$
$$U \subseteq r^{-*}D \iff r^{-}U \subseteq D \tag{2.4.1.3}$$

for all $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$.

Proof. Let $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$. Then

$$\begin{aligned} rD \between U &\iff (\exists a \in S)\, a \in rD \,\&\, a \in U \\ &\iff (\exists a \in S)\, ((\exists x \in X)\, x \in D \,\&\, x\, r\, a) \,\&\, a \in U \\ &\iff (\exists x \in X)\, x \in D \,\&\, ((\exists a \in S)\, x\, r\, a \,\&\, a \in U) \\ &\iff (\exists x \in X)\, x \in D \,\&\, x \in r^{-}U \\ &\iff D \between r^{-}U. \end{aligned}$$

Thus (2.4.1.1) holds. Also we have

$$\begin{aligned} D \subseteq r^{*}U &\iff (\forall x \in X)\, x \in D \to x \in r^{*}U \\ &\iff (\forall x \in X)\, x \in D \to ((\forall a \in S)\, x\, r\, a \to a \in U) \\ &\iff (\forall a \in S)\, (\forall x \in X)\, x \in D \to (x\, r\, a \to a \in U) \\ &\iff (\forall a \in S)\, ((\exists x \in X)\, x \in D \,\&\, x\, r\, a) \to a \in U \\ &\iff (\forall a \in S)\, a \in rD \to a \in U \\ &\iff rD \subseteq U. \end{aligned}$$

Thus (2.4.1.2) holds. The proof of (2.4.1.3) is similar to that of (2.4.1.2). □

Corollary 2.4.4. *For any relation $r \subseteq X \times S$, the operators r and r^{-} are left adjoint to r^{*} and r^{-*} respectively.*

Proof. Since the operators r, r^{-}, r^{*} and r^{-*} are all monotone, (2.4.1.2) and (2.4.1.3) assert that the pairs (r, r^{*}) and (r^{-}, r^{-*}) are Galois connections between partially ordered classes $(\mathrm{Pow}(X), \subseteq)$ and $(\mathrm{Pow}(S), \subseteq)$. Hence $r \dashv r^{*}$ and $r^{-} \dashv r^{-*}$. □

Notation. We write $r \cdot\!|\!\cdot r^{-}$, $r \dashv r^{*}$ and $r^{-} \dashv r^{-*}$ to refer to (2.4.1.1), (2.4.1.2) and (2.4.1.3), respectively.

Corollary 2.4.5. *For any relation $r \subseteq X \times S$,*

1. *rr^{*} and $r^{-}r^{-*}$ are interior operators on $\mathrm{Pow}(S)$.*

2. *$r^{*}r$ and $r^{-*}r^{-}$ are closure operators on $\mathrm{Pow}(X)$.*

3. The following equations hold.

$$r^*rr^* = r^*, \quad rr^*r = r, \quad r^-r^{-*}r^- = r^-, \quad r^{-*}r^-r^{-*} = r^{-*}.$$

4. r and r^- preserve unions, i.e.

$$r\bigcup_{i\in I} D_i = \bigcup_{i\in I} rD_i, \quad r^-\bigcup_{j\in J} U_j = \bigcup_{j\in J} r^-U_j$$

for any set-indexed families $(D_i)_{i\in I}$ and $(U_j)_{j\in J}$ of subsets of X and S respectively.

5. r^* and r^{-*} preserve intersections, i.e.

$$r^*\bigcap_{i\in I} U_i = \bigcap_{i\in I} r^*U_i, \quad r^{-*}\bigcap_{j\in J} D_j = \bigcap_{j\in J} r^{-*}D_j$$

for any set-indexed families $(D_i)_{i\in I}$ and $(U_j)_{j\in J}$ of subsets of X and S respectively.

Proof. (1) and (2) follow from Proposition 2.3.80. (3) is the triangular identities (2.3.6.5) stated in terms of Galois connection between partially ordered classes.

(4) and (5) can be checked directly, but they also follow from the following observation: Since the unions and intersections are joins and meets in $(\mathrm{Pow}(X), \subseteq)$ and $(\mathrm{Pow}(S), \subseteq)$, and since $r \dashv r^*$ and $r^- \dashv r^{-*}$, it follows from Proposition 2.3.71 that r and r^- preserve unions and r^* and r^{-*} preserve intersections. ☐

Lemma 2.4.6. *For any relations $r \subseteq X \times Y$ and $s \subseteq Y \times Z$,*

$$(s \circ r)^- = r^-s^-, \quad (s \circ r)^* = r^*s^*, \quad (s \circ r)^{-*} = s^{-*}r^{-*}.$$

Proof. For any $V \in \mathrm{Pow}(Z)$ and $x \in X$, we have

$$\begin{aligned} x \in (s \circ r)^- V &\iff (s \circ r)\, x \between V && \text{by } s \circ r \cdot|\cdot (s \circ r)^- \\ &\iff srx \between V \\ &\iff rx \between s^-V && \text{by } s \cdot|\cdot s^- \\ &\iff \{x\} \between r^-s^-V && \text{by } r \cdot|\cdot r^- \\ &\iff x \in r^-s^-V. \end{aligned}$$

Thus, $(s \circ r)^- = r^-s^-$. Also, we have

$$\begin{aligned} x \in (s \circ r)^* V &\iff (s \circ r)\, x \subseteq V && \text{by } (s \circ r) \dashv (s \circ x)^* \\ &\iff srx \subseteq V \\ &\iff rx \subseteq s^*V && \text{by } s \dashv s^* \\ &\iff \{x\} \subseteq r^*s^*V && \text{by } r \dashv r^* \\ &\iff x \in r^*s^*V. \end{aligned}$$

Thus, $(s \circ r)^* = r^*s^*$. The proof of $(s \circ r)^{-*} = s^{-*}r^{-*}$ is similar. ☐

Proposition 2.4.7. *For any relations $r \subseteq X \times S$ and $s \subseteq X \times S$, the following are equivalent.*

1. *$r = s$ as relations between X and S.*
2. *$r = s$ as operators from $\mathrm{Pow}(X)$ to $\mathrm{Pow}(S)$.*
3. *$r^- = s^-$ as the inverse relations of r and s respectively.*
4. *$r^- = s^-$ as operators from $\mathrm{Pow}(S)$ to $\mathrm{Pow}(X)$.*
5. *$r^* = s^*$.*
6. *$r^{-*} = s^{-*}$.*

Proof. The equivalence between (1), (2), (3), and (4) are immediate from the definitions of inverse relation and the operators r, s, r^- and s^-. The equivalence (2) $\leftrightarrow$ (5) and (4) $\leftrightarrow$ (6) follows from the fact that right and left adjoints are unique; see Proposition 2.3.70. □

Definition 2.4.8. Let $r \subseteq X \times S$ be a relation.

1. r is *total* if $r \in \mathbf{mv}(X, S)$.
2. r is *single-valued* if $(\forall x \in X)\,(\forall a, a' \in S)\, x\, r\, a \,\&\, x\, r\, a' \to a = a'$.

A *function* is a total and single-valued relation.

Lemma 2.4.9. *Let X be a set. For any $D \in \mathrm{Pow}(X)$, the following are equivalent.*

1. *D is a singleton.*
2. *$(\forall E \in \mathrm{Pow}(X))\, D \between E \iff D \subseteq E$.*

Proof. Suppose that $D = \{x\}$ for some $x \in X$. Then, for any $E \in \mathrm{Pow}(X)$, we have $\{x\} \between E \iff x \in E \iff \{x\} \subseteq E$. Conversely, suppose that $D \between E \iff D \subseteq E$ for all $E \in \mathrm{Pow}(X)$. Put $E = D$. Then we have $D \between D$, so there is $x \in D$. Putting $E = \{x\}$, we have $D \subseteq \{x\}$. Therefore, $D = \{x\}$. □

Proposition 2.4.10. *A relation $r \subseteq X \times S$ is a function iff $r^* = r^-$.*

Proof. By the above lemma, we have

$$\begin{aligned} r \text{ is a function} &\iff (\forall x \in X)\,(\exists a \in S)\, x\, r\, a \,\&\, (\forall b \in S)\, x\, r\, b \to b = a \\ &\iff (\forall x \in X)\,(\exists a \in S)\, rx = \{a\} \\ &\iff (\forall x \in X)\,(\forall E \in \mathrm{Pow}(S))\, rx \between E \leftrightarrow rx \subseteq E \\ &\iff (\forall x \in X)\,(\forall E \in \mathrm{Pow}(S))\, x \in r^- E \leftrightarrow x \in r^* E \\ &\iff r^- = r^*. \end{aligned}$$

□

2.4.2 Suplattices and complete lattices

In this section, we introduce the notion of suplattice, complete lattice and frame. In this thesis, complete lattices arise as lattices of open and closed subsets of basic pairs and frames arise as lattices of open subsets of concrete spaces. This section is largely based on [21, Section 0.3].

Definition 2.4.11. A *suplattice* is a partially ordered class $(S, \leq, \bigvee)$ which has a join $\bigvee_{i \in I} a_i$ for any set-indexed family $(a_i)_{i \in I}$ of elements of S.

Dually, an *inflattice* is a partially ordered class $(S, \leq, \bigwedge)$ which has a meet $\bigwedge_{i \in I} a_i$ for any set-indexed family $(a_i)_{i \in I}$ of elements of S.

A *complete lattice* is a partially ordered class $(S, \leq, \bigvee, \bigwedge)$ which has a meet and join for every set-indexed family of elements of S.

If S and S' are suplattices (or inflattices), a function $h : S \to S'$ which preserves all joints (respectively meets), i.e. $h\left(\bigvee_{i \in I} a_i\right) = \bigvee h(a_i)$ (respectively $h\left(\bigwedge_{i \in I} a_i\right) = \bigwedge h(a_i)$) for every set-indexed family $(a_i)_{i \in I}$, is called a *homomorphism* of suplattice (respectively inflattice). A homomorphism $h : S \to S'$ is an *isomorphism* if there is a homomorphism $h' : S' \to S$ such that $h' \circ h = 1_S$ and $h \circ h' = 1_{S'}$.

A *frame* is a partially ordered class $(S, \leq, T, \wedge, \bigvee)$ which has the top T, i.e. an element $T \in S$ such that $a \leq T$ for all $a \in S$, the meet $a \wedge b$ for every $a, b \in S$ and the join $\bigvee_{i \in I} a_i$ for every set-indexed family of elements of S, such that meets distribute over joins, i.e. $a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} a \wedge a_i$ for any $a \in S$ and any set-indexed family $(a_i)_{i \in I}$ of elements of S. A frame homomorphism $f : S \to S'$ is a function which preserves top, and all binary meets and set-indexed joins.

Remark 2.4.12. A suplattice is a partially ordered class which is cocomplete as a category. Dually, an inflattice is a partially ordered class which is complete as a category. A frame is a partially ordered class which is cocomplete and finitely complete as a category.

Definition 2.4.13. Let S be a partially ordered class and $c : S \to S$ be an operator on S. We say that an element $a \in S$ is *c-fixed* if $a = c(a)$. If *cl* and *int* are closure and interior operators on S respectively, we write

$$Sat(cl) = \{a \in S \mid cl(a) = a\},$$
$$Red(int) = \{a \in S \mid int(a) = a\}$$

for the class of *cl*-fixed elements and *int*-fixed elements of S respectively.

Remark 2.4.14. Since closure and interior operators are idempotent, we can define $Sat(cl)$ and $Red(int)$ by

$$Sat(cl) = \{cl(a) \mid a \in S\},$$
$$Red(int) = \{int(a) \mid a \in S\}.$$

Proposition 2.4.15. *Let $(S, \leq, \bigvee, \bigwedge)$ be a complete lattice, and let cl and int be closure and interior operators respectively. Then for any set-indexed family $(a_i)_{i\in I}$ of elements of S, we have*

$$cl\left(\bigvee_{i\in I} cl(a_i)\right) = cl\left(\bigvee_{i\in I} a_i\right), \qquad cl\left(\bigwedge_{i\in I} cl(a_i)\right) = \bigwedge_{i\in I} cl(a_i),$$
$$int\left(\bigwedge_{i\in I} int(a_i)\right) = int\left(\bigwedge_{i\in I} a_i\right), \qquad int\left(\bigvee_{i\in I} int(a_i)\right) = \bigvee_{i\in I} int(a_i).$$

Proof. We just give a proof for the closure operator. The proof for the interior operator is dual.

For the first equation, since $a_i \leq \bigvee_{i\in I} a_i$ for all $i \in I$, $cl(a_i) \leq cl\left(\bigvee_{i\in I} a_i\right)$ for all $i \in I$ by monotonicity of cl. Since this is equivalent to $\bigvee_{i\in I} cl(a_i) \leq cl\left(\bigvee_{i\in I} a_i\right)$, we have $cl\left(\bigvee_{i\in I} cl\left(a_i\right)\right) \leq clcl\left(\bigvee_{i\in I} a_i\right) \leq cl\left(\bigvee_{i\in I} a_i\right)$ by monotonicity and idempotency of cl. Since cl is expansive, we have $cl\left(\bigvee_{i\in I} cl\left(a_i\right)\right) = cl\left(\bigvee_{i\in I} a_i\right)$.

For the second equation, since $\bigwedge_{i\in I} cl(a_i) \leq cl(a_i)$ for all $i \in I$, $cl\left(\bigwedge_{i\in I} cl(a_i)\right) \leq clcl(a_i) \leq cl(a_i)$ for all $i \in I$ by monotonicity and idempotency of cl. Therefore, $cl\left(\bigwedge_{i\in I} cl(a_i)\right) \leq \bigwedge_{i\in I} cl(a_i)$, and hence $cl\left(\bigwedge_{i\in I} cl(a_i)\right) = \bigwedge_{i\in I} cl(a_i)$ since cl is expansive. □

Proposition 2.4.16. *Let $(S, \leq, \bigvee, \bigwedge)$ be a complete lattice and $c : S \to S$ be a monotone and idempotent operator on S. Then the class $Fix(c) = \{c(a) \mid a \in S\}$ of all c-fixed elements of S forms a complete lattice with a join $\bigvee^c$ and a meet $\bigwedge^c$ defined by*

$$\bigvee\nolimits^c_{i\in I} c(a_i) = c\left(\bigvee_{i\in I} c(a_i)\right), \qquad \bigwedge\nolimits^c_{i\in I} c(a_i) = c\left(\bigwedge_{i\in I} c(a_i)\right)$$

for any set-indexed family $(a_i)_{i\in I}$ of elements of S.

If c is a closure operator, then $\bigwedge^c$ coincides with $\bigwedge$, and if c is an interior operator, then $\bigvee^c$ coincides with $\bigvee$.

Proof. Let $(a_i)_{i\in I}$ be a set-indexed family of elements of S. To prove the first equation, it suffices to show that

$$c\left(\bigvee_{i\in I} c(a_i)\right) \leq c(b) \iff (\forall i \in I)\, c(a_i) \leq c(b)$$

for all $b \in S$. Since $c(a_i) \leq \bigvee_{i\in I} c(a_i)$ for all $i \in I$, $c(a_i) = cc(a_i) \leq c\left(\bigvee_{i\in I} c(a_i)\right)$ for all $i \in I$ by monotonicity and idempotency of c. Therefore if $c\left(\bigvee_{i\in I} c(a_i)\right) \leq c(b)$, then $c(a_i) \leq c(b)$ for all $i \in I$. Conversely, if $c(a_i) \leq c(b)$ for all $i \in I$, then $\bigvee_{i\in I} c(a_i) \leq c(b)$, and hence $c\left(\bigvee_{i\in I} c(a_i)\right) \leq cc(b) = c(b)$ by monotonicity and idempotency of c. The proof for the meet $\bigwedge^c$ is dual.

The second part of the proposition follows from the previous proposition. □

Chapter 3

Basic Pairs

A basic pair is a triple $(X, \Vdash, S)$ where X and S are sets and $\Vdash \subseteq X \times S$ is a relation. In this chapter, we first see that this simple structure allows us to define the notion of open and closed subsets both on X and S. Then we introduce the notion of map between basic pairs, a relation pair, and define the notion of equality between these maps. Finally, we introduce the category **BP** which consists of basic pairs and relation pairs between them. This chapter is largely based on Chapter 2 of [21].

3.1 Basic Pairs

Definition 3.1.1. A *basic pair* is a triple $(X, \Vdash, S)$ where X and S are sets and $\Vdash \subseteq X \times S$ is a relation. If $\Vdash$ is the relation associated with a basic pair $(X, \Vdash, S)$, we define

$$\begin{aligned}
&\Diamond = \Vdash, \quad \Box = \Vdash^{-*}, \quad \mathsf{ext} = \Vdash^{-}, \quad \mathsf{rest} = \Vdash^{*},\\
&\mathsf{int} = \mathsf{ext}\,\Box, \qquad \mathcal{A} = \Box\,\mathsf{ext},\\
&\mathsf{cl} = \mathsf{rest}\,\Diamond, \qquad \mathcal{J} = \Diamond\,\mathsf{rest}
\end{aligned}$$

for the operators $\Vdash, \Vdash^{-*} \colon \mathrm{Pow}(X) \to \mathrm{Pow}(S)$ and $\Vdash^{-}, \Vdash^{*} \colon \mathrm{Pow}(S) \to \mathrm{Pow}(X)$.

Remark 3.1.2. By Corollary 2.4.5, int and $\mathcal{J}$ are interior operators on $\mathrm{Pow}(X)$ and $\mathrm{Pow}(S)$ respectively, and cl and $\mathcal{A}$ are closure operators on $\mathrm{Pow}(X)$ and $\mathrm{Pow}(S)$ respectively. By the triangular identities, we have the following equations.

$$\begin{aligned}
\Box\,\mathsf{ext}\,\Box = \mathcal{A}\,\Box = \Box\,\mathsf{int} = \Box, &\qquad \mathsf{ext}\,\Box\,\mathsf{ext} = \mathsf{int}\,\Box = \mathsf{ext}\,\mathcal{A} = \mathsf{ext},\\
\Diamond\,\mathsf{rest}\,\Diamond = \mathcal{J}\,\Diamond = \Diamond\,\mathsf{cl} = \Diamond, &\qquad \mathsf{rest}\,\Diamond\,\mathsf{rest} = \mathsf{cl}\,\Diamond = \mathsf{rest}\,\mathcal{J} = \mathsf{rest}\,.
\end{aligned}$$

Notations. In the following, letters $\mathcal{X}, \mathcal{Y}, \ldots$ denote basic pairs. If $\mathcal{X}$ denotes a basic pair, then X, S and $\Vdash$ denote the underlying sets and the relation of $\mathcal{X}$, i.e. unless otherwise noted, we assume that $\mathcal{X} = (X, \Vdash, S)$. Any subscripts, primes etc. added to the name of a basic pair matches those of underlying sets and the relation, and any other operators defined in terms of the relation. For example, $\mathcal{X}_1$ denotes a basic pair $(X_1, \Vdash_1, S_1)$, and $\Diamond_1$ denotes the operator determined by $\Vdash_1$. We shall omit subscripts of operators associated with basic pairs, ext etc., whenever the context makes clear.

Definition 3.1.3. Let $(X, \Vdash, S)$ be a basic pair. A subset $D \in \mathrm{Pow}(X)$ is

- *concrete open* iff $\mathsf{int}\, D = D$,
- *concrete closed* iff $\mathsf{cl}\, D = D$.

Dually, a subset $U \in \mathrm{Pow}(S)$ is

- *formal open* iff $\mathcal{A}\, U = U$,
- *formal closed* iff $\mathcal{J}\, U = U$.

Notations. Since int and cl are an interior and closure operators on $\mathrm{Pow}(X)$ respectively, and $\mathcal{A}$ and $\mathcal{J}$ are a closure and interior operators on $\mathrm{Pow}(S)$ respectively, we shall write

$$\begin{aligned} Red(\mathsf{int}) &= \{\mathsf{int}\, D \in \mathrm{Pow}(X) \mid D \in \mathrm{Pow}(X)\} \\ Sat(\mathsf{cl}) &= \{\mathsf{cl}\, D \in \mathrm{Pow}(X) \mid D \in \mathrm{Pow}(X)\} \\ Sat(\mathcal{A}) &= \{\mathcal{A}\, U \in \mathrm{Pow}(S) \mid U \in \mathrm{Pow}(S)\} \\ Red(\mathcal{J}) &= \{\mathcal{J}\, U \in \mathrm{Pow}(S) \mid U \in \mathrm{Pow}(S)\} \end{aligned}$$

for the classes of concrete (formal) open (closed) subsets of $\mathrm{Pow}(X)$ and $\mathrm{Pow}(S)$ (See Definition 2.4.13).

Proposition 3.1.4. *Let $(X, \Vdash, S)$ be a basic pair. For any $D \in \mathrm{Pow}(X)$ and $U \in \mathrm{Pow}(S)$*

1. *D is concrete open iff $(\exists V \in \mathrm{Pow}(S))\, D = \mathsf{ext}\, V$,*
2. *D is concrete closed iff $(\exists V \in \mathrm{Pow}(S))\, D = \mathsf{rest}\, V$,*
3. *U is formal open iff $(\exists D \in \mathrm{Pow}(X))\, U = \Box D$,*
4. *U is formal closed iff $(\exists D \in \mathrm{Pow}(X))\, U = \Diamond\, D$.*

Proof. We prove only (1). The proofs for the others are similar.
Let $D \in \mathrm{Pow}(X)$. If D is concrete open, then

$$D = \mathsf{int}\, D = \mathsf{ext}\, \Box D.$$

Conversely, if $D = \mathsf{ext}\, U$ for some $U \in \mathrm{Pow}(S)$, then

$$\mathsf{int}\, D = \mathsf{int}\,\mathsf{ext}\, U = \mathsf{ext}\, U = D.$$

□

Proposition 3.1.5. *Let $(X, \Vdash, S)$ be a basic pair. Then the operators $\mathsf{ext} : \mathrm{Pow}(S) \to \mathrm{Pow}(X)$ and $\Box : \mathrm{Pow}(X) \to \mathrm{Pow}(S)$ restrict to isomorphisms between complete lattices $Sat(\mathcal{A})$ and $Red(\mathsf{int})$. Dually, the operators $\mathsf{rest} : \mathrm{Pow}(S) \to \mathrm{Pow}(X)$ and $\Diamond : \mathrm{Pow}(X) \to \mathrm{Pow}(S)$ restrict to isomorphisms between complete lattices $Red(\mathcal{J})$ and $Sat(\mathsf{cl})$.*

Proof. Since $\mathsf{ext} \dashv \Box$, by Corollary 2.3.82, ext and $\Box$ restrict to bijections between $Sat(\mathcal{A})$ and $Red(\mathsf{int})$. Dually, since $\Diamond \dashv \mathsf{rest}$, $\Diamond$ and rest restrict to bijections between $Red(\mathcal{J})$ and $Sat(\mathsf{cl})$.

It remains to be shown that those operators preserve joins and meets. We just show that ext preserves joins and meets of $Sat(\mathcal{A})$. The proofs for the other operators are similar. Recall that, by Proposition 2.4.16, joins and meets of $Sat(\mathcal{A})$ and $Red(\mathsf{int})$ are given by

$$\begin{array}{cc} \bigvee_{i \in I} \mathcal{A}\, U_i = \mathcal{A} \bigcup_{i \in I} U_i, & \bigwedge_{i \in I} \mathcal{A}\, U_i = \bigcap_{i \in I} \mathcal{A}\, U_i, \\ \bigvee_{j \in J} \mathsf{int}\, D_j = \bigcup_{j \in J} \mathsf{int}\, D_j, & \bigwedge_{j \in J} \mathsf{int}\, D_j = \mathsf{int} \bigcap_{j \in J} D_j \end{array}$$

for any set-indexed families $(U_i)_{i \in I}$ and $(D_j)_{j \in J}$ of subsets of S and X respectively. To see that ext preserves joins, let $(U_i)_{i \in I}$ be a set-indexed family of subsets of S. Then we have

$$\begin{aligned} \mathsf{ext} \bigvee_{i \in I} \mathcal{A}\, U_i &= \mathsf{ext}\, \mathcal{A} \bigcup_{i \in I} U_i \\ &= \mathsf{ext} \bigcup_{i \in I} U_i \\ &= \bigcup_{i \in I} \mathsf{ext}\, U_i \\ &= \bigvee_{i \in I} \mathsf{ext}\, U_i \\ &= \bigvee_{i \in I} \mathsf{ext}\, \mathcal{A}\, U_i. \end{aligned}$$

For the meets in $Sat(\mathcal{A})$, we have

$$\begin{aligned} \mathsf{ext} \bigwedge_{i \in I} \mathcal{A}\, U_i &= \mathsf{ext} \bigcap_{i \in I} \Box\, \mathsf{ext}\, U_i \\ &= \mathsf{ext}\, \Box \bigcap_{i \in I} \mathsf{ext}\, U_i \\ &= \mathsf{int} \bigcap_{i \in I} \mathsf{ext}\, U_i \\ &= \bigwedge_{i \in I} \mathsf{ext}\, U_i \\ &= \bigwedge_{i \in I} \mathsf{ext}\, \mathcal{A}\, U_i. \end{aligned}$$

□

3.2 Relation Pairs

In this section, we introduce the notion of a relation pair between basic pairs and define the notion of equality on them.

Definition 3.2.1. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be basic pairs. A *relation pair* from $\mathcal{X}_1$ to $\mathcal{X}_2$ is a pair (r, s) of relations $r \subseteq X_1 \times X_2$ and $s \subseteq S_1 \times S_2$ such that

$$\Vdash_2 \circ r = s \circ \Vdash_1,$$

where $\circ$ is the composition of relations. That is, (r, s) is a pair of relations which makes the following diagram commute.

$$\begin{array}{ccc} X_1 & \xrightarrow{\Vdash_1} & S_1 \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X_2 & \xrightarrow{\Vdash_2} & S_2 \end{array}$$

We write $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ to assert that the pair (r, s) is a relation pair between basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$.

By Lemma 2.4.6 and Proposition 2.4.7, we have several equivalent characterizations for relation pairs.

Proposition 3.2.2. *For any basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$, the following are equivalent:*

1. $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ *is a relation pair,*
2. $rx \between \mathsf{ext}\, b \iff \Diamond\, x \between s^- b$ *for all* $x \in X_1$ *and* $b \in S_2$*,*
3. $\Diamond\, rx = s \Diamond\, x$ *for all* $x \in X_1$*,*
4. $r^- \,\mathsf{ext}\, b = \mathsf{ext}\, s^- b$ *for all* $b \in S_2$*,*
5. $r^* \,\mathsf{rest}\, b = \mathsf{rest}\, s^* b$ *for all* $U \in \mathrm{Pow}(S_2)$*,*
6. $\Box r^{-*} D = s^{-*} \Box D$ *for all* $D \in \mathrm{Pow}(X_1)$*.*

Proof. $(1 \leftrightarrow 2)$ In general, for any relations $R : X \to Y$ and $S : Y \to Z$ we have

$$x\,(S \circ R)\,z \iff Rx \between S^- z$$

for all $x \in X$ and $z \in Z$. Thus, we have

$$\begin{aligned}
(r, s) \text{ is a relation pair } &\iff \Vdash_2 \circ r = s \circ \Vdash_1 \\
&\iff x\,(\Vdash_2 \circ r)\,b \leftrightarrow x\,(s \circ \Vdash_1)\,b \qquad \text{for all } x \in X_1 \text{ and } b \in S_2 \\
&\iff rx \between \mathsf{ext}\, b \leftrightarrow \Diamond\, x \between s^- b \qquad \text{for all } x \in X_1 \text{ and } b \in S_2.
\end{aligned}$$

The equivalence between (1), (3), (4), (5) and (6) follows from Lemma 2.4.6 and Proposition 2.4.7. □

Remark 3.2.3.
By Proposition 3.1.4, it follows from (3), (4), (5) and (6) that

- s preserves formal closed subsets, i.e. s is formal closed.
- r^- preserves concrete open subsets, i.e. r^- is concrete open.
- r^* preserves concrete closed subsets, i.e. r^* is concrete closed.
- s^{-*} preserves formal open subsets, i.e. s^{-*} is formal open.

At this point, we introduce the notion of continuous relation between basic pairs, which is a natural generalization of the notion of continuous function between topological spaces. We will see that a continuous relation gives rise to a relation pairs, and conversely the relation r of any relation pair $(r, s) : \mathcal{X} \to \mathcal{Y}$ is a continuous relation.

Definition 3.2.4. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be basic pairs. A relation $r \subseteq X_1 \times X_2$ is *continuous* if

$$r^- \operatorname{\mathsf{ext}} b = \operatorname{\mathsf{int}}(r^- \operatorname{\mathsf{ext}} b)$$

for all $b \in S_2$.

Proposition 3.2.5. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be basic pairs and let $r \subseteq X_1 \times X_2$ be a relation. Then, the following are equivalent.*

1. *r is continuous.*
2. *There is a relation $s \subseteq S_1 \times S_2$ such that*

$$rx \between \operatorname{\mathsf{ext}} b \iff \Diamond\, x \between s^- b$$

for all $x \in X_1$ and $b \in S_2$, i.e. (r, s) is a relation pair from $\mathcal{X}_1$ to $\mathcal{X}_2$.

Proof. Assume 1. Define a relation $s \subseteq S_1 \times S_2$ by

$$a\, s\, b \iff a \in \Box r^- \operatorname{\mathsf{ext}} b$$

for all $a \in S_1$ and $b \in S_2$. Then

$$\begin{aligned} r \text{ is continuous} &\iff (\forall b \in S_2)\,(\forall x \in X_1)\, x \in r^- \operatorname{\mathsf{ext}} b \leftrightarrow x \in \operatorname{\mathsf{int}} r^- \operatorname{\mathsf{ext}} b \\ &\iff (\forall b \in S_2)\,(\forall x \in X_1)\, rx \between \operatorname{\mathsf{ext}} b \leftrightarrow \Diamond\, x \between \Box r^- \operatorname{\mathsf{ext}} b \\ &\iff (\forall b \in S_2)\,(\forall x \in X_1)\, rx \between \operatorname{\mathsf{ext}} b \leftrightarrow \Diamond\, x \between s^- b. \end{aligned}$$

Conversely, if $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ is a relation pair, then r^- is concrete open by Remark 3.2.3, and hence it is continuous. $\square$

Definition 3.2.6. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be basic pairs. Two relation pairs $(r_1, s_1), (r_2, s_2) : \mathcal{X}_1 \to \mathcal{X}_2$ are said to be *equivalent* (or *equal*), denoted $(r_1, s_1) \sim (r_2, s_2)$, if

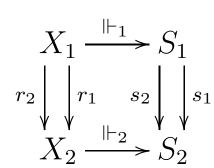
$$\Vdash_2 \circ r_1 = \Vdash_2 \circ r_2.$$

Clearly, $\sim$ is an equivalence relation on the relation pairs from $\mathcal{X}_1$ to $\mathcal{X}_2$. We shall often identify an equivalence class of relation pairs with its arbitrary representative.

There are many characterizations for equal relation pairs. We list some of them below.

Proposition 3.2.7. *For any relation pairs $(r_1, s_1), (r_2, s_2) : \mathcal{X}_1 \to \mathcal{X}_2$, the following are equivalent.*

1. *$(r_1, s_1) \sim (r_2, s_2)$,*
2. *All square commutes,*



3. $\Diamond_2 r_1 = \Diamond_2 r_2$,

4. $s_1 \Diamond_1 = s_2 \Diamond_1$,

5. $r_1^- \,\mathsf{ext}_2 = r_2^- \,\mathsf{ext}_2$,

6. $\mathsf{ext}_1 \, s_1^- = \mathsf{ext}_1 \, s_2^-$,

7. $\Box_2 r_1^{-*} = \Box_2 r_2^{-*}$,

8. $s_1^{-*} \Box_1 = s_2^{-*} \Box_1$,

9. $r_1^* \,\mathsf{rest}_2 = r_2^* \,\mathsf{rest}_2$,

10. $\mathsf{rest}_1 \, s_1^* = \mathsf{rest}_1 \, s_2^*$.

Proof. By the definitions of relation pairs, equal relation pairs, Lemma 2.4.6 and Proposition 2.4.7. □

By the triangular identities, we have still more characterizations of equal relation pairs.

Proposition 3.2.8. *For any relation pairs $(r_1, s_1), (r_2, s_2) : \mathcal{X}_1 \to \mathcal{X}_2$, the following are equivalent.*

1. $(r_1, s_1) \sim (r_2, s_2)$,

2. $\mathsf{cl}_2 \, r_1 D = \mathsf{cl}_2 \, r_2 D$ *for all* $D \in \mathrm{Pow}(X_1)$,

3. $\mathsf{cl}_2 \, r_1 x = \mathsf{cl}_2 \, r_2 x$ *for all* $x \in X_1$,

4. $s_1 \mathcal{J}_1 = s_2 \mathcal{J}_1$,

5. $r_1^- \,\mathsf{int}_2 = r_2^- \,\mathsf{int}_2$,

6. $\mathcal{A}_1 \, s_1^- U = \mathcal{A}_1 \, s_2^- U$ *for all* $U \in \mathrm{Pow}(S_2)$,

7. $\mathcal{A}_1 \, s_1^- a = \mathcal{A}_1 \, s_2^- a$ *for all* $a \in S_2$,

8. $\mathsf{int}_2 \, r_1^{-*} = \mathsf{int}_2 \, r_2^{-*}$,

9. $s_1^{-*} \mathcal{A}_1 = s_2^{-*} \mathcal{A}_1$,

10. $r_1^* \,\mathsf{cl}_2 = r_2^* \,\mathsf{cl}_2$,

11. $\mathcal{J}_1 \, s_1^* = \mathcal{J}_1 \, s_2^*$.

Proof. By the previous proposition, the triangular identities and Remark 2.4.2. □

3.3 The category of basic pairs (BP)

Basic pairs and relation pairs between them naturally form a category.

Proposition 3.3.1. *Basic pairs and the equivalence classes of relation pairs between them form a category* **BP**.

Proof. The composition of relation pairs $(r_1, s_1) : \mathcal{X}_1 \to \mathcal{X}_2$ and $(r_2, s_2) : \mathcal{X}_2 \to \mathcal{X}_3$ is defined by

$$(r_2, s_2) \circ (r_1, s_1) = (r_2 \circ r_1, s_2 \circ s_1).$$

The identity morphism on a basic pair $\mathcal{X}$ is (id_X, id_S), where id_X and id_S are diagonal relations on X and S.

For any relation pairs $(r_1, s_1) : \mathcal{X}_1 \to \mathcal{X}_2$ and $(r_2, s_2) : \mathcal{X}_2 \to \mathcal{X}_3$, the two squares in the diagram below commute, and hence so does the outer rectangle.

$$\begin{array}{ccc} X_1 & \xrightarrow{\Vdash_1} & S_1 \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle s_1} \\ X_2 & \xrightarrow{\Vdash_2} & S_2 \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle s_2} \\ X_3 & \xrightarrow{\Vdash_3} & S_3 \end{array}$$

Therefore, the composite $(r_2 \circ r_1, s_2 \circ s_1)$ is a relation pair from $\mathcal{X}_1$ to $\mathcal{X}_3$. The composition respects the equivalence of relation pairs. In fact, if (r_1, s_1) and (r_1', s_1') are equivalent relation pairs from $\mathcal{X}_1$ to $\mathcal{X}_2$, and if (r_2, s_2) and (r_2', s_2') are equivalent relation pairs from $\mathcal{X}_2$ to $\mathcal{X}_3$, then

$$\begin{aligned} \Vdash_3 \circ r_2 \circ r_1 &= \Vdash_3 \circ r_2' \circ r_1 \\ &= s_2' \circ \Vdash_2 \circ r_1 \\ &= s_2' \circ \Vdash_2 \circ r_1' \\ &= \Vdash_3 \circ r_2' \circ r_1'. \end{aligned}$$

Therefore $(r_1 \circ r_2, s_1 \circ s_2), \sim (r_1' \circ r_2', s_1' \circ s_2')$. The composition is associative since the composition of relations is associative. The identity law is obvious. □

The following expresses, in a categorical term, the symmetry of basic pairs.

Proposition 3.3.2. *The category of basic pair is isomorphic to its own dual, i.e.*

$$\mathbf{BP} \cong \mathbf{BP}^{\mathrm{op}}.$$

Proof. The isomorphism $(-)^- : \mathbf{BP} \to \mathbf{BP}^{\mathrm{op}}$ is defined by,

$$(X, \Vdash, S)^- = (S, \Vdash^-, X)$$

for each basic pair $(X, \Vdash, S)$, and

$$(r, s)^- = (s^-, r^-)$$

for each relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$. Then $(S, \Vdash^-, X)$ is a basic pair. If $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ is a relation pair, then $\Vdash_2 \circ r = s \circ \Vdash_1$, which is equivalent to $r^- \circ \Vdash_2^- = \Vdash_1^- \circ s^-$. Thus $(r, s)^-$ is a relation pair from $\mathcal{X}_2^-$ to $\mathcal{X}_1^-$. Also, we have

$$(id_X, id_S)^- = (id_S^-, id_X^-) = (id_S, id_X)$$

for the identity on $\mathcal{X}$, and

$$
\begin{aligned}
((u,v)\circ(r,s))^- &= ((u\circ r),(v\circ s))^- \\
&= ((v\circ s)^-,(u\circ r)^-) \\
&= (s^-\circ v^-, r^-\circ u^-) \\
&= (s^-,r^-)\circ(v^-,u^-) \\
&= (r,s)^-\circ(u,v)^-
\end{aligned}
$$

for any relation pairs $\mathcal{X}_1 \xrightarrow{(r,s)} \mathcal{X}_2 \xrightarrow{(u,v)} \mathcal{X}_3$. Therefore $(-)^-$ is a contravariant functor from $\mathbf{BP}$ to $\mathbf{BP}$. Moreover, we have $\mathcal{X}^{--} = \mathcal{X}$ for any basic pair $\mathcal{X}$ and $(r,s)^{--} = (r,s)$ for any relation pair (r,s). Therefore, $(-)^-$ is its own inverse, and hence $\mathbf{BP} \cong \mathbf{BP}^{\mathrm{op}}$. □

Finally, we give sufficient conditions for two basic pairs to be isomorphic in $\mathbf{BP}$.

Proposition 3.3.3. *For any basic pairs $\mathcal{X}_1$ and $\mathcal{X}_2$ if*

1. *$X_1 = X_2$ and $\mathsf{int}_1 = \mathsf{int}_2$ or*
2. *$S_1 = S_2$ and $\mathcal{J}_1 = \mathcal{J}_2$,*

then $\mathcal{X}_1 \cong \mathcal{X}_2$.

Proof. 1. Suppose that $X = X_1 = X_2$ and $\mathsf{int}_1 = \mathsf{int}_2$. Since

$$\mathsf{ext}_2\, b = \mathsf{int}_2\, \mathsf{ext}_2\, b = \mathsf{int}_1\, \mathsf{ext}_2\, b$$

for all $b \in S_2$, and similarly, $\mathsf{ext}_1\, a = \mathsf{int}_2\, \mathsf{ext}_1\, a$ for all $a \in S_1$, id_X is a continuous relation from $\mathcal{X}_1$ to $\mathcal{X}_2$ and from $\mathcal{X}_2$ to $\mathcal{X}_1$. Hence, there exist relations $s \subseteq S_1 \times S_2$ and $v \subseteq S_2 \times S_1$ such that (id_X, s) and (id_X, v) are relation pairs. Trivially, they are mutual inverse. Thus $\mathcal{X}_1 \cong \mathcal{X}_2$.

2. Since we have $\mathbf{BP} \cong \mathbf{BP}^{\mathrm{op}}$, if $S_1 = S_2$ and $\mathcal{J}_1 = \mathcal{J}_2$, then $\mathcal{X}_1^- \cong \mathcal{X}_2^-$ by 1. Therefore $\mathcal{X}_1 \cong \mathcal{X}_2$ by Proposition 3.3.2. □

Chapter 4

Concrete Spaces

The notion of concrete space is obtained from that of basic pair by adding the conditions that the set S of a basic pair $\mathcal{X}$ forms a base for a topology. Thus a concrete space is just a topological space. However, a map between concrete spaces is a relation pair which satisfies certain conditions which, in a sense, preserves the structure of a concrete space. So the notion of concrete space differs from that of topological space. In this chapter, we introduce the notion of concrete space, convergent subset and ideal point of a basic pair and convergent relation pair between basic pairs. We see that concrete spaces and convergent relation pairs form a coreflective subcategory **CSpa** of **BP**. We also introduce the notions of weak separation axiom T_0 and sobriety of basic pairs, and consider relations between **BP** and the subcategory of T_0 basic pairs and between the category of sober concrete spaces and that of sober topological spaces. In particular, we see that the notion of concrete space and topological space coincide when we restrict our attention to sober concrete spaces and convergent relation pairs.

Except for Section 4.6, this chapter is largely based on Chapter 3 and Chapter 4 of [21].

4.1 Concrete spaces

Definition 4.1.1. A *concrete space* is a basic pair $\mathcal{X} = (X, \Vdash, S)$ which satisfies the following two conditions:

(B1) $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}(a \downarrow b)$,

(B2) $X = \mathsf{ext}\, S$

for all $a, b \in S$ where

$$a \downarrow b = \{c \in S \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}.$$

Note that in (B1) and (B2), the inclusions from right to left always hold.

The notion of a concrete space is exactly that of a small ct-space by Aczel [4].

Definition 4.1.2. For any basic pair $\mathcal{X}$, define a preorder $\leq$ on S by

$$a \leq b \iff \mathsf{ext}\, a \subseteq \mathsf{ext}\, b$$

for all $a, b \in S$. For any $U, V \in \mathrm{Pow}(S)$, define sets $\downarrow U$ and $U \downarrow V$ by

$$\begin{aligned} \downarrow U &= \{a \in S \mid (\exists c \in U)\, a \leq c\}, \\ U \downarrow V &= \downarrow U \cap \downarrow V. \end{aligned}$$

We often write $\downarrow a$ for $\downarrow \{a\}$. Note that $a \downarrow b = \downarrow a \cap \downarrow b$ and $\downarrow a = \mathcal{A}\, a$.

Lemma 4.1.3. *For any basic pairs $\mathcal{X}$,*

1. $\downarrow \left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \downarrow U_i$,
2. $\left(\bigcup_{i \in I} U_i\right) \downarrow W = \bigcup_{i \in I} (U_i \downarrow W)$

for any $W \in \mathrm{Pow}(S)$ and set-indexed family $(U_i)_{i \in I}$ of $\mathrm{Pow}(S)$.

Proof. Let $W \in \mathrm{Pow}(S)$ and let $(U_i)_{i \in I}$ be a set-indexed family of $\mathrm{Pow}(S)$.
1. For any $a \in S$, we have

$$\begin{aligned} a \in \downarrow \left(\bigcup_{i \in I} U_i\right) &\iff (\exists b \in S)\, (\exists i \in I)\, b \in U_i \,\&\, a \leq b \\ &\iff (\exists i \in I)\, a \in \downarrow U_i \\ &\iff a \in \bigcup_{i \in I} \downarrow U_i. \end{aligned}$$

2. By 1 and the definition of $\downarrow$, we have

$$\begin{aligned} \left(\bigcup_{i \in I} U_i\right) \downarrow W &= \downarrow \left(\bigcup_{i \in I} U_i\right) \cap \downarrow W \\ &= \left(\bigcup_{i \in I} \downarrow U_i\right) \cap \downarrow W \\ &= \bigcup_{i \in I} \left(\downarrow U_i \cap \downarrow W\right) \\ &= \bigcup_{i \in I} (U_i \downarrow W). \end{aligned}$$

□

We list some of the equivalents of the condition (B1).

Proposition 4.1.4. *For any basic pair $\mathcal{X}$, the following are equivalent.*

1. $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}\, (a \downarrow b)$ *for all* $a, b \in S$,
2. $\mathsf{ext}\, U \cap \mathsf{ext}\, V = \mathsf{ext}\, (U \downarrow V)$ *for all* $U, V \in \mathrm{Pow}(S)$,
3. $\mathsf{ext}\, a \cap \mathsf{ext}\, b$ *is open for all* $a, b \in S$,
4. $\mathsf{ext}\, U \cap \mathsf{ext}\, V$ *is open for all* $U, V \in \mathrm{Pow}(S)$,
5. $\mathsf{int}\, D \cap \mathsf{int}\, E = \mathsf{int}\, (D \cap E)$ *for all* $D, E \in \mathrm{Pow}(X)$.

Proof. $(1 \to 2)$ For any $U, V \in \mathrm{Pow}(S)$, we have

$$\begin{aligned}\mathsf{ext}\, U \cap \mathsf{ext}\, V &= \textstyle\bigcup_{a \in U} \mathsf{ext}\, a \cap \bigcup_{b \in V} \mathsf{ext}\, b \\ &= \textstyle\bigcup_{a \in U, b \in V} (\mathsf{ext}\, a \cap \mathsf{ext}\, b) \\ &= \textstyle\bigcup_{a \in U, b \in V} \mathsf{ext}\, (a \downarrow b) \\ &= \mathsf{ext} \textstyle\bigcup_{a \in U, b \in V} (a \downarrow b) \\ &= \mathsf{ext}\, (U \downarrow V)\,.\end{aligned}$$

$(2 \to 1)$ 1 is just a restriction of (2) to the singletons of S.
$(1 \to 3)$ For any $a, b \in S$, we have $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \mathsf{ext}\, (a \downarrow b) = \mathsf{int}\, \mathsf{ext}\, (a \downarrow b)$. Thus $\mathsf{ext}\, a \cap \mathsf{ext}\, b$ is open.
$(3 \to 4)$ For any $U, V \in \mathrm{Pow}(S)$, we have

$$\mathsf{ext}\, U \cap \mathsf{ext}\, V = \textstyle\bigcup_{a \in U} \mathsf{ext}\, a \cap \bigcup_{b \in V} \mathsf{ext}\, b = \bigcup_{a \in U, b \in V} (\mathsf{ext}\, a \cap \mathsf{ext}\, b)\,.$$

Since a union of opens is open, $\mathsf{ext}\, U \cap \mathsf{ext}\, V$ is open.
$(4 \to 5)$ Since $\mathsf{int} = \mathsf{ext}\, \Box$, we have

$$\mathsf{int}\, D \cap \mathsf{int}\, E = \mathsf{int}\, (\mathsf{int}\, D \cap \mathsf{int}\, E) = \mathsf{int}\, (D \cap E)$$

for any $D, E \in \mathrm{Pow}(X)$ by Proposition 2.4.15.
$(5 \to 1)$: Since $\Box$ preserves $\cap$, we have

$$\begin{aligned}\mathsf{ext}\, a \cap \mathsf{ext}\, b &= \mathsf{int}\, \mathsf{ext}\, a \cap \mathsf{int}\, \mathsf{ext}\, b \\ &= \mathsf{int}\, (\mathsf{ext}\, a \cap \mathsf{ext}\, b) \\ &= \mathsf{ext}\, (\Box\, \mathsf{ext}\, a \cap \Box\, \mathsf{ext}\, b) \\ &= \mathsf{ext}\, (\mathcal{A}\, a \cap \mathcal{A}\, b) \\ &= \mathsf{ext}\, (a \downarrow b)\end{aligned}$$

for any $a, b \in S$. □

By Proposition 4.1.4.5, (B1) is equivalent to saying that the class of open subsets, namely *Red*(int), is closed under the finite intersections. Also, (B2) says that X is in *Red*(int). Moreover, since intersections distribute over unions, meets distribute over all joins in *Red*(int). Thus, in a concrete space the class of open sets *Red*(int) forms a frame.

4.2 Convergent subsets, ideal points

The notions of convergent subset and ideal point of a basic pair are obtained by abstracting the notion of point of a space to that of subset of a space as follows: every point $x \in X$ of a concrete space $\mathcal{X}$ must satisfy $x \in \mathsf{ext}\, a \,\&\, x \in \mathsf{ext}\, b \to x \in \mathsf{ext}(a \downarrow b)$ and $x \in \mathsf{ext}\, S$, that is $\{x\} \between \mathsf{ext}\, a \,\&\, \{x\} \between \mathsf{ext}\, b \to \{x\} \between \mathsf{ext}(a \downarrow b)$ and $\{x\} \between \mathsf{ext}\, S$. Moreover, this is equivalent to $a, b \in \Diamond x \to \Diamond x \between (a \downarrow b)$ and $\Diamond x \between S$. These observation motivates the following definitions.

Definition 4.2.1. Let $\mathcal{X}$ be a basic pair. A subset $D \in \mathrm{Pow}(X)$ is called a *convergent subset* of $\mathcal{X}$ if

(D1) $D \between \mathsf{ext}\, a \,\&\, D \between \mathsf{ext}\, b \to D \between (a \downarrow b)$,

(D2) $D \between \mathsf{ext}\, S$

for all $a, b \in S$. The class of convergent subsets of $\mathcal{X}$ will be denoted by $Conv(\mathcal{X})$.

A subset $\alpha \in \mathrm{Pow}(S)$ is called an *ideal point* of $\mathcal{X}$ if

(P1) $\alpha \between S$,

(P2) $a, b \in \alpha \to \alpha \between (a \downarrow b)$,

(P3) $\alpha = \mathcal{J}\, \alpha$

for all $a, b \in S$. The class of ideal points of $\mathcal{X}$ will be denoted by $Pt(\mathcal{X})$. Note that, by the condition (P3), every ideal point is formal closed.

Remark 4.2.2. Since $\mathsf{cl}\, D \between \mathsf{ext}\, a \iff D \between \mathsf{ext}\, a$, it follows that

$$D \in Conv(\mathcal{X}) \iff \mathsf{cl}\, D \in Conv(\mathcal{X}).$$

The following proposition summarizes the remark preceding the above definition.

Proposition 4.2.3. *For any basic pair $\mathcal{X}$, the following are equivalent.*

1. *$\mathcal{X}$ is a concrete space.*
2. *$\{x\} \in Conv(\mathcal{X})$ for all $x \in X$.*
3. *$\Diamond\, x \in Pt(\mathcal{X})$ for all $x \in X$.*

Proposition 4.2.4. *For any basic pair $\mathcal{X}$, the bijections $\Diamond : Sat(\mathsf{cl}) \to Red(\mathcal{J})$ and $\mathsf{rest} : Red(\mathcal{J}) \to Sat(\mathsf{cl})$ restrict to bijections between the classes of closed convergent subsets and $Pt(\mathcal{X})$.*

Proof. For any $D \in Conv(\mathcal{X})$, since $D \between \mathsf{ext}\, a \leftrightarrow a \in \Diamond\, D$ for all $a \in S$ and $\Diamond\, D \in Red(\mathcal{J})$, we have $\Diamond\, D \in Pt(\mathcal{X})$. Conversely, for any $\alpha \in Pt(\mathcal{X})$, since $\alpha \between S \leftrightarrow \mathsf{rest}\, \alpha \between \mathsf{ext}\, S$ and $a \in \alpha \leftrightarrow \mathsf{ext}\, a \between \mathsf{rest}\, \alpha$ by (P3), we have $\mathsf{rest}\, \alpha \in Conv(\mathcal{X})$. The restrictions of $\Diamond$ and rest to the class of closed convergent subsets and $Pt(\mathcal{X})$ are bijective: if $D \in Conv(\mathcal{X})$ is closed, then $\mathsf{rest}\, \Diamond\, D = \mathsf{cl}\, D = D$, and conversely, $\Diamond\, \mathsf{rest}\, \alpha = \mathcal{J}\, \alpha = \alpha$ for any $\alpha \in Pt(\mathcal{X})$. □

A closed convergent subset (or equivalently an ideal point) of a concrete space $\mathcal{X}$ can be characterized by a filter on $Red(\mathsf{int})$ fixed by a closed subset D of the set X.

Definition 4.2.5. Let $\mathcal{X} = (X, \Vdash, S)$ be a concrete space. A *filter* on $Red(\mathsf{int})$ is a subclass $\mathcal{F}$ of $Red(\mathsf{int})$ such that

(F1) $\exists D \in \mathcal{F}$,

(F2) $D \in \mathcal{F} \,\&\, D \subseteq E \Longrightarrow E \in \mathcal{F}$,

(F3) $D, E \in \mathcal{F} \Longrightarrow D \cap E \in \mathcal{F}$

for all $D, E \in \mathit{Red}(\mathsf{int})$. A filter $\mathcal{F}$ on $\mathit{Red}(\mathsf{int})$ is *fixed by a subset* $D \subseteq X$ if

$$E \in \mathcal{F} \iff D \between E$$

for all $E \in \mathit{Red}(\mathsf{int})$.

Note that if $\mathcal{F}$ is a filter on $\mathit{Red}(\mathsf{int})$ fixed by a set $D \in \mathrm{Pow}(X)$, then $\mathcal{F}$ can be written as $\mathcal{F} = \{\mathsf{ext}\, U \mid \mathsf{ext}\, U \between D, U \in \mathrm{Pow}(S)\}$.

Lemma 4.2.6. *If a filter $\mathcal{F}$ on $\mathit{Red}(\mathsf{int})$ is fixed by subsets D_1 and D_2 of X, then $\mathsf{cl}\, D_1 = \mathsf{cl}\, D_2$.*

Proof. Let D_1 and D_2 be subsets of X fixing $\mathcal{F}$. Then, since $\mathsf{ext}\, a \in \mathit{Red}(\mathsf{int})$ for all $a \in S$, we have $D_1 \between \mathsf{ext}\, a \iff D_2 \between \mathsf{ext}\, a$ for all $a \in S$, i.e. $a \in \Diamond D_1 \iff a \in \Diamond D_2$ for all $a \in S$; hence $\Diamond D_1 = \Diamond D_2$. Thus $\mathsf{cl}\, D_1 = \mathsf{cl}\, D_2$. □

Proposition 4.2.7. *In every concrete space $\mathcal{X}$, there is bijective correspondence between any two of the following:*

1. *a filter $\mathcal{F}$ on $\mathit{Red}(\mathsf{int})$ fixed by $D \in \mathit{Sat}(\mathsf{cl})$,*
2. *a closed convergent subset of X,*
3. *an ideal point of $\mathcal{X}$.*

Proof. By Proposition 4.2.4, it suffices to show the correspondence between (1) and (2). First, it is straightforward to see that $D \in \mathit{Sat}(\mathsf{cl})$ is a closed subset of X fixing a filter on $\mathit{Red}(\mathsf{int})$ iff D satisfies the conditions (D1) and (D2) of convergent subset of X. Moreover, the mapping $D \mapsto \{\mathsf{ext}\, U \mid \mathsf{ext}\, U \between D, U \in \mathrm{Pow}(S)\}$ which assigns to every closed convergent subset D a filter on $\mathit{Red}(\mathsf{int})$ fixed by D is injective by the above lemma, and hence it is bijective. □

4.3 Convergent relation pairs

A convergent relation pair is a relation pair which satisfies certain conditions. Of particular interest is a convergent relation pair between concrete spaces, since it can be characterized by the preservation of the convergent subsets of its domain, or by the preservation of the frame structure of the concrete open subsets of its codomain.

Definition 4.3.1. A relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ between basic pairs is *convergent* if

(C1) $r^- \mathsf{ext}\,(a \downarrow b) = \mathsf{ext}\,(s^- a \downarrow s^- b)$, (C2) $r^{-}\,\mathsf{ext}\,S_2 = \mathsf{ext}\,S_1$

for all $a, b \in S_2$.

Remark 4.3.2. In (C2), the inclusion from left to right always holds, since

$$r^{-}\,\mathsf{ext}\,S_2 = \mathsf{ext}\,s^{-}S_2 \subseteq \mathsf{ext}\,S_1.$$

Moreover, assuming (B1), the inclusion from left to right in (C1) holds: for since $\mathsf{ext}(a \downarrow b) \subseteq \mathsf{ext}\,a$, we have $r^{-}\,\mathsf{ext}\,(a \downarrow b) \subseteq r^{-}\,\mathsf{ext}\,a = \mathsf{ext}\,s^{-}a$, and similarly $r^{-}\,\mathsf{ext}\,(a \downarrow b) \subseteq \mathsf{ext}\,s^{-}b$. Hence, we have

$$r^{-}\,\mathsf{ext}\,(a \downarrow b) \subseteq \mathsf{ext}\,s^{-}a \cap \mathsf{ext}\,s^{-}b = \mathsf{ext}\left(s^{-}a \downarrow s^{-}b\right)$$

by (B1). Since any relation pair (r, s) satisfies $\Vdash_2 \circ r = s \circ \Vdash_1$, the conditions (C1) and (C2) are equivalent to the following:

$$\begin{aligned} &\mathsf{ext}\,s^{-}\,(a \downarrow b) = \mathsf{ext}\left(s^{-}a \downarrow s^{-}b\right), \\ &\mathsf{ext}\,s^{-}S_2 = \mathsf{ext}\,S_1. \end{aligned}$$

Moreover, (C1) and the first of the above are equivalent to their formulation with subsets; for example (C1) is equivalent to

$$r^{-}\,\mathsf{ext}\,(U \downarrow V) = \mathsf{ext}\left(s^{-}U \downarrow s^{-}V\right),$$

for, assuming (C1), we have

$$\begin{aligned} r^{-}\,\mathsf{ext}\,(U \downarrow V) &= r^{-}\,\mathsf{ext}\textstyle\bigcup_{a\in U, b\in V}(a \downarrow b) \\ &= \textstyle\bigcup_{a\in U, b\in V} r^{-}\,\mathsf{ext}\,(a \downarrow b) \\ &= \textstyle\bigcup_{a\in U, b\in V} \mathsf{ext}\,(s^{-}a \downarrow s^{-}b) \\ &= \mathsf{ext}\textstyle\bigcup_{a\in U, b\in V}(s^{-}a \downarrow s^{-}b) \\ &= \mathsf{ext}\left(\textstyle\bigcup_{a\in U} s^{-}b \downarrow \bigcup_{a\in V} s^{-}b\right) \\ &= \mathsf{ext}\left(s^{-}U \downarrow s^{-}V\right) \end{aligned}$$

by Lemma 4.1.3. The converse is trivial.

Proposition 4.3.3. *For any convergent relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$, $rD \in Conv(\mathcal{X}_2)$ for all $D \in Conv(\mathcal{X}_1)$.*

Proof. Let $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ be a convergent relation pair and let $D \in Conv(\mathcal{X}_1)$. Then we have

$$\begin{aligned} rD \between \mathsf{ext}\,a \,\&\, rD \between \mathsf{ext}\,b &\iff D \between r^{-}\,\mathsf{ext}\,a \,\&\, D \between r^{-}\,\mathsf{ext}\,b \\ &\iff D \between \mathsf{ext}\,s^{-}a \,\&\, D \between \mathsf{ext}\,s^{-}b \\ &\iff D \between \mathsf{ext}\left(s^{-}a \downarrow s^{-}b\right) && \text{by (D1)} \\ &\iff D \between r^{-}\,\mathsf{ext}\,(a \downarrow b) && \text{by (C1)} \\ &\iff rD \between \mathsf{ext}\,(a \downarrow b) \end{aligned}$$

for any $a, b \in S_2$, and hence rD satisfies (D1). Since D satisfies (D2) and we have

$$\begin{aligned} D \between \mathsf{ext}\, S_1 &\iff D \between r^- \mathsf{ext}\, S_2 && \text{by (C2)} \\ &\iff rD \between \mathsf{ext}\, S_2, \end{aligned}$$

rD satisfies (D2). Thus $rD \in Conv(\mathcal{X}_2)$. □

Proposition 4.3.4. *For any relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ between basic pairs, r preserves convergent subsets iff s preserves ideal points, i.e. $rD \in Conv(\mathcal{X}_2)$ for all $D \in Conv(\mathcal{X}_1)$ iff $s\alpha \in Pt(\mathcal{X}_2)$ for all $\alpha \in Pt(\mathcal{X}_1)$.*

Proof. Let $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ be a relation pair. Suppose that r preserves convergent subsets, and let $\alpha \in Pt(\mathcal{X}_1)$. Then, by Proposition 4.2.4, we have $s\alpha = s\,\mathcal{J}\,\alpha = s \Diamond \mathsf{rest}\, \alpha = \Diamond\, r\, \mathsf{rest}\, \alpha \in Pt(\mathcal{X}_2)$. Conversely, suppose that s preserves ideal points, and let $D \in Conv(\mathcal{X}_1)$. Then $\mathsf{cl}\, rD = \mathsf{rest} \Diamond rD = \mathsf{rest}\, s \Diamond D \in Conv(\mathcal{X}_2)$. Thus, $rD \in Conv(\mathcal{X}_2)$ by Remark 4.2.2. □

For a relation pair between concrete spaces, the converse of Proposition 4.3.3 holds as we see below. Thus the preservation of convergent subsets (or ideal points) characterizes convergent relation pairs. In the following, we list several equivalent characterizations for convergent relation pairs between concrete spaces.

Proposition 4.3.5. *For any relation pairs $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ between concrete spaces, the following are equivalent.*

1. *(r, s) is convergent, i.e. it satisfies (C1) and (C2).*
2. *(r, s) satisfies*

   (C1') $rx \between \mathsf{ext}\, a \;\&\; rx \between \mathsf{ext}\, b \to rx \between \mathsf{ext}\,(a \downarrow b)$,

   (C2') $rx \between \mathsf{ext}\, S_2$

   for all $x \in X_1$ and $a, b \in S_2$.
3. *(r, s) satisfies*

   (E1) $r^- (\mathsf{ext}\, U \cap \mathsf{ext}\, V) = r^- \mathsf{ext}\, U \cap r^- \mathsf{ext}\, V$,

   (E2) $r^- X_2 = X_1$

   for all $U, V \in \mathrm{Pow}(S_2)$.
4. *$D \in Conv(\mathcal{X}_1)$ implies $rD \in Conv(\mathcal{X}_2)$.*
5. *$\alpha \in Pt(\mathcal{X}_1)$ implies $s\alpha \in Pt(\mathcal{X}_2)$.*

Proof. (1) → (4) and (4) ↔ (5) follows from Proposition 4.3.3 and Proposition 4.3.4 respectively. Since all singletons are convergent in a concrete space, we have (4) → (2).

For (1) ↔ (2), since $\mathcal{X}_1$ and $\mathcal{X}_2$ are concrete spaces, we have

$$\begin{aligned}
\text{(C1')} &\iff r^- \,\mathsf{ext}\, a \cap r^- \,\mathsf{ext}\, b = r^- \,\mathsf{ext}\,(a \downarrow b) \\
&\iff \mathsf{ext}\, s^- a \cap \mathsf{ext}\, s^- b = \mathsf{ext}\, s^- \,(a \downarrow b) \\
&\iff \mathsf{ext}\left(s^- a \downarrow s^- b\right) = \mathsf{ext}\, s^- \,(a \downarrow b) && \text{by (B1)} \\
&\iff \text{(C1)}
\end{aligned}$$

for all $a, b \in S_2$, and also

$$\begin{aligned}
\text{(C2')} &\iff X_1 = r^- \,\mathsf{ext}\, S_2 \\
&\iff \mathsf{ext}\, S_1 = \mathsf{ext}\, s^- S_2 && \text{by (B2)} \\
&\iff \text{(C2)}.
\end{aligned}$$

Similarly, for (2 ↔ 3), we have

$$\begin{aligned}
\text{(C1')} &\iff r^- \,\mathsf{ext}\, U \cap r^- \,\mathsf{ext}\, V = r^- \,\mathsf{ext}\,(U \downarrow V) \\
&\iff r^- \,\mathsf{ext}\, U \cap r^- \,\mathsf{ext}\, V = r^- \,(\mathsf{ext}\, U \cap \mathsf{ext}\, V) && \text{by (B1)} \\
&\iff \text{(E1)}
\end{aligned}$$

for all $U, V \in \mathrm{Pow}(S_2)$, and also

$$\begin{aligned}
\text{(C2')} &\iff X_1 = r^- \,\mathsf{ext}\, S_2 \\
&\iff X_1 = r^- X_2 && \text{by (B2)} \\
&\iff \text{(E2)}.
\end{aligned}$$

□

Remark 4.3.6. (E1) and (E2) say that r^- preserves finite intersections and the top of the frame $Red(\mathsf{int})$. Also it is easy to see that r^- preserves arbitrary set-indexed joins of $Red(\mathsf{int})$. Hence, a relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ between concrete spaces is convergent iff r^- is a frame homomorphism from $Red(\mathsf{int}_2)$ to $Red(\mathsf{int}_1)$. Note that any relation pair $(f, s) : \mathcal{X}_1 \to \mathcal{X}_2$ between concrete spaces where f is (the graph of) a function is necessarily convergent as f^- always preserves finite intersections and the top of $Red(\mathsf{int}_2)$.

4.4 The category of concrete space (CSpa)

Concrete spaces and convergent relations pairs naturally form a category.

Proposition 4.4.1. *Concrete spaces and (the equivalence classes of) convergent relation pairs between them form a subcategory* **CSpa** *of category* **BP**.

Proof. We must show that every equivalence class of convergent relation pairs in **CSpa** is an equivalence class of relation pairs in **BP**. To see this, it suffices to show that for any equivalent relation pairs $(r, s), (u, r) : \mathcal{X}_1 \to \mathcal{X}_2$ between concrete spaces, if (r, s) is convergent, so is (u, v). But since $(r, s) \sim (u, v)$ iff $r^- \mathsf{ext}_2 = u^- \mathsf{ext}_2$, the conclusion follows from Proposition 4.3.5 (2).

It remains to be shown that the composition of convergent relation pairs is convergent and identity morphisms are convergent. The latter is trivial. To see the former, let $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ and $(u, v) : \mathcal{X}_2 \to \mathcal{X}_3$ be convergent relation pairs. Then we have

$$\begin{aligned}(u \circ r)^- \mathsf{ext}\,(a \downarrow b) &= r^- u^- \mathsf{ext}\,(a \downarrow b)\\ &= r^- \mathsf{ext}\left(v^- a \downarrow v^- b\right)\\ &= \mathsf{ext}\left(s^- v^- a \downarrow s^- v^- b\right)\\ &= \mathsf{ext}\left((v \circ s)^- a \downarrow (v \circ s)^- b\right)\end{aligned}$$

for any $a, b \in S_3$, and

$$\begin{aligned}(u \circ r)^- \mathsf{ext}\, S_3 &= r^- u^- \mathsf{ext}\, S_3\\ &= r^- \mathsf{ext}\, S_2\\ &= \mathsf{ext}\, S_1.\end{aligned}$$

Thus, the composite $(u \circ r, v \circ s)$ is convergent. □

The following suggests that in **CSpa**, an appropriate notion of point of a concrete space $\mathcal{X}$ is not an element of X but an ideal point.

Proposition 4.4.2. *In* **CSpa**, *there is a bijective correspondence*

$$Pt(\mathcal{X}) \cong \mathbf{CSpa}(\mathbf{1}, \mathcal{X})$$

for any concrete space $\mathcal{X}$, *where* $\mathbf{1}$ *is a concrete space such that* $\mathbf{1} = \left(\{*\}, id_{\{*\}}, \{*\}\right)$.

Proof. Let $\mathcal{X}$ be a concrete space and let $(r, s) : \mathbf{1} \to \mathcal{X}$ be a convergent relation pair. Since r preserves convergent subsets and r is completely determined by $r\{*\}$, we can define a mapping $\theta : \mathbf{CSpa}(\mathbf{1}, \mathcal{X}) \to Pt(\mathcal{X})$ by $\theta(r) = \Diamond\, r\{*\}$. Obviously, θ respects the equality of relation pairs. It is also injective. Conversely, for any $\alpha \in Pt(\mathcal{X})$, defined a relation $r_\alpha \subseteq \{*\} \times X$ by $r_\alpha = \{*\} \times \mathsf{rest}\,\alpha$. Then r_α is trivially continuous since any subset of $\{*\}$ is open in $\mathbf{1}$. Therefore, there is a relation $s \subseteq \{*\} \times S$ such that (r, s) is a relation pair from $\mathbf{1}$ to $\mathcal{X}$. Then (r, s) is convergent; since r is clearly a total relation, it satisfies (C2). Moreover, we have

$$\begin{aligned}* \in r_\alpha^- \mathsf{ext}\, a \cap r_\alpha^- \mathsf{ext}\, b &\iff a, b \in \Diamond\, r_\alpha\{*\}\\ &\iff a, b \in \alpha\\ &\iff \alpha \between (a \downarrow b)\\ &\iff * \in r_\alpha^- \mathsf{ext}\,(a \downarrow b)\end{aligned}$$

for any $a, b \in S$, so (r, s) satisfies (C1). Since α is formal closed, we have $\theta(r_\alpha) = \Diamond\, \mathsf{rest}\,\alpha = \alpha$. Therefore the mapping θ is surjective, and hence it is bijective. □

4.5 T_0 spaces and Sobriety

We introduce the notion of weak separation axiom T_0 and sobriety of basic pairs and concrete spaces, and consider a relation between **BP** and the subcategory of T_0 basic pairs and a relation between the category of sober concrete spaces and that of sober topological spaces.

4.5.1 T_0 basic pairs

Definition 4.5.1. A basic pair $\mathcal{X}$ is T_0 if

$$\Diamond x = \Diamond y \Rightarrow x = y$$

for all $x, y \in X$.

Let $\mathbf{BP}_0$ be a full subcategory **BP** whose objects are T_0 basic pairs.

Proposition 4.5.2. **BP** *and* $\mathbf{BP}_0$ *are equivalent.*

Proof. For any basic pair $\mathcal{X}$, define a relation $\equiv_0$ on X by

$$x \equiv_0 x' \iff \Diamond x = \Diamond x'$$

for all $x, x' \in X$. Clearly, $\equiv_0$ is an equivalence relation on X. For any basic pair $\mathcal{X}$, define a basic pair $\widetilde{\mathcal{X}} = (\widetilde{X}, \widetilde{\Vdash}, S)$ by

$$\begin{aligned} \widetilde{X} &= X/\equiv_0, \\ [x]\,\widetilde{\Vdash} a &\iff a \in \Diamond x \end{aligned}$$

for any $[x] \in \widetilde{X}$ and $a \in S$. Note that $\widetilde{\Vdash}$ is well-defined by the definition of $\equiv_0$. Evidently, $\widetilde{\mathcal{X}}$ is T_0. Since we have

$$\begin{aligned} a \in \mathcal{J}\,U &\iff \mathsf{ext}\,a \between \mathsf{rest}\,U \\ &\iff (\exists x \in X)\, a \in \Diamond x \subseteq U \\ &\iff (\exists x \in X)\, a \in \widetilde{\Diamond}\,[x] \subseteq U \\ &\iff \widetilde{\mathsf{ext}} a \between \widetilde{\mathsf{rest}} U \\ &\iff a \in \widetilde{\mathcal{J}} U \end{aligned}$$

for any $a \in S$ and $U \in \mathrm{Pow}(S)$, it follows that $\mathcal{J} = \widetilde{\mathcal{J}}$. Thus, by Proposition 3.3.3, there exists a relation $r \subseteq X \times \widetilde{X}$ such that $(r, id_S) : \mathcal{X} \to \widetilde{\mathcal{X}}$ is an isomorphism. Write $\eta_{\mathcal{X}} = (r, id_S)$. Then we have a family $(\langle \widetilde{\mathcal{X}}, \eta_{\mathcal{X}} \rangle)_{\mathcal{X} \in \mathrm{Ob}(\mathbf{BP})}$ such that $\widetilde{\mathcal{X}} \in \mathrm{Ob}(\mathbf{BP_0})$ and $\eta_{\mathcal{X}} : \mathcal{X} \cong \widetilde{\mathcal{X}}$. Moreover, since $\mathbf{BP}_0$ is full in **BP**, the insertion functor $I : \mathbf{BP}_0 \to \mathbf{BP}$ is full and faithful. Therefore, by Proposition 2.3.73 (in the dual formulation), I is an equivalence between $\mathbf{BP}_0$ and **BP**. ☐

4.5.2 Sober concrete spaces

Definition 4.5.3. Concrete space $\mathcal{X}$ is *sober* if for any $D \in Conv(\mathcal{X})$ there is a unique $x \in X$ such that $\mathsf{cl}\, D = \mathsf{cl}\{x\}$.

Proposition 4.5.4. *A concrete space $\mathcal{X}$ is sober iff for any $\alpha \in Pt(\mathcal{X})$ there is a unique $x \in X$ such that $\alpha = \Diamond\{x\}$.*

Proof. Note that $\mathsf{cl}\, D = \mathsf{cl}\{x\} \iff \Diamond D = \Diamond\{x\}$ in Definition 4.5.3; the equivalence follows from Proposition 4.2.4. $\square$

Note that a sober space characterized by the condition of the above proposition is called a weakly sober space in [5].

Let $\mathbf{CSpa}_s$ be the full subcategory of $\mathbf{CSpa}$ whose objects are sober concrete spaces. Let $\mathbf{Top}$ be the category whose objects are concrete spaces and whose morphisms are continuous functions between concrete spaces, where a function $f : X_1 \to X_2$ between concrete spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ is *continuous* if

$$(\forall x \in X)\,(\forall b \in S_2)\, f(y) \in \mathsf{ext}\, b \to (\exists a \in S_1)\left[x \in \mathsf{ext}\, a \,\&\, \mathsf{ext}\, a \subseteq f^{-}\, \mathsf{ext}\, b\right]$$

i.e. if f is a continuous relation from $\mathcal{X}_1$ to $\mathcal{X}_2$. Let $\mathbf{Top}_s$ be a full subcategory of $\mathbf{Top}$ whose objects are sober concrete spaces.

Proposition 4.5.5. *$\mathbf{Top}_s$ and $\mathbf{CSpa}_s$ are isomorphic.*

We first prove the following lemma.

Lemma 4.5.6. *A concrete space $\mathcal{X}$ is sober if and only if for any concrete space $\mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$ and convergent relation pair $(r, s) : \mathcal{Y} \to \mathcal{X}$, there is a unique function $f : Y \to X$ such that $(f, s) \sim (r, s)$.*

Proof. Let $\mathcal{X}$ be a concrete space, and suppose that $\mathcal{X}$ is sober. Let $\mathcal{Y}$ be a concrete space and $(r, s) : \mathcal{Y} \to \mathcal{X}$ be a convergent relation pair. Note that since (r, s) is convergent, we have $s \Diamond\{y\} \in Pt(\mathcal{X})$ for each $y \in Y$. Since $\mathcal{X}$ is sober, there exists a unique $x \in X$ such that $\Diamond\{x\} = s \Diamond\{y\}$. Define a function $f : Y \to X$ by

$$f(y) = \text{unique } x \in X \text{ such that } \Diamond\{x\} = s \Diamond\{y\}$$

for each $y \in Y$. Then, we have

$$\Diamond f(y) = s \Diamond\{y\} = \Diamond r\{y\}$$

for any $y \in Y$. Therefore $(f, s) \sim (r, s)$. Given any function $g : Y \to X$ such that $(g, s) \sim (r, s)$, then $\Diamond g(y) = s \Diamond\{y\}$ for all $y \in Y$, and hence $g(y) = f(y)$ for all $y \in Y$ by sobriety of $\mathcal{X}$. Therefore $f = g$.

Conversely, let $\alpha \in Pt(\mathcal{X})$. By Proposition 4.4.2, there is a unique convergent relation pair $(r, s) : \mathbf{1} \to \mathcal{X}$ such that $\alpha = \Diamond r\{*\}$. Thus there is a unique function $f : \{*\} \to X$ such that $\Diamond f(*) = \Diamond r\{*\}$. Suppose that there is $x \in X$ such that $\Diamond x = \alpha$. Then $g : * \mapsto x$ is a function $g : \{*\} \to X$ such that $(g, s) \sim (r, s)$, so we must have $x = g(*) = f(*)$. Therefore $\mathcal{X}$ is sober. $\square$

Proof of Proposition 4.5.5. Define a functor $F : \mathbf{Top}_s \to \mathbf{CSpa}_s$ by

$$F(\mathcal{X}) = \mathcal{X}$$

for each concrete space $\mathcal{X}$, and

$$F(f) = (f, s)$$

for each continuous function $f : \mathcal{X}_1 \to \mathcal{X}_2$, where in (f, s), f is the graph of function f and $s \subseteq S_1 \times S_2$ is a relation defined by

$$a \, s \, b \iff \mathsf{ext}\, a \subseteq f^{-} \mathsf{ext}\, b$$

for all $a \in S_1$ and $b \in S_2$. Then (f, s) is a relation pair by Proposition 3.2.5, and it is also convergent by Remark 4.3.6. Thus $(f, s) : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism in $\mathbf{CSpa}_s$. That F preserves the composition and the identity is obvious from the definition of the composition and identity of **CSpa**.

Conversely, define a functor $G : \mathbf{CSpa}_s \to \mathbf{Top}_s$ by

$$G(\mathcal{X}) = \mathcal{X}$$

for each concrete space $\mathcal{X}$ and

$$G(r, s) = \text{unique } g : X_1 \to X_2 \text{ such that } (g, s) \sim (r, s)$$

for each convergent relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$. Note that such g exists by the above lemma. Then G preserves compositions of morphisms. To see this, let $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ and $(u, v) : \mathcal{X}_2 \to \mathcal{X}_3$ be morphisms of $\mathbf{CSpa}_s$, and let $f = G(r, s)$ and $g = G(u, v)$. Since the composition respects the equality of relation pairs, we have $(g \circ f, v \circ s) = (u \circ r, v \circ s)$. Then, have $G((u, v) \circ (r, s)) = G(u \circ r, v \circ s) = g \circ f = G(u, v) \circ G(r, s)$. Since diagonal relations on a set are identity functions, G preserves identities. Now, the object part of F and G are trivially bijective. By the definition of F and G, we have $GF(f) = f$ for any continuous function f. Also, we have $FG(r, s) \sim (r, s)$ for any convergent relation pair (r, s), i.e. $GF(r, s) = (r, s)$ as a morphism in $\mathbf{CSpa}_s$. Therefore, F and G are mutual inverses. Thus $\mathbf{Top}_s \cong \mathbf{CSpa}_s$. ☐

4.6 Relation between BP and CSpa

We show that **CSpa** is a coreflective subcategory of **BP**; see Definition 2.3.75 for the notion of coreflection.

Theorem 4.6.1. **CSpa** *is coreflective in* **BP**.

Proof. Given a basic pair $\mathcal{X}$, define a basic pair $\overline{\mathcal{X}}$ by

$$\overline{\mathcal{X}} = \left(\mathrm{Fin}(X), \overline{\Vdash}, \mathrm{Fin}(S)\right),$$

where

$$D \,\overline{\Vdash}\, A \iff (\forall a \in A)\, D \between \mathsf{ext}_{\mathcal{X}}\{a\}$$

for each $D \in \mathrm{Fin}(X)$ and $A \in \mathrm{Fin}(S)$. We show that $\overline{\mathcal{X}}$ a concrete space. To this end, let $A, B \in \mathrm{Fin}(S)$ and $D \in \mathrm{Fin}(X)$. Since $A \cup B \in \{A\} \overline{\downarrow} \{B\}$ for all $A, B \in \mathrm{Fin}(S)$, we have

$$\begin{aligned} D \in \overline{\mathsf{ext}}\{A\} \cap \overline{\mathsf{ext}}\{B\} &\iff (\forall a \in A \cup B)\, D \between \mathsf{ext}_{\mathcal{X}}\{a\} \\ &\iff D \in \overline{\mathsf{ext}}\{A \cup B\} \\ &\implies D \in \overline{\mathsf{ext}}\left(\{A\} \overline{\downarrow} \{B\}\right). \end{aligned}$$

So $\overline{\mathcal{X}}$ satisfies (B1). Since $D \in \overline{\mathsf{ext}}\,\emptyset$ for all $D \in \mathrm{Fin}(X)$, $\overline{\mathcal{X}}$ satisfies (B2).

Define a relation pair $(p, q) : \overline{\mathcal{X}} \to \mathcal{X}$ by

$$\begin{aligned} D\, p\, x &\iff x \in D, \\ A\, q\, a &\iff a \in A \end{aligned}$$

for all $D \in \mathrm{Fin}(X), x \in X, A \in \mathrm{Fin}(S)$ and $a \in S$. The pair (p, q) is indeed a relation pair; for we have

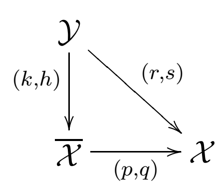
$$\begin{aligned} D\, (\Vdash_{\mathcal{X}} \circ p)\, a &\iff D \between \mathsf{ext}_{\mathcal{X}}\{a\} \\ &\iff D \,\overline{\Vdash}\, \{a\} \\ &\iff (\exists A \in \mathrm{Fin}(S))\, D \overline{\Vdash} A \,\&\, a \in A \\ &\iff D \left(q \circ \overline{\Vdash}\right) a \end{aligned}$$

for any $D \in \mathrm{Fin}(X)$ and $a \in S$.

Now, given any relation pair $(r, s) : \mathcal{Y} \to \mathcal{X}$ where $\mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$ is a concrete space, define a convergent relation pair $(k, h) : \mathcal{Y} \to \overline{\mathcal{X}}$ by

$$\begin{aligned} y\, k\, D &\iff D \subseteq r\{y\}, \\ b\, h\, A &\iff \mathsf{ext}\, b \subseteq \textstyle\bigcap_{a \in A} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{a\} \end{aligned}$$

for all $D \in \mathrm{Fin}(X), y \in Y, A \in \mathrm{Fin}(S)$ and $b \in T$. We show that (k, h) is indeed a convergent relation pair, and moreover, it is a unique convergent relation pair which makes the following diagram commute.



First, since $\mathcal{Y}$ satisfies (B1), we have

$$
\begin{aligned}
& y\,(h \circ \Vdash_{\mathcal{Y}})\,A \\
\iff & y \in \textstyle\bigcap_{a \in A} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{a\} \\
\iff & y \in \textstyle\bigcap_{a \in A} r^{-}\, \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & (\forall a \in A)\, r\{y\} \between \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & (\exists D \in \mathrm{Fin}(X))\, D \subseteq r\{y\} \;\&\; (\forall a \in A)\, D \between \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & (\exists D \in \mathrm{Fin}(X))\, y\, k\, D \;\&\; D \overline{\Vdash} A \\
\iff & y \left(\overline{\Vdash} \circ k\right) A
\end{aligned}
$$

for any $y \in Y$ and $A \in \mathrm{Fin}(S)$, and hence (k, h) is a relation pair. Since we have

$$
\begin{aligned}
& y \in \mathsf{ext}_{\mathcal{Y}}\, h^{-}\{A\} \cap \mathsf{ext}_{\mathcal{Y}}\, h^{-}\{B\} \\
\iff & y \in \left(\textstyle\bigcap_{a \in A} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{a\}\right) \cap \left(\textstyle\bigcap_{b \in B} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{b\}\right) \\
\iff & y \in \textstyle\bigcap_{a \in A \cup B} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{a\} \\
\iff & y \in \mathsf{ext}_{\mathcal{Y}}\, h^{-}\{A \cup B\} \\
\implies & y \in \mathsf{ext}_{\mathcal{Y}}\, h^{-}\left(A \downarrow B\right)
\end{aligned}
$$

for any $y \in Y$ and $A, B \in \mathrm{Fin}(S)$, (k, h) satisfies (C1). Since $\emptyset \in \mathrm{Fin}(X)$, we have $\emptyset \subseteq r\{y\}$ for any $y \in Y$, i.e. $y\, k\, \emptyset$. So k is total, and thus (k, h) satisfies (C2). Since we have

$$
\begin{aligned}
& y\,(\Vdash_{\mathcal{X}} \circ p \circ k)\, a \\
\iff & (\exists D \in \mathrm{Fin}(X))\, D \subseteq r\{y\} \;\&\; D \between \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & r\{y\} \between \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & y\,(\Vdash_{\mathcal{X}} \circ r)\, a
\end{aligned}
$$

for any $y \in Y$ and $a \in S$, the diagram commutes.

Given any convergent relation pair $\left(\bar{k}, \bar{h}\right) : \mathcal{Y} \to \overline{\mathcal{X}}$ which makes the above diagram commute, we have

$$
\begin{aligned}
y\,(h \circ \Vdash_{\mathcal{Y}})\,A \iff & y \in \textstyle\bigcap_{a \in A} \mathsf{ext}_{\mathcal{Y}}\, s^{-}\{a\} \\
\iff & y \in \textstyle\bigcap_{a \in A} \bar{k}^{-} p^{-}\, \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & y \in \bar{k}^{-} \textstyle\bigcap_{a \in A} p^{-}\, \mathsf{ext}_{\mathcal{X}}\{a\} \\
\iff & (\exists D \in \mathrm{Fin}(X))\, y\, \bar{k}\, D \,\&\, (\forall a \in A)\, D \between \mathsf{ext}\{a\} \\
\iff & (\exists D \in \mathrm{Fin}(X))\, y\, \bar{k}\, D \,\&\, D \overline{\Vdash} A \\
\iff & y \left(\overline{\Vdash} \circ \bar{k}\right) A
\end{aligned}
$$

for any $y \in Y$ and $A \in \mathrm{Fin}(S)$. Hence $\left(\bar{k}, \bar{h}\right) \sim (k, h)$, and so (k, h) is a unique morphism in **CSpa** which make the diagram commute. □

Chapter 5

Categorical constructions in BP and CSpa

In this section, we show that **BP** and **CSpa** are both complete and cocomplete by constructing (co)products and (co)equalisers in both categories. The constructions of coequalisers of **BP** and **CSpa** and products of **CSpa** is not straightforward. This has already been observed in the construction of coequalisers for the category of set-presented formal topologies [20]. The crucial point of the construction in both cases is in showing that a certain class is set-generated in the sense of Definition 2.2.1. Instead of directly constructing a generating subset, we exploit the notion of a *generalized geometric theory* [6] (cf. Section 2.2) to show that the class has a generating subset.

5.1 Completeness and cocompleteness of BP

The main result of this section is the following.

Theorem 5.1.1. **BP** *is cocomplete.*

Note that by Proposition 3.3.2, we immediately obtain the following.

Corollary 5.1.2. **BP** *is complete.*

By Corollary 2.3.63, it suffices show that **BP** has arbitrary small coproducts and coequalisers. In the following two sections, we give construction of coproducts and coequalisers in **BP**.

5.1.1 Coproducts

In this section, we show that **BP** has arbitrary small, i.e. set-sized, coproducts.

Proposition 5.1.3. **BP** *has small coproducts.*

Proof. Given a set-indexed family $(\mathcal{X}_i)_{i\in I}$ of basic pairs, define a basic pair $\coprod_{i\in I}\mathcal{X}_i = (\sum_{i\in I} X_i, \Vdash_\Sigma, \sum_{i\in I} S_i)$ by

$$(i,x) \Vdash_\Sigma (j,a) \iff i = j \;\&\; x \Vdash_i a$$

for all $(i,x) \in \sum_{i\in I} X_i$ and $(j,a) \in \sum_{i\in I} S_i$. Also, define a family of relation pairs $\big((r_i,s_i) : \mathcal{X}_i \to \coprod_{i\in I}\mathcal{X}_i\big)_{i\in I}$ by

$$\begin{aligned} x\, r_i\, (j,x') &\iff i = j \;\&\; x = x', \\ a\, s_i\, (j,a') &\iff i = j \;\&\; a = a' \end{aligned}$$

for all $i \in I, x \in X_i, (j,x') \in \sum_{i\in I} X_i, a \in S_i$ and $(j,a') \in \sum_{i\in I} S_i$. We claim that $\coprod_{i\in I}\mathcal{X}_i$ is a coproduct of $(\mathcal{X}_i)_{i\in I}$. Since we have

$$\begin{aligned} x(s_i \circ \Vdash_i)(j,a) &\iff x \Vdash_i a \,\&\, i = j \\ &\iff x(\Vdash_\Sigma \circ r_i)(j,a) \end{aligned}$$

for any $x \in X_i$ and $(j,a) \in \sum_{i\in I} S_i$, (r_i,s_i) is a relation pair from $\mathcal{X}_i$ to $\coprod_{i\in I}\mathcal{X}_i$ for each $i \in I$. Given any family of relation pairs $((u_i,v_i) : \mathcal{X}_i \to \mathcal{Y})_{i\in I}$ where $\mathcal{Y} = (Y, \Vdash_\mathcal{Y}, T)$, define a relation pair $(k,h) : \coprod_{i\in I}\mathcal{X}_i \to \mathcal{Y}$ by

$$\begin{aligned} (i,x)\, k\, y &\iff x\, u_i\, y, \\ (i,a)\, h\, b &\iff a\, v_i\, b \end{aligned}$$

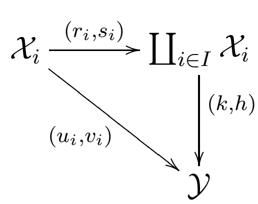
for all $(i,x) \in \coprod_{i\in I} X_i, y \in Y, (i,a) \in \coprod_{i\in I} S_i$ and $b \in T$. Since we have

$$\begin{aligned} (i,x)\,(\Vdash_\mathcal{Y} \circ k)\, b &\iff (\exists y \in Y)\,(i,x)\; k\; y \;\&\; y \Vdash_\mathcal{Y} b \\ &\iff (\exists y \in Y)\, x\; u_i\; y \;\&\; y \Vdash_\mathcal{Y} b \\ &\iff x\,(\Vdash_\mathcal{Y} \circ u_i)\, b \\ &\iff x\,(v_i \circ \Vdash_i)\, b \\ &\iff (\exists a \in S_i)\, x \Vdash_i a \;\&\; a\; v_i\; b \\ &\iff (\exists a \in S_i)\,(i,x) \Vdash_\Sigma (i,a) \;\&\; (i,a)\; h\; b \\ &\iff (i,x)\,(h \circ \Vdash_\Sigma)\, b \end{aligned}$$

for any $(i,x) \in \sum_{i\in I} X_i$ and $b \in T$, (k,h) is indeed a relation pair. Since we have

$$\begin{aligned} x\,(\Vdash_\mathcal{Y} \circ k \circ r_i)\, b &\iff (i,x)\,(\Vdash_\mathcal{Y} \circ k)\; b \\ &\iff (\exists y \in Y)\,(i,x)\; k\; y \;\&\; y \Vdash_\mathcal{Y} b \\ &\iff (\exists y \in Y)\, x\; u_i\; y \;\&\; y \Vdash_\mathcal{Y} b \\ &\iff x\,(\Vdash_\mathcal{Y} \circ u_i)\, b \end{aligned}$$

for any $i \in I$, $x \in X_i$ and $b \in T$, the following diagram commutes



for each $i \in I$. Finally, given any relation pair (k', h') which makes the above diagram commute, we have

$$\begin{aligned} (i,x)\,(\Vdash_{\mathcal{Y}} \circ k')\, b &\iff x\,(\Vdash_{\mathcal{Y}} \circ k' \circ r_i)\, b \\ &\iff x\,(\Vdash_{\mathcal{Y}} \circ u_i)\, b \\ &\iff x\,(\Vdash_{\mathcal{Y}} \circ k \circ r_i)\, b \\ &\iff (i,x)\,(\Vdash_{\mathcal{Y}} \circ k)\, b \end{aligned}$$

for any $(i,x) \in \sum_{i \in I} X_i$ and $b \in T$. Therefore $(k', h') \sim (k, h)$, and hence (k, h) is a unique morphism which makes the diagram commute. □

As a corollary, **BP** has an initial object and a terminal object. They are given by $\mathbf{0} = (\emptyset, \emptyset, \emptyset)$, and by duality $\mathbf{1} = \mathbf{0}$.

5.1.2 Coequalisers

Proposition 5.1.4. BP *has coequalisers for any parallel pair of morphisms.*

Proof. Given any parallel pair of relation pairs $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$, define a class Q by

$$Q = \left\{ U \in \mathrm{Pow}(S_2) \mid \mathsf{ext}_1\, s_1^- U \;=\; \mathsf{ext}_1\, s_2^- U \right\}.$$

Then, Q is the class of models of the following theory of rank 2 over the set S_2.

$$\left\{ a \to \textstyle\bigwedge_{x \in \mathsf{ext}_1 s_1^-\{a\}} \bigvee_{b \in s_2 \Diamond \{x\}} b \mid a \in S_2 \right\} \cup \left\{ a \to \textstyle\bigwedge_{x \in \mathsf{ext}_1 s_2^-\{a\}} \bigvee_{b \in s_1 \Diamond \{x\}} b \mid a \in S_2 \right\}.$$

Therefore, the class Q has a generating subset G by Theorem 2.2.13. We show that

$$\mathcal{X}_2 \xrightarrow{(id_{X_2}, \in)} \mathcal{G} = (X_2, \Vdash_{\mathcal{G}}, G)$$

is a coequaliser for (r_1, s_1) and (r_2, s_2), where

$$x \Vdash_{\mathcal{G}} U \iff x \in \mathsf{ext}_2\, U$$

for all $x \in X_2$ and $U \in G$, and $\in$ is the standard set membership relation.

First, since we have

$$\begin{aligned} x\,(\Vdash_{\mathcal{G}} \circ id_{X_2})\, U &\iff x \Vdash_{\mathcal{G}} U \\ &\iff x \in \mathsf{ext}_2\, U \\ &\iff (\exists a \in S_2)\, x \Vdash_2 a \;\&\; a \in U \\ &\iff x\,(\Vdash_2 \circ \in)\, U \end{aligned}$$

for any $x \in X_2$ and $U \in G$, $(id_{X_2}, \in)$ is a relation pair. Next, since we have

$$\begin{aligned} x\,(\in \circ s_1 \circ \Vdash_1)\,U &\iff x \in \ \mathsf{ext}_1 s_1^- \in^- \{U\} \\ &\iff x \in \ \mathsf{ext}_1 s_1^- U \\ &\iff x \in \ \mathsf{ext}_1 s_2^- U \\ &\iff x\,(\in \circ s_2 \circ \Vdash_1)\,U \end{aligned}$$

for any $x \in X_1$ and $U \in G$, the diagram $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2 \xrightarrow{(id_{X_2},\in)} \mathcal{G}$ commutes. Given any relation pair $(k, h) : \mathcal{X}_2 \to \mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$ which makes the diagram $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2 \xrightarrow{(k,h)} \mathcal{Y}$ commute, define a relation pair $(\bar{k}, \bar{h}) : \mathcal{G} \to \mathcal{Y}$ by

$$\begin{aligned} &\bar{k} = k, \\ &U\,\bar{h}\,b \iff U \subseteq h^-\{b\} \end{aligned}$$

for all $U \in G$ and $b \in T$. Note that $h^-\{b\} \in Q$ for all $b \in T$ by the commutativity of the diagram. Since G generates Q, we have

$$\begin{aligned} x\left(\Vdash_{\mathcal{Y}} \circ \bar{k}\right) b &\iff x\,(\Vdash_{\mathcal{Y}} \circ k)\,b \\ &\iff x\,(h \circ \Vdash_2)\,b \\ &\iff (\exists\, a \in S_2)\, x \Vdash_2 a \;\&\; a\,h\,b \\ &\iff (\exists\, a \in S_2)\, x \Vdash_2 a \;\&\; a \in h^-\{b\} \\ &\iff (\exists\, a \in S_2)\,(\exists U \in G)\;\; x \Vdash_2 a \;\&\; a \in U \subseteq h^-\{b\} \\ &\iff (\exists\, U \in G)\, x \in \ \mathsf{ext}_2\, U \;\&\; U \subseteq h^-\{b\} \\ &\iff (\exists\, U \in G)\, x \Vdash_{\mathcal{G}} U \;\&\; U\bar{h}\ b \\ &\iff x\left(\bar{h} \circ \Vdash_{\mathcal{G}}\right) b \end{aligned}$$

for any $x \in X_2$ and $b \in T$, and so $(\bar{k}, \bar{h})$ is a relation pair. Since $k = k \circ id_{X_2} = \bar{k} \circ id_{X_2}$, the diagram

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{(id_{X_2},\in)} & \mathcal{G} \\ & \searrow^{(k,h)} & \downarrow{\scriptstyle (\bar{k},\bar{h})} \\ & & \mathcal{Y} \end{array}$$

commutes. Finally, since $(id_{X_2}, \in)$ is an epimorphism, $(\bar{k}, \bar{h})$ is a unique relation pair which makes the above diagram commute. $\square$

5.2 Completeness and cocompleteness of CSpa

In this section, we show completeness and cocompleteness of **CSpa**. For cocompleteness, the same construction of coproducts and coequalisers for **BP** carries over to **CSpa**. For completeness, we show that **CSpa** has equalisers and arbitrary products.

5.2.1 Cocompleteness

In this section we show completeness of **CSpa** by observing the following fact.

Proposition 5.2.1. *The insertion functor I : **CSpa** $\to$ **BP** creates colimits, i.e. the colimits in **CSpa** are exactly the colimits in the underlying category **BP**.*

To see this, it suffices to show that the coproduct, coequaliser and the unique morphisms constructed in Propositions 5.1.3 and 5.1.4, respectively, satisfy (B1), (B2), (C1) and (C2) when the objects and morphisms involved in the construction satisfy them. We verify each construction in the following two lemmas.

Lemma 5.2.2. *For any set-indexed family $(\mathcal{X}_i)_{i\in I}$ of concrete spaces, the basic pair $\coprod_{i\in I}\mathcal{X}_i = \left(\sum_{i\in I} X_i, \Vdash_{\coprod}, \sum_{i\in I} S_i\right)$ and the family of relation pairs $\left((r_i, s_i) : \mathcal{X}_i \to \coprod_{i\in I}\mathcal{X}_i\right)_{i\in I}$ constructed in Propositions 5.1.3 is a coproduct of $(\mathcal{X}_i)_{i\in I}$.*

Proof. First, we show that $\coprod_{i\in I}\mathcal{X}_i$ satisfies (B1) and (B2). To this end, let $(i,x) \in \sum_{i\in I} X_i$ and $(j,a),(k,b) \in \sum_{i\in I} S_i$. Since $\mathcal{X}_i$ satisfies (B1), we have

$$\begin{aligned}(i,x) \in \mathsf{ext}_{\coprod}\{(j,a)\} \cap \mathsf{ext}_{\coprod}\{(k,b)\} &\iff x \in \mathsf{ext}_i\{a\} \cup \mathsf{ext}_i\{b\} \;\&\; i = j = k\\ &\iff x \in \mathsf{ext}_i(a \downarrow_i b) \;\&\; i = j = k\\ &\iff (i,x) \in \mathsf{ext}_{\coprod}\left((j,a) \downarrow_{\coprod} (k,b)\right),\end{aligned}$$

and hence $\coprod_{i\in I}\mathcal{X}_i$ satisfies (B1). Now let $(i,x) \in \sum_{i\in I} X_i$. Since $\mathcal{X}_i$ satisfies (B2), there exists $a \in S_i$ such that $x \Vdash_i a$, and hence $(i,x) \Vdash_{\coprod} (i,a)$. Thus $\coprod_{i\in I}\mathcal{X}_i$ satisfies (B2). Next, we see that each injection $(r_i,s_i) : \mathcal{X}_i \to \coprod \mathcal{X}_i$ satisfies (C1) and (C2). To this end, let $(j,a),(k,b) \in \sum_{i\in I} S_i$ and $x \in X_i$. Since $\coprod \mathcal{X}_i$ satisfies (B1), we have

$$\begin{aligned}x \in r_i^- \,\mathsf{ext}_{\coprod}\{(j,a)\} \cap r_i^- \,\mathsf{ext}_{\coprod}\{(k,b)\} &\iff (i,x) \in \mathsf{ext}_{\coprod}\{(j,a)\} \cap \mathsf{ext}_{\coprod}\{(k,b)\}\\ &\iff (i,x) \in \mathsf{ext}_{\coprod}((j,a) \downarrow_{\coprod} (k,b))\\ &\iff x \in r_i^- \,\mathsf{ext}_{\coprod}((j,a) \downarrow_{\coprod} (k,b))\end{aligned}$$

and hence $(r_i,s_i) : \mathcal{X}_i \to \coprod \mathcal{X}_i$ satisfies (C1). Also, for any $x \in X_i$, we have $x r_i (i,x)$, and thus $(r_i,s_i) : \mathcal{X}_i \to \coprod \mathcal{X}_i$ satisfies (C2).

Now, given any family of convergent relation pairs $((u_i,v_i) : \mathcal{X}_i \to \mathcal{Y})_{i\in I}$ where $\mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$, let $(k,h) : \coprod_{i\in I}\mathcal{X}_i \to \mathcal{Y}$ be a relation pair constructed in Proposition 5.1.3. We must show that (k,h) is convergent. To this end, let $a,b \in Y$ and $(i,x) \in \sum X_i$. Since u_i satisfies (C1), we have

$$\begin{aligned}(i,x) \in k^- \,\mathsf{ext}_{\mathcal{Y}}\{a\} \cap k^- \,\mathsf{ext}_{\mathcal{Y}}\{b\} &\iff x \in u_i^- \,\mathsf{ext}_{\mathcal{Y}}\{a\} \cap u_i^- \,\mathsf{ext}_{\mathcal{Y}}\{b\}\\ &\iff x \in u_i^- \,\mathsf{ext}_{\mathcal{Y}}(a \downarrow b)\\ &\iff (i,x) \in k^- \,\mathsf{ext}_{\mathcal{Y}}(a \downarrow b).\end{aligned}$$

Hence (k,h) satisfies (C1). To see that (k,h) satisfies (C2), let $(i,x) \in \sum X_i$. Since $x \in X_i$ and (u_i,v_i) satisfies (C2), there exists $y \in Y$ such that $x\, u_i\, y$, and thus $(i,x)\, k\, y$. Therefore u_i satisfies (C2). ∎

Lemma 5.2.3. *For any pair of convergent relation pairs* $\mathcal{X}_1 \overset{(r_1,s_1)}{\underset{(r_2,s_2)}{\rightrightarrows}} \mathcal{X}_2$ *between concrete spaces, the basic pair and the morphism*

$$\mathcal{X}_2 \xrightarrow{(id_{X_2},\in)} \mathcal{G} = (X_2, \Vdash_{\mathcal{G}}, G)$$

constructed in Propositions 5.1.4 is a coequaliser for (r_1, s_1) *and* (r_2, s_2)*.*

Proof. First, we show that $\mathcal{G}$ satisfies (B1) and (B2). Since (r_1, s_1) and (r_2, s_2) satisfy (C1), we have

$$\begin{aligned} \mathsf{ext}_1\, s_1^-(U \downarrow V) &= \mathsf{ext}_1\, s_1^- U \cap \mathsf{ext}_1\, s_1^- V \\ &= \mathsf{ext}_1\, s_2^- U \cap \mathsf{ext}_1\, s_2^- V \\ &= \mathsf{ext}_1\, s_2^-(U \downarrow V) \end{aligned}$$

for any $U, V \in Q$, and so $(U \downarrow V) \in Q$ for any $U, V \in Q$. Hence, since $\mathcal{X}_2$ satisfies (B1) and G generates Q, we have

$$\begin{aligned} x \in \mathsf{ext}_{\mathcal{G}}\{U\} \cap \mathsf{ext}_{\mathcal{G}}\{V\} &\iff x \in \mathsf{ext}_2\, U \cap \mathsf{ext}_2\, V \\ &\iff x \in \mathsf{ext}_2(U \downarrow V) \\ &\iff (\exists a \in S_2) x \Vdash_2 a \;\&\; a \in U \downarrow V \\ &\iff (\exists W \in G)\, (\exists a \in S_2)\, x \Vdash_2 a \;\&\; a \in W \subseteq \left(U \downarrow_2 V\right) \\ &\iff (\exists W \in G)\, x \in \mathsf{ext}_2\, W \;\&\; W \subseteq \left(U \downarrow_2 V\right) \\ &\implies (\exists W \in G)\, x \Vdash_{\mathcal{G}} W \;\&\; W \in \left(\{U\} \downarrow_{\mathcal{G}} \{V\}\right) \\ &\iff x \in \mathsf{ext}_{\mathcal{G}} \left(\{U\} \downarrow_{\mathcal{G}} \{V\}\right) \end{aligned}$$

for any $U, V \in G$ and $x \in X_2$. Therefore $\mathcal{G}$ satisfies (B1). For any $x \in X_2$, there exists $a \in S_2$ such that $x \in \mathsf{ext}_2\{a\}$ by (B2) for $\mathcal{X}_2$. Since $S_2 \in Q$ and G generates Q, there exists $U \in G$ such that $a \in U$, and thus $x \Vdash_{\mathcal{G}} U$, and so $\mathcal{G}$ satisfies (B2). Since the first component of the relation pair $(id_{X_2}, \ni)$ is the diagonal relation, it satisfies (C1) and (C2).

Suppose that we are given a concrete space $\mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$ and a convergent relation pair $(k, h) : \mathcal{X}_2 \to \mathcal{Y}$ which makes the diagram

$$\mathcal{X}_1 \overset{(r_1,s_1)}{\underset{(r_2,s_2)}{\rightrightarrows}} \mathcal{X}_2 \xrightarrow{(k,h)} \mathcal{Y}$$

commute, let $(\bar{k}, \bar{h}) : \mathcal{G} \to \mathcal{Y}$ be a relation pair which is defined as in the proof of Proposition 5.1.4. Then, since $\bar{k} = k$ and (k, h) satisfies (C1) and (C2), $(\bar{k}, \bar{h})$ satisfies (C1) and (C2). □

Thus we obtained the following result.

Theorem 5.2.4. CSpa *is cocomplete.*

5.2.2 Completeness

The main result of this section is as follows:

Theorem 5.2.5. **CSpa** *is complete.*

We show the above theorem by showing that **CSpa** has equalisers and arbitrary products. The construction of equalisers and products are given in the following two sections. By Proposition 2.3.62, the existence of these two suffices for completeness of **CSpa**. In the construction, we exploit the notion of an ideal point (cf. Definition 4.2.1). The idea is to lift the notion of a point of a space to that of an ideal point, i.e. we perform construction of equalisers and products well-known in **Top** in terms of concrete spaces and ideal points. However, since the class of ideal points of a concrete space is not known to form a set, the resulting object is usually a class-size concrete space. We deal with this difficulty by using the notion of a generalized geometric theory.

Equalisers

Proposition 5.2.6. **BP** *has equalisers for any parallel pair of morphisms.*

Proof. Given any pair of convergent relation pairs $\mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$, define a class E by

$$E = \{\alpha \in Pt(\mathcal{X}_1) \mid s_1\alpha \ = \ s_2\alpha\}.$$

Then, E is the class of models of the following theory of rank 2 over the set S_1.

$$\begin{aligned}&\left\{a \to \textstyle\bigwedge_{b \in s_1 a} \bigvee_{c \in s_2^- b} c \mid a \in S_1\right\} \cup \left\{a \to \textstyle\bigwedge_{b \in s_2 a} \bigvee_{c \in s_1^- b} c \mid a \in S_1\right\} \cup \left\{\textstyle\bigvee_{a \in S_1} a\right\} \\ \cup &\left\{\textstyle\bigwedge \{a,b\} \to \bigvee_{c \in (a \downarrow b)} c \mid a, b \in S_1\right\} \cup \left\{a \to \textstyle\bigvee_{x \in \mathsf{ext}\, a} \bigwedge_{c \in \Diamond x} c \mid a \in S_1\right\}.\end{aligned}$$

Hence, the class E has a generating subset G by Theorem 2.2.13. Define a basic pair and a relation pair as in

$$\mathcal{E} = (G, \Vdash_{\mathcal{E}}, S_1) \xrightarrow{(e, id_{S_1})} \mathcal{X}_1$$

where $\Vdash_{\mathcal{E}}$ and e are defined by

$$\begin{aligned}\alpha \Vdash_{\mathcal{E}} a &\iff a \in \alpha, \\ \alpha\, e\, x &\iff \Diamond x \subseteq \alpha\end{aligned}$$

for all $\alpha \in G$, $a \in S_1$ and $x \in X_1$. First, we show that $\mathcal{E}$ is a concrete space. Since $G \subseteq Pt(\mathcal{X}_1)$, $\mathcal{E}$ satisfies (B2). To see that $\mathcal{E}$ satisfies (B1), note that

$$(a \downarrow_1 b) \subseteq (a \downarrow_{\mathcal{E}} b) \qquad (5.2.2.1)$$

holds: if $c \in (a \downarrow b)$ and $\alpha \in \mathsf{ext}_{\mathcal{E}}\{c\}$, then $c \in \alpha = \mathcal{J}\,\alpha$ by (P3), and so $\mathsf{ext}\, c \between \mathsf{rest}\,\alpha$. Hence $\mathsf{ext}\, a \cup \mathsf{ext}\, b \between \mathsf{rest}\,\alpha$, and so $a, b \in \alpha$, i.e. $\alpha \in (a \downarrow_{\mathcal{E}} b)$. Thus, we have

$$\begin{aligned} \alpha \in \mathsf{ext}_{\mathcal{E}}\{a\} \cap \mathsf{ext}_{\mathcal{E}}\{b\} &\iff a, b \in \alpha \\ &\iff \alpha \between (\alpha \downarrow_1 b) \\ &\Longrightarrow \alpha \between (\alpha \downarrow_{\mathcal{E}} b) \\ &\iff \alpha \in \mathsf{ext}_{\mathcal{E}}\left(a \downarrow_{\mathcal{E}} b\right) \end{aligned}$$

for any $\alpha \in G$ and $a, b \in S_1$ by (P2) and (5.2.2.1). Therefore $\mathcal{E}$ satisfies (B1). Next, we show that (e, id_{S_1}) is a convergent relation pair. Since we have

$$\begin{aligned} \alpha\,(\Vdash_1 \circ e) &\iff (\exists x \in X_1)\, x \in \mathsf{ext}\, a \,\&\, \Diamond\, x \subseteq \alpha \\ &\iff a \in \mathcal{J}\,\alpha \\ &\iff \alpha\,(id_{S_1} \circ \in)\, a \end{aligned}$$

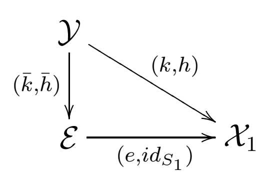
for any $\alpha \in G$ and $a \in S_1$, (e, id_{S_1}) is a relation pair, and it is straightforward to see that (e, id_{S_1}) satisfies (C1) and (C2) by the definition of $Pt(\mathcal{X}_1)$. We claim that $(e, id_{S_1}) : \mathcal{E} \to \mathcal{X}_1$ is an equaliser for (r_1, s_1) and (r_2, s_2). First, since the second component of (e, id_{S_1}) is id_{S_1}, the diagram $\mathcal{E} \xrightarrow{(e,id_{S_1})} \mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$ trivially commutes. Given any convergent relation pair $(k, h) : \mathcal{Y} \to \mathcal{X}_1$ which makes the diagram $\mathcal{Y} \xrightarrow{(k,h)} \mathcal{X}_1 \underset{(r_2,s_2)}{\overset{(r_1,s_1)}{\rightrightarrows}} \mathcal{X}_2$ commute, define a relation pair $(\bar{k}, \bar{h}) : \mathcal{Y} \to \mathcal{E}$ by

$$\begin{aligned} &y\,\bar{k}\,\alpha \iff \alpha \subseteq h\,\Diamond\{y\}, \\ &\bar{h} = h \end{aligned}$$

for all $y \in Y$ and $\alpha \in G$. Note that $h\,\Diamond\{y\} \in E$ for all $y \in Y$ by Proposition 4.3.5 and the commutativity of the diagram. Since G generates E, we have

$$\begin{aligned} y(\Vdash_{\mathcal{E}} \circ \bar{k})a &\iff (\exists \alpha \in G)\, \alpha \subseteq h\,\Diamond\{y\} \,\&\, a \in \alpha \\ &\iff a \in h\,\Diamond\{y\} \\ &\iff y\,(h \circ \Vdash_{\mathcal{Y}})\, a \end{aligned}$$

for any $y \in Y$ and $a \in S_1$, and so $(\bar{k}, \bar{h})$ is a relation pair. Convergence of $(\bar{k}, \bar{h})$ follows from convergence of (k, h) and 5.2.2.1. Finally, since the right component of $(e, id_{S_1}) \circ (\bar{k}, \bar{h})$ and (k, h) are equal, the diagram



commutes. The uniqueness of $(\bar{k}, \bar{h})$ follows from the fact that (e, id_{S_1}) is a monomorphism. □

Products

Proposition 5.2.7. CSpa *has small products.*

Proof. Let $(\mathcal{X}_i)_{i\in I}$ be a family of concrete spaces indexed by a set I. Define a class P by

$$P = \left\{\sum_{i\in I}\alpha(i) \subseteq \sum_{i\in I} S_i \mid \alpha \in \prod_{i\in I} Pt(\mathcal{X}_i)\right\}$$

Then, P is the class of models of the following theory of rank 1 over the set $\sum_{i\in I} S_i$.

$$\begin{aligned}
&\left\{\bigvee_{a\in S_1}(i,a) \mid i \in I\right\}\\
&\cup \left\{\bigwedge\{(i,a),(i,b)\} \to \bigvee_{c\in(a\downarrow_i b)}(i,c) \mid a,b,\in S_i, i\in I\right\}\\
&\cup \left\{(i,a) \to \bigvee_{x\in\mathsf{ext}_i\, a}\bigwedge_{c\in\Diamond_i x}(i,c) \mid a\in S_i, i\in I\right\}.
\end{aligned}$$

Hence, the class P has a generating subset G by Theorem 2.2.13. Define a basic pair $\prod_{i\in I}\mathcal{X}_i = (G, \Vdash_\Pi, S_\Pi)$ by

$$\begin{aligned}
S_\Pi &= \mathrm{Fin}\left(\sum_{i\in I} S_i\right),\\
\alpha \Vdash_\Pi A &\iff A \subseteq \alpha
\end{aligned}$$

for all $i\in I, \alpha\in G$ and $A\in S_\Pi$. Also, define a family $\left((r_i,s_i) : \prod_{i\in I} X_i \to \mathcal{X}_i\right)_{i\in I}$ of relation pairs by

$$\begin{aligned}
\alpha\, p_i\, x &\iff \Diamond_i\, x \in \alpha(i),\\
A\, q_i\, a &\iff (i,a)\in A
\end{aligned}$$

for all $i\in I, \alpha\in G, x\in X_i, A\in S_\Pi$ and $a\in S_i$. First, we see that $\prod_{i\in I}\mathcal{X}_i$ satisfies (B1) and (B2). Since $\emptyset\in S_\Pi$, we have $\alpha\Vdash_\Pi\emptyset$ for any $\alpha\in G$, and so $\prod_{i\in I}\mathcal{X}_i$ satisfies (B2). Also, since $A\cup B\in\{A\}\downarrow_\Pi\{B\}$ for any $A,B\in S_\Pi$, we have

$$\begin{aligned}
\alpha\in\mathsf{ext}_\Pi\{A\}\cap\mathsf{ext}_\Pi\{B\} &\iff A\cup B\subseteq\alpha\\
&\iff \alpha\in\mathsf{ext}_\Pi\{A\cup B\}\\
&\implies \alpha\in\mathsf{ext}_\Pi\left(\{A\}\downarrow_\Pi\{B\}\right)
\end{aligned}$$

for any $A,B\in S_\Pi$ and $\alpha\in G$, and hence $\prod_{i\in I}\mathcal{X}_i$ satisfies (B1). Next, we see that (p_i,q_i) is a convergent relation pair for all $i\in I$. Since we have

$$\begin{aligned}
\alpha\,(\Vdash_i\circ p_i)\,a &\iff (\exists x\in X_i)\,\Diamond_i\,x\subseteq\alpha(i)\,\&\,x\in\mathsf{ext}_i\,a\\
&\iff a\in\mathcal{J}_i\,\alpha(i)\\
&\iff \alpha\Vdash_\Pi\{(i,a)\}\\
&\iff (\exists A\in S_\Pi)\,\alpha\Vdash_\Pi A\,\&\,(i,a)\in A\\
&\iff \alpha\,(q_i\circ\Vdash_\Pi)\,a
\end{aligned}$$

for any $\alpha \in G$ and $a \in S_i$, (p_i, q_i) is a relation pair for each $i \in I$. To see that (p_i, q_i) is convergent, let $a, b \in S_i$ and $\alpha \in G$. Then

$$\begin{aligned} \alpha \in p_i^- \,\mathsf{ext}_i\{a\} \cap p_i^- \,\mathsf{ext}_i\{b\} &\iff a, b, \in \alpha(i) \\ &\iff \alpha(i) \between (a \downarrow b) \\ &\iff \alpha(i) \in p_i^- \,\mathsf{ext}_i\,(a \downarrow b)\,. \end{aligned}$$

Hence (p_i, q_i) satisfies (C1). The condition (C2) for (p_i, q_i) follows from (P1) and (P3) for $\alpha(i)$ for each $i \in I$.

Given any family $((u_i, v_i) : \mathcal{Y} \to \mathcal{X}_i)_{i \in I}$ of convergent relation pairs, where $\mathcal{Y} = (Y, \Vdash_{\mathcal{Y}}, T)$, define a relation pair $(k, h) : \mathcal{Y} \to \prod_{i \in I} \mathcal{X}_i$ by

$$\begin{aligned} y\, k\, \alpha &\iff \alpha \subseteq \textstyle\sum_{i \in I} v_i \diamond \{y\} \\ b\, h\, A &\iff \mathsf{ext}\{b\} \subseteq \textstyle\bigcap_{(i,a) \in A} \mathsf{ext}_{\mathcal{Y}}\, v_i^- \{a\} \end{aligned}$$

for all $y \in Y, \alpha \in G, b \in T$ and $A \in S_\Pi$. We show that (k, h) is a convergent relation pair. To this end, let $y \in Y$ and $A \in S_\Pi$. Then, we have

$$\begin{aligned} y\,(h \circ \Vdash_{\mathcal{Y}})\, A &\iff (\exists b \in T)\, y \in \mathsf{ext}\{b\} \,\&\, \mathsf{ext}\{b\} \subseteq \textstyle\bigcap_{(i,a) \in A} \mathsf{ext}_{\mathcal{Y}}\, v_i^- \{a\} \\ &\iff y \in \textstyle\bigcap_{(i,a) \in A} \mathsf{ext}_{\mathcal{Y}}\, v_i^- \{a\} \\ &\iff (\forall (i, a) \in A)\, a \in v_i \diamond \{y\} \\ &\iff A \subseteq \textstyle\sum_{i \in I} v_i \diamond \{y\}. \end{aligned}$$

Since $\sum_{i \in I} v_i \diamond \{y\} \in P$ by Proposition 4.3.5 and A is finitely enumerable, we have

$$\begin{aligned} y\,(h \circ \Vdash_{\mathcal{Y}})\, A &\iff A \subseteq \textstyle\sum_{i \in I} v_i \diamond \{y\} \\ &\iff (\exists \alpha \in G)\, A \subseteq \alpha \subseteq \textstyle\sum_{i \in I} v_i \diamond \{y\} \\ &\iff (\exists \alpha \in G)\, \alpha \Vdash_\Pi A \,\&\, \alpha\, k\, y \\ &\iff y\,(\Vdash_\Pi \circ k)\, A. \end{aligned}$$

Thus (k, h) is a relation pair. Since we have

$$\begin{aligned} &y \in \mathsf{ext}_{\mathcal{Y}}\, h^-\{A\} \cap \mathsf{ext}_{\mathcal{Y}}\, h^-\{B\} \\ \iff &y \in \left(\textstyle\bigcap_{(i,a) \in A} \mathsf{ext}_{\mathcal{Y}}\, v_i^- \{a\}\right) \cap \left(\textstyle\bigcap_{(j,b) \in B} \mathsf{ext}_{\mathcal{Y}}\, v_j^- \{b\}\right) \\ \iff &y \in \textstyle\bigcap_{(i,a) \in A \cup B} \mathsf{ext}_{\mathcal{Y}}\, v_i^- \{a\} \\ \iff &y \in \mathsf{ext}_{\mathcal{Y}}\, h^- \{A \cup B\} \\ \implies &y \in \mathsf{ext}_{\mathcal{Y}}\, h^- \left(A \downarrow_\Pi B\right) \end{aligned}$$

for any $A, B \in S_\Pi$ and $y \in Y$, (k, h) satisfies (C1). Also, for any $y \in Y$, since $\sum_{i \in I} v_i \diamond \{y\} \in P$, there exists $\alpha \in G$ such that $\alpha \subseteq \sum_{i \in I} v_i \diamond \{y\}$. Therefore $y\, k\, \alpha$, and hence (k, h) satisfies (C2). Since we have

$$\begin{aligned} y(\Vdash_i \circ p_i \circ k)a &\iff (\exists \alpha \in G)\, y\, k\, \alpha \,\&\, \alpha\, (\Vdash_i \circ p_i)\, a \\ &\iff (\exists \alpha \in G)\, (i, a) \in \alpha \subseteq \textstyle\sum_{i \in I} v_i \diamond \{y\} \\ &\iff (i, a) \in \textstyle\sum_{i \in I} v_i \diamond \{y\} \\ &\iff y\,(v_i \circ \Vdash_{\mathcal{Y}})\, a \end{aligned}$$

for any $y \in Y$ and $a \in S_i$, the diagram

$$\begin{array}{ccc} & & \mathcal{Y} \\ & \overset{(u_i,v_i)}{\swarrow} & \big\downarrow{\scriptstyle (k,h)} \\ \mathcal{X}_i & \underset{(p_i,q_i)}{\longleftarrow} & \prod_{i\in I} \mathcal{X}_i \end{array}$$

commutes for each $i \in I$. Finally, suppose that $(\bar{k}, \bar{h}) : \mathcal{Y} \to \prod_{i\in I} \mathcal{X}_i$ is a convergent relation pair which makes the above diagram commute. Since $(\bar{k}, \bar{h})$ satisfies (C1), we have

$$\begin{aligned} y(h\circ \Vdash_{\mathcal{Y}})A &\iff y \in \bigcap\nolimits_{(i,a)\in A} \mathsf{ext}_{\mathcal{Y}}\, v_i^{-}\{a\} \\ &\iff y \in \bigcap\nolimits_{(i,a)\in A} \bar{k}^{-} p_i^{-}\, \mathsf{ext}_i\{a\} \\ &\iff y \in \bar{k}^{-} \bigcap\nolimits_{(i,a)\in A} p_i^{-}\, \mathsf{ext}_i\{a\} \\ &\iff (\exists \alpha \in G)\, y\, \bar{k}\, \alpha \,\&\, (\forall (i,a) \in A)\, a \in \alpha(i) \\ &\iff (\exists \alpha \in G)\, y\, \bar{k}\, \alpha \,\&\, A \subseteq \alpha \\ &\iff y \left(\Vdash_{\Pi} \circ \bar{k}\right) A \end{aligned}$$

for any $y \in Y$ and $A \in S_{\Pi}$, and hence $(\bar{k}, \bar{h}) \sim (k, h)$. Therefore, (k, h) is a unique morphism in **CSpa** which makes the above diagram commute. □

As a corollary, **CSpa** has a terminal object, which is given by $\mathbf{1} = \left(\{*\}, id_{\{*\}}, \{*\}\right)$, where $\{*\}$ is any singleton.

Chapter 6

Concluding remarks

With completeness and cocompleteness of **BP** and **CSpa**, we can now undertake the actual development of general topology, for example the construction of product and quotient spaces, in the setting of basic pairs and concrete spaces that is analogous to the classical general topology. Currently, very little work has been done on the actual development of general topology using the notion of basic pair and concrete space; this must be done to test the validity of these notions.

Other aspects of basic pairs and concrete spaces which must be developed further are categorical relations between **BP** and **CSpa**, and their formal counterparts, basic topologies and formal topologies with binary positivity predicate (also called balanced formal topologies) respectively; see [23] for the notions of basic topology and formal topology with binary positivity predicate. It is well-known that there exist embeddings from **BP** and **CSpa** to those of basic topologies and balanced formal topologies respectively, however, it is not known whether there are functors from these formal counterparts to **BP** and **CSpa** which form adjoints with those embeddings. We even hope to see that those adjunctions restrict to equivalences between **BP** and **CSpa** and certain subcategories of basic topologies and balanced formal topologies respectively, as in the case of the categories of formal topologies and constructive topological spaces [4].

As the final remark, we present one way of viewing the notion of concrete space. Impredicatively, it can easily be seen that every concrete space $\mathcal{X}$ is isomorphic to a sober space, namely the space of its ideal points $(Pt(\mathcal{X}), \ni, S)$. Thus one can think of a concrete space as a predicative way to deal with such class-size sober concrete space whose class of points has a generating subset, namely a set $G \subseteq Pt(\mathcal{X})$ such that $(\forall \alpha \in Pt(\mathcal{X}))\,(\forall a \in S)\, a \in \alpha \to (\exists \beta \in G)\, a \in \beta \subseteq \alpha$. Indeed, one can easily see that every concrete space $\mathcal{X}$ is isomorphic to a concrete space $\widetilde{\mathcal{X}} = (\{\Diamond\, x \mid x \in X\}, \ni, S)$ and $\{\Diamond\, x \mid x \in X\}$ is a generating subset of $Pt(\mathcal{X})$, and that every convergent relation pair $(r, s) : \mathcal{X}_1 \to \mathcal{X}_2$ extends to a continuous class function $f : Pt(\mathcal{X}_1) \to Pt(\mathcal{X}_2)$ between sober spaces $(Pt(\mathcal{X}_1), \ni, S)$ and $(Pt(\mathcal{X}_2), \ni, S)$ given by $f(\alpha) = s\alpha$ for $\alpha \in Pt(\mathcal{X}_1)$. Conversely, with every class-size sober concrete space $\mathcal{X} = (X, \Vdash, S)$ where X is a class, S is a set and there is a generating subset G of $Pt(\mathcal{X})$, one can associate a concrete space $(G, \ni, S)$ whose space of ideal points is impredicatively isomorphic to $(X, \Vdash, S)$. Moreover, by identifying each sober space $(X, \Vdash, S)$ with the space of its ideal points, every continuous function $f : Pt(\mathcal{X}_1) \to Pt(\mathcal{X}_2)$ between sober spaces $(Pt(\mathcal{X}_1), \ni, S_1)$ and $(Pt(\mathcal{X}_2), \ni, S_2)$ with generating subsets $G_1 \subseteq Pt(\mathcal{X}_1)$ and $G_2 \subseteq Pt(\mathcal{X}_2)$ respectively gives rise to a convergent relation pair $(r, s) : (G_1, \ni, S_1) \to (G_2, \ni, S_2)$ where

$$
\begin{aligned}
a\, s\, b &\iff (\forall \alpha \in G_1)\, a \in \alpha \to b \in f(\alpha), \\
\alpha\, r\, \beta &\iff \beta \subseteq s\alpha
\end{aligned}
$$

for $a \in S_1, b \in S_2, \alpha \in G_1$ and $\beta \in G_2$. Thus, one can say that the theory of concrete space is a predicative theory of set-generated sober topological spaces.

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