A predicative completion of a uniform space

Josef Berger, Hajime Ishihara, Erik Palmgren
and Peter Schuster

November 9, 2010

Abstract

We give a predicative construction of a completion of a uniform space in the constructive Zermelo-Fraenkel set theory.

Keywords: constructive mathematics, uniform space, completion, constructive set theory.
2010 Mathematics Subject Classification: 03F65, 54E15.

1 Introduction

In [6, Problems 17 to 21 of Chapter 4], Bishop introduced a constructive concept of a uniform space with a set of pseudometrics, and showed basic theorems, such as, that arbitrary uniform space has a completion (the set of Cauchy filters); see also [7, Problems 22 to 26 of Chapter 4], and [8, 10] for Bishop's constructive mathematics. Although, apparently, Bishop did not actually say explicitly that the completion should have been constructed in this way, since we have to think of the *set* of Cauchy filters, the construction of a completion is problematic from a predicative point of view, such as in the constructive Zermelo-Fraenkel set theory (**CZF**), founded by Aczel [1, 2, 3], without the powerset axiom and the full separation axiom.

Schuster et al. [19] and Bridges and Vîţă [9] employed a set of entourages with an extra condition to define a uniformity. If the discrete uniformity on $\mathbf{R}$ were defined by a set D of pseudometrics, then there would exist $d_1, \ldots, d_n \in D$ and $\epsilon > 0$ such that $\sum_{k=1}^{n} d_k(x, y) < \epsilon$ implies $x = y$ for each

$x, y \in \mathbf{R}$, and hence we would have the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x, y \in \mathbf{R}[x = y \vee \neg(x = y)],$$

which is provably false both in intuitionistic mathematics and in constructive recursive mathematics. Therefore their approach seems more general than the approach with a set of pseudometrics by Bishop; see also a discussion in [6, Appendix A], and [16]. However their approach for uniform spaces has a problem from a predicative point of view, and the extra condition leads to a phenomenon that we find unsatisfactory: namely, that if the real line, taken with the discrete uniform structure, satisfies it, then one can derive the non-constructive principle WLPO; see [13, Remark 3.1].

In this paper, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in a subsystem $\mathbf{CZF}^-$ of the constructive set theory **CZF**; see [12] for a construction of a completion of a uniform space in terms of formal topology [17, 18].

There are other constructive treatments of uniformity: for example, see [11] for uniform spaces in formal topology; see also [4] for general topology and formal topology in **CZF**.

2 The constructive set theory CZF

The constructive set theory **CZF**, founded by Aczel [1, 2, 3], grew out of Myhill's constructive set theory [15] as a formal system for Bishop's constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory [14].

Definition 1. The language of **CZF** contains variables for sets, a constant ω, and the binary predicates $=$ and $\in$. The axioms and rules are the axioms and rules of intuitionistic predicate logic with equality, and the following set theoretic axioms:

1. **Extensionality:** $\forall a \forall b(\forall x(x \in a \Longleftrightarrow x \in b) \Longrightarrow a = b)$.

2. **Pairing:** $\forall a \forall b \exists c \forall x(x \in c \Longleftrightarrow x = a \vee x = b)$.

3. **Union:** $\forall a \exists b \forall x(x \in b \Longleftrightarrow \exists y \in a(x \in y))$.

4. **Restricted Separation:**

$$\forall a \exists b \forall x (x \in b \Longleftrightarrow x \in a \wedge \varphi(x))$$

for every *restricted* formula $\varphi(x)$, where a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

5. **Strong Collection:**

$$\forall a(\forall x \in a \exists y \varphi(x, y) \Longrightarrow \exists b(\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y)))$$

for every formula $\varphi(x, y)$.

6. **Subset Collection:**

$$\forall a \forall b \exists c \forall u(\forall x \in a \exists y \in b \varphi(x, y, u) \Longrightarrow \\ \exists d \in c(\forall x \in a \exists y \in d \varphi(x, y, u) \wedge \forall y \in d \exists x \in a \varphi(x, y, u)))$$

for every formula $\varphi(x, y, u)$.

7. **Infinity:**

$$(\omega 1) \qquad 0 \in \omega \wedge \forall x(x \in \omega \Longrightarrow x + 1 \in \omega),$$
$$(\omega 2) \qquad \forall y(0 \in y \wedge \forall x(x \in y \Longrightarrow x + 1 \in y) \Longrightarrow \omega \subseteq y),$$

where $x + 1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset = \{x \in \omega \mid \bot\}$.

8. **$\in$-Induction:**

$$(\mathrm{IND}_\in) \qquad \forall a(\forall x \in a \varphi(x) \Longrightarrow \varphi(a)) \Longrightarrow \forall a \varphi(a)$$

for every formula $\varphi(a)$.

A subsystem $\mathbf{CZF}^-$ is obtained by removing $\in$-Induction from $\mathbf{CZF}$. Let a and b be sets. Using Strong Collection, the *cartesian product* $a \times b$ of a and b consisting of the ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in $\mathbf{CZF}^-$. A *relation* r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is *total* (or is a *multivalued function*) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$. The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$, or more formally

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

A *function* from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$. The class of functions from a to b is denoted by b^a, or more formally

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \implies y = z).$$

In $\mathbf{CZF}^-$, we can prove

Fullness: $\forall a \forall b \exists c(c \subseteq \mathrm{mv}(a, b) \wedge \forall r \in \mathrm{mv}(a, b) \exists s \in c(s \subseteq r)),$

and, as a corollary, we see that b^a is a set, that is

Exponentiation: $\forall a \forall b \exists c \forall f(f \in c \Longleftrightarrow f \in b^a).$

For more details of **CZF**, see [5].

3 A completion of a uniform space

In this section, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in $\mathbf{CZF}^-$.

A *uniform space* $(X, \mathcal{U})$ is a pair of a set X and a set $\mathcal{U}$ of subsets of $X \times X$ such that

Ub1. $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U}(W \subseteq U \cap V),$

Ub2. $\forall U \in \mathcal{U}(\Delta \subseteq U),$

Ub3. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \subseteq U^{-1}),$

Ub4. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \circ V \subseteq U).$

Here $\Delta = \{(x, x) \mid x \in X\}$, and $U^{-1} = \{(x, y) \mid (y, x) \in U\}$ and $U \circ V = \{(x, z) \mid \exists y((x, y) \in V \wedge (y, z) \in U)\}$ for each $U, V \subseteq X \times X$. Note that $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. We set $U^0 = \Delta$ and $U^{n+1} = U^n \circ U$.

A uniform space $(X, \mathcal{U})$ is T_1 if

$$\forall x, y \in X[\forall U \in \mathcal{U}((x, y) \in U) \implies x = y].$$

Remark 2. Let D be a set of pseudometrics on a set X, and let $\mathcal{U}_D$ be the set of subsets of $X \times X$ of the form

$$U_{d_1,\ldots,d_n}(\epsilon) = \{(x,y) \in X \times X \mid \sum_{k=1}^{n} d_k(x,y) < \epsilon\},$$

where $d_1, \ldots, d_n \in D$ ($n \geq 0$) and $\epsilon > 0$. Then it is straightforward to see that the pair $(X, \mathcal{U}_D)$ forms a uniform space, and it is T_1 if

$$\forall x, y \in X[\forall d \in D(d(x,y) = 0) \Longrightarrow x = y].$$

Especially, for a metric space (X, d), the pair $(X, \mathcal{U}_d)$ forms a T_1 uniform space, where $\mathcal{U}_d = \{U_n \mid n \in \mathbf{N}\}$ and $U_n = \{(x,y) \in X \times X \mid d(x,y) < 2^{-n}\}$.

Let $\ll_n$ be a relation on $\mathcal{U}$ defined by

$$V \ll_n U \Leftrightarrow \exists W \in \mathcal{U}(V \subseteq W \cap W^{-1} \wedge W^n \subseteq U).$$

Lemma 3. *For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \ll_n U$, and if $V \ll_n U$, then $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq U$ for each $k_1, \ldots, k_n \in \{-1, 1\}$.*

Proof. Let $U \in \mathcal{U}$, and let m be a natural number with $n \leq 2^m$. Then, using (Ub4) m times, there exists $W \in \mathcal{U}$ such that $W^{2^m} \subseteq U$, and hence we have $W^n \subseteq W^{2^m} \subseteq U$, by using (Ub2) if necessary. There exists $W' \in \mathcal{U}$ such that $W' \subseteq W^{-1}$, by (Ub3), and hence there exists $V \in \mathcal{U}$ such that $V \subseteq W \cap W' \subseteq W \cap W^{-1}$, by (Ub1). If $V \ll_n U$, then there exists $W \in \mathcal{U}$ such that $V \subseteq W \cap W^{-1}$ and $W^n \subseteq U$, and therefore, since $V^k \subseteq W$ for each $k \in \{-1, 1\}$, we have $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq W^n \subseteq U$ for each $k_1, \ldots, k_n \in \{-1, 1\}$. □

A set $\mathcal{F}$ of subsets of X is a *filter* if

Fb1. $\forall A \in \mathcal{F} \exists x \in X(x \in A)$,

Fb2. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F}(C \subseteq A \cap B)$.

A filter $\mathcal{F}$ on X *converges* to x in X if for each $U \in \mathcal{U}$ there exists $A \in \mathcal{F}$ such that $A \subseteq U(x) = \{y \in X \mid (x,y) \in U\}$. A filter $\mathcal{F}$ on X is a *Cauchy filter* if

FbC. $\forall U \in \mathcal{U} \exists A \in \mathcal{F}(A \times A \subseteq U)$.

A uniform space $(X, \mathcal{U})$ is *complete* if every Cauchy filter on X converges.

Let $(X, \mathcal{U})$ be a T_1 uniform space. Then, since X and $\mathcal{U}$ are sets, by Fullness, there exists a set R such that $R \subseteq \mathrm{mv}(\mathcal{U}, X)$ and

$$\forall r \in \mathrm{mv}(\mathcal{U}, X) \exists s \in R (s \subseteq r). \tag{1}$$

Let φ be a restricted formula defined by

$$\varphi(r) \Leftrightarrow \forall U, V \in \mathcal{U} \forall x, y \in X[(U, x) \in r \wedge (V, y) \in r \implies (x, y) \in V^{-1} \circ U].$$

Note that

$$\varphi(r) \wedge s \subseteq r \implies \varphi(s). \tag{2}$$

Using Restricted separation, define a set $\widetilde{X}$ by

$$\widetilde{X} = \{r \in R \mid \varphi(r)\}.$$

For each $U \in \mathcal{U}$, define a subset $\widetilde{U}$ of $\widetilde{X} \times \widetilde{X}$, using Restricted Separation, as follows:

$$\widetilde{U} = \{(r, s) \mid \exists U_1, U_2 \in \mathcal{U} \exists x_1, x_2 \in X (U_1 \subseteq U \wedge U_2 \subseteq U \wedge (U_1, x_1) \in r \wedge (U_2, x_2) \in s \wedge (x_1, x_2) \in U)\}.$$

By Strong Collection, let

$$\widetilde{\mathcal{U}} = \{\widetilde{U} \mid U \in \mathcal{U}\}.$$

The equality $=_{\widetilde{X}}$ on $\widetilde{X}$ is defined by

$$r =_{\widetilde{X}} s \Leftrightarrow \forall \widetilde{U} \in \widetilde{\mathcal{U}}((r, s) \in \widetilde{U}).$$

Remark 4. We may think of a multivalued function $r \in \mathrm{mv}(\mathcal{U}, X)$ as a multivalued *net* in X indexed by the directed set $\mathcal{U}$, and the formula $\varphi(r)$ as expressing a *regularity* of r. Then the set $\widetilde{X}$ is a set of regular multivalued nets in X indexed by the specific directed set $\mathcal{U}$; a similar trick can be found in the proof that the class of points of a complete uniform formal topology is a set in [11]. If $\mathcal{U}$ is countable, then, in the presence of the axiom of countable choice, we may define $\widetilde{X}$ as the set of regular *sequences* (singlevalued functions on $\mathbf{N}$) in X. In the uniform space $(X, \mathcal{U}_d)$ induced by a metric space (X, d), each regular sequence $(x_n)_n$ in $(X, \mathcal{U}_d)$ is a regular sequence in the metric space (X, d) in the sense that

$$d(x_m, x_n) < 2^{-m} + 2^{-n}$$

for each $m, n \in \mathbf{N}$. On the other hand, for each regular sequence $(x_n)_n$ in (X, d), the sequence $(x_{n+1})_n$ is a regular sequence in $(X, \mathcal{U}_d)$.

Proposition 5. $(\widetilde{X}, \widetilde{\mathcal{U}})$ *is a* T_1 *uniform space.*

Proof. (Ub1): Let $U, V \in \mathcal{U}$. Then there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$, and it is straightforward to see that $\widetilde{W} \subseteq \widetilde{U} \cap \widetilde{V}$.

(Ub2): Let $U \in \mathcal{U}$ and $r \in \widetilde{X}$. Then, since $r \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(U, x) \in r$, and therefore, since $(x, x) \in U$, we have $(r, r) \in \widetilde{U}$.

(Ub3): Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \subseteq U^{-1}$, and it is straightforward to see that $\widetilde{V} \subseteq \widetilde{U}^{-1}$.

(Ub4): Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_4 U$, by Lemma 3. Let $(r, s) \in \widetilde{V}$ and $(s, t) \in \widetilde{V}$. Then there exist $V_1, V_2, W_1, W_2 \in \mathcal{U}$ and $x_1, x_2, y_1, y_2 \in X$ such that $V_1, V_2, W_1, W_2 \subseteq V$, $(V_1, x_1) \in r$, $(V_2, x_2) \in s$, $(W_1, y_1) \in s$, $(W_2, y_2) \in t$, $(x_1, x_2) \in V$ and $(y_1, y_2) \in V$. Since $(V_2, x_2) \in s$, $(W_1, y_1) \in s$ and $\varphi(s)$, we have $(x_2, y_1) \in W_1^{-1} \circ V_2$, and hence

$$(x_1, y_2) \in V \circ W_1^{-1} \circ V_2 \circ V \subseteq V \circ V^{-1} \circ V \circ V \subseteq U,$$

by Lemma 3. Therefore, since $V_1, W_2 \subseteq V \subseteq U$, we have $(r, t) \in \widetilde{U}$.

The uniform space $(\widetilde{X}, \widetilde{\mathcal{U}})$ is T_1 by the definition of equality. ☐

Let $\mathcal{F}$ be a Cauchy filter on $\widetilde{X}$. Define a subset r of $\mathcal{U} \times X$, by Restricted Separation, as follows:

$$r = \{(U, x) \mid \exists V \in \mathcal{U} \exists A \in \mathcal{F} \exists s \in A (V \ll_4 U \wedge A \times A \subseteq \widetilde{V} \wedge (V, x) \in s)\}.$$

Lemma 6. $r \in \mathrm{mv}(\mathcal{U}, X)$ *and* $\varphi(r)$.

Proof. Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_4 U$, by Lemma 3. Since $\mathcal{F}$ is a Cauchy filter, there exists $A \in \mathcal{F}$ such that $A \times A \subseteq \widetilde{V}$, by (FbC), and hence there exists $s \in A$, by (Fb1). Since $s \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(V, x) \in s$, and hence $(U, x) \in r$. Therefore $r \in \mathrm{mv}(\mathcal{U}, X)$.

Let $(U, x) \in r$ and $(V, y) \in r$. Then there exist $U_0, V_0 \in \mathcal{U}$, $A, B \in \mathcal{F}$, $s \in A$ and $s' \in B$ such that $U_0 \ll_4 U$, $V_0 \ll_4 V$, $A \times A \subseteq \widetilde{U_0}$, $B \times B \subseteq \widetilde{V_0}$, $(U_0, x) \in s$ and $(V_0, y) \in s'$. Since $\mathcal{F}$ is a filter, there exist $C \in \mathcal{F}$ and $t \in C$ such that $t \in C \subseteq A \cap B$, by (Fb2) and (Fb1). Since $(s, t) \in \widetilde{U_0}$ and $(s', t) \in \widetilde{V_0}$, there exist $U_1, U_2, V_1, V_2 \in \mathcal{U}$ and $x_1, x_2, y_1, y_2 \in X$ such that $U_1, U_2 \subseteq U_0$, $V_1, V_2 \subseteq V_0$, $(U_1, x_1) \in s$, $(U_2, x_2) \in t$, $(V_1, y_1) \in s'$, $(V_2, y_2) \in t$, $(x_1, x_2) \in U_0$ and $(y_1, y_2) \in V_0$. Since $(U_0, x), (U_1, x_1) \in s$, $(V_1, y_1), (V_0, y) \in s'$ and $(U_2, x_2), (V_2, y_2) \in t$, we have $(x, x_1) \in U_1^{-1} \circ U_0$, $(y_1, y) \in V_0^{-1} \circ V_1$, and $(x_2, y_2) \in V_2^{-1} \circ U_2$, and hence

$$
\begin{aligned}
(x, y) &\in V_0^{-1} \circ V_1 \circ V_0^{-1} \circ V_2^{-1} \circ U_2 \circ U_0 \circ U_1^{-1} \circ U_0 \\
&\subseteq (V_0 \circ V_0 \circ V_0^{-1} \circ V_0)^{-1} \circ (U_0 \circ U_0 \circ U_0^{-1} \circ U_0) \subseteq V^{-1} \circ U,
\end{aligned}
$$

by Lemma 3. Therefore $\varphi(r)$. □

By (1), there exists $r_{\mathcal{F}} \in R$ such that $r_{\mathcal{F}} \subseteq r$. Since $\varphi(r_{\mathcal{F}})$, by Lemma 6 and (2), we have $r_{\mathcal{F}} \in \widetilde{X}$.

Lemma 7. *$\mathcal{F}$ converges to $r_{\mathcal{F}}$.*

Proof. Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$ such that $V \ll_2 U$, and there exists $W \in \mathcal{U}$ such that $W \ll_4 V$, by Lemma 3. Since $\mathcal{F}$ is a Cauchy filter, there exists $A \in \mathcal{F}$ such that $A \times A \subseteq \widetilde{W}$. Let $s \in A$. Since $s \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x \in X$ such that $(W, x) \in s$, and hence $(V, x) \in r$. Since $r_{\mathcal{F}} \in \mathrm{mv}(\mathcal{U}, X)$, there exists $x' \in X$ such that $(V, x') \in r_{\mathcal{F}} \subseteq r$. Since $\varphi(r)$ by Lemma 6, we have $(x', x) \in V^{-1} \circ V \subseteq U$, and therefore, since $V, W \subseteq U$, we have $(r_{\mathcal{F}}, s) \in \widetilde{U}$. Thus $A \subseteq \widetilde{U}(r_{\mathcal{F}})$. □

Thus we have the following proposition.

Proposition 8. *$(\widetilde{X}, \widetilde{\mathcal{U}})$ is complete.*

For each $x \in X$, define a subset $\tilde{x}$ of $\mathcal{U} \times X$ by

$$\tilde{x} = \{(U, x) \mid U \in \mathcal{U}\}.$$

Then $\tilde{x}$ is a constant function on $\mathcal{U}$, and, since for each $(U, x), (V, x) \in \tilde{x}$, we have $(x, x) \in V^{-1} \circ U$, we have $\tilde{x} \in \widetilde{X}$.

Lemma 9. *For each $U \in \mathcal{U}$ and $x, y \in X$, $(x, y) \in U$ if and only if $(\tilde{x}, \tilde{y}) \in \widetilde{U}$.*

Proof. Since $(U,x) \in \tilde{x}$ and $(U,y) \in \tilde{y}$, if $(x,y) \in U$, then $(\tilde{x},\tilde{y}) \in \widetilde{U}$. If $(\tilde{x},\tilde{y}) \in \widetilde{U}$, then there exist $V, W \in \mathcal{U}$ such that $V, W \subseteq U$, $(V,x) \in \tilde{x}$, $(W,y) \in \tilde{y}$ and $(x,y) \in U$, and so $(x,y) \in U$. □

A mapping f between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{U}')$ is *uniformly continuous* if for each $V \in \mathcal{U}'$ there exists $U \in \mathcal{U}$ such that

$$(x,y) \in U \Longrightarrow (f(x),f(y)) \in V$$

for each $x, y \in X$.

Let i be the mapping from $(X,\mathcal{U})$ into $(\widetilde{X},\widetilde{\mathcal{U}})$ such that

$$i : x \mapsto \tilde{x}.$$

Thus, by Lemma 9, we immediately have the following proposition.

Proposition 10. *$i : (X,\mathcal{U}) \to (\widetilde{X},\widetilde{\mathcal{U}})$ is a uniformly continuous injection.*

Let $(Y,\mathcal{V})$ be a complete T_1 uniform space, and let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be uniformly continuous. Let $r \in \widetilde{X}$. For each $U \in \mathcal{U}$, define a subset A^r_U of Y by

$$A^r_U = \{f(x) \mid \exists V \in \mathcal{U}(V \subseteq U \wedge (V,x) \in r)\}.$$

By Strong Collection, let

$$\mathcal{F}_r = \{A^r_U \mid U \in \mathcal{U}\}.$$

Lemma 11. *$\mathcal{F}_r$ is a Cauchy filter on Y.*

Proof. For each $U \in \mathcal{U}$, since $(U,x) \in r$ for some $x \in X$, we have $f(x) \in A^r_U$. Since for each $U, V \in \mathcal{U}$ if $V \subseteq U$, then $A^r_V \subseteq A^r_U$, we have for each $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $A^r_W \subseteq A^r_U \cap A^r_V$ by (Ub1). Let $U \in \mathcal{V}$. Then, since f is uniformly continuous, there exists $V \in \mathcal{U}$ such that $(x,y) \in V \Longrightarrow (f(x),f(y)) \in U$ for each $x, y \in X$, and there exists $W \in \mathcal{U}$ such that $W \ll_2 V$. Suppose that $(f(x),f(y)) \in A^r_W \times A^r_W$. Then there exists $W_1, W_2 \in \mathcal{U}$ such that $W_1, W_2 \subseteq W$, $(W_1,x) \in r$ and $(W_2,y) \in r$, and hence $(x,y) \in W_2^{-1} \circ W_1 \subseteq W^{-1} \circ W \subseteq V$. Thus $(f(x),f(y)) \in U$. Therefore $A^r_W \times A^r_W \subseteq U$. □

Since $(Y,\mathcal{V})$ is complete, $\mathcal{F}_r$ converges to a point $\tilde{f}(r)$ in Y.

Lemma 12. *For each $U \in \mathcal{V}$ there exists $V \in \mathcal{U}$ such that*

$$(r, s) \in \widetilde{V} \Longrightarrow (\tilde{f}(r), \tilde{f}(s)) \in U$$

for each $r, s \in \widetilde{X}$.

Proof. Let $U \in \mathcal{V}$. Then there exists $U_0 \in \mathcal{V}$ such that $U_0 \ll_3 U$, and, since f is uniformly continuous, there exists $V_0 \in \mathcal{U}$ such that $(x, y) \in V_0 \Longrightarrow (f(x), f(y)) \in U_0$ for each $x, y \in X$. By Lemma 3, there exists $V \in \mathcal{U}$ such that $V \ll_5 V_0$. Suppose that $(r, s) \in \widetilde{V}$. Then there exist $V_1, V_2 \in \mathcal{U}$ and $x_1, x_2 \in X$ such that $V_1, V_2 \subseteq V$, $(V_1, x_1) \in r$, $(V_2, x_2) \in s$ and $(x_1, x_2) \in V$. Since $\mathcal{F}_r$ and $\mathcal{F}_s$ converge to $\tilde{f}(r)$ and $\tilde{f}(s)$, respectively, we can find $W \in \mathcal{U}$ such that $W \subseteq V$, $A^r_W \subseteq U_0(\tilde{f}(r))$ and $A^s_W \subseteq U_0(\tilde{f}(s))$, and, since A^r_W and A^s_W are inhabited, there exist $x, y \in X$ such that $f(x) \in A^r_W$ and $f(y) \in A^s_W$. Hence there exist $W_1, W_2 \in \mathcal{U}$ such that $W_1, W_2 \subseteq W$, $(W_1, x) \in r$ and $(W_2, y) \in s$. Since $(W_1, x), (V_1, x_1) \in r$ and $(V_2, x_2), (W_2, y) \in s$, we have $(x, x_1) \in V_1^{-1} \circ W_1$ and $(x_2, y) \in W_2^{-1} \circ V_2$, and therefore

$$\begin{aligned} (x, y) &\in W_2^{-1} \circ V_2 \circ V \circ V_1^{-1} \circ W_1 \subseteq W^{-1} \circ V \circ V \circ V^{-1} \circ W \\ &\subseteq V^{-1} \circ V \circ V \circ V^{-1} \circ V \subseteq V_0. \end{aligned}$$

Thus $(f(x), f(y)) \in U_0$. Since $(\tilde{f}(r), f(x)) \in U_0$ and $(\tilde{f}(s), f(y)) \in U_0$, we have $(\tilde{f}(r), \tilde{f}(s)) \in U_0^{-1} \circ U_0 \circ U_0 \subseteq U$. ☐

Since $(Y, \mathcal{V})$ is T_1, we have $\tilde{f}(r) = \tilde{f}(s)$ whenever $r =_{\widetilde{X}} s$, by Lemma 12. Hence $\tilde{f}$ is a function on $\widetilde{X}$, and it is uniformly continuous, by Lemma 12. Since $A^{\tilde{x}}_U = \{f(x)\}$ for each $U \in \mathcal{U}$, $\mathcal{F}_{\tilde{x}}$ converges to $f(x)$, and therefore we have the following lemma.

Lemma 13. $f = \tilde{f} \circ i$.

The function $\tilde{f}$ is unique in the following sense.

Lemma 14. *If $h : (\widetilde{X}, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ is uniformly continuous with $f = h \circ i$, then $h = \tilde{f}$.*

Proof. Let $r \in \widetilde{X}$, and let $U \in \mathcal{V}$. Then there exists $U_0 \in \mathcal{V}$ such that $U_0 \ll_2 U$, and since h is uniformly continuous, there exists $V \in \mathcal{U}$ such that $(s, t) \in \widetilde{V} \Longrightarrow (h(s), h(t)) \in U_0$ for each $s, t \in \widetilde{X}$. Since $\mathcal{F}_r$ converges to $\tilde{f}(r)$, we can find $W \in \mathcal{U}$ such that $W \subseteq V$ and $A^r_W \subseteq U_0(\tilde{f}(r))$, and, since A^r_W is inhabited, there exists $x \in X$ such that $f(x) \in A^r_W$. Hence there exists $W' \in \mathcal{U}$ such that $W' \subseteq W \subseteq V$ and $(W', x) \in r$, and therefore, since $(V, x) \in \tilde{x}$ and $(x, x) \in V$, we have $(r, \tilde{x}) \in \widetilde{V}$. Thus $(h(r), h(\tilde{x})) = (h(r), f(x)) \in U_0$. Since $(\tilde{f}(r), f(x)) \in U_0$, we have $(h(r), \tilde{f}(r)) \in U_0^{-1} \circ U_0 \subseteq U$. Therefore, since $(Y, \mathcal{V})$ is T_1, we have $h(r) = \tilde{f}(r)$. □

Now we have shown the following theorem.

Theorem 15. *Let $(Y, \mathcal{V})$ be a complete T_1 uniform space, and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly continuous. Then there exists a unique uniformly continuous $\tilde{f} : (\widetilde{X}, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ such that $f = \tilde{f} \circ i$.*

Remark 16. Let $\mathcal{F}$ be a Cauchy filter on a uniform space $(X, \mathcal{U})$, and let

$$r = \{(U, x) \mid \exists A \in \mathcal{F}(A \times A \subseteq U \wedge x \in A)\}.$$

Then $r \in \mathrm{mv}(\mathcal{U}, X)$ and $\varphi(r)$, and hence there exists $r_{\mathcal{F}} \in \widetilde{X}$ such that $r_{\mathcal{F}} \subseteq r$. On the other hand, for each $r \in \widetilde{X}$, let $\mathcal{F}_r = \{B^r_U \mid U \in \mathcal{U}\}$, where

$$B^r_U = \{x \mid \exists V \in \mathcal{U}(V \subseteq U \wedge (V, x) \in r)\}.$$

Then $\mathcal{F}_r$ is a Cauchy filter on $(X, \mathcal{U})$. In the presence of the powerset axiom, it is straightforward to show that these correspondences $r \mapsto \mathcal{F}_r$ and $\mathcal{F} \mapsto r_{\mathcal{F}}$ between the completion $(\widetilde{X}, \widetilde{\mathcal{U}})$ and the uniform space of the set of all Cauchy filters on $(X, \mathcal{U})$ (the completion of $(X, \mathcal{U})$ in the sense of Bishop) form a uniform isomorphism.

For a metric space (X, d), as mentioned in Remark 4, in the presence of the axiom of countable choice, there is a uniform isomorphism between the completion $(\widetilde{X}, \widetilde{\mathcal{U}_d})$ and Bishop's metric completion.

Acknowledgements. A part of research leading to this paper was done in February 2008 when Palmgren and Schuster were visiting Japan Advanced Institute of Science and Technology (JAIST), and Berger was staying at JAIST as a postdoctoral fellow of the Japan Society for the Promotion of Science (JSPS). Palmgren and Schuster are grateful to JAIST for supporting their visit, and Berger is grateful to JSPS for supporting his stay. Also Ishihara was supported by JSPS (Grant-in-Aid for Scientific Research (C) No.19500012), and Palmgren was supported by Swedish Research Council (VR). All authors are grateful to the anonymous referees, whose comments and suggestions were helpful for bringing this paper into its final form.

References

[1] Peter Aczel, *The type theoretic interpretation of constructive set theory*, In: A. Macintyre, L. Pacholski, J. Paris, eds., *Logic Colloquium '77*, North-Holland, Amsterdam, 1978, 55–66.

[2] Peter Aczel, *The type theoretic interpretation of constructive set theory: choice principles*, In: A.S. Troelstra, D. van Dalen, eds., *The L.E.J. Brouwer Centenary Symposium*, North-Holland, Amsterdam, 1982, 1–40.

[3] Peter Aczel, *The type theoretic interpretation of constructive set theory: inductive definitions*, In: R.B. Marcus et al. eds., *Logic, Methodology, and Philosophy of Science VII*, North-Holland, Amsterdam, 1986, 17–49.

[4] Peter Aczel, *Aspects of general topology in constructive set theory*, Ann. Pure Appl. Logic **137** (2006), 3–29.

[5] Peter Aczel and Michael Rathjen, *Notes on constructive set theory*, Report No. 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.

[6] Errett Bishop, *Foundations of Constructive Mathematics*, McGraw-Hill, New York, 1967.

[7] Errett Bishop and Douglas Bridges, *Constructive Analysis*, Springer, Berlin, 1985.

[8] Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, Cambridge University Press, Cambridge, 1987.

[9] Douglas Bridges and Luminiţa Vîţă, *Strong and uniform continuity – the uniform space case*, LMS J. Comput. Math. **6** (2003), 326–334.

[10] Douglas Bridges and Luminiţa Vîţă, *Techniques of Constructive Analysis*, Springer, New York, 2006.

[11] Giovanni Curi, *On the collection of points of a formal space*, Ann. Pure Appl. Logic **137** (2006), 126–146.

[12] Christopher Fox, *Point-Set and Point-Free Topology in Constructive Set Theory*, Dissertation, University of Manchester, 2005.

[13] Hajime Ishihara, *Two subcategories of apartness spaces*, to appear in Ann. Pure Appl. Logic.

[14] Per Martin-Löf, *Intuitionistic Type Theory*, Studies in Proof Theory **1**, Bibliopolis, Naples, 1984.

[15] J. Myhill, *Constructive set theory*, J. Symbolic Logic **40** (1975), 347–382.

[16] Fred Richman, *Uniform spaces*, unpublished note, 8 July, 2002.

[17] Giovanni Sambin, *Intuitionistic formal spaces – a first communication*, In: D. Skordev ed., *Mathematical logic and its applications*, Plenum, New York, 1987, 187–204.

[18] Giovanni Sambin, *Some points in formal topology*, Theoret. Comput. Sci. **305** (2003), 347–408.

[19] Peter Schuster, Luminiţa Vîţă and Douglas Bridges, *Apartness as a relation between subsets*, In: C.S. Calude, M.J. Dinneen and S. Sburlan eds., *Combinatorics, Computability and Logic*, Springer, London, 2001, 203–214.

Josef Berger
Mathematisches Institut
Ludwig-Maximilians Universität München
Theresienstr. 39, D-80333 München, Germany
E-mail: `Josef.Berger@mathematik.uni-muenchen.de`

Hajime Ishihara
School of Information Science
Japan Advanced Institute of Science and Technology
Nomi, Ishikawa 923-1292, Japan
E-mail: `ishihara@jaist.ac.jp`
Tel: +81-761-51-1206
Fax: +81-761-51-1149

Erik Palmgren
Department of Mathematics

Uppsala University
PO Box 480, SE-751 06 Uppsala, Sweden
E-mail: palmgren@math.uu.se

Peter Schuster
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT, England, UK
E-mail: pschust@maths.leeds.ac.uk