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Author(s)	Berger, Josef; Ishihara, Hajime; Palmgren, Erik; Schuster, Peter
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Japan Advanced Institute of Science and Technology

# A predicative completion of a uniform space

Josef Berger, Hajime Ishihara, Erik Palmgren and Peter Schuster

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### Abstract

We give a predicative construction of a completion of a uniform space in the constructive Zermelo-Fraenkel set theory.

*Keywords:* constructive mathematics, uniform space, completion, constructive set theory.

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# 1 Introduction

In [6, Problems 17 to 21 of Chapter 4], Bishop introduced a constructive concept of a uniform space with a set of pseudometrics, and showed basic theorems, such as, that arbitrary uniform space has a completion (the set of Cauchy filters); see also [7, Problems 22 to 26 of Chapter 4], and [8, 10] for Bishop's constructive mathematics. Although, apparently, Bishop did not actually say explicitly that the completion should have been constructed in this way, since we have to think of the *set* of Cauchy filters, the construction of a completion is problematic from a predicative point of view, such as in the constructive Zermelo-Fraenkel set theory (**CZF**), founded by Aczel [1, 2, 3], without the powerset axiom and the full separation axiom.

Schuster et al. [19] and Bridges and Vîţă [9] employed a set of entourages with an extra condition to define a uniformity. If the discrete uniformity on **R** were defined by a set D of pseudometrics, then there would exist  $d_1, \ldots, d_n \in D$  and  $\epsilon > 0$  such that  $\sum_{k=1}^n d_k(x, y) < \epsilon$  implies x = y for each  $x, y \in \mathbf{R}$ , and hence we would have the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x, y \in \mathbf{R}[x = y \lor \neg (x = y)],$$

which is provably false both in intuitionistic mathematics and in constructive recursive mathematics. Therefore their approach seems more general than the approach with a set of pseudometrics by Bishop; see also a discussion in [6, Appendix A], and [16]. However their approach for uniform spaces has a problem from a predicative point of view, and the extra condition leads to a phenomenon that we find unsatisfactory: namely, that if the real line, taken with the discrete uniform structure, satisfies it, then one can derive the non-constructive principle WLPO; see [13, Remark 3.1].

In this paper, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in a subsystem  $\mathbf{CZF}^-$  of the constructive set theory  $\mathbf{CZF}$ ; see [12] for a construction of a completion of a uniform space in terms of formal topology [17, 18].

There are other constructive treatments of uniformity: for example, see [11] for uniform spaces in formal topology; see also [4] for general topology and formal topology in  $\mathbf{CZF}$ .

# 2 The constructive set theory CZF

The constructive set theory **CZF**, founded by Aczel [1, 2, 3], grew out of Myhill's constructive set theory [15] as a formal system for Bishop's constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory [14].

**Definition 1.** The language of **CZF** contains variables for sets, a constant  $\omega$ , and the binary predicates = and  $\in$ . The axioms and rules are the axioms and rules of intuitionistic predicate logic with equality, and the following set theoretic axioms:

- 1. Extensionality:  $\forall a \forall b (\forall x (x \in a \iff x \in b) \implies a = b).$
- 2. Pairing:  $\forall a \forall b \exists c \forall x (x \in c \iff x = a \lor x = b).$
- 3. Union:  $\forall a \exists b \forall x (x \in b \iff \exists y \in a (x \in y)).$

#### 4. Restricted Separation:

$$\forall a \exists b \forall x (x \in b \Longleftrightarrow x \in a \land \varphi(x))$$

for every restricted formula  $\varphi(x)$ , where a formula  $\varphi(x)$  is restricted, or  $\Delta_0$ , if all the quantifiers occurring in it are bounded, i.e. of the form  $\forall x \in c \text{ or } \exists x \in c.$ 

#### 5. Strong Collection:

 $\forall a (\forall x \in a \exists y \varphi(x, y) \Longrightarrow \exists b (\forall x \in a \exists y \in b \varphi(x, y) \land \forall y \in b \exists x \in a \varphi(x, y)))$ 

for every formula  $\varphi(x, y)$ .

#### 6. Subset Collection:

$$\forall a \forall b \exists c \forall u (\forall x \in a \exists y \in b \varphi(x, y, u) \Longrightarrow \\ \exists d \in c (\forall x \in a \exists y \in d \varphi(x, y, u) \land \forall y \in d \exists x \in a \varphi(x, y, u)))$$

for every formula  $\varphi(x, y, u)$ .

### 7. Infinity:

$$\begin{array}{ll} (\omega 1) & 0 \in \omega \land \forall x (x \in \omega \Longrightarrow x + 1 \in \omega), \\ (\omega 2) & \forall y (0 \in y \land \forall x (x \in y \Longrightarrow x + 1 \in y) \Longrightarrow \omega \subseteq y), \end{array}$$

where x + 1 is  $x \cup \{x\}$ , and 0 is the empty set  $\emptyset = \{x \in \omega \mid \bot\}$ .

### 8. $\in$ -Induction:

$$(IND_{\epsilon}) \qquad \forall a (\forall x \in a\varphi(x) \Longrightarrow \varphi(a)) \Longrightarrow \forall a\varphi(a)$$

for every formula  $\varphi(a)$ .

A subsystem  $\mathbb{CZF}^-$  is obtained by removing  $\in$ -Induction from  $\mathbb{CZF}$ . Let a and b be sets. Using Strong Collection, the cartesian product  $a \times b$  of a and b consisting of the ordered pairs  $(x, y) = \{\{x\}, \{x, y\}\}$  with  $x \in a$  and  $y \in b$  can be introduced in  $\mathbb{CZF}^-$ . A relation r between a and b is a subset of  $a \times b$ . A relation  $r \subseteq a \times b$  is total (or is a multivalued function) if for every  $x \in a$  there exists  $y \in b$  such that  $(x, y) \in r$ . The class of total relations between a and b is denoted by  $\mathrm{mv}(a, b)$ , or more formally

$$r \in mv(a, b) \Leftrightarrow r \subseteq a \times b \land \forall x \in a \exists y \in b((x, y) \in r).$$

A function from a to b is a total relation  $f \subseteq a \times b$  such that for every  $x \in a$  there is exactly one  $y \in b$  with  $(x, y) \in f$ . The class of functions from a to b is denoted by  $b^a$ , or more formally

 $f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \land \forall x \in a \forall y, z \in b((x, y) \in f \land (x, z) \in f \Longrightarrow y = z).$ 

In  $\mathbf{CZF}^-$ , we can prove

**Fullness:**  $\forall a \forall b \exists c (c \subseteq mv(a, b) \land \forall r \in mv(a, b) \exists s \in c(s \subseteq r)),$ 

and, as a corollary, we see that  $b^a$  is a set, that is

**Exponentiation:**  $\forall a \forall b \exists c \forall f (f \in c \iff f \in b^a).$ 

For more details of  $\mathbf{CZF}$ , see [5].

# **3** A completion of a uniform space

In this section, we define a notion of a uniform space using a base of uniformity as in [13], and construct a completion of a uniform space in  $\mathbf{CZF}^-$ .

A uniform space  $(X, \mathcal{U})$  is a pair of a set X and a set  $\mathcal{U}$  of subsets of  $X \times X$  such that

- Ub1.  $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U}(W \subseteq U \cap V),$
- Ub2.  $\forall U \in \mathcal{U}(\Delta \subseteq U),$
- Ub3.  $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \subseteq U^{-1}),$
- Ub4.  $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \circ V \subseteq U).$

Here  $\Delta = \{(x, x) \mid x \in X\}$ , and  $U^{-1} = \{(x, y) \mid (y, x) \in U\}$  and  $U \circ V = \{(x, z) \mid \exists y((x, y) \in V \land (y, z) \in U)\}$  for each  $U, V \subseteq X \times X$ . Note that  $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$ . We set  $U^0 = \Delta$  and  $U^{n+1} = U^n \circ U$ .

A uniform space  $(X, \mathcal{U})$  is  $T_1$  if

$$\forall x, y \in X [\forall U \in \mathcal{U}((x, y) \in U) \Longrightarrow x = y].$$

Remark 2. Let D be a set of pseudometrics on a set X, and let  $\mathcal{U}_D$  be the set of subsets of  $X \times X$  of the form

$$U_{d_1,...,d_n}(\epsilon) = \{(x,y) \in X \times X \mid \sum_{k=1}^n d_k(x,y) < \epsilon\},\$$

where  $d_1, \ldots, d_n \in D$   $(n \ge 0)$  and  $\epsilon > 0$ . Then it is straightforward to see that the pair  $(X, \mathcal{U}_D)$  forms a uniform space, and it is  $T_1$  if

$$\forall x, y \in X [\forall d \in D(d(x, y) = 0) \Longrightarrow x = y].$$

Especially, for a metric space (X, d), the pair  $(X, \mathcal{U}_d)$  forms a  $T_1$  uniform space, where  $\mathcal{U}_d = \{U_n \mid n \in \mathbf{N}\}$  and  $U_n = \{(x, y) \in X \times X \mid d(x, y) < 2^{-n}\}.$ 

Let  $\ll_n$  be a relation on  $\mathcal{U}$  defined by

$$V \ll_n U \Leftrightarrow \exists W \in \mathcal{U}(V \subseteq W \cap W^{-1} \land W^n \subseteq U).$$

**Lemma 3.** For each  $U \in \mathcal{U}$  there exists  $V \in \mathcal{U}$  such that  $V \ll_n U$ , and if  $V \ll_n U$ , then  $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq U$  for each  $k_1, \ldots, k_n \in \{-1, 1\}$ .

Proof. Let  $U \in \mathcal{U}$ , and let m be a natural number with  $n \leq 2^m$ . Then, using (Ub4) m times, there exists  $W \in \mathcal{U}$  such that  $W^{2^m} \subseteq U$ , and hence we have  $W^n \subseteq W^{2^m} \subseteq U$ , by using (Ub2) if necessary. There exists  $W' \in \mathcal{U}$ such that  $W' \subseteq W^{-1}$ , by (Ub3), and hence there exists  $V \in \mathcal{U}$  such that  $V \subseteq W \cap W' \subseteq W \cap W^{-1}$ , by (Ub1). If  $V \ll_n U$ , then there exists  $W \in \mathcal{U}$ such that  $V \subseteq W \cap W^{-1}$  and  $W^n \subseteq U$ , and therefore, since  $V^k \subseteq W$  for each  $k \in \{-1, 1\}$ , we have  $V^{k_1} \circ \ldots \circ V^{k_n} \subseteq W^n \subseteq U$  for each  $k_1, \ldots, k_n \in \{-1, 1\}$ .

A set  $\mathcal{F}$  of subsets of X is a *filter* if

- Fb1.  $\forall A \in \mathcal{F} \exists x \in X (x \in A),$
- Fb2.  $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F}(C \subseteq A \cap B).$

A filter  $\mathcal{F}$  on X converges to x in X if for each  $U \in \mathcal{U}$  there exists  $A \in \mathcal{F}$  such that  $A \subseteq U(x) = \{y \in X \mid (x, y) \in U\}$ . A filter  $\mathcal{F}$  on X is a Cauchy filter if

FbC.  $\forall U \in \mathcal{U} \exists A \in \mathcal{F}(A \times A \subseteq U).$ 

A uniform space  $(X, \mathcal{U})$  is *complete* if every Cauchy filter on X converges.

Let  $(X, \mathcal{U})$  be a  $T_1$  uniform space. Then, since X and  $\mathcal{U}$  are sets, by Fullness, there exists a set R such that  $R \subseteq mv(\mathcal{U}, X)$  and

$$\forall r \in \mathrm{mv}(\mathcal{U}, X) \exists s \in R(s \subseteq r).$$
(1)

Let  $\varphi$  be a restricted formula defined by

$$\varphi(r) \Leftrightarrow \forall U, V \in \mathcal{U} \forall x, y \in X[(U, x) \in r \land (V, y) \in r \Longrightarrow (x, y) \in V^{-1} \circ U].$$

Note that

$$\varphi(r) \land s \subseteq r \Longrightarrow \varphi(s). \tag{2}$$

Using Restricted separation, define a set  $\widetilde{X}$  by

$$\widetilde{X} = \{ r \in R \mid \varphi(r) \}.$$

For each  $U \in \mathcal{U}$ , define a subset  $\widetilde{U}$  of  $\widetilde{X} \times \widetilde{X}$ , using Restricted Separation, as follows:

$$\widetilde{U} = \{ (r, s) \mid \exists U_1, U_2 \in \mathcal{U} \exists x_1, x_2 \in X (U_1 \subseteq U \land U_2 \subseteq U \land (U_1, x_1) \in r \land (U_2, x_2) \in s \land (x_1, x_2) \in U) \}.$$

By Strong Collection, let

$$\widetilde{\mathcal{U}} = \{ \widetilde{U} \mid U \in \mathcal{U} \}.$$

The equality  $=_{\widetilde{X}}$  on  $\widetilde{X}$  is defined by

$$r =_{\widetilde{X}} s \Leftrightarrow \forall \widetilde{U} \in \widetilde{\mathcal{U}}((r,s) \in \widetilde{U}).$$

Remark 4. We may think of a multivalued function  $r \in \operatorname{mv}(\mathcal{U}, X)$  as a multivalued *net* in X indexed by the directed set  $\mathcal{U}$ , and the formula  $\varphi(r)$  as expressing a *regularity* of r. Then the set  $\widetilde{X}$  is a set of regular multivalued nets in X indexed by the specific directed set  $\mathcal{U}$ ; a similar trick can be found in the proof that the class of points of a complete uniform formal topology is a set in [11]. If  $\mathcal{U}$  is countable, then, in the presence of the axiom of countable choice, we may define  $\widetilde{X}$  as the set of regular *sequences* (singlevalued functions on **N**) in X. In the uniform space  $(X, \mathcal{U}_d)$  induced by a metric space (X, d), each regular sequence  $(x_n)_n$  in  $(X, \mathcal{U}_d)$  is a regular sequence in the metric space (X, d) in the sense that

$$d(x_m, x_n) < 2^{-m} + 2^{-n}$$

for each  $m, n \in \mathbf{N}$ . On the other hand, for each regular sequence  $(x_n)_n$  in (X, d), the sequence  $(x_{n+1})_n$  is a regular sequence in  $(X, \mathcal{U}_d)$ .

**Proposition 5.**  $(\widetilde{X}, \widetilde{\mathcal{U}})$  is a  $T_1$  uniform space.

*Proof.* (Ub1): Let  $U, V \in \mathcal{U}$ . Then there exists  $W \in \mathcal{U}$  such that  $W \subseteq U \cap V$ , and it is straightforward to see that  $\widetilde{W} \subseteq \widetilde{U} \cap \widetilde{V}$ .

(Ub2): Let  $U \in \mathcal{U}$  and  $r \in \widetilde{X}$ . Then, since  $r \in \operatorname{mv}(\mathcal{U}, X)$ , there exists  $x \in X$  such that  $(U, x) \in r$ , and therefore, since  $(x, x) \in U$ , we have  $(r, r) \in \widetilde{U}$ .

(Ub3): Let  $U \in \mathcal{U}$ . Then there exists  $V \in \mathcal{U}$  such that  $V \subseteq U^{-1}$ , and it is straightforward to see that  $\widetilde{V} \subseteq \widetilde{U}^{-1}$ .

(Ub4): Let  $U \in \mathcal{U}$ . Then there exists  $V \in \mathcal{U}$  such that  $V \ll_4 U$ , by Lemma 3. Let  $(r, s) \in \widetilde{V}$  and  $(s, t) \in \widetilde{V}$ . Then there exist  $V_1, V_2, W_1, W_2 \in \mathcal{U}$ and  $x_1, x_2, y_1, y_2 \in X$  such that  $V_1, V_2, W_1, W_2 \subseteq V$ ,  $(V_1, x_1) \in r$ ,  $(V_2, x_2) \in s$ ,  $(W_1, y_1) \in s$ ,  $(W_2, y_2) \in t$ ,  $(x_1, x_2) \in V$  and  $(y_1, y_2) \in V$ . Since  $(V_2, x_2) \in s$ ,  $(W_1, y_1) \in s$  and  $\varphi(s)$ , we have  $(x_2, y_1) \in W_1^{-1} \circ V_2$ , and hence

$$(x_1, y_2) \in V \circ W_1^{-1} \circ V_2 \circ V \subseteq V \circ V^{-1} \circ V \circ V \subseteq U,$$

by Lemma 3. Therefore, since  $V_1, W_2 \subseteq V \subseteq U$ , we have  $(r, t) \in \widetilde{U}$ .

The uniform space  $(\widetilde{X}, \widetilde{\mathcal{U}})$  is  $T_1$  by the definition of equality.

Let  $\mathcal{F}$  be a Cauchy filter on  $\widetilde{X}$ . Define a subset r of  $\mathcal{U} \times X$ , by Restricted Separation, as follows:

$$r = \{ (U, x) \mid \exists V \in \mathcal{U} \exists A \in \mathcal{F} \exists s \in A (V \ll_4 U \land A \times A \subseteq \widetilde{V} \land (V, x) \in s) \}.$$

**Lemma 6.**  $r \in mv(\mathcal{U}, X)$  and  $\varphi(r)$ .

*Proof.* Let  $U \in \mathcal{U}$ . Then there exists  $V \in \mathcal{U}$  such that  $V \ll_4 U$ , by Lemma 3. Since  $\mathcal{F}$  is a Cauchy filter, there exists  $A \in \mathcal{F}$  such that  $A \times A \subseteq \widetilde{V}$ , by (FbC), and hence there exists  $s \in A$ , by (Fb1). Since  $s \in \operatorname{mv}(\mathcal{U}, X)$ , there exists  $x \in X$  such that  $(V, x) \in s$ , and hence  $(U, x) \in r$ . Therefore  $r \in \operatorname{mv}(\mathcal{U}, X)$ .

Let  $(U, x) \in r$  and  $(V, y) \in r$ . Then there exist  $U_0, V_0 \in \mathcal{U}$ ,  $A, B \in \mathcal{F}$ ,  $s \in A$  and  $s' \in B$  such that  $U_0 \ll_4 U$ ,  $V_0 \ll_4 V$ ,  $A \times A \subseteq \widetilde{U_0}$ ,  $B \times B \subseteq \widetilde{V_0}$ ,  $(U_0, x) \in s$  and  $(V_0, y) \in s'$ . Since  $\mathcal{F}$  is a filter, there exist  $C \in \mathcal{F}$  and  $t \in C$  such that  $t \in C \subseteq A \cap B$ , by (Fb2) and (Fb1). Since  $(s, t) \in \widetilde{U_0}$  and  $(s', t) \in \widetilde{V_0}$ , there exist  $U_1, U_2, V_1, V_2 \in \mathcal{U}$  and  $x_1, x_2, y_1, y_2 \in X$  such that  $U_1, U_2 \subseteq U_0, V_1, V_2 \subseteq V_0, (U_1, x_1) \in s, (U_2, x_2) \in t, (V_1, y_1) \in s', (V_2, y_2) \in t,$   $(x_1, x_2) \in U_0$  and  $(y_1, y_2) \in V_0$ . Since  $(U_0, x), (U_1, x_1) \in s, (V_1, y_1), (V_0, y) \in s'$ and  $(U_2, x_2), (V_2, y_2) \in t$ , we have  $(x, x_1) \in U_1^{-1} \circ U_0, (y_1, y) \in V_0^{-1} \circ V_1$ , and  $(x_2, y_2) \in V_2^{-1} \circ U_2$ , and hence

$$\begin{array}{rcl} (x,y) & \in & V_0^{-1} \circ V_1 \circ V_0^{-1} \circ V_2^{-1} \circ U_2 \circ U_0 \circ U_1^{-1} \circ U_0 \\ & \subseteq & (V_0 \circ V_0 \circ V_0^{-1} \circ V_0)^{-1} \circ (U_0 \circ U_0 \circ U_0^{-1} \circ U_0) \subseteq V^{-1} \circ U, \end{array}$$

by Lemma 3. Therefore  $\varphi(r)$ .

By (1), there exists  $r_{\mathcal{F}} \in R$  such that  $r_{\mathcal{F}} \subseteq r$ . Since  $\varphi(r_{\mathcal{F}})$ , by Lemma 6 and (2), we have  $r_{\mathcal{F}} \in \widetilde{X}$ .

### **Lemma 7.** $\mathcal{F}$ converges to $r_{\mathcal{F}}$ .

Proof. Let  $U \in \mathcal{U}$ . Then there exists  $V \in \mathcal{U}$  such that  $V \ll_2 U$ , and there exists  $W \in \mathcal{U}$  such that  $W \ll_4 V$ , by Lemma 3. Since  $\mathcal{F}$  is a Cauchy filter, there exists  $A \in \mathcal{F}$  such that  $A \times A \subseteq \widetilde{W}$ . Let  $s \in A$ . Since  $s \in \operatorname{mv}(\mathcal{U}, X)$ , there exists  $x \in X$  such that  $(W, x) \in s$ , and hence  $(V, x) \in r$ . Since  $r_{\mathcal{F}} \in \operatorname{mv}(\mathcal{U}, X)$ , there exists  $x' \in X$  such that  $(V, x') \in r_{\mathcal{F}} \subseteq r$ . Since  $\varphi(r)$  by Lemma 6, we have  $(x', x) \in V^{-1} \circ V \subseteq U$ , and therefore, since  $V, W \subseteq U$ , we have  $(r_{\mathcal{F}}, s) \in \widetilde{U}$ . Thus  $A \subseteq \widetilde{U}(r_{\mathcal{F}})$ .

Thus we have the following proposition.

## **Proposition 8.** $(\widetilde{X}, \widetilde{\mathcal{U}})$ is complete.

For each  $x \in X$ , define a subset  $\tilde{x}$  of  $\mathcal{U} \times X$  by

$$\tilde{x} = \{ (U, x) \mid U \in \mathcal{U} \}.$$

Then  $\tilde{x}$  is a constant function on  $\mathcal{U}$ , and, since for each  $(U, x), (V, x) \in \tilde{x}$ , we have  $(x, x) \in V^{-1} \circ U$ , we have  $\tilde{x} \in \tilde{X}$ .

**Lemma 9.** For each  $U \in \mathcal{U}$  and  $x, y \in X$ ,  $(x, y) \in U$  if and only if  $(\tilde{x}, \tilde{y}) \in \widetilde{U}$ .

Proof. Since  $(U, x) \in \tilde{x}$  and  $(U, y) \in \tilde{y}$ , if  $(x, y) \in U$ , then  $(\tilde{x}, \tilde{y}) \in \tilde{U}$ . If  $(\tilde{x}, \tilde{y}) \in \tilde{U}$ , then there exist  $V, W \in \mathcal{U}$  such that  $V, W \subseteq U$ ,  $(V, x) \in \tilde{x}$ ,  $(W, y) \in \tilde{y}$  and  $(x, y) \in U$ , and so  $(x, y) \in U$ .

A mapping f between uniform spaces  $(X, \mathcal{U})$  and  $(Y, \mathcal{U}')$  is uniformly continuous if for each  $V \in \mathcal{U}'$  there exists  $U \in \mathcal{U}$  such that

$$(x, y) \in U \Longrightarrow (f(x), f(y)) \in V$$

for each  $x, y \in X$ .

Let *i* be the mapping from  $(X, \mathcal{U})$  into  $(\widetilde{X}, \widetilde{\mathcal{U}})$  such that

 $i: x \mapsto \tilde{x}.$ 

Thus, by Lemma 9, we immediately have the following proposition.

**Proposition 10.**  $i: (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$  is a uniformly continuous injection.

Let  $(Y, \mathcal{V})$  be a complete  $T_1$  uniform space, and let  $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly continuous. Let  $r \in \widetilde{X}$ . For each  $U \in \mathcal{U}$ , define a subset  $A^r_U$  of Y by

$$A_U^r = \{ f(x) \mid \exists V \in \mathcal{U} (V \subseteq U \land (V, x) \in r) \}.$$

By Strong Collection, let

$$\mathcal{F}_r = \{ A_U^r \mid U \in \mathcal{U} \}.$$

**Lemma 11.**  $\mathcal{F}_r$  is a Cauchy filter on Y.

Proof. For each  $U \in \mathcal{U}$ , since  $(U, x) \in r$  for some  $x \in X$ , we have  $f(x) \in A_U^r$ . Since for each  $U, V \in \mathcal{U}$  if  $V \subseteq U$ , then  $A_V^r \subseteq A_U^r$ , we have for each  $U, V \in \mathcal{U}$  there exists  $W \in \mathcal{U}$  such that  $A_W^r \subseteq A_U^r \cap A_V^r$  by (Ub1). Let  $U \in \mathcal{V}$ . Then, since f is uniformly continuous, there exists  $V \in \mathcal{U}$  such that  $(x, y) \in V \Longrightarrow (f(x), f(y)) \in U$  for each  $x, y \in X$ , and there exists  $W \in \mathcal{U}$  such that  $W \ll_2 V$ . Suppose that  $(f(x), f(y)) \in A_W^r \times A_W^r$ . Then there exists  $W_1, W_2 \in \mathcal{U}$  such that  $W_1, W_2 \subseteq W$ ,  $(W_1, x) \in r$  and  $(W_2, y) \in r$ , and hence  $(x, y) \in W_2^{-1} \circ W_1 \subseteq W^{-1} \circ W \subseteq V$ . Thus  $(f(x), f(y)) \in U$ . Therefore  $A_W^r \times A_W^r \subseteq U$ .

Since  $(Y, \mathcal{V})$  is complete,  $\mathcal{F}_r$  converges to a point  $\tilde{f}(r)$  in Y.

**Lemma 12.** For each  $U \in \mathcal{V}$  there exists  $V \in \mathcal{U}$  such that

$$(r,s) \in \widetilde{V} \Longrightarrow (\widetilde{f}(r), \widetilde{f}(s)) \in U$$

for each  $r, s \in \widetilde{X}$ .

Proof. Let  $U \in \mathcal{V}$ . Then there exists  $U_0 \in \mathcal{V}$  such that  $U_0 \ll_3 U$ , and, since f is uniformly continuous, there exists  $V_0 \in \mathcal{U}$  such that  $(x,y) \in V_0 \Longrightarrow (f(x), f(y)) \in U_0$  for each  $x, y \in X$ . By Lemma 3, there exists  $V \in \mathcal{U}$ such that  $V \ll_5 V_0$ . Suppose that  $(r, s) \in \widetilde{V}$ . Then there exist  $V_1, V_2 \in \mathcal{U}$  and  $x_1, x_2 \in X$  such that  $V_1, V_2 \subseteq V$ ,  $(V_1, x_1) \in r$ ,  $(V_2, x_2) \in s$  and  $(x_1, x_2) \in V$ . Since  $\mathcal{F}_r$  and  $\mathcal{F}_s$  converge to  $\widetilde{f}(r)$  and  $\widetilde{f}(s)$ , respectively, we can find  $W \in \mathcal{U}$ such that  $W \subseteq V$ ,  $A_W^r \subseteq U_0(\widetilde{f}(r))$  and  $A_W^s \subseteq U_0(\widetilde{f}(s))$ , and, since  $A_W^r$  and  $A_W^s$  are inhabited, there exist  $x, y \in X$  such that  $f(x) \in A_W^r$  and  $f(y) \in A_W^s$ . Hence there exist  $W_1, W_2 \in \mathcal{U}$  such that  $W_1, W_2 \subseteq W$ ,  $(W_1, x) \in r$  and  $(W_2, y) \in s$ . Since  $(W_1, x), (V_1, x_1) \in r$  and  $(V_2, x_2), (W_2, y) \in s$ , we have  $(x, x_1) \in V_1^{-1} \circ W_1$  and  $(x_2, y) \in W_2^{-1} \circ V_2$ , and therefore

$$(x,y) \in W_2^{-1} \circ V_2 \circ V \circ V_1^{-1} \circ W_1 \subseteq W^{-1} \circ V \circ V \circ V^{-1} \circ W \subseteq V^{-1} \circ V \circ V \circ V^{-1} \circ V \subseteq V_0.$$

Thus  $(f(x), f(y)) \in U_0$ . Since  $(\tilde{f}(r), f(x)) \in U_0$  and  $(\tilde{f}(s), f(y)) \in U_0$ , we have  $(\tilde{f}(r), \tilde{f}(s)) \in U_0^{-1} \circ U_0 \circ U_0 \subseteq U$ .

Since  $(Y, \mathcal{V})$  is  $T_1$ , we have  $\tilde{f}(r) = \tilde{f}(s)$  whenever  $r =_{\tilde{X}} s$ , by Lemma 12. Hence  $\tilde{f}$  is a function on  $\tilde{X}$ , and it is uniformly continuous, by Lemma 12. Since  $A_U^{\tilde{x}} = \{f(x)\}$  for each  $U \in \mathcal{U}$ ,  $\mathcal{F}_{\tilde{x}}$  converges to f(x), and therefore we have the following lemma.

### Lemma 13. $f = \tilde{f} \circ i$ .

The function  $\tilde{f}$  is unique in the following sense.

**Lemma 14.** If  $h: (\widetilde{X}, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$  is uniformly continuous with  $f = h \circ i$ , then  $h = \widetilde{f}$ .

*Proof.* Let  $r \in \widetilde{X}$ , and let  $U \in \mathcal{V}$ . Then there exists  $U_0 \in \mathcal{V}$  such that  $U_0 \ll_2 U$ , and since h is uniformly continuous, there exists  $V \in \mathcal{U}$  such that  $(s,t) \in \widetilde{V} \Longrightarrow (h(s),h(t)) \in U_0$  for each  $s,t \in \widetilde{X}$ . Since  $\mathcal{F}_r$  converges to  $\widetilde{f}(r)$ , we can find  $W \in \mathcal{U}$  such that  $W \subseteq V$  and  $A_W^r \subseteq U_0(\widetilde{f}(r))$ , and, since  $A_W^r$  is

inhabited, there exists  $x \in X$  such that  $f(x) \in A_W^r$ . Hence there exists  $W' \in \mathcal{U}$  such that  $W' \subseteq W \subseteq V$  and  $(W', x) \in r$ , and therefore, since  $(V, x) \in \tilde{x}$ and  $(x, x) \in V$ , we have  $(r, \tilde{x}) \in \tilde{V}$ . Thus  $(h(r), h(\tilde{x})) = (h(r), f(x)) \in U_0$ . Since  $(\tilde{f}(r), f(x)) \in U_0$ , we have  $(h(r), \tilde{f}(r)) \in U_0^{-1} \circ U_0 \subseteq U$ . Therefore, since  $(Y, \mathcal{V})$  is  $T_1$ , we have  $h(r) = \tilde{f}(r)$ .

Now we have shown the following theorem.

**Theorem 15.** Let  $(Y, \mathcal{V})$  be a complete  $T_1$  uniform space, and let  $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$  be uniformly continuous. Then there exists a unique uniformly continuous  $\tilde{f} : (\tilde{X}, \tilde{\mathcal{U}}) \to (Y, \mathcal{V})$  such that  $f = \tilde{f} \circ i$ .

*Remark* 16. Let  $\mathcal{F}$  be a Cauchy filter on a uniform space  $(X, \mathcal{U})$ , and let

$$r = \{ (U, x) \mid \exists A \in \mathcal{F}(A \times A \subseteq U \land x \in A) \}.$$

Then  $r \in \operatorname{mv}(\mathcal{U}, X)$  and  $\varphi(r)$ , and hence there exists  $r_{\mathcal{F}} \in \widetilde{X}$  such that  $r_{\mathcal{F}} \subseteq r$ . On the other hand, for each  $r \in \widetilde{X}$ , let  $\mathcal{F}_r = \{B_U^r \mid U \in \mathcal{U}\}$ , where

$$B_U^r = \{ x \mid \exists V \in \mathcal{U} (V \subseteq U \land (V, x) \in r) \}.$$

Then  $\mathcal{F}_r$  is a Cauchy filter on  $(X, \mathcal{U})$ . In the presence of the powerset axiom, it is straightforward to show that these correspondences  $r \mapsto \mathcal{F}_r$  and  $\mathcal{F} \mapsto r_{\mathcal{F}}$ between the completion  $(\widetilde{X}, \widetilde{\mathcal{U}})$  and the uniform space of the set of all Cauchy filters on  $(X, \mathcal{U})$  (the completion of  $(X, \mathcal{U})$  in the sense of Bishop) form a uniform isomorphism.

For a metric space (X, d), as mentioned in Remark 4, in the presence of the axiom of countable choice, there is a uniform isomorphism between the completion  $(\widetilde{X}, \widetilde{\mathcal{U}}_d)$  and Bishop's metric completion.

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Josef Berger Mathematisches Institut Ludwig-Maximilians Universität München Theresienstr. 39, D-80333 München, Germany E-mail: Josef.Berger@mathematik.uni-muenchen.de

Hajime Ishihara

School of Information Science Japan Advanced Institute of Science and Technology Nomi, Ishikawa 923-1292, Japan E-mail: ishihara@jaist.ac.jp Tel: +81-761-51-1206 Fax: +81-761-51-1149

Erik Palmgren Department of Mathematics Uppsala University PO Box 480, SE-751 06 Uppsala, Sweden E-mail: palmgren@math.uu.se

Peter Schuster Department of Pure Mathematics University of Leeds Leeds LS2 9JT, England, UK E-mail: pschust@maths.leeds.ac.uk