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# The uniform boundedness theorem and a boundedness principle

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#### Abstract

We deal with a form of the uniform boundedness theorem (or the Banach-Steinhaus theorem) for topological vector spaces in Bishop's constructive mathematics, and show that the form is equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics.

*Keywords:* constructive mathematics, topological vector space, the uniform boundedness theorem, boundedness principle. 2010 Mathematics Subject Classification: 03F60, 46S30.

## 1 Introduction

The notion of a topological vector space, as a generalization of the notion of a normed space, is a very important notion to investigate function spaces, such as the space of test functions, which do not form Banach spaces (see, for example, [23]). Nonetheless little investigation on topological vector spaces has been done in Bishop's constructive mathematics [6, 7, 8, 10]; see also a discussion in [6, Appendix A]. We can only find, noting that a topological vector space is a uniform space, a constructive concept of a uniform space with a set of pseudometrics, and basic theorems, such as, that arbitrary uniform space has a completion, in [6, Problems 17 to 21 of Chapter 4]; see also [7, Problems 22 to 26 of Chapter 4], and [9, 24, 12, 11, 17, 5] for other constructive treatments of a uniform space.

However, using the notion of a neighbourhood space [6, 3.3] (see also [7, 3.3]) introduced by Bishop, we can naturally define a notion of a topological vector space in Bishop's constructive mathematics as follow.

A neighbourhood space is a pair  $(X, \tau)$  consisting of a set X and a set  $\tau$  of subsets of X such that

NS1. 
$$\forall x \in X \exists U \in \tau (x \in U),$$

NS2. 
$$\forall x \in X \forall U, V \in \tau [x \in U \cap V \implies \exists W \in \tau (x \in W \subseteq U \cap V)].$$

The set  $\tau$  is an open base on X, and an element of  $\tau$  is a basic open set. A subset of X is open if it is a union of basic open sets. A neighbourhood of a point  $x \in X$  is a subset  $A \subseteq X$  such that  $x \in U \subseteq A$  for some  $U \in \tau$ . An open base  $\sigma$  on X is compatible with  $\tau$  if each neighbourhood in  $\sigma$  is a neighbourhood in  $\tau$ , and vice versa. An open base is compatible with a metric d if it is compatible with the open base induced by open balls. A function f between neighbourhood spaces  $(X, \tau)$  and  $(Y, \sigma)$  is continuous if  $f^{-1}(V)$  is open for each  $V \in \sigma$ .

A topological vector space is a vector space E equipped with an open base  $\tau$  such that the vector space operations, addition  $(x, y) \mapsto x + y$  and scalar multiplication  $(a, x) \mapsto ax$ , are continuous, that is, if U is a neighbourhood of x + y, then there exist neighbourhoods V and V' of x and y, respectively, such that  $V + V' = \{v + v' \mid v \in V, v' \in V'\} \subseteq U$ , and if U is a neighbourhood of ax, then for some  $\delta > 0$  and some neighbourhood V of x we have  $bV = \{bv \mid v \in V\} \subseteq U$  whenever  $|a - b| < \delta$ . It is metrizable if  $\tau$  is compatible with some metric d, and is an F-space if its open base  $\tau$  is compatible with a complete invariant metric d. Here a metric d on a vector space E is invariant if d(x + z, y + z) = d(x, y) for all  $x, y, z \in E$ .

In this paper, we deal with the following form of the uniform boundedness theorem (or the Banach-Steinhaus theorem) for topological vector spaces [23, 2.6] in Bishop's constructive mathematics.

**The Uniform Boundedness Theorem.** If  $(T_m)_m$  is a sequence of continuous linear mappings from an *F*-space *E* into a topological vector space *F* such that the set

 $\{T_m x \mid m \in \mathbf{N}\}$ 

is bounded in F for each  $x \in E$ , then  $(T_m)_m$  is equicontinuous.

Here a subset A of a topological vector space E is bounded if for each neighbourhood V of 0 in E there exists a positive integer K such that  $A \subseteq tV$  for each  $t \geq K$ , and a set  $\Gamma$  of continuous linear mappings between topological vector spaces E and F is equicontinuous if for each neighbourhood V of 0 in F there exists a neighbourhood U of 0 in E such that  $T(U) \subseteq V$  for each  $T \in \Gamma$ .

We know that a (contrapositive) form of the uniform boundedness theorem for normed spaces has a constructive proof [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]), and a corollary [23, Theorem 2.8] of the uniform boundedness theorem for a sequence of *sequentially continuous* linear mappings from a separable Banach space into a normed space holds constructively [14, Theorem 7]. However, the corollary for a sequence of *continuous* linear mappings not only implies, but also is equivalent to the following boundedness principle (BD-N) [15, Theorem 21].

BD-N. Every pseudobounded countable subset of N is bounded.

Here a subset S of N is countable if it is a range of N, pseudobounded if  $\lim_{n\to\infty} s_n/n = 0$  for each sequence  $(s_n)_n$  in S, and bounded if there exists a positive integer K such that s < K for each  $s \in S$ ; see [13, 18, 22] for pseudobounded sets.

The boundedness principle BD-N is equivalent to the statement "every sequentially continuous mapping from a separable metric space into a metric space is continuous" [13, Theorem 4], is derivable in intuitionistic mathematics with a continuity principle [13, Proposition 3] and in constructive recursive mathematics with Church's thesis and Markov's principle [13, Proposition 4], and is not provable in  $\mathbf{HA}^{\omega}$  with axiom of choice for all finite types [19].

In the following, we show that the uniform boundedness theorem for topological vector spaces with a sequence of continuous linear mappings is also equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics [16, 20, 25].

Although the result is presented in informal Bishop-style constructive mathematics, it is possible to formalize it in constructive Zermero-Fraenkel set theory (**CZF**), founded by Aczel [1, 2, 3], with the dependent choice axiom (DC), which permits a quite natural interpretation in Martin-Löf type theory [21]. Note that the axiom of countable choice (AC<sub> $\omega$ </sub>) follows from the dependent choice axiom in **CZF**; see [4, Section 8].

### 2 The main results

A topological vector space E is *separated* if for each neighbourhood U of 0 there exist a neighbourhood V of 0 and an open set W such that  $E = U \cup W$  and  $V \cap W = \emptyset$ .

Each topological vector space E whose open base is compatible with a set  $\{d_i \mid i \in I\}$  of pseudometrics, that is, compatible with the open base consisting of the sets  $B_{i_1,\ldots,i_n}(x,\epsilon) = \{y \in E \mid \sum_{k=1}^n d_{i_k}(x,y) < \epsilon\}$ , is separated. In fact, for each neighbourhood U of 0, there exist  $i_1,\ldots,i_n \in I$  and  $\epsilon > 0$  such that  $B_{i_1,\ldots,i_n}(0,\epsilon) \subseteq U$ , and hence, taking  $V = B_{i_1,\ldots,i_n}(0,\epsilon/2)$  and  $W = \{y \in E \mid \epsilon/2 < \sum_{k=1}^n d_{i_k}(0,y)\}$ , we have  $E = U \cup W$  and  $V \cap W = \emptyset$ .

On the other hand, suppose that a vector space E with the discrete topology is separated. Then, since  $\{0\}$  is a neighbourhood of 0, there exist a neighbourhood V of 0 and an open set W such that  $E = \{0\} \cup W$  and  $V \cap W = \emptyset$ . Hence for each  $x \in E$ , either  $x \in \{0\}$  or  $x \in W$ : in the former case, we have x = 0; in the latter case, we have  $\neg(x = 0)$ . If  $E = \mathbf{R}$ , then this is equivalent to the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x \in \mathbf{R}[x = 0 \lor \neg (x = 0)].$$

Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot find out whether the topological vector space  $\mathbf{R}$  with the discrete topology is separated.

A subset A of a topological vector space E is unbounded if there exits a neighbourhood V of 0 in E such that for each positive integer k there exist  $t \ge k$  and  $x \in A$  such that  $x \notin tV$ .

The following theorem generalizes the constructive version of the uniform boundedness theorem [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]) to topological vector spaces.

**Theorem 1.** Let  $(T_n)_n$  be a sequence of continuous linear mappings from an *F*-space *E* into a separated topological vector space *F*. If there exists a bounded sequence  $(x_n)_n$  in *E* such that  $\{T_nx_n \mid n \in \mathbf{N}\}$  is unbounded, then  $\{T_nx \mid n \in \mathbf{N}\}$  is unbounded for some  $x \in E$ .

*Proof.* Suppose that  $(x_n)_n$  is a bounded sequence in E such that  $\{T_nx_n \mid n \in \mathbb{N}\}$  is unbounded. Then there exists a neighbourhood  $V_0$  of 0 in F such that for each  $k, T_nx_n \notin tV_0$  for some  $t \geq k$  and n. Since  $(a, x) \mapsto ax$  is continuous at (0,0), there exist N and a neighbourhood  $V_1$  of 0 in F such that  $a'V_1 \subseteq V_0$ 

for each a' with  $|a'| \leq 1/N$ . Furthermore, since  $(x, y) \mapsto x - y$  is continuous at (0,0), there exists a neighbourhood  $V_2$  of 0 in F such that  $V_2 - V_2 \subseteq V_1$ . Since F is separated, there exist a neighbourhood  $V_3$  of 0 and an open set W in F such that  $F = V_2 \cup W$  and  $V_3 \cap W = \emptyset$ .

For each  $m \ge 1$ , define a subset  $G_m$  of E by

$$G_m = \{ x \in E \mid T_n x \in mW \text{ for some } n \}.$$

Then  $G_m$  is open. Let  $y \in E$  and let U be a neighbourhood of y in E. Then, since the addition is continuous, there exists a neighbourhood U' of 0 in Esuch that  $y+U' = \{y+u \mid u \in U'\} \subseteq U$ . Since  $(x_n)_n$  is bounded, there exists  $k_0$  such that  $x_n \in k_0 U'$  for all n. Set  $k = mk_0 N$ . Then there exist  $t \geq k$  and n such that  $T_n x_n \notin tV_0$ . Either  $T_n y/m \in W$  or  $T_n y/m \in V_2$ . In the former case, setting z = y, we have  $z \in G_m \cap U$ . In the latter case, if  $T_n z/m \in V_2$ where  $z = y + x_n/k_0$ , then, since  $mk_0/t \leq mk_0/k = 1/N$ , we have

$$\frac{T_n x_n}{t} = \frac{mk_0}{t} \left( \frac{T_n z}{m} - \frac{T_n y}{m} \right) \in \frac{mk_0}{t} (V_2 - V_2) \subseteq \frac{mk_0}{t} V_1 \subseteq V_0,$$

a contradiction; whence  $T_n z/m \in W$ , and therefore, since  $z \in y + U' \subseteq U$ , we have  $z \in G_m \cap U$ . Thus  $G_m$  is dense in E.

By applying the constructive version of Baire's theorem [6, Thorem 4 of Chapter 4] (see also [7, Theorem 3.9 of Chapter 4] and [8, Theorem 1.3 of Chapter 2]), we can find a point  $x \in E$  such that  $x \in G_m$  for all m, that is, for each m there exists n such that  $T_n x \notin mV_3$ . Therefore  $\{T_n x \mid n \in \mathbf{N}\}$  is unbounded.

We will need the following general lemma later.

#### Lemma 2. Each convergent sequence in a topological vector space is bounded.

Proof. Let  $(x_n)_n$  be a sequence in a topological vector space E converging to a limit x in E, and let V be a neighbourhood of 0 in E. Since  $(a, x) \mapsto ax$ is continuous at (0,0), there exist a positive integer M and a neighbourhood  $V_0$  such that  $a'V_0 \subseteq V$  for each a' with  $|a'| \leq 1/M$ . Since  $(x_n)_n$  converges to x, there exists a positive integer N such that  $x_n - x \in V_0$  for each  $n \geq N$ . Note that for each  $y \in E$ , since  $a \mapsto ay$  is continuous at 0, there exists a positive integer M' such that  $y/M' \in V_0$ . Then there exist  $M', M'_1, \ldots, M'_{N-1}$ such that  $x/M' \in V_0$  and  $x_n/M'_n \in V_0$  for  $n = 1, \ldots, N - 1$ . For each  $n \geq N$ , since  $x_n = (x_n - x) + x \in V_0 + M'V_0 \subseteq (M' + 1)V_0$ , we have  $x_n/(M'+1) \in V_0$ . Let  $K = M \max\{M'+1, M'_1, \dots, M'_{N-1}\}$ , and let  $t \ge K$ . Then for each n, if  $n \ge N$ , then, since  $t^{-1}(M'+1) \le 1/M$ , we have  $t^{-1}x_n = (t^{-1}(M'+1))(x_n/(M'+1)) \in V$ ; or else n < N and, since  $t^{-1}M'_n \le 1/M$ , we have  $t^{-1}x_n = (t^{-1}M'_n)(x_n/M'_n) \in V$ . Therefore  $x_n \in tV$  for each n.

Let E be a separable F-space with a dense sequence  $(z_i)_i$ , and F be a separated topological vector space. Let  $(T_m)_m$  be a sequence of continuous linear mappings from E into F such that for each  $x \in E$  its orbit  $\{T_m x \mid m \in \mathbf{N}\}$  is bounded. Then for each neighbourhood V of 0 in F, since F is separated, there exist a neighbourhood  $V_0$  of 0 and an open set  $W_0$  such that  $F = V \cup W_0$  and  $V_0 \cap W_0 = \emptyset$ , and there exist a neighbourhood  $V_1$  of 0 and an open set  $W_1$  such that  $F = V_0 \cup W_1$  and  $V_1 \cap W_1 = \emptyset$ . Construct a binary triple sequence  $\alpha$  such that

$$\begin{aligned} \alpha(i,m,k) &= 0 \quad \Longrightarrow \quad 1/(k+1)^2 < d(0,z_i) \lor T_m z_i \in V_0 \\ \alpha(i,m,k) &= 1 \quad \Longrightarrow \quad d(0,z_i) < 1/k^2 \land T_m z_i \in W_1, \end{aligned}$$

and define a countable subset S of  $\mathbf{N}$  by

$$S = \{k \mid \alpha(i, m, k) = 1 \text{ for some } i \text{ and } m\} \cup \{0\}.$$

**Lemma 3.** For each sequence  $(s_n)_n$  in S, either  $s_n < n$  for all n or  $s_n \ge n$  for some n.

*Proof.* Let  $(s_n)_n$  be a sequence in S, and construct an increasing binary sequence  $(\lambda_n)$  such that

$$\lambda_n = 0 \implies s_{n'} < n' \text{ for all } n' \le n$$
  
$$\lambda_n = 1 \implies s_{n'} \ge n' \text{ for some } n' \le n.$$

We may assume that  $\lambda_1 = 0$ . Define a sequence  $(x_n)_n$  in E as follow: if  $\lambda_n = 0$ , then set  $x_n = 0$ ; if  $\lambda_n = 1 - \lambda_{n-1}$ , then pick i and m such that  $d(0, z_i) < 1/s_n^2 \le 1/n^2$  and  $T_m z_i \in W_1$ , and set  $x_j = n z_i$  for all  $j \ge n$  (and note that  $d(0, x_j) = d(0, n z_i) \le \sum_{k=1}^n d((k-1)z_i, k z_i) = nd(0, z_i) < 1/n$ ). Then  $(x_n)$  is a Cauchy sequence, and hence converges to a limit x in E. Since  $\{T_m x \mid m \in \mathbf{N}\}$  is bounded, there exists a positive integer K such that  $\{T_m x \mid m \in \mathbf{N}\} \subseteq tV_1$  for each  $t \ge K$ . If  $\lambda_n = 1 - \lambda_{n-1}$  for some  $n \ge K$ , then there exist i and m such that  $x = n z_i$  and  $T_m z_i \in W_1$ , and hence  $T_m x \notin nV_1$ , a contradiction. Therefore  $\lambda_n = \lambda_{n-1}$  for all  $n \ge K$ , and so either  $\lambda_n = 1$  for some n or  $\lambda_n = 0$  for all n.

**Lemma 4.** For each sequence  $(s_n)_n$  in S, either  $s_n < n$  for all sufficiently large n or  $s_n \ge n$  for infinitely many n.

*Proof.* Let  $(s_n)_n$  be a sequence in S, and, by applying Lemma 3 to subsequences  $(s_{n'})_{n'>n}$ , construct an increasing binary sequence  $(\lambda_n)$  such that

$$\lambda_n = 0 \implies s_{n'} \ge n' \text{ for some } n' \ge n$$
  
$$\lambda_n = 1 \implies s_{n'} < n' \text{ for all } n' \ge n.$$

We may assume that  $\lambda_1 = 0$ . Define a sequence  $(x_n)_n$  in E as follow: if  $\lambda_n = 0$ , then pick  $n' \geq n$ , i and m such that  $d(0, z_i) < 1/s_{n'}^2 \leq 1/n^2$  and  $T_m z_i \in W_1$ , and set  $x_n = n z_i$  (and note that  $d(0, x_n) < 1/n$ ); if  $\lambda_n = 1$ , then set  $x_n = x_{n-1}$ . Then  $(x_n)$  is a Cauchy sequence, and hence converges to a limit x in E. Since  $\{T_m x \mid m \in \mathbf{N}\}$  is bounded, there exists a positive integer K such that  $\{T_m x \mid m \in \mathbf{N}\} \subseteq tV_1$  for each  $t \geq K$ . If  $\lambda_n = 1 - \lambda_{n-1}$  for some n > K, then there exist i and m such that  $x = (n-1)z_i$  and  $T_m z_i \in W_1$ , and hence  $T_m x \notin (n-1)V_1$ , a contradiction. Therefore  $\lambda_n = \lambda_{n-1}$  for all n > K, and so either  $\lambda_n = 1$  for some n or  $\lambda_n = 0$  for all n.

#### **Proposition 5.** The set S is pseudobounded.

Proof. By [18, Lemma 3], it suffices to show that for each sequence  $(s_n)_n$  in  $S, s_n < n$  for all sufficiently large n. Let  $(s_n)_n$  be a sequence in S. Then, by Lemma 4, either  $s_n < n$  for all sufficiently large n or  $s_n \ge n$  for infinitely many n. Suppose that  $s_n \ge n$  for infinitely many n. Then for each n there exists  $n' \ge n$  such that  $s_{n'} \ge n'$ , and hence there exist  $i_n$  and  $m_n$  such that  $d(0, z_{i_n}) < 1/s_{n'}^2 \le 1/n^2$  and  $T_{m_n} z_{i_n} \in W_1$ . Therefore  $nT_{m_n} z_{i_n} \notin nV_1$  for each n, and so  $\{T_{m_n}(nz_{i_n}) \mid n \in \mathbb{N}\}$  is unbounded. Since  $(nz_{i_n})_n$  converges to 0, it is bounded, by Lemma 2, and hence  $\{T_{m_n} x \mid n \in \mathbb{N}\}$  is unbounded for some  $x \in E$ , by Theorem 1. This contradicts with the fact that  $\{T_m x \mid m \in \mathbb{N}\}$  is bounded for all  $x \in E$ . Therefore  $s_n < n$  for all sufficiently large n.

Suppose that S is bounded, that is, there exists a positive integer K such that k < K for each  $k \in S$ . Let  $x \in U = \{x \in E \mid d(0, x) < 1/(K + 1)^2\}$ and assume that  $T_m x \in W_0$  for some m. Then, since  $T_m$  is continuous and  $W_0$  is open, there exists i such that  $d(0, z_i) < 1/(K + 1)^2$  and  $T_m z_i \in W_0$ , and hence  $T_m z_i \notin V_0$ . Therefore  $\alpha(i, m, K)$  must be 1, and so  $K \in S$ , a contradiction. Thus  $T_m x \in V$  for each  $x \in U$  and  $m \in \mathbb{N}$ .

We have shown the following theorem.

**Theorem 6.** Assume BD-N. If  $(T_m)_m$  is a sequence of continuous linear mappings from a separable F-space E into a separated topological vector space F such that the set

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded for each  $x \in E$ , then  $(T_m)_m$  is equicontinuous.

Let  $(T_m)_m$  be a sequence of continuous linear mappings from a separable *F*-space *E* into a separated topological vector space *F* such that the limit

$$Tx = \lim_{m \to \infty} T_m x$$

exists for each  $x \in E$ . Then for each  $x \in E$  its orbit  $\{T_m x \mid m \in \mathbf{N}\}$  is bounded, by Lemma 2. Therefore, by applying the uniform boundedness theorem,  $(T_m)_m$  is equicontinuous.

Let V be a neighbourhood of 0 in F. Then, since  $(x, y) \mapsto x - y$  is continuous at (0, 0), there exists a neighbourhood W of 0 in F such that  $W - W \subseteq V$ . Note that  $\overline{W} \subseteq V$ : in fact, if  $x \in \overline{W}$ , then there exist  $y, z \in W$  such that x + y = z, and hence  $x = z - y \in W - W \subseteq V$ . Since  $(T_m)_m$  is equicontinuous, there exists a neighbourhood U of 0 in E such that  $T_m(U) \subseteq W$  for each m, and hence  $T(U) \subseteq \overline{W} \subseteq V$ . Therefore (being obviously linear) T is continuous.

Let  $S = \{s_n \mid n \in \mathbf{N}\}$  be a pseudobounded countable subset of  $\mathbf{N}$ , and define a sequence  $(T_m)_m$  of continuous linear mappings from the Hilbert space  $l_2$  of square summable sequences into itself by

$$T_m x = \sum_{n=1}^m s_n \langle x, e_n \rangle e_n,$$

where  $(e_n)_n$  is an orthonormal basis of  $l_2$ . Then, since S is pseudobounded, as in the proof of [15, Lemma 20], we can show that the limit

$$Tx = \lim_{m \to \infty} T_m x = \sum_{n=1}^{\infty} s_n \langle x, e_n \rangle e_n$$

exists for each  $x \in l_2$ . If T is continuous, then, since  $s_n = ||Te_n||$  for each n, we see that S is bounded.

Thus we have the following theorem.

**Theorem 7.** The following are equivalent

- 1. BD-N.
- 2. If  $(T_m)_m$  is a sequence of continuous linear mappings from a separable F-space E into a separated topological vector space F such that the set

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded for each  $x \in E$ , then  $(T_m)_m$  is equicontinuous.

3. If  $(T_m)_m$  is a sequence of continuous linear mappings from a separable F-space E into a separated topological vector space F such that

$$Tx = \lim_{m \to \infty} T_m x$$

exists for each  $x \in E$ , then T is continuous.

Remark 8. The sequence  $(T_m)_m$  of continuous linear mappings from  $l_2$  into itself constructed from a countable subset S of  $\mathbf{N}$  before Theorem 7 is a sequence of *compact self-adjoint* operators on  $l_2$ . Therefore BD-N actually follows from the weaker statement: If  $(T_m)_m$  is a sequence of compact selfadjoint operators on  $l_2$  such that  $Tx = \lim_{m\to\infty} T_m x$  exists for each  $x \in l_2$ , then T is continuous.

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## References

- Peter Aczel, The type theoretic interpretation of constructive set theory, In: A. Macintyre, L. Pacholski, J. Paris, eds., Logic Colloquium '77, North-Holland, Amsterdam, 1978, 55–66.
- [2] Peter Aczel, The type theoretic interpretation of constructive set theory: choice principles, In: A.S. Troelstra, D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium, North-Holland, Amsterdam, 1982, 1– 40.

- [3] Peter Aczel, The type theoretic interpretation of constructive set theory: inductive definitions, In: R.B. Marcus et al. eds., Logic, Methodology, and Philosophy of Science VII, North-Holland, Amsterdam, 1986, 17– 49.
- [4] Peter Aczel and Michael Rathjen, Notes on constructive set theory, Report No. 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.
- [5] Josef Berger, Hajime Ishihara, Erik Palmgren and Peter Schuster, A predicative completion of a uniform space, preprint, 2010.
- [6] Errett Bishop, Foundations of Constructive Mathematics, McGraw-Hill, New York, 1967.
- [7] Errett Bishop and Douglas Bridges, Constructive Analysis, Springer, Berlin, 1985.
- [8] Douglas Bridges and Fred Richman, Varieties of Constructive Mathematics, London Math. Soc. Lecture Notes 97, Cambridge Univ. Press, London, 1987.
- [9] Douglas Bridges and Luminiţa Vîţă, Strong and uniform continuity the uniform space case, LMS J. Comput. Math. 6 (2003), 326–334.
- [10] Douglas Bridges and Luminiţa Vîţă, Techniques of Constructive Analysis, Springer, New York, 2006.
- [11] Giovanni Curi, On the collection of points of a formal space, Ann. Pure Appl. Logic 137 (2006), 126–146.
- [12] Christopher Fox, Point-Set and Point-Free Topology in Constructive Set Theory, Dissertation, University of Manchester, 2005.
- [13] Hajime Ishihara, Continuity properties in constructive mathematics, J. Symbolic Logic 57 (1992), 557–565.
- [14] Hajime Ishihara, Sequential continuity of linear mappings in constructive mathematics, J. UCS 3 (1997), 1250–1254.

- [15] Hajime Ishihara, Sequential continuity in constructive mathematics, In: C.S. Calude, M.J. Dinneen and S. Sburlan eds., Combinatorics, Computability and Logic, Springer-Verlag, London, 2001, 5–12.
- [16] Hajime Ishihara, Constructive reverse mathematics: compactness properties, In: L. Crosilla and P. Schuster eds., From Sets and Types to Analysis and Topology, Oxford Logic Guides 48, Oxford Univ. Press, 2005, 245-267.
- [17] Hajime Ishihara, Two subcategories of apartness spaces, to appear in Ann. Pure Appl. Logic.
- [18] Hajime Ishihara and Satoru Yoshida, A constructive look at the completeness of  $\mathcal{D}(\mathbf{R})$ , J. Symbolic Logic **67** (2002), 1511–1519.
- [19] Peter Lietz, From Constructive Mathematics to Computable Analysis via Realizability Interpretation, Dissertation, Technische Universität Darmstadt, 2004.
- [20] Iris Loeb, Equivalents of the (weak) fan theorem, Ann. Pure Appl. Logic 132 (2005), 51–66.
- [21] Per Martin-Löf, Intuitionistic Type Theory, Studies in Proof Theory 1, Bibliopolis, Naples, 1984.
- [22] Fred Richman, Intuitionistic notions of boundedness in N, MLQ Math. Log. Q. 55 (2009), 31–36.
- [23] Walter Rudin, Functional Analysis, McGraw-Hill, New York, 1991.
- [24] Peter Schuster, Luminiţa Vîţă and Douglas Bridges, Apartness as a relation between subsets, In: C.S. Calude, M.J. Dinneen and S. Sburlan eds., Combinatorics, Computability and Logic, Springer, London, 2001, 203-214.
- [25] Wim Veldman, Brouwer's fan theorem as an axiom and as a contrast to Kleene's alternative, preprint, Radboud University, Nijmegen, 2005.

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