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Description	

Two subcategories of apartness spaces

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Abstract

We introduce the notion of a topological quasi-apartness space and the notion of a uniform quasi-apartness space, and construct an adjunction between the category of topological quasi-apartness spaces and the category of neighbourhood spaces and an adjunction between the category of uniform spaces and the category of uniform quasi-apartness spaces.

Keywords: constructive mathematics, neighbourhood space, uniform space, apartness space, adjunction.
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1 Introduction

Bridges and Vîţă [11] proposed the theory of apartness spaces as an alternative approach to topology from a constructive point of view. The theory of apartness spaces consists of a theory of point-set apartness spaces [8] and a theory of set-set apartness spaces [18, 7]. Ishihara et al [13] introduced the notion of a (point-set) quasi-apartness space, and constructed an adjunction between the category of (point-set) quasi-apartness spaces with (point-set) quasi-apartness spaces as objects and continuous functions as morphisms and the category **Nbh** of neighbourhood spaces with neighbourhood spaces as objects and continuous functions as morphisms.

A *(point-set) quasi-apartness space*, $\langle X, - \rangle$, is a set X with an operation, $-$, on the subsets of X satisfying the following axioms, in which we let $\neg S = \{x \in X \mid \neg(x \in S)\}$:

QA1. $-\emptyset = X$,

QA2. $-S \subset \neg S$,

QA3. $-(S \cup T) = -S \cap -T$,

QA4. $-S \subset \neg T \Longrightarrow -S \subset -T$.

A *neighbourhood space* [3] is a pair (X, τ) consisting of a set X and a set τ of subsets of X such that

NS1. $\forall x \in X \exists U \in \tau (x \in U)$,

NS2. $\forall x \in X \forall U, V \in \tau [x \in U \cap V \Longrightarrow \exists W \in \tau (x \in W \subset U \cap V)]$.

A function f between (point-set) quasi-apartness spaces $\langle X, - \rangle$ and $\langle Y, -' \rangle$ is *continuous* if $f(x) \in -' f(S) \Longrightarrow x \in -S$ for each $x \in X$ and $S \subset X$, and a function f between neighbourhood spaces (X, τ) and (Y, τ') is *continuous* if $f(x) \in V \Longrightarrow \exists U \in \tau (x \in U \subset f^{-1}(V))$ for each $x \in X$ and $V \in \tau'$.

Schuster et al [18] introduced the notion of a (set-set) apartness space in relation to the notion of a metric space and the notion of a uniform space; see also [6] for recent treatment of (set-set) apartness spaces. A *(set-set) apartness space*, $\langle X, \bowtie \rangle$, consists of a set X equipped with an *inequality* $\neq$ (that is, a binary relation on X such that $x \neq y \Longrightarrow y \neq x$ and $x \neq y \Longrightarrow \neg(x = y)$) and a binary relation, $\bowtie$, on the subsets of X satisfying the following axioms, in which we let $\sim S = \{x \in X \mid \forall y \in S (x \neq y)\}$ and $-S = \{x \in X \mid \{x\} \bowtie S\}$:

B1. $X \bowtie \emptyset$,

B2. $S \bowtie T \Longrightarrow S \cap T = \emptyset$,

B3. $S \bowtie (T \cup T') \Longleftrightarrow S \bowtie T \wedge S \bowtie T'$,

B4. $-S \subset \sim T \Longrightarrow -S \subset -T$,

B5. $x \in -S \Longrightarrow \forall y \in X (x \neq y \vee y \in -S)$,

B6. $S \bowtie T \Longrightarrow T \bowtie S$,

B7. $S \bowtie T \Longrightarrow \forall x \in X \exists T' (x \in -T' \wedge (\exists y \in -T' (y \in S) \Longrightarrow \neg T' \bowtie T)$;

see [18, 7] for more details. A function f between (set-set) apartness spaces $\langle X, \bowtie \rangle$ and $\langle Y, \bowtie' \rangle$ is *strongly continuous* if

$$f(S) \bowtie' f(T) \Longrightarrow S \bowtie T$$

for each $S, T \subset X$. Although various conditions on a (set-set) apartness space have been introduced to try to characterize various spaces, any adjunction has not been constructed between a natural category, such as the category of neighbourhood spaces or the category of uniform spaces, and a subcategory of the category of (set-set) apartness spaces consisting of (set-set) apartness spaces as objects and strongly continuous functions as morphisms.

In this paper, we introduce the notion of a (set-set) quasi-apartness space by dropping axioms B5-B7 from the definition of a (set-set) apartness space. We also drop the inequality. We define the notion of a topological quasi-apartness space, and construct an adjunction between the full subcategory of (set-set) quasi-apartness spaces with topological quasi-apartness spaces as objects and strongly continuous functions as morphisms and the category **Nbh** of neighbourhood spaces. Then we introduce the notion of a completely left joinable quasi-apartness space, show that it is equivalent to the notion of a topological quasi-apartness space, and construct an adjoint equivalence between the category of (point-set) quasi-apartness spaces and the category of topological quasi-apartness spaces. Finally, we define the notion of a uniform quasi-apartness space, and construct an adjunction between the category of uniform spaces and the full subcategory of (set-set) quasi-apartness spaces consisting of uniform quasi-apartness spaces as objects and strongly continuous functions as morphisms.

Although the results are presented in informal Bishop-style constructive mathematics [3, 4, 5, 19, 10], it is possible to formalize them (except the results in category theory) in Aczel's constructive Zermelo-Fraenkel set theory (CZF) [2] together with the power set axiom. However, we would like to follow the minimalist brad [16, 1] of constructivism as far as possible. In the section 4, the readers will find some remarks on avoiding use of the power set axiom. We do not use any choice axioms. Therefore the work in this paper holds in an arbitrary topos (with a natural number object).

2 Topological quasi-apartness spaces

A *(set-set) quasi-apartness space*, $\langle X, \bowtie \rangle$, is a set X with a binary relation, $\bowtie$, on the subsets of X such that for each sets $S, S', T, T' \subset X$,

QB1. $X \bowtie \emptyset$,

QB2. $S \bowtie T \Longrightarrow S \cap T = \emptyset$,

QB3$_m$. $S \bowtie T \wedge S' \subset S \wedge T' \subset T \Longrightarrow S' \bowtie T'$,

QB3$_r$. $S \bowtie T \wedge S \bowtie T' \Longrightarrow S \bowtie (T \cup T')$,

QB4. $-S \subset \neg T \Longrightarrow -S \subset -T$.

Here $-$ is a unary operation on the subsets of X defined by

$$-S = \{x \in X \mid \{x\} \bowtie S\};$$

we sometimes write $-_{\bowtie}$ for $-$. We say that $\bowtie$ is a *(set-set) quasi-apartness* on X.

Let $\bowtie$ and $\bowtie'$ be two quasi-apartness relations on a set X. Then we say that $\bowtie$ *is weaker than* $\bowtie'$ (or $\bowtie'$ *is stronger than* $\bowtie$) and write $\bowtie \preceq \bowtie'$ if $S \bowtie T \Longrightarrow S \bowtie' T$ for each sets $S, T \subset X$.

A quasi-apartness space $\langle X, \bowtie \rangle$ is *weakly topological* if

QBT1. $-S \bowtie \neg - S$

for each set $S \subset X$, and *topological* if it is weakly topological and

QBT2. $\forall i \in I (S_i \bowtie \neg S_i) \Longrightarrow (\bigcup_{i \in I} S_i) \bowtie \neg (\bigcup_{i \in I} S_i)$

for each family $\{S_i \mid i \in I\}$ of subsets of X.

Lemma 2.1. (QB3$_m$) *and* (QBT1) *imply* (QB4).

Proof. Suppose that $-S \subset \neg T$. Then $T \subset \neg - S$. Since $-S \bowtie \neg - S$, by (QBT1), we have $-S \bowtie T$ by (QB3$_m$), and hence, again by (QB3$_m$), $\{x\} \bowtie T$ for each $x \in -S$. Therefore $-S \subset -T$. $\square$

Let $\langle X, \bowtie \rangle$ be a quasi-apartness space. Then we can define a family $\tau_{\bowtie}$ of subsets of X by

$$\tau_{\bowtie} = \{U \subset X \mid U \bowtie \neg U\}.$$

Lemma 2.2. *Let $\langle X, \bowtie\rangle$ be a weakly topological quasi-apartness space. Then $S \bowtie T \Longrightarrow S \bowtie \neg\neg T$ for each $S, T \subset X$.*

Proof. Suppose that $S \bowtie T$. Then $S \subset -T$. Since $-T \subset \neg T$, we have $\neg\neg T \subset \neg - T$. Therefore, since $-T \bowtie \neg - T$, by (QBT1), we have $S \bowtie \neg\neg T$, by (QB3_m). □

Proposition 2.3. *If $\langle X, \bowtie\rangle$ is a weakly topological quasi-apartness space, then $(X, \tau_{\bowtie})$ is a neighbourhood space.*

Proof. Since $\neg X = \emptyset$, we have $X \in \tau_{\bowtie}$, by (QB1). Suppose that $U, V \in \tau_{\bowtie}$. Then, since $U \bowtie \neg U$ and $V \bowtie \neg V$, we have $(U \cap V) \bowtie \neg U$ and $(U \cap V) \bowtie \neg V$, by (QB$3_m$), and hence $(U \cap V) \bowtie (\neg U \cup \neg V)$, by (QB$3_r$). Therefore $(U \cap V) \bowtie \neg\neg(\neg U \cup \neg V)$, by Lemma 2.2, and, since $\neg\neg(\neg U \cup \neg V) = \neg(U \cap V)$, we have $(U \cap V) \bowtie \neg(U \cap V)$. Thus $U \cap V \in \tau_{\bowtie}$. □

Proposition 2.4. *Let $\langle X, \bowtie\rangle$ and $\langle Y, \bowtie'\rangle$ be weakly topological apartness spaces. If $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous, then $f : (X, \tau_{\bowtie}) \to (Y, \tau_{\bowtie'})$ is continuous.*

Proof. Suppose that $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous and $V \in \tau_{\bowtie'}$. Then, letting $T = f^{-1}(\neg V)$, since $f(f^{-1}(\neg V)) \subset \neg V$, we have $f(T) \subset \neg V$, and therefore, since $f(f^{-1}(V)) \subset V$ and $V \bowtie' \neg V$, we have $f(f^{-1}(V)) \bowtie' f(T)$, by (QB$3_m$). Hence $f^{-1}(V) \bowtie T$, and therefore, since $T = f^{-1}(\neg V) = \neg f^{-1}(V)$, we have $f^{-1}(V) \bowtie \neg f^{-1}(V)$. Thus $f^{-1}(V) \in \tau_{\bowtie}$. □

Let (X, τ) be a neighbourhood space. Then we can define a binary relation $\bowtie_\tau$ on the subsets of X by

$$S \bowtie_\tau T \Longleftrightarrow \forall x \in S \exists U \in \tau (x \in U \subset \neg T).$$

Proposition 2.5. *If (X, τ) is a neighbourhood space, then $\langle X, \bowtie_\tau\rangle$ is a topological quasi-apartness space.*

Proof. It is straightforward to show that $\bowtie_\tau$ satisfies (QB1), (QB2), and (QB3_m).

(QB3_r): Suppose that $S \bowtie_\tau T$ and $S \bowtie_\tau T'$. Then for each $x \in S$ there exist $U, V \in \tau$ such that $x \in U \subset \neg T$ and $x \in V \subset \neg T'$, and hence there exists $W \in \tau$ such that $x \in W \subset U \cap V \subset \neg T \cap \neg T' = \neg(T \cup T')$. Therefore $S \bowtie_\tau (T \cup T')$.

(QBT1): Suppose that $x \in -S$. Then $\{x\} \bowtie_\tau S$, and hence there exists $U \in \tau$ such that $x \in U \subset \neg S$. Since $y \in U \subset \neg S$ for each $y \in U$, we have $U \subset -S$, and therefore $x \in U \subset \neg\neg U \subset \neg\neg - S$. Thus $-S \bowtie_\tau \neg - S$.

(QBT2): Supposet that $S_i \bowtie_\tau \neg S_i$ for each $i \in I$ and $x \in \bigcup_{i\in I} S_i$. Then there exists $i \in I$ such that $x \in S_i$, and hence there exists $U \in \tau$ such that $x \in U \subset \neg\neg S_i \subset \neg\neg(\bigcup_{i\in I} S_i)$. Therefore $(\bigcup_{i\in I} S_i) \bowtie_\tau \neg(\bigcup_{i\in I} S_i)$. □

Proposition 2.6. *Let (X, τ) and (Y, τ') be neighbourhood spaces. If $f : (X, \tau) \to (Y, \tau')$ is continuous, then $f : \langle X, \bowtie_\tau\rangle \to \langle Y, \bowtie_{\tau'}\rangle$ is strongly continuous.*

Proof. Suppose that $f : (X, \tau) \to (Y, \tau')$ is continuous and $f(S) \bowtie_{\tau'} f(T)$. Then for each $x \in S$ there exists $V \in \tau'$ such that $f(x) \in V \subset \neg f(T)$, and hence there exists $U \in \tau$ such that $x \in U \subset f^{-1}(V) \subset f^{-1}(\neg f(T)) = \neg f^{-1}(f(T)) \subset \neg T$. Therefore $S \bowtie_\tau T$. □

Proposition 2.7. *Let (X, τ) be a neighbourhood space. Then $\tau \subset \tau_{\bowtie_\tau}$.*

Proof. Let $U \in \tau$. Then, since $x \in U \subset \neg\neg U$ for each $x \in U$, we have $U \bowtie_\tau \neg U$, and hence $U \in \tau_{\bowtie_\tau}$. □

Proposition 2.8. *Let $\langle X, \bowtie\rangle$ be a weakly topological quasi-apartness space. Then $\bowtie \preceq \bowtie_{\tau_\bowtie}$. Moreover, if $\langle X, \bowtie\rangle$ is topological, then $\bowtie_{\tau_\bowtie} \preceq \bowtie$.*

Proof. Suppose that $S \bowtie T$. Then $S \subset -T \subset \neg T$. Since $-T \in \tau_\bowtie$, by (QBT1), we have $S \bowtie_{\tau_\bowtie} T$. Suppose that $\langle X, \bowtie\rangle$ is topological and $S \bowtie_{\tau_\bowtie} T$. Then for each $x \in S$ there exists $U_x \subset X$ such that $U_x \bowtie \neg U_x$ and $x \in U_x \subset \neg T$, and hence $(\bigcup_{x\in S} U_x) \bowtie \neg(\bigcup_{x\in S} U_x)$, by (QBT2). Therefore, since $S \subset \bigcup_{x\in S} U_x$ and $T \subset \neg(\bigcup_{x\in S} U_x)$, we have $S \bowtie T$. □

An *adjunction* $\langle F, G, \eta, \varepsilon\rangle$ between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, and natural transformations $\eta : 1_\mathbf{C} \to GF$ and $\varepsilon : FG \to 1_\mathbf{D}$ such that $\varepsilon_F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta_G = 1_G$. The functor F is the *left-adjoint*, and the functor G is the *right-adjoint*. The natural transformation η is the *unit*, and the natural transformation ε is the *counit*. The adjunction $\langle F, G, \eta, \varepsilon\rangle$ is called an *adjoint equivalence* if both the unit η and the counit ϵ are natural isomorphisms. For basic notions and results in category theory, we refer the reader to [12, 14, 15, 17].

Let $\mathbf{Qap}^\bowtie$ denote the category of (set-set) quasi-apartness spaces with (set-set) quasi-apartness spaces as objects and strongly continuous functions as morphisms, and let $\mathbf{Qap}^\bowtie_T$ denote the full subcategory of $\mathbf{Qap}^\bowtie$

whose objects are topological quasi-apartness spaces and whose morphisms are strongly continuous functions.

Theorem 2.9. *There exists an adjunction between* $\mathbf{Qap}_T^{\bowtie}$ *and* $\mathbf{Nbh}$ *whose unit is a natural isomorphism.*

Proof. Define a functor F_T from $\mathbf{Qap}_T^{\bowtie}$ to $\mathbf{Nbh}$ by $F_T\langle X, \bowtie\rangle = (X, \tau_{\bowtie})$ and $F_T f = f$, and define a functor G_T from $\mathbf{Nbh}$ to $\mathbf{Qap}_T^{\bowtie}$ by $G_T(X,\tau) = \langle X, \bowtie_\tau\rangle$ and $G_T f = f$. Then F_T and G_T are faithful functors, by Proposition 2.4 and 2.6.

Furthermore, by Proposition 2.7 and 2.8, we see that if we let $\eta_{T(X,\tau)}$ and $\epsilon_{T\langle X,\bowtie\rangle}$ denote the indentity map on the set X, then $\eta_{T\langle X,\bowtie\rangle} : \langle X, \bowtie\rangle \to \langle X, \bowtie_{\tau_{\bowtie}}\rangle$ and $\epsilon_{T(X,\tau)} : (X, \tau_{\bowtie_\tau}) \to (X,\tau)$ are morphisms in the category $\mathbf{Qap}_T^{\bowtie}$ and $\mathbf{Nbh}$, respectively. Hence $\eta_T : 1_{\mathbf{Qap}_T^{\bowtie}} \to G_T F_T$ is a natural isomorphism and $\epsilon_T : F_T G_T \to 1_{\mathbf{Nbh}}$ is a natural transformation satisfying $\epsilon_{T F_T} \circ F_T \eta_T = 1_{F_T}$ and $G_T \epsilon_T \circ \eta_{T G_T} = 1_{G_T}$. Therefore $\langle F_T, G_T, \eta_T, \epsilon_T\rangle$ forms an adjunction between $\mathbf{Qap}_T^{\bowtie}$ and $\mathbf{Nbh}$. □

A quasi-apartness space $\langle X, \bowtie\rangle$ is *left joinable* if

QB3$_l$. $S \bowtie T \wedge S' \bowtie T \Longrightarrow (S \cup S') \bowtie T$

for each sets $S, S', T \subset X$, and *completely left joinable* if

QBC$_l$. $\forall i \in I (S_i \bowtie T) \Longrightarrow (\bigcup_{i\in I} S_i) \bowtie T$

for each family $\{S_i \mid i \in I\}$ of subsets of X and set $T \subset X$.

Note that a quasi-apartness space is always *right joinable* by (QB3$_r$).

Lemma 2.10. *If* (X,τ) *is a neighbourhood space, then* $\langle X, \bowtie_\tau\rangle$ *is a completely left joinable quasi-apartness space.*

Proof. It is enough to show that $\bowtie_\tau$ satisfies (QBC$_l$), by Proposition 2.5. Suppose that $S_i \bowtie_\tau T$ for each $i \in I$ and $x \in \bigcup_{i\in I} S_i$. Then there exists $i \in I$ such that $x \in S_i$, and hence there exists $U \in \tau$ such that $x \in U \subset \neg T$. Therefore we have $(\bigcup_{i\in I} S_i) \bowtie_\tau T$. □

Proposition 2.11. *A quasi-apartness space is topological if and only if it is completely left joinable.*

Proof. Let $\langle X, \bowtie \rangle$ be a topological quasi-apartness space. Then for each $S, T \subset X$, $S \bowtie T$ if and only if $S \bowtie_{\tau_{\bowtie}} T$, by Proposition 2.8, and therefore, since $\bowtie_{\tau_{\bowtie}}$ is completely left joinable by Lemma 2.10, so is $\bowtie$.

Conversely, let $\langle X, \bowtie \rangle$ be a completely left joinable quasi-apartness space. Since $-S \subset \neg\neg - S$, we have $-S \subset -\neg - S$, by (QB4), and hence $\{x\} \bowtie \neg - S$ for each $x \in -S$. Therefore $-S \bowtie \neg - S$, by (QBC$_l$). Suppose that $S_i \bowtie \neg S_i$ for each $i \in I$. Then, since $\neg(\bigcup_{i \in I} S_i) \subset \neg S_j$ for each $j \in I$, we have $S_j \bowtie \neg(\bigcup_{i \in I} S_i)$ for each $j \in I$, and hence $(\bigcup_{i \in I} S_i) \bowtie \neg(\bigcup_{i \in I} S_i)$, by (QBC$_l$). □

After acquiring the equivalence between the notion of a topological quasi-apartness space and the notion of a completely left joinable quasi-apartness space, we can construct an adjoint equivalence between the category of (point-set) quasi-apartness spaces and the category of topological quasi-apartness spaces.

Let $\langle X, - \rangle$ be a (point-set) quasi-apartness space. Then we can define a binary relation $\bowtie_-$ on the subsets of X by

$$S \bowtie_- T \Longleftrightarrow S \subset -T.$$

Proposition 2.12. *If $\langle X, - \rangle$ is a (point-set) quasi-apartness space, then $\langle X, \bowtie_- \rangle$ is a completely left joinable (set-set) quasi-apartness on X.*

Proof. It is straightforward to see that $\bowtie_-$ satisfies (QB1), (QB2), (QB3$_m$), (QB3$_r$), (QB4) and (QBC$_l$). □

Proposition 2.13. *Let $\langle X, - \rangle$ and $\langle Y, -' \rangle$ be (point-set) quasi-apartness spaces. Then $f : \langle X, - \rangle \to \langle Y, -' \rangle$ is continuous if and only if $f : \langle X, \bowtie_- \rangle \to \langle Y, \bowtie_{-'} \rangle$ is strongly continuous.*

Proof. Suppose that $f : \langle X, - \rangle \to \langle Y, -' \rangle$ is continuous and $f(S) \bowtie_{-'} f(T)$. Then $f(x) \in -' f(T)$ for each $x \in S$, and hence $x \in -T$ for each $x \in S$. Therefore $S \bowtie_- T$. Conversely suppose that $f : \langle X, \bowtie_- \rangle \to \langle Y, \bowtie_{-'} \rangle$ is strongly continuous and $f(x) \in -' f(S)$. Then $f(\{x\}) \bowtie_{-'} f(S)$, and hence $\{x\} \bowtie_- S$. Therefore $x \in -S$. □

Proposition 2.14. *If $\langle X, \bowtie \rangle$ is a (set-set) quasi-apartness space, then $\langle X, -_{\bowtie} \rangle$ is a (point-set) quasi-apartness space.*

Proof. It is straightforward to see that $-_{\bowtie}$ satisfies (QA1), (QA2), (QA3) and (QA4). □

Proposition 2.15. *Let $\langle X, \bowtie\rangle$ and $\langle Y, \bowtie'\rangle$ be quasi-apartness spaces. If $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous, then $f : \langle X, -_{\bowtie}\rangle \to \langle Y, -_{\bowtie'}\rangle$ is continuous. Moreover, if $\langle X, \bowtie\rangle$ is completely left joinable, then the converse holds.*

Proof. Suppose that $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous and $f(x) \in -_{\bowtie'} f(S)$. Then $f(\{x\}) \bowtie' f(S)$, and hence $\{x\} \bowtie S$. Threrefore $x \in -_{\bowtie} S$. Suppose that $\langle X, \bowtie\rangle$ is completely left joinable and $f : \langle X, -_{\bowtie}\rangle \to \langle Y, -_{\bowtie}\rangle$ is continuous. Assume further that $f(S) \bowtie' f(T)$. Then $f(x) \in -_{\bowtie'} f(T)$ for each $x \in S$. Hence $x \in -_{\bowtie} T$ for each $x \in S$, and therefore $\{x\} \bowtie T$ for each $x \in S$. Thus $S \bowtie T$, by (QBC$_l$). □

Proposition 2.16. *Let $\langle X, -\rangle$ be a (point-set) quasi-apartness space. Then $-S = -_{\bowtie_-} S$ for each set $S \subset X$.*

Proof. $x \in -S \Longleftrightarrow \{x\} \bowtie_- S \Longleftrightarrow x \in -_{\bowtie_-} S$ for each $x \in X$ and $S \subset X$. □

Proposition 2.17. *Let $\langle X, \bowtie\rangle$ be a quasi-apartness space. Then $\bowtie \preceq \bowtie_{-_{\bowtie}}$. Moreover if $\langle X, \bowtie\rangle$ is completely left joinable, then $\bowtie_{-_{\bowtie}} \preceq \bowtie$.*

Proof. Suppose that $S \bowtie T$. Then $\{x\} \bowtie T$ for each $x \in S$, and hence $x \in -_{\bowtie} T$ for each $x \in S$. Therefore $S \bowtie_{-_{\bowtie}} T$. Conversely, suppose that $\langle X, \bowtie\rangle$ is completely left joinable and $S \bowtie_{-_{\bowtie}} T$. Then $x \in -_{\bowtie} T$ for each $x \in S$, and hence $\{x\} \bowtie T$ for each $x \in S$. Therefore $S \bowtie T$, by (QBC$_l$). □

Let $\mathbf{Qap}^-$ denote the category of (point-set) quasi-apartness spaces with (point-set) quasi-apartness spaces as objects and continuous functions as morphisms.

Theorem 2.18. *There exists an adjunction between $\mathbf{Qap}^-$ and $\mathbf{Qap}^{\bowtie}$ whose unit is a natural isomorphism, and there exists an adjoint equivalence between $\mathbf{Qap}^-$ and $\mathbf{Qap}^{\bowtie}_T$.*

Proof. Define a functor F from $\mathbf{Qap}^-$ to $\mathbf{Qap}^{\bowtie}_T$ by $F\langle X, -\rangle = \langle X, \bowtie_-\rangle$ and $Ff = f$, and define a functor G from $\mathbf{Qap}^{\bowtie}$ to $\mathbf{Qap}^-$ by $G\langle X, \bowtie\rangle = \langle X, -_{\bowtie}\rangle$ and $Gf = f$. Then F is a full and faithful functor and G is a faithful functor (and a full and faithful functor from $\mathbf{Qap}^{\bowtie}_T$ to $\mathbf{Qap}^-$), by Proposition 2.13 and 2.15.

By Proposition 2.16 and 2.17, we see that if we let $\eta_{\langle X,-\rangle}$ and $\epsilon_{\langle X,\bowtie\rangle}$ denote the indentity map on the set X, then $\eta_{\langle X,-\rangle} : \langle X, -\rangle \to \langle X, -_{\bowtie_-}\rangle$ and $\epsilon_{\langle X,\bowtie\rangle} : \langle X, \bowtie_{-_{\bowtie}}\rangle \to \langle X, \bowtie\rangle$ are morphisms in the category $\mathbf{Qap}^-$ and

$\mathbf{Qap}^{\bowtie}$, respectively. Hence $\eta : 1_{\mathbf{Qap}^-} \to GF$ is a natural isomorphism and $\epsilon : FG \to 1_{\mathbf{Qap}^{\bowtie}}$ is a natural transformation satisfying $\epsilon_F \circ F\eta = 1_F$ and $G\epsilon \circ \eta_G = 1_G$. Therefore $\langle F, G, \eta, \epsilon \rangle$ forms an adjunction between $\mathbf{Qap}^-$ and $\mathbf{Qap}^{\bowtie}$.

Furthermore, by Proposition 2.17, the natural transformation $\epsilon : FG \to 1_{\mathbf{Qap}^{\bowtie}_T}$ is a natural isomorphism, and hence $\langle F, G, \eta, \epsilon \rangle$ forms an adjoint equivalence between $\mathbf{Qap}^-$ and $\mathbf{Qap}^{\bowtie}_T$. □

Now we can reprove Theorem 2.9 as a corollary of Theorem 2.18.

Corollary 2.19. *There exists an adjunction between* $\mathbf{Qap}^{\bowtie}_T$ *and* $\mathbf{Nbh}$ *whose unit is a natural isomorphism.*

Proof. By Theorem 2.18 and Theorem 4.1 in [13]. □

A quasi-apartness space $\langle X, \bowtie \rangle$ is *symmetric* if

QBS. $S \bowtie T \Longrightarrow T \bowtie S$

for each sets $S, T \subset X$.

The following lemma shows that a weakly topological symmetric quasi-apartness space has a peculiar property.

Lemma 2.20. *Let* $\langle X, \bowtie \rangle$ *be a weakly topological symmetric quasi-apartness space. Then*

1. $S \subset \neg - S$,
2. $\neg - S \bowtie \neg\neg - S$,
3. $S \bowtie T \Longleftrightarrow \neg - S \cap \neg - T = \emptyset$.

Proof. (1): Since $-S \subset \neg S$, we have $S \subset \neg - S$.

(2): Since $-S \bowtie \neg - S$, we have $\neg - S \bowtie -S$, by (QBS), and hence $\neg - S \subset - - S \subset \neg - S$. Therefore $\neg - S = - - S$. Since $- - S \bowtie \neg - -S$, by (QBT1), we have $\neg - S \bowtie \neg\neg - S$.

(3): Suppose that $S \bowtie T$ and $x \in \neg - S \cap \neg - T$. Then, since $\neg - T = - - T$, we have $\{x\} \bowtie -T$, and therefore, since $S \subset -T$, we have $\{x\} \bowtie S$. Hence $x \in -S$, a contradiction. Conversely, suppose that $\neg - S \cap \neg - T = \emptyset$. Then, since $T \subset \neg - T$, we have $T \subset \neg\neg - S$, and, since $S \subset \neg - S$ and $\neg - S \bowtie \neg\neg - S$, we have $S \bowtie T$. □

The following corollary is a straightforward consequence of Lemma 2.20.

Corollary 2.21. *A weakly topological quasi-apartness space* $\langle X, \bowtie \rangle$ *is symmetric if and only if* $S \bowtie T \Longleftrightarrow \neg - S \cap \neg - T = \emptyset$ *for each* $S, T \subset X$.

3 Uniform quasi-apartness spaces

A *uniform space* $(X, \mathcal{U})$ is pair of a set X and a set $\mathcal{U}$ of subsets of $X \times X$ such that

Ub1. $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U}(W \subset U \cap V)$,

Ub2. $\forall U \in \mathcal{U}(\Delta \subset U)$,

Ub3. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \subset U^{-1})$,

Ub4. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \circ V \subset U)$.

Here $\Delta = \{(x, x) \mid x \in X\}$, and $U^{-1} = \{(x, y) \mid (y, x) \in U\}$ and $U \circ V = \{(x, z) \mid \exists y((x, y) \in V \wedge (y, z) \in U)\}$ for each $U, V \subset X \times X$. A function f between uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ is *uniformly continuous* if for each $V \in \mathcal{U}'$ there exists $U \in \mathcal{U}$ such that $(x, y) \in U \Longrightarrow (f(x), f(y)) \in V$ for each $x, y \in X$.

Remark 3.1. Schuster et al [18] and Bridges and Vîţă [9] defined a uniformity on a set X as a set $\mathcal{U}$ of $X \times X$ such that

U1. $\forall V \subset X \forall U \in \mathcal{U}(U \subset V \Longrightarrow V \in \mathcal{U}) \wedge \forall U, V \in \mathcal{U}(U \cap V \in \mathcal{U})$,

U2. $\forall U \in \mathcal{U}(\Delta \subset U) \wedge \forall U \in \mathcal{U} \exists V \in \mathcal{U}(V = V^{-1} \wedge V \subset U)$,

U3. $\forall U \in \mathcal{U} \exists V \in \mathcal{U}(V \circ V \subset U)$,

U4 $\forall U \in \mathcal{U} \exists V \in \mathcal{U} \forall x, y \in X[(x, y) \in U \vee (x, y) \in \neg V]$,

that is, they employed a set of entourages with an extra condition (U4) to define a uniformity. The following shows why we adopt a base for uniformity without the extra condition as the definition of a uniformity.

Consider a family $\mathcal{U}_d$ of entourages satisfying (U1), (U2) and (U3) for the discrete uniformity on a set X. Then, since $\Delta \in \mathcal{U}_d$, it consists of all subsets of $X \times X$ containing Δ. The family $\mathcal{U}_d$ is not a set in a predicative system, such as CZF [2]. But the singleton $\{\Delta\}$ which is a set in CZF forms a base for the discrete uniformity.

Furthermore, suppose that the condition (U4) holds for $\mathcal{U}_d$. Then, since $\Delta \in \mathcal{U}_d$, there exists $V \in \mathcal{U}_d$ such that

$$\forall x, y \in X[(x, y) \in \Delta \vee (x, y) \in \neg V],$$

and hence $x = y \vee \neg(x = y)$ for all $x, y \in X$. If $X = \mathbf{R}$, then this is equivalent to the *weak limited principle of omniscience* (WLPO) [5]:

$$\forall x \in \mathbf{R}[x = 0 \vee \neg(x = 0)].$$

Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot find one of (U4) for the discrete uniformity on $\mathbf{R}$.

A quasi-apartness space $\langle X, \bowtie \rangle$ is *weakly uniform* if it is symmetric and

QBU. $S \bowtie T \Longrightarrow \exists S', T' \subset X(S \bowtie \neg S' \wedge \neg T' \bowtie T \wedge S' \cap T' = \emptyset)$

for each sets $S, T \subset X$.

Lemma 3.2. (QB3_m) *and* (QBU) *imply* (QB4).

Proof. Suppose that $-S \subset \neg T$ and $x \in -S$. Then $\{x\} \bowtie S$, and hence, by (QBU), there exist $A, S' \subset X$ such that $\{x\} \bowtie \neg A$, $\neg S' \bowtie S$ and $A \cap S' = \emptyset$. Since $\neg S' \subset -S \subset \neg T$, we have $T \subset \neg\neg S'$, and, since $A \subset \neg S'$, we have $\neg\neg S' \subset \neg A$. Therefore $T \subset \neg A$, and so $\{x\} \bowtie T$, by (QB3_m). Thus $x \in -T$. □

Lemma 3.3. *A weakly topological symmetric quasi-apartness space is weakly uniform.*

Proof. Let $\langle X, \bowtie \rangle$ be a weakly topological symmetric quasi-apartness space. Let $S' = \neg - S$ and $T' = \neg - T$. Then, by Lemma 2.20 (1), (2) and (QBS), we have $S \bowtie \neg S'$ and $\neg T' \bowtie T$, and, by Lemma 2.20 (3), if $S \bowtie T$, then $S' \cap T' = \emptyset$. □

A quasi-apartness space $\langle X, \bowtie \rangle$ is *joinable* if

QBJ. $\forall i \in \{1, \dots, n\}(S_i \bowtie T_i) \wedge S \times T \subset \bigcup_{i=1}^{n}(S_i \times T_i) \Longrightarrow S \bowtie T$

for each sets $S, S_1, \dots, S_n, T, T_1, \dots, T_n \subset X$, and *strongly joinable* if

QBJ$_s$. $\forall i \in \{1, \dots, n\}(S_i \bowtie T_i) \wedge S \times T \subset \neg\neg \bigcup_{i=1}^{n}(S_i \times T_i) \Longrightarrow S \bowtie T$

for each sets $S, S_1, \dots, S_n, T, T_1, \dots, T_n \subset X$.

Lemma 3.4. (QBJ) *implies* (QB3_l) *and* (QB3_r).

Proof. Suppose that $S \bowtie T$ and $S' \bowtie T$. Then, since $(S \cup S') \times T = (S \times T) \cup (S' \times T)$, we have $(S \cup S') \bowtie T$, by (QBJ). Similarly, we have (QB3_r). □

A quasi-apartness space $\langle X, \bowtie \rangle$ is *uniform* if it is weakly uniform and strongly joinable.

Lemma 3.5. *There is a uniform quasi-apartness space which is not weakly topological.*

Proof. Let $\bowtie$ be a quasi-apartness relation on $\mathbf{R}$ defined by

$$S \bowtie T \Longleftrightarrow \exists r > 0 \forall x \in S \forall y \in T(|x - y| > r).$$

Then it is straightforward to see that $\bowtie$ satisfies (QB1), (QB2), (QB3$_m$), and (QBS).

(QBU): Suppose that $S \bowtie T$. Then there exists $r > 0$ such that $|x-y| > r$ for each $x \in S$ and $y \in T$. Let $S' = \{z \in X \mid \exists x \in S(|x - z| < r/2)\}$ and $T' = \{z \in X \mid \exists y \in T(|y - z| < r/2)\}$. Then $S' \cap T' = \emptyset$. If $|x - z| < r/2$ for some $x \in S$ and $z \in \neg S'$, then $z \in S'$, a contradiction. Hence $|x - z| > r/3$ for each $x \in S$ and $z \in \neg S'$, and therefore $S \bowtie \neg S'$. Similarly, we have $\neg T' \bowtie T$.

(QBJ$_s$): Suppose that $S_i \bowtie T_i$ for each $i = 1, \ldots, n$ and $S \times T \subset \neg\neg \bigcup_{i=1}^{n}(S_i \times T_i)$. Then there exist $r_1, \ldots, r_n > 0$ such that $\forall x \in S_i \forall y \in T_i(|x - y| > r_i)$ for each $i = 1, \ldots, n$. Let $r = \min\{r_1, \ldots, r_n\}$, and suppose that $|x - y| < r$ for some $x \in S$ and $y \in T$. If $(x, y) \in S_i \times T_i$ for some $i = 1, \ldots, n$, then $r_i < |x - y| < r \leq r_i$, a contradiction, and hence $(x, y) \in \neg \bigcup_{i=1}^{n}(S_i \times T_i)$. This contradiction entails that $|x - y| > r/2$ for each $x \in S$ and $y \in T$. Thus $S \bowtie T$.

Hence $(\mathbf{R}, \bowtie)$ is a uniform quasi-apartness space. Let $S = [0, \infty)$. Then $-S = (-\infty, 0)$ and $\neg - S = S$, and hence $-S \bowtie \neg - S$ is impossible. Therefore $(\mathbf{R}, \bowtie)$ is not a weakly topological quasi-apartness space. □

Let $(X, \mathcal{U})$ be a uniform space. Then we can define a binary relation $\bowtie_{\mathcal{U}}$ on the subsets of X by

$$S \bowtie_{\mathcal{U}} T \Longleftrightarrow \exists U \in \mathcal{U}(S \times T \subset \neg U).$$

Proposition 3.6. *If $(X, \mathcal{U})$ is a uniform space, then $\langle X, \bowtie_{\mathcal{U}} \rangle$ is a uniform quasi-apartness space.*

Proof. It is straightforward to show that $\bowtie_{\mathcal{U}}$ satisfies (QB1), (QB2), (QB3$_m$), and (QBS).

(QBU): Suppose that $S \bowtie_{\mathcal{U}} T$. Then there exists $U \in \mathcal{U}$ such that $S \times T \subset \neg U$, and hence there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$. Let

$S' = \{y \in X \mid \exists x \in S((x, y) \in V)\}$ and $T' = \{y \in X \mid \exists z \in T((y, z) \in V)\}$. If $(x, y) \in S \times \neg S'$ and $(x, y) \in V$, then $y \in S'$ and $y \in \neg S'$, a contradiction. Hence $S \times \neg S' \subset \neg V$, and therefore $S \bowtie_{\mathcal{U}} \neg S'$. Similarly, we have $\neg T' \bowtie_{\mathcal{U}} T$. Assume that $y \in S' \cap T'$. Then there exist $x \in S$ and $z \in T$ such that $(x, y) \in V$ and $(y, z) \in V$, and hence $(x, z) \in S \times T$ and $(x, z) \in V \circ V \subset U$, a contradiction. Thus $S' \cap T' = \emptyset$.

(QBJ$_s$): Suppose that $S_i \bowtie_{\mathcal{U}} T_i$ for each $i = 1, \ldots, n$ and $S \times T \subset \neg\neg \bigcup_{i=1}^{n}(S_i \times T_i)$. Then there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $S_i \times T_i \subset \neg U_i$ for each $i = 1, \ldots, n$, and hence there exists $V \in \mathcal{U}$ such that $V \subset U_1 \cap \ldots \cap U_n$. Therefore $S \times T \subset \neg\neg \bigcup_{i=1}^{n}(S_i \times T_i) \subset \neg\neg \bigcup_{i=1}^{n} \neg U_i = \neg \bigcap_{i=1}^{n} U_i \subset \neg V$. Thus $S \bowtie_{\mathcal{U}} T$. □

Proposition 3.7. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ be uniform spaces. If $f : (X, \mathcal{U}) \to (Y, \mathcal{U}')$ is uniformly continuous, then $f : \langle X, \bowtie_{\mathcal{U}} \rangle \to \langle Y, \bowtie_{\mathcal{U}'} \rangle$ is strongly continuous.*

Proof. Suppose that $f : (X, \mathcal{U}) \to (Y, \mathcal{U}')$ is uniformly continuous and $f(S) \bowtie_{\mathcal{U}'} f(T)$. Then there exists $V \in \mathcal{U}'$ such that $f(S) \times f(T) \subset \neg V$. Since f is uniformly continuous, there exists $U \in \mathcal{U}$ such that $(x, y) \in U \Longrightarrow (f(x), f(y)) \in V$ for each $x, y \in X$. If $(x, y) \in S \times T$ and $(x, y) \in U$, then $(f(x), f(y)) \in f(S) \times f(T)$ and $(f(x), f(y)) \in V$, a contradiction. Hence $S \times T \subset \neg U$, and therefore $S \bowtie_{\mathcal{U}} T$. □

Let $\langle X, \bowtie \rangle$ be a quasi-apartness space. Then we can define a family $\mathcal{U}_{\bowtie}$ of subsets of $X \times X$ by

$$\mathcal{U}_{\bowtie} = \left\{ \bigcap_{i=1}^{n} \neg(S_i \times T_i) \mid S_i \bowtie T_i \text{ for each } i = 1, \ldots, n \right\}.$$

Lemma 3.8. *Let S, S', T and T' be subsets of a set X such that $S' \cap T' = \emptyset$. Then $\neg(\neg T' \times T) \circ \neg(S \times \neg S') \subset \neg(S \times T)$.*

Proof. Let $(x, y) \in \neg(S \times \neg S')$ and $(y, z) \in \neg(\neg T' \times T)$, and suppose that $(x, z) \in S \times T$. Then $y \in \neg\neg S'$ and $y \in \neg\neg T'$. Assume that $y \in S'$. Then, since $S' \subset \neg T'$, we have $y \in \neg T'$, a contradiction. Hence $y \in \neg S'$. This contradiction entails that $(x, z) \in \neg(S \times T)$. □

Proposition 3.9. *If $\langle X, \bowtie \rangle$ is a weakly uniform quasi-apartness space, then $(X, \mathcal{U}_{\bowtie})$ is a uniform space.*

Proof. It is straightforward to show that $\mathcal{U}_{\bowtie}$ satisfies (Ub1), (Ub2) and (Ub3). To show (Ub4), let $U \in \mathcal{U}_{\bowtie}$. Then there exist subsets $S_1, \ldots, S_n, T_1, \ldots, T_n$ of X such that $U = \bigcap_{i=1}^n \neg(S_i \times T_i)$ and $S_i \bowtie T_i$ for each $i = 1, \ldots, n$. By (QBU), there exist subsets $S'_1, \ldots, S'_n, T'_1, \ldots, T'_n$ of X such that $S_i \bowtie \neg S'_i$, $\neg T'_i \bowtie T_i$ and $S'_i \cap T'_i = \emptyset$ for each $i = 1, \ldots, n$. Let $V = \bigcap_{i=1}^n \neg(S_i \times \neg S'_i) \cap \bigcap_{i=1}^n \neg(\neg T'_i \times T_i)$. Then $V \in \mathcal{U}_{\bowtie}$ and, for each $i = 1, \ldots, n$, we have

$$V \circ V \subset \neg(\neg T'_i \times T_i) \circ \neg(S_i \times \neg S'_i) \subset \neg(S_i \times T_i),$$

by Lemma 3.8. Hence $V \circ V \subset U$. □

Proposition 3.10. *Let $\langle X, \bowtie\rangle$ and $\langle Y, \bowtie'\rangle$ be weakly uniform quasi-apartness spaces. If $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous, then $f : (X, \mathcal{U}_{\bowtie}) \to (Y, \mathcal{U}_{\bowtie'})$ is uniformly continuous.*

Proof. Suppose that $f : \langle X, \bowtie\rangle \to \langle Y, \bowtie'\rangle$ is strongly continuous, and let $V \in \mathcal{U}_{\bowtie'}$. Then there exist subsets $S_1, \ldots, S_n, T_1, \ldots, T_n$ of Y such that $V = \bigcap_{i=1}^n \neg(S_i \times T_i)$ and $S_i \bowtie' T_i$ for each $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, since $f(f^{-1}(S_i)) \subset S_i$ and $f(f^{-1}(T_i)) \subset T_i$, we have $f(f^{-1}(S_i)) \bowtie' f(f^{-1}(T_i))$. Hence $f^{-1}(S_i) \bowtie f^{-1}(T_i)$ for each $i = 1, \ldots, n$. Let $U = \bigcap_{i=1}^n \neg(f^{-1}(S_i) \times f^{-1}(T_i))$. Then $U \in \mathcal{U}_{\bowtie}$. If $(x, y) \in U$, then $(f(x), f(y)) \in \neg(S_i \times T_i)$ for each $i = 1, \ldots, n$, and hence $(f(x), f(y)) \in V$. Thus $f : (X, \mathcal{U}_{\bowtie}) \to (Y, \mathcal{U}_{\bowtie'})$ is uniformly continuous. □

Proposition 3.11. *Let $(X, \mathcal{U})$ be a uniform space. Then for each $U \in \mathcal{U}_{\bowtie_{\mathcal{U}}}$ there exists $V \in \mathcal{U}$ such that $V \subset U$.*

Proof. Let $U \in \mathcal{U}_{\bowtie_{\mathcal{U}}}$. Then there exist subsets $S_1, \ldots, S_n, T_1, \ldots, T_n$ of X such that $U = \bigcap_{i=1}^n \neg(S_i \times T_i)$ and $S_i \bowtie_{\mathcal{U}} T_i$ for each $i = 1, \ldots, n$, and hence there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $S_i \times T_i \subset \neg U_i$ for each $i = 1, \ldots, n$. Choose $V \in \mathcal{U}$ such that $V \subset \bigcap_{i=1}^n U_i$. Then, since $U_i \subset \neg(S_i \times T_i)$ for each $i = 1, \ldots, n$, we have $V \subset U$. □

Proposition 3.12. *Let $\langle X, \bowtie\rangle$ be a weakly uniform quasi-apartness space. Then $\bowtie \preceq \bowtie_{\mathcal{U}_{\bowtie}}$. Moreover, if $\langle X, \bowtie\rangle$ is uniform, then $\bowtie_{\mathcal{U}_{\bowtie}} \preceq \bowtie$.*

Proof. Suppose that $S \bowtie T$. Then, since $\neg(S \times T) \in \mathcal{U}_{\bowtie}$ and $S \times T \subset \neg\neg(S \times T)$, we have $S \bowtie_{\mathcal{U}_{\bowtie}} T$. Suppose that $\langle X, \bowtie\rangle$ is uniform and $S \bowtie_{\mathcal{U}_{\bowtie}} T$. Then there exist $S_1, \ldots, S_n, T_1, \ldots, T_n$ such that $S \times T \subset \neg \bigcap_{i=1}^n \neg(S_i \times T_i) = \neg\neg \bigcup_{i=1}^n (S_i \times T_i)$ and $S_i \bowtie T_i$ for each $i = 1, \ldots, n$. Therefore $S \bowtie T$, by (QBJ_s). □

Let **Uni** denote the category of uniform spaces with uniform spaces as objects and uniformly continuous functions as morphisms, and let $\mathbf{Qap}_U^{\bowtie}$ denote the full subcategory of $\mathbf{Qap}^{\bowtie}$ whose objects are uniform quasi-apartness spaces and whose morphisms are strongly continuous functions.

Theorem 3.13. *There exists an adjunction between* **Uni** *and* $\mathbf{Qap}_U^{\bowtie}$ *whose counit is a natural isomorphism.*

Proof. Define a functor F_U from **Uni** to $\mathbf{Qap}_U^{\bowtie}$ by $F_U(X,\mathcal{U}) = \langle X, \bowtie_{\mathcal{U}}\rangle$ and $F_U f = f$, and define a functor G_U from $\mathbf{Qap}_U^{\bowtie}$ to **Uni** by $G_U\langle X, \bowtie\rangle = (X, \mathcal{U}_{\bowtie})$ and $G_U f = f$. Then F_U and G_U are faithful functors, by Proposition 3.7 and 3.10.

Furthermore, by Proposition 3.11 and 3.12, we see that if we let $\eta_{U(X,\mathcal{U})}$ and $\epsilon_{U\langle X,\bowtie\rangle}$ denote the indentity map on the set X, then $\eta_{U(X,\mathcal{U})} : (X,\mathcal{U}) \to (X, \mathcal{U}_{\bowtie_{\mathcal{U}}})$ and $\epsilon_{U\langle X,\bowtie\rangle} : \langle X, \bowtie_{\mathcal{U}_{\bowtie}}\rangle \to \langle X, \bowtie\rangle$ are morphisms in the category **Uni** and $\mathbf{Qap}_U^{\bowtie}$, respectively. Hence $\eta_U : 1_{\mathbf{Uni}} \to G_U F_U$ is a natural transformation and $\epsilon_U : F_U G_U \to 1_{\mathbf{Qap}_U^{\bowtie}}$ is a natural isomorphism satisfying $\epsilon_{U F_U} \circ F_U \eta_U = 1_{F_U}$ and $G_U \epsilon_U \circ \eta_{U G_U} = 1_{G_U}$. Therefore $\langle F_U, G_U, \eta_U, \epsilon_U\rangle$ forms an adjunction between **Uni** and $\mathbf{Qap}_U^{\bowtie}$. □

Note that $\langle F_U, G_U, \eta_U, \epsilon_U\rangle$ constructed in the proof of Theorem 3.13 also forms an adjunction between **Uni** and the category of weakly uniform quasi-apartness spaces with weakly uniform quasi-apartness spaces as objects and strongly continuous functions as morphisms.

4 Concluding remarks

A (point-set) quasi-apartness space $\langle X, -\rangle$ is *set-presented* if there exists a subset C of $Pow(X)$ such that

$$x \in -S \iff \exists U \in C (x \in U \subset \neg S)$$

for each $x \in X$ and $S \subset X$. Then the (point-set) quasi-apartness $-_\tau$, induced by an open base τ [13, Proposition 2.3], is set-presented. Furthermore, the open base τ_-^s, induced by a set-presented (point-set) quasi-apartness $-$ [13, Proposition 3.6], is generated by the subbase C, and hence it forms a set. Hence there is a possibility to avoid use of the power set axiom in the results of [13] by introducing the notion of a set-presented (point-set) quasi-apartness.

Although we have not arrived at an appropriate notion of a set-presented (set-set) quasi-apartness, we hope that we will be able to avoid use of the power set axiom in this paper by introducing a similar notion. Therefore we have refrained from freely using the power set axiom, and inclined to follow the minimalist brand of constructivism as far as possible.

Of course, this attitude has caused a bit of problem when stating some category theoretic results in this paper. Since there are many open bases which generate the same open sets, the category **Top** of topological spaces equipped with the open sets (by using the power set axiom) and continuous mappings and the category **Nbh** of neighbourhood spaces equipped with the basic opens and continuous mappings are not *isomorphic*, but *equivalent*.

In the following, we summarise the category theoretic results of this paper.

A subcategory **D** of a category **C** is *reflective* and *coreflective* if the inclusion functor $I : \mathbf{D} \to \mathbf{C}$ has a left adjoint and a right adjoint, respectively.

Let us call a neighbourhood space (X, τ) is *negative* if $U \subset (\neg\neg U)^\circ$ implies $U \subset U^\circ$ for each subset U of X, where S° denotes the interior of S. Since $U \in \tau_{\bowtie_\tau}$ if and only if $U \subset (\neg\neg U)^\circ$, a negative neighbourhood space (X, τ) is isomorphic to the neighbourhood space $(X, \tau_{\bowtie_\tau})$. Therefore, by Theorem 2.9, we see that the full subcategory $\mathbf{Nbh}^\neg$ of negative neighbourhood spaces is a coreflective subcategory of **Nbh**.

On the other hand, Theorem 2.18 shows that the category $\mathbf{Qap}_T^\bowtie$ is a full coreflective subcategory of the category $\mathbf{Qap}^\bowtie$. Furthermore, calling a uniform space $(X, \mathcal{U})$ *strongly uniform* if $(X, \mathcal{U}_{\bowtie_\mathcal{U}})$ is isomorphic to the original space $(X, \mathcal{U})$, the category **stUni** of strongly uniform spaces is a full reflective subcategory of the category **Uni** of uniform spaces and uniformly continuous mappings, by Theorem 3.13.

Let **Set** be the category of sets and functions. Then a category **C** is a *category over* **Set** if there exists a faithful functor, called a *forgetful functor*, $K : \mathbf{C} \to \mathbf{Set}$. Categories **C** and **D** over **Set** with forgetful functors $K : \mathbf{C} \to \mathbf{Set}$ and $J : \mathbf{D} \to \mathbf{Set}$, respectively, is *isomorphic* (over **Set**) if there exists an isomorphism $I : \mathbf{C} \to \mathbf{D}$ such that $K = J \circ I$. Theorem 2.18 shows that the categories $\mathbf{Qap}^-$ and $\mathbf{Qap}_T^\bowtie$ are isomorphic over **Set**. Note that, in the presence of the power set axiom, $\mathbf{Qap}^-$, $\mathbf{Qap}_T^\bowtie$ and the category $\mathbf{Top}^\neg$ of negative topological spaces and continuous maps are isomorphic over **Set**, and, by identifying uniform spaces that generate the same uniformities, the categories **stUni** and $\mathbf{Qap}_U^\bowtie$ are isomorphic over **Set**.

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