A Semantic Investigation of Orthologic and Orthomodular Logic

Yutaka Miyazaki

School of Information Science,
Japan Advanced Institute of Science and Technology

February 14, 1997

Keywords: quantum logic, ortholattice, orthomodular lattice, Rickart*semigroup.

1 Introduction

The study of quantum logic had started from the foundational work of G.Birkhoff and J.von Neumann in 1936 ([1]). They pointed out that according to Hilbert-space formalism of quantum mechanics, physical propositions of a quantum system, which are represented by closed subspaces of a Hilbert space, form an orthomodular lattice. Since then, orthomodular logic, in which the truth values of formulas are interpreted by elements of an orthomodular lattice, is considered to be the most hopeful candidate of quantum logic. But orthomodular logic turned out to be rather intractable, then orthologic, whose algebraic model is an ortholattice, has been discussed in this research area.

In both orthologic and orthomodular logic, two different notions of logical consequences are considered: weak logical consequence and strong logical consequence and they determine different logics: weak logic and strong logic ([5]). In 1974, R.I.Goldblatt introduced a Kripke-style semantics for strong orthologic, and by applying filtration technique to his Kripke-style model, he proved that it has the finite model property and hence that strong orthologic is decidable ([3]).

On the other hand, the decision problem for orthomodular logic still remains open. Goldblatt also proposed in the same paper a Kripke-style semantics for strong orthomodular logic, but this semantics was not successful in solving its decision problem. In lattice theory, there is a representation theorem of orthomodular lattice proved by D.J.Foulis in 1960 ([2],[4]). This representation theorem may be helpful when we investigate the orthomodular logic semantically.

The present paper is devided into two parts. In Part I, we will try to extend the methods of Goldblatt to weak orthologic. Then we can show that it enjoys the Kripke-style semantics and that it is also decidable. In Part II, we will construct a new semantics for orthomodular logic by using the Foulis representation theorem and show that orthomodular logic is complete with respect to this semantics.

2 Semantics of weak orthologic

In Part I, we take only orthologics into account. We interpret the truth values of our formulas by elements of an ortholattice A by way of an orthovaluation v.

Strong orthologic SOL is a set of pairs of formulas (α, β) such that for any ortholattice A and for any orthovaluation v, $v(\alpha) \leq v(\beta)$ holds. On the other hand, weak orthologic WOL is a set of pairs of formulas (α, β) such that for any ortholattice A and for any orthovaluation v, if $v(\alpha) = \mathbf{1}$, then $v(\beta) = \mathbf{1}$. This $\mathbf{1}$ is the maximum element of the ortholattice A.

Goldblatt proved the decidabilty of strong orthologic in the following way.

First a Kripke-style model $\mathcal{M}$ (called an orthomodel) was introduced, which consists of a non-empty set, an irreflexive and symmetric binary relation, and a valuation function. Then it is proved that strong orthologic is complete with respect to the above model. More precisely, for any formulas α and β, the following two statements are equivalent.

(P_1) for any ortholattice A and for any orthovaluation v, $v(\alpha) \leq v(\beta)$ holds.

(Q_1) for any orthomodel $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$. (This means "for any x in the model $\mathcal{M}$, α implies β at x in $\mathcal{M}$ ".)

The filtration technique is applied to this models and it is showed that strong orthologic has the finite model property, that is, the above statement (Q_1) is shown to be equivalent to the following statement (R_1).

(R_1) for any orthomodel $\mathcal{N}$ which has at most 2^{k+l} points, $\mathcal{N} : \alpha \models \beta$, where k is the number of subformulas included in α and β together and l is the number of propositional variables included in α and β together.

We will modify the proof of Goldblatt, and will prove that for any formulas α, β, σ and τ, the following three statements are equivalent.

(P_2) for any ortholattice A and for any orthovaluation v, if $v(\sigma) \leq v(\alpha)$, then $v(\tau) \leq v(\beta)$.

(Q_2) for any orthomodel $\mathcal{M}$, if $\mathcal{M} : \sigma \models \alpha$, then $\mathcal{M} : \tau \models \beta$.

(R_2) for any orthomodel $\mathcal{N}$ which has at most 2^{k+l} points, if $\mathcal{N} : \sigma \models \alpha$, then $\mathcal{N} : \tau \models \beta$, where k is the number of subformulas included in α, β, σ and τ together and l is the number of propositional variables included in α, β, σ and τ together.

It is easy to see that there exists an algorithm which can decide whether (R_2) holds or not. Now take $\neg(\chi \wedge \neg\chi)$ for both σ and τ. Then this algorithm gives us a procedure which decide whether $(\alpha, \beta) \in$ WOS or not for given formulas α and β.

Therefore we have the following theorem.

Theorem Weak orthologic is decidable.

□

3 Semigroup semantics for orthomodular logic

In Part II, we discuss only strong orthomodular logic. This time, we interpret the truth values of our formulas by elements of an orthomodular lattice A by way of an orthomodular valuation$v$. Then orthomodular logic is defined as a set of pairs of formulas (α, β) such that for any orthomodular lattice A, and for any orthomodular valuation v, $v(\alpha) \leq v(\beta)$ holds.

Apart from Goldblatt's Kripke-style semantics for orthomodular logic, we consider a bit different algebraic structure $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ called a *Rickart * semigroup*, in which $\langle \mathrm{G}, \cdot \rangle$ is a semigroup containing the zero element 0 and such a binary operation $*$ on G that satisfies the following (a): $(x^*)^* = x$ and (b): $(x \cdot y)^* = y^* \cdot x^*$.
Moreover $\mathcal{G}$ satisfies the following condition, that is: for any $x \in \mathrm{G}$, there exists a projection e such that $\{ x \}^{(\mathrm{r})} = e \cdot \mathrm{G} = \{ e \cdot y \mid y \in \mathrm{G} \}$.

Here it is needed to explain some terms.

- An element $e \in \mathrm{G}$ is called a *projection* iff it satisfies $e^* = e \cdot e = e$.
  The set of all projections in G is denoted by P(G).
- For an element $x \in \mathrm{G}$, the set $\{ x \}^{(\mathrm{r})} := \{ y \in \mathrm{G} \mid x \cdot y = 0 \}$ is called the *right annihilator* for x.
- A projection f is called *closed* iff for f there exists some $x \in \mathrm{G}$ such that $\{ x \}^{(\mathrm{r})} = f \cdot \mathrm{G}$. The set of all closed projections in G is denoted by $\mathrm{P}_c(\mathrm{G})$

Based on Rickart * semigroups, an *orthomodular model* $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ for orthomodular logic is constructed as follows: That is, $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ is a Rickart * semigroup, and u is a function assigning to each propositional variable p_i an element $\mathrm{u}(p_i)$ of $\mathrm{P}_c(\mathrm{G})$.

The notion of truth for our formulas is defined inductively as foloows: The symbols $(\mathcal{M}, x) \models \alpha$ mean " a formula α is true at x in a model $\mathcal{M}$".

(i) $(\mathcal{M}, x) \models p_i$ iff $p_i \in \mathrm{u}(p_i) \cdot \mathrm{G}$.

(ii) $(\mathcal{M}, x) \models \alpha \wedge \beta$ iff $(\mathcal{M}, x) \models \alpha$ and $(\mathcal{M}, x) \models \beta$.

(iii) $(\mathcal{M}, x) \models \neg\alpha$ iff $\forall y \in \mathrm{G}$, [$(\mathcal{M}, y) \models \alpha$ only if $y^* \cdot x = 0$].

We write $\mathcal{M} : \alpha \models \beta$, iff for any x in a model $\mathcal{M}$, either $(\mathcal{M}, x) \not\models \alpha$ or $(\mathcal{M}, x) \models \beta$ holds. We can prove the completeness theorem.

Theorem (Completeness Theorem for orthomodular logic) For any formulas α and β, the following statemants are equivalent.

(S) For any orthomodular lattice A, and for any orthomodular valuation V, $v(\alpha) \leq v(\beta)$.

(T) For any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

$\square$

Proof of this theorem consists of two parts,
1): (S) implies (T), and 2): (T) implies (S).

1): It is not so hard to show this direction. We can define a partially order on P(G) and we can prove that $\mathrm{P}_c(\mathrm{G})$ forms an orthomodular lattice with respect to this order. Moreover we can define a suitable orthomodular valuation from Φ to $\mathrm{P}_c(\mathrm{G})$. By these facts together with other simple observations, we can prove that if (T) does not hold, then (S) does not hold.

2): It is a bit complicated to show this direction. We use the following notion to construct a Rickart * semigroup from an orthomodular lattice.

For a given ordered set $\langle \mathrm{A}, \leq \rangle$, consider such monotone map φ from A to A that for *varphi*, there exists some monotone map $\varphi^\sharp$ which satisfies the following: for any $x \in \mathrm{A}$,

$$\varphi^\sharp(\varphi(x)) \geq x \qquad \text{and} \qquad \varphi(\varphi^\sharp(x)) \leq x.$$

We call this map $\varphi^\sharp$ a *residual map* for φ, and denote the set of all such monotone maps as φ on A by G(A).

It is easily check that this G(A) is a semigroup with respect to the composition operation of maps. And particularly, if A is an orthomodular lattice, then we can define such * operation on G(A) that $\mathcal{G}_\mathrm{A} = \langle \mathrm{G(A)}, \cdot, * \rangle$ is a Rickart * semigroup. Thus we can build up the suitable orthomodular model from a given orthomodular lattice and a given orthomodular valuation by using the above construction. Therefore we can show that if (S) does not hold, then (T) does not hold.

Basically, we make use of the following theorem in the above proof.

Theorem (Foulis's representation theorem for orthomodular lattice) Let A be an orthomodular lattice. Then $\mathcal{G}_\mathrm{A} = \langle \mathrm{G(A)}, \cdot, * \rangle$ is a Rickart * semigroup, and A is isomorphic to $\mathrm{P}_c(\mathrm{G(A)})$.

References

[1] G.Birkhoff and J.von Neumann, *The logic of quantum mechanics*, Annals of Mathematics Vol.37 No.4, 823–843 (1936).

[2] D.J.Foulis, *Baer *-semigroups*, Proceedings of American Mathematical Society 11, 648–654 (1960).

[3] R.I.Goldblatt, *Semantic analysis of orthologic* Journal of Philosophical Logic 3, 19–35 (1974).

[4] S.Maeda, *Lattice Theory and Quantum Logic* (in Japanese), Makishoten, Tokyo (1980).

[5] J.Malinowski, *Strong Versus Weak Quantum Consequence Operations*, Studia Logica 51, 113–123 (1992).