On Prioritized Weighted Aggregation in Multi-criteria Decision Making

Hong-Bin Yan [a] Van-Nam Huynh [a] Yoshiteru Nakamori [a] Tetsuya Murai [b]

[a] *School of Knowledge Science, Japan Advanced Institute of Science and Technology*
1-1 Asahidai, Nomi City, Ishikawa, 923-1292 Japan

[b] *Graduate School of Information Science and Technology, Hokkaido University*
Kita-ku, Sapporo 060-0814, Japan

Abstract

This paper deals with multi-criteria decision making (MCDM) problems with multiple priorities, in which priority weights associated with the lower priority criteria are related to the satisfactions of the higher priority criteria. To do so, we first propose a prioritized weighted aggregation operator based on ordered weighted averaging (OWA) operator and triangular norms (t-norms). In order to preserve the tradeoffs among the criteria in the same priority level, we suggest that the satisfaction degree regarding each priority level is viewed as a pseudo criterion. On the other hand, t-norms are used to model the priority relationships between the criteria in different priority levels. In particular, we show that strict Archimedean t-norms perform better in inducing priority weights. As Hamacher family of t-norms provides a wide class of strict Archimedean t-norms ranging from the product to weakest t-norm, Hamacher parameterized t-norms are used to induce the priority weight for each priority level. Furthermore, considering decision maker (DM)'s requirement toward higher priority levels, a *benchmark* based approach is proposed to induce priority weight for each priority level. In particular, Łukasiewicz implication is used to compute benchmark achievement for crisp requirements; target-oriented decision analysis is utilized to obtain the benchmark achievement for fuzzy requirements. Finally, some numerical examples are used to illustrate the proposed prioritized aggregation technique as well as to compare with previous researches.

Key words: Multi-criteria decision making (MCDM), Prioritized aggregation, Ordered weighted averaging (OWA), Triangular norms (T-norms), Benchmark.

1. Introduction

Multi-criteria decision making (MCDM) problems are encountered under various situations where a number of alternatives and actions or candidates need to be chosen based on a set of criteria or attributes [22]. In MCDM problems, the final solution must be obtained from a synthesis of degree of satisfactions for all criteria, per decision alternative [20]. Central to these problems is the task of aggregation operations [7]. In general, the type of aggregation used should reflect the decision maker (DM)'s imperative and behavior of individual choice [22,38]. Consideration of different relative importances of different criteria is important as some criteria are more important than others. In this case, the DM associates different importance weights with different criteria [7,19,22,24,28,32,34]. There are several approaches to incorporating and/or assigning weights to different criteria. Typical is some form of weighted arithmetic mean, such as quasi-arithmetic means, weighted arithmetic means, weighted quasi-arithmetic means [7]. These aggregation operations work well in situations in which any differences are viewed as being in conflict because the operator reflects a form of compromise behavior among the various criteria [22,29]. In general, the importance information associated with different criteria plays a fundamental role in the comparison between alternatives by overseeing tradeoffs between respective satisfactions of different criteria [38,39].

A concept closely related to the importance of criteria is the priority of criteria [11,38]. In practical decision making situations, it is usual for DM(s) to consider different priorities of criteria in MCDM. A typical example is in the case of buying a car based upon the criteria of *safety* and *cost*. In this case, usually we may not allow compensation between *cost* and *safety*. Simply speaking, by saying criterion *safety* has a higher priority than criterion *cost*, it indicates that we are not willing to tradeoff satisfaction of criterion

Email addresses: hby19790214@gmail.com (Hong-Bin Yan), huynh@jaist.ac.jp (Van-Nam Huynh), nakamori@jaist.ac.jp (Yoshiteru Nakamori), murahiko@main.ist.hokudai.ac.jp (Tetsuya Murai).

cost until perhaps we attain some level of satisfaction of *safety* [35,38,39]. This kind of MCDM in which a prioritization of criteria exists, so-called prioritized MCDM, will be studied in this paper.

Many studies have attempted to include different priorities of criteria into MCDM problems in the literature. Generally speaking, approaches to prioritized MCDM can be classified into two categories according to our knowledge. Approaches belonging to the first class aim to use non-monotonic intersection operator [15,33] and triangular norms (t-norms) to model the priority relationships among criteria. For example, Yager [35] uses the weighted conjunction of fuzzy sets and fuzzy modeling to develop the operators in fuzzy information structures. Chen and Chen [9] extend the non-monotonic intersection operator to present a prioritized multi-criteria fuzzy decision making problems based on the similarity measure of generalized fuzzy numbers. Luo et al [23] give five methods to construct the priority operators that are used for calculating the global degree of satisfaction of a prioritized fuzzy constraint problem based on Dubois et al [12]. The second class of approaches tend to use weighted aggregation operators to model the prioritized MCDM. For example, Yager [38] shows that the prioritization of criteria can be modeled by using importance weights in which the weights associated with the lower priority criteria are related to the satisfaction of the higher priority criteria. Moreover, they provide some models that allow for the formalization of these prioritized MCDM problems using both the Bellman-Zadeh paradigm [4] for MCDM and the ordered weighted averaging (OWA) operator. To develop this concept further, Yager [39] proposes a prioritized averaging/scoring aggregation operator with a strict/weak priority order by means of the product t-norm. Furthermore, taking DM's requirements into account, Wang and Chen [10] suggest that the weights of the lower priority criteria depend on *whether* each alternative satisfies the requirements of all the higher priority criteria *or not*.

In this study, we focus on the second class of prioritized MCDM, i.e., priority weighted MCDM [10,38,39]. Although previous research has greatly advanced the priority weighted MCDM, there are still some limitations and drawbacks in previous works. First of all, in prioritized MCDM, we will have a prioritization of criteria. Criteria in the same priority level should allow different tradeoffs. However, as we shall see in Section 3, Yager's method [38,39] does not preserve this property. Furthermore, as suggested by Yager [38,39], the product triangular norm is used to induce the priority weight for each priority level. However, as there are many types of t-norms available, can any t-norm be used to induce the priority weight? If so, which type of t-norms are better? Finally, DM(s) may have a requirement toward the higher priority levels. The method of inclusion of DM's requirements into satisfaction function proposed by Wang and Chen [10] will be too strict for DM to make decision under prioritized environments. In addition, due to the vagueness or impreciseness of knowledge, it is difficult for DMs to estimate their requirements with precision.

Motivated by the above observations, the objective of this paper is to propose a prioritized aggregation operator to overcome the limitations and drawbacks of previous works [10,38,39]. Toward this end, the OWA operator is first be used to obtain the degree of satisfaction for each priority level. To preserve the tradeoffs among the criteria in the same priority level, the degree of satisfaction for each priority level is viewed as a pseudo criterion. Second, we suggest that roughly speaking any t-norm can be used to model the priority relationships between the criteria in different priority levels. To keep the slight change of priority weight, strict Archimedean t-norms perform better in inducing priority weight. As Hamacher family of t-norms provide a wide class of strict Archimedean t-norms ranging from the product to weakest t-norm [26], Hamacher t-norms are selected to induce the priority weight for each priority level. Finally, considering DM's requirement toward the higher priority levels, a *benchmark* based approach is proposed to induce priority weight for each priority level, i.e., "*the satisfactions of the higher priority criteria are larger than or equal to the DM's requirements*". We suggest that the weights of lower priority level should depend on the benchmark achievement of all the higher priority levels. In particular, Łukasiewicz implication is utilized to compute benchmark achievement for crisp requirements. In case of fuzzy uncertain requirements, fuzzy target-oriented decision analysis is utilized to obtain the benchmark achievement.

The rest of this paper is organized as follows. Sec. 2 introduces some basic aspects of MCDM and t-norms. In Sec. 3 we propose a prioritized weighted aggregation operator based on OWA operator and t-norms, we also compare our method with Yager's prioritized aggregation operator [38,39]. In Sec. 4, we propose a benchmark based approach to inducing the priority weight for each priority level by taking DM's requirement toward higher priority levels in account. Considering the uncertainties of DM's requirements, crisp and fuzzy uncertain benchmarks are studied. Comparative analysis with [10] are also given to show the effectiveness and advantages of our proposed approach. Finally, we provide some concluding remarks and future work in Sec. 5.

2. Theoretical Background

2.1. *Basic Concepts of Multi-criteria Decision Making*

A MCDM problem consists of a set of alternatives $\mathcal{A} = \{A_1, \cdots, A_m, \cdots, A_M\}$ and a set of criteria $\mathcal{C} = \{C_1, \cdots, C_n, \cdots, C_N\}$ to evaluate each alternative and rank or select the best alternatives. In their pioneering work on MCDM, Bellman and Zadeh [4] suggested that each criterion C_n can be represented as a fuzzy subset over the alternatives. In particular, if C_n is a criterion we can represent this as a fuzzy subset C_n over A_m such that $C_n(A_m)$ is the degree to which this criterion is satisfied,

where $\forall C_n(A_m) \in [0,1]$. By using this, an aggregation function F is used to aggregate each $C_n(A_m)$ into an overall degree of satisfaction $V(A_m)$ with respect to the set of criteria C such that

$$V(A_m) = F\left(C_1(A_m), \cdots, C_n(A_m), \cdots, C_N(A_m)\right) \quad (1)$$

For denotational simplicity, from now on we shall denote $V(\cdot)$ and $C_n(\cdot)$ to represent $V(A_m)$ and $C_n(A_m)$, respectively.

The choice of the form for F models the DM's desired imperative and individual preference for combining the criteria [22,38]. As suggested by Bellman and Zadeh, if the relationship is that we desire **all** criteria be satisfied then we can use $V(\cdot) = \min_n[C_n(\cdot)]$. If we need **only one** criterion satisfied then we can model this as $V(\cdot) = \max_n[C_n(\cdot)]$.

Yager [32] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating multiple inputs that lie between the min and max operators. An OWA operator of dimension N is a mapping $F : R^N \to R$ that has an associated weighting N vector $W = (w_1, \cdots, w_n, \cdots, w_N)$ such that

$$\text{OWA}\left(C_1(\cdot), \cdots, C_n(\cdot), \cdots, C_N(\cdot)\right) = \sum\nolimits_{n=1}^{N} b_n(\cdot) w_n$$

where $w_n \in [0,1], \sum_{n=1}^{N} w_n = 1,$ for $n = 1, 2, \cdots, N$, $b_n(\cdot)$ is the n-th largest element in the collection $\mathcal{C}$.

The OWA operator provides a class of averaging operators parameterized by the weighting vector W. The type of average is determined by the weighting vector W. Some notable examples are

(i) If $W = W_*$ where $w_N = 1$ and $w_n = 0$ for $n \neq N$, then

$$\text{OWA}\left(C_1(\cdot), \cdots, C_n(\cdot), \cdots, C_N(\cdot)\right) = \min_n[C_n(\cdot)]$$

(ii) If $W = W^*$ where $w_1 = 1$ and $w_n = 0$ for $n \neq 1$, then

$$\text{OWA}\left(C_1(\cdot), \cdots, C_n(\cdot), \cdots, C_N(\cdot)\right) = \max_n[C_n(\cdot)]$$

(iii) If $W = W_N$ where $w_n = \frac{1}{N}$, then

$$\text{OWA}\left(C_1(\cdot), \cdots, C_n(\cdot), \cdots, C_N(\cdot)\right) = \frac{1}{N}\sum\nolimits_{n=1}^{N} C_n(\cdot)$$

Central to the OWA operator is how to obtain OWA weights. Many techniques are available to calculate the OWA weights [14]. We could resolve a mathematical programming problem [1,14,30,31], associate it with a linguistic quantifier [14,32], or obtain OWA weights via analytic method [13]. In the first part, Yager [32] introduced two characterizing measures associated with the weighting vector W of an OWA operator. The first one, *orness measure* of the aggregation, is defined as

$$\Omega = \text{orness}(W) = \sum\nolimits_{n=1}^{N} \frac{N-n}{N-1} \cdot w_n$$

and it characterizes the degree to which the aggregation is like an ***or*** operation. It is clear that $\Omega \in [0,1]$ holds for any weighting vector. Recently, the "orness" of OWA operator is also called "attitudinal character" [37], as it associates with the subjective preference in decision making and can be viewed as a measure of optimism of a DM. In this study, we prefer using "attitudinal character". The closer Ω to zero, the more pessimistic of the DM; the closer Ω to one, the more optimistic of the DM. Three special cases are

- $\Omega = 0$ indicates min operation for aggregation, used to represent DM's pessimistic attitudinal character;
- $\Omega = 0.5$ indicates average operation for aggregation, used to represent DM's neutral attitudinal character;
- $\Omega = 1.0$ indicates max operation for aggregation, used to represent DM's optimistic attitudinal character;

In real applications, it is not easy to specify an Ω value. Ahn [1] discussed the issue of obtaining OWA operator weights with constant level of orness/Ω, in which some commonly used Ω values are $\{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}$.

The second one, the *dispersion measure* of the aggregation, is defined as

$$\text{disp}(W) = -\sum\nolimits_{n=1}^{N} w_n \cdot \ln w_n$$

and it measures the degree to which W takes into account all information in the aggregation. O'Hagan [25] suggested a maximum entropy method to determined OWA operator weights, which formulates the OWA operator weight problem as a constrained nonlinear optimization model with a predefined degree of orness (attitudinal character) as its constraint and the entropy as its objective function. This approach is based on the solution of the following mathematical programming problem:

$$\text{Maximize} - \sum\nolimits_{n=1}^{N} w_n \cdot \ln w_n \quad (2a)$$

$$\text{subject to} \sum\nolimits_{n=1}^{N} \frac{N-n}{N-1} \cdot w_n = \Omega,\ 0 \leq \Omega \leq 1 \quad (2b)$$

$$\sum\nolimits_{n=1}^{N} w_n = 1,\ w_n \in [0,1],\ n = 1, 2, \cdots, N. \quad (2c)$$

An Operations Research software package called *LINDO* [1] can be used to solve this mathematical programming problem.

2.2. *Triangular Norms*

Definition A triangular norm (t-norm for short) T is a mapping from $[0,1]^2$ to $[0,1]$, which is increasing in both arguments, commutative, associative and fulfilling the boundary condition: $\forall x \in [0,1], \text{T}(x,1) = x$ [2,11,26].

The definition of t-norms does not imply any kind of continuity. Nevertheless, such a property is desirable from theoretical as well as practical points of view. A t-norm is is said to be continuous if it is continuous as a two-place function. T-norms can be classified as follows:

- A t-norm T is called Archimedean if it is continuous and $\text{T}(x,x) < x$, for all $x \in (0,1)$.
- An Archimedean t-norm T is called strict if it is strictly increasing in each variable for $x, y \in (0,1)$.
- An Archimedean t-norm T is called nilpotent if it is not strictly increasing in each variable for $x, y \in (0,1)$.

[1] http://www.lindo.com/.

Typical examples of t-norm operators are listed as below [18,19,23].

(i) Minimum operator: $\mathrm{T}_M(x, y) = \min(x, y)$

(ii) Product operator: $\mathrm{T}_P(x, y) = x \cdot y$

(iii) Łukasiewicz operator: $\mathrm{T}_L(x, y) = \max(x + y - 1, 0)$

These basic t-norms have some remarkable properties. The minimum t-norm T_M is the largest t-norm. The product t-norm T_P and the Łukasiewicz t-norm T_L are prototypical examples of two important subclasses of t-norms (of strict Archimedean and nilpotent Archimedean t-norms, respectively).

3. Prioritized Weighted Aggregation Based on OWA Operator and T-norms

Assume that a set of criteria $\mathcal{C}$ are partitioned into Q distinct priority levels, $H = \{H_1, \cdots, H_q, \cdots, H_Q\}$, such that $H_q = \{C_{q1}, \cdots, C_{qk}, \cdots, C_{qN_q}\}$, where N_q is the criteria number in priority level H_q, and C_{qk} is the k-th criterion in priority level H_q. We also assume a prioritization of these priority levels is $H_1 \succ \cdots \succ H_q \succ \cdots > H_Q$. The total set of criteria is $\mathcal{C} = \bigcup_{q=1}^{Q} H_q$. We have for each criterion C_{qk}, a value $C_{qk}(\cdot) \in [0, 1]$ indicating degree of satisfaction of a given alternative regarding criterion C_{qk}. Table 1 shows the priority hierarchy structure of the set of criteria $\mathcal{C}$. Yager [38,39] classified this priority hierarchy into two cases:

– ***strict priority order***, if each priority level has only one criterion this type, i.e. $N_q = 1$ for $q = 1, \cdots, Q$;

– otherwise the priority order is called ***weakly ordered prioritization***.

Table 1
Priority hierarchy of a set of criteria $\mathcal{C}$

Priority level	Criteria
H_1	$C_{11}, \cdots, C_{1k}, \cdots, C_{1N_1}$
$\vdots$	$\vdots$
H_q	$C_{q1}, \cdots, C_{qk}, \cdots, C_{qN_q}$
$\vdots$	$\vdots$
H_Q	$C_{Q1}, \cdots, C_{Qk}, \cdots, C_{QN_Q}$

3.1. *A Prioritized Weighted Aggregation Operator*

As an OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way [19,21]. Thus, contrary to the weighted means, the weights are not associated with concrete variables. Consequently, OWA operators satisfy symmetry. Moreover, OWA operators generalize the arithmetic mean and the median, and they also exhibit some other interesting properties such as monotonicity, idempotence, and compensativeness (i.e., the value of an OWA operator is located between the minimum and the maximum values of the variables).

Due to these properties, the OWA operator will be used to obtain degree of satisfaction for each priority level. Given DM's attitudinal character Ω_q toward priority level H_q, according to O'Hagan's OWA weight determination method as shown in Eq. (2), we can associate with priority level H_q an OWA weighting vector such that $U_q = \{u_{q1}, \cdots, u_{qk}, \cdots, u_{qN_q}\}$, where $u_{qk} \in [0, 1]$ and $\sum_{k=1}^{N_q} u_{qk} = 1$. In addition, let $B_q(\cdot) = \{b_{q1}(\cdot), \cdots, b_{qN_q}(\cdot)\}$ be the reordered vector of $C_q(\cdot) = \{C_{q1}(\cdot), \cdots, C_{qN_q}(\cdot)\}$, where $b_{qk}(\cdot)$ is the k-th largest in priority level H_q. Using this we can calculate the degree of satisfaction in priority level H_q as

$$\mathrm{Sat}_q(\cdot) = \mathrm{OWA}_{\Omega_q}[H_q] = \sum_{k=1}^{N_q} b_{qk}(\cdot) u_{qk} \tag{3}$$

To model the priority relationship, as suggested by Yager [38,39], the lower priority criteria will become important with the higher degree of satisfaction of higher priority level, i.e., the priority weights are dependent upon the satisfaction of higher priority levels. Motivated by this observation, we will associate with each priority level a **priority weight** $Z_q(\cdot)$, which is derived from the degree of satisfaction of all the higher priority levels. Furthermore, as t-norms do not allow low values to be compensated by high values [2,11,26], t-norms are used to induce the priority weight $Z_q(\cdot)$ for each priority level.

In particular, for priority level H_1, we have $Z_1(\cdot) = 1$. For priority level H_2, we express the priority weight as $Z_2(\cdot) = \mathrm{T}(Z_1(\cdot), \mathrm{Sat}_1(\cdot))$. For priority level H_3, we express the priority weight as $Z_3(\cdot) = \mathrm{T}(Z_2(\cdot), \mathrm{Sat}_2(\cdot))$. More succinctly and generally, we can induce the priority weight for priority level H_q as

$$Z_q(\cdot) = \mathrm{T}\left(Z_{q-1}(\cdot), \mathrm{Sat}_{q-1}(\cdot)\right) = \mathrm{T}_{l=0}^{q-1} \mathrm{Sat}_l(\cdot) \tag{4}$$

with the understanding that $Z_0(\cdot) = \mathrm{Sat}_0(\cdot) = 1$.

We now see that for priority level H_q, we have a priority weight $Z_q(\cdot)$. In addition, for each criterion in priority level H_q, we have a local OWA weight. To preserve the tradeoffs between criteria in the same priority level, we shall view the degree of satisfaction of each priority level as a pseudo criterion. In this way, we can get an aggregated value for each alternative under these prioritized criteria as

$$\mathrm{V}(\cdot) = \sum_{q=1}^{Q} Z_q(\cdot) \mathrm{Sat}_q(\cdot) \tag{5}$$

No matter what type of t-norms is selected, the priority weight $Z_q(\cdot)$ of a priority level depends upon the satisfaction of all the higher priority levels, such that $Z_q(\cdot) = \mathrm{T}_{l=0}^{q-1} \mathrm{Sat}_l(\cdot)$, thus poor satisfaction of all the higher priority levels leads to lower priority weights for the current priority level. In addition, the OWA operator are used to aggregate the criteria in the same priority level. Based on these two features, we shall call the proposed aggregation operator as *Prioritized OWA operator.* [2]

[2] It should be noted that each priority level H_q may have a different attitudinal character Ω_q. Here for purposes of simplicity, we assume that each priority level H_q has the same attitudinal character $\Omega_q = \Omega$.

The prioritization of the criteria induces a priority weighting schema such that the criteria gains more importance only if all the higher priority criteria are higher satisfied. If one wants to raise the global degree of satisfaction of all criteria, a criterion with a relatively high priority level must be sufficiently satisfied prior to the criteria in relatively low priority levels. This is accordance with the meaning of the word *priority* in English dictionaries [27]. In fact, the concept of priority has the following two characteristics [23]:

(i) It measures the relative importance among things in a group to determine only their relative precedence, and

(ii) the higher the priority of one thing, the earlier the thing should be handled or the more preferred is the thing.

3.2. *Properties of Proposed Prioritized Aggregation Operator*

Proposition 3.1 *The proposed prioritized aggregation operator is monotonic regarding any criteria C_{lk}.*

Proof For monotonicity to hold, we have to prove $\frac{\partial V(\cdot)}{\partial C_{lk}(\cdot)} \geq 0$. Not all the Sat_q and Z_q change with C_{lk}, thus we express $V(\cdot)$ as

$$V(\cdot) = \sum_{i=1}^{l-1} Z_i(\cdot)\mathrm{Sat}_i(\cdot) + Z_l(\cdot)\mathrm{Sat}_l(\cdot) + \sum_{j=l+1}^{Q} Z_j(\cdot)\mathrm{Sat}_j(\cdot)$$

And then we can obtain $\frac{\partial V(\cdot)}{\partial C_{lk}(\cdot)}$ as

$$\begin{aligned}\frac{\partial V(\cdot)}{\partial C_{lk}(\cdot)} &= 0 + \frac{\partial \mathrm{Sat}_l(\cdot)}{\partial C_{lk}} Z_l(\cdot) + \sum_{j=l+1}^{Q}\left[\frac{\partial Z_j(\cdot)}{\partial C_{lk}(\cdot)}\mathrm{Sat}_j(\cdot)\right] \\ &= \frac{\partial \mathrm{Sat}_l(\cdot)}{\partial C_{lk}} Z_l(\cdot) + \sum_{j=l+1}^{Q}\left[\frac{\partial Z_j(\cdot)}{\partial \mathrm{Sat}_l(\cdot)}\frac{\partial \mathrm{Sat}_l(\cdot)}{\partial C_{lk}(\cdot)}\mathrm{Sat}_j(\cdot)\right]\end{aligned}$$

Since we use the OWA operator to obtain the degree of satisfaction for each priority level and OWA is monotonic, thus we know that $\frac{\partial \mathrm{Sat}_l(\cdot)}{\partial C_{lk}(\cdot)} \geq 0$.

Since $Z_j(\cdot) = \mathrm{T}(Z_{j-1}(\cdot), \mathrm{Sat}_{j-1}(\cdot)) \geq 0$ and t-norm increases in both arguments, thus $\frac{\partial Z_j(\cdot)}{\partial \mathrm{Sat}_l(\cdot)} \geq 0$. The product of monotonic operators is also monotonic, hence we know that $\frac{\partial V(\cdot)}{\partial C_{lk}(\cdot)} \geq 0$. □

Proposition 3.2 *Our proposed prioritized OWA operator guarantees monotonicity regarding DM's attitudinal character Ω.*

Proof To prove our proposed prioritized OWA operator is monotonic regarding DM's attitudinal character, we have to prove that

$$\frac{\partial V(\cdot)}{\partial \Omega} = \frac{\partial\left(\sum_{q=1}^{Q} Z_q(\cdot)\mathrm{Sat}_q(\cdot)\right)}{\partial \Omega} \geq 0$$

We know that

$$\frac{\partial\left(\sum_{q=1}^{Q} Z_q(\cdot)\mathrm{Sat}_q(\cdot)\right)}{\partial \Omega} = \sum_{q=1}^{Q}\left(\mathrm{Sat}_q(\cdot)\frac{\partial Z_q(\cdot)}{\partial \Omega} + Z_q(\cdot)\frac{\partial \mathrm{Sat}_q(\cdot)}{\partial \Omega}\right)$$

According to the properties of OWA operator, it is clear that $\frac{\partial \mathrm{Sat}_q(\cdot)}{\partial \Omega} \geq 0$. In addition, we know $Z_q(\cdot) = \mathrm{T}(Z_{q-1}(\cdot), \mathrm{Sat}_{q-1}(\cdot))$ and t-norm increases in both arguments, thus $\frac{\partial Z_q(\cdot)}{\partial \Omega} \geq 0$, hence $\frac{\partial V(\cdot)}{\partial \Omega} \geq 0$. □

3.3. *Illustrative Examples*

We shall apply the proposed prioritized OWA aggregation operator to deal with a car selection problem, adapted from [10].

Example Assume that John wants to buy a new car considering the following criteria "C_1 ***Safety***", "C_2 ***Price***", "C_3 ***Appearance***" and "C_4 ***Performance***". We also assume that there are four alternatives of cars A_1, A_2, A_3, A_4 and the degrees in which each alternative satisfies each criterion are shown in Table 2. We also assume that the priority hierarchy specified by John is $H_1 = \{C_1\}$, $H_2 = \{C_2\}$, $H_3 = \{C_3, C_4\}$, and $H_1 \succ H_2 \succ H_3$.

Table 2
Satisfaction degree of each criterion regarding each alternative: car selection

Alt.	Criteria			
	C_1	C_2	C_3	C_4
A_1	0.95	0.60	0.70	0.80
A_2	0.91	0.75	0.50	0.90
A_3	0.95	0.70	0.80	0.70
A_4	0.945	0.75	0.30	0.70

For purposes of simplicity, for each priority level H_q we shall specify the same attitudinal character, such that $\Omega_q = \Omega$, where $q = 1, 2, 3$. We assume that $\Omega = 0.5$. The minimum t-norm T_M is the largest t-norm. The product t-norm T_P and the Łukasiewicz t-norm T_L are prototypical examples of two important subclasses of t-norms (of strict Archimedean and nilpotent Archimedean t-norms, respectively). In this example, these three prototypical t-norms are used to induce the priority weights. Taking car A_1 as an example. We first consider product t-norm T_P, we proceed as follows:

(i) We first calculate the degree of satisfaction for each priority level via OWA operator as follows:

$$\begin{aligned}\mathrm{Sat}_1(A_1) &= \mathrm{OWA}_{0.5}\{0.95\} = 0.95 \\ \mathrm{Sat}_2(A_1) &= \mathrm{OWA}_{0.5}\{0.60\} = 0.6 \\ \mathrm{Sat}_3(A_1) &= \mathrm{OWA}_{0.5}\{0.70, 0.80\} = 0.75\end{aligned}$$

(ii) We then calculate the priority weight for each priority level by using product t-norm via Eq. (4):

$$\begin{aligned}Z_1(A_1) &= \mathrm{T}_P(Z_0(A_1), \mathrm{Sat}_0(A_1)) = \mathrm{T}_P(1, 1) = 1 \\ Z_2(A_1) &= \mathrm{T}_P(Z_1(A_1), \mathrm{Sat}_1(A_1)) = \mathrm{T}_P(1, 0.95) = 0.95 \\ Z_3(A_1) &= \mathrm{T}_P(Z_2(A_1), \mathrm{Sat}_2(A_1)) = \mathrm{T}_P(0.95, 0.6) = 0.57\end{aligned}$$

(iii) Finally, we obtain the global prioritized aggregated value via Eq. (5):

$$V(A_1) = \sum_{q=1}^{3} Z_q(A_1) \cdot \mathrm{Sat}_q(A_1)$$
$$= 1 \cdot 0.95 + 0.95 \cdot 0.6 + 0.57 \cdot 0.75 = 1.9475$$

Similarly, the prioritized aggregation values for cars A_2, A_3, and A_4 by product t-norm can be obtained. We can also obtain the aggregated value with Minimum t-norm and Łukasiewicz t-norm, as shown in Table 3. From Table 3, it is clearly seen that car A_3 is the best choice whatever the t-norm is. The ranking order of prioritized aggregation values are as $A_3 \succ A_2 \succ A_4 \succ A_1$, where $\succ$ denotes "*prefer to*".

Table 3
Prioritized aggregation with different t-norms under attitudinal character $\Omega = 0.5$

T-norms	Alt.			
	A_1	A_2	A_3	A_4
Minimum t-norm T_M	1.9700	2.1175	**2.1400**	2.0286
Product t-norm T_P	1.9475	2.0703	**2.1137**	2.0080
Łukasiewicz t-norm T_L	1.9325	2.0545	**2.1025**	2.0012

3.4. *Choosing Suitable T-norms*

In the previous example, we used minimum t-norm, Łukasiewicz t-norm and product t-norm to induce the priority weight for each priority level. As mentioned previously, in general any t-norm can be used to induce the priority weight. Are there any differences between these t-norms? Which type of t-norms perform better in inducing priority weight? To illustrate this point, we shall discuss this topic from the following two aspects.

A Special Case

Firstly, let us consider a special case, where only two levels of priority hierarchy exists, i.e. $Q = 2$. We observed that in this case, no matter which t-norm is used, we always obtain the priority weight $Z_1(\cdot) = 1$ for priority level H_1 and a priority weight $Z_2(\cdot) = \mathrm{Sat}_1(\cdot)$ for priority level H_2. The main reasons for this are as follows:

(i) we assume there is a pseudo hierarchy level H_0 with $Z_0(\cdot) = \mathrm{Sat}_0(\cdot) = 1$.

(ii) Moreover, any t-norm has the property such that $\mathrm{T}(1, x) = x$.

An Example

Assume that there are two alternatives A_1 and A_2. For the q-th priority level, the priority weight and degree of satisfactions are shown in Columns 2-3 of Table 4. According to Eq. (4) and the three typical t-norms, we can calculate three induced priority weights for priority level $q + 1$, as shown in Columns 4-6 of Table 4.

Table 4
The priority weight and degree of satisfactions of q-th priority level and its induced priority weights

Alt.	q-th priority level		Induced weights Z_{q+1}		
	Weight Z_q	Satisfaction degree	T_M	T_L	T_P
A_1	0.3	0.5	0.3	0	0.15
A_2	0.3	0.6	0.3	0	0.18

It is clear that the priority weights of $q + 1$-th priority level induced by Minimum t-norm and Łukasiewicz t-norm do not reflect the changes of the priority weight and degree of satisfactions of q-th priority level. We want to preserve the slight change of priority weight as well as do not want to ignore the slight change, thus non-Archimedean t-norms and nilpotent t-norms are not suitable to induce the priority weight for each priority level. The product t-norm is the prototypical example of strict Archimedean t-norms and can catch the slight change of of priority weight. This is perhaps the main reason why Yager [39] and Wang and Chen [10] both use the product t-norm to induce the priority weight.

Based on the above observations, strict Archimedean t-norms perform well in reducing the priority weight. As Hamacher family of t-norms provide a wide class of strict Archimedean t-norms ranging from the product to weakest t-norm [26], we shall use Hamacher parameterized t-norm to induce the priority weight such that

$$\mathrm{T}_H^{\gamma} = \frac{Z_{q-1}(\cdot)\mathrm{Sat}_{q-1}(\cdot)}{\gamma + (1-\gamma)(Z_{q-1}(\cdot) + \mathrm{Sat}_{q-1}(\cdot) - Z_{q-1}(\cdot)\mathrm{Sat}_{q-1}(\cdot))}$$
$$= \mathrm{T}_{H\,l=0}^{\gamma\;\;q-1}\mathrm{Sat}_l(\cdot), \gamma \geq 0. \tag{6}$$

If $\gamma = 0$, then we can obtain the priority weight inducing method as

$$Z_q(\cdot) = \frac{Z_{q-1}(\cdot)\mathrm{Sat}_{q-1}(\cdot)}{Z_{q-1}(\cdot) + \mathrm{Sat}_{q-1}(\cdot) - Z_{q-1}(\cdot)\mathrm{Sat}_{q-1}(\cdot)} \tag{7}$$

When $\gamma = 1$, the induced priority weight is represented as

$$Z_q(\cdot) = Z_{q-1}(\cdot)\mathrm{Sat}_{q-1}(\cdot) = \prod_{l=0}^{q-1} \mathrm{Sat}_l(\cdot) \tag{8}$$

with the understanding that $Z_0(\cdot) = \mathrm{Sat}_0(\cdot) = 1$.

Now let us reconsider the example as shown in Table 2 via the Hamacher parameterized t-norm. According to the three steps of our proposed prioritized aggregation, we obtain the prioritized aggregated values as shown in Fig. 1, where γ is set to $\in [0, 1000]$. Obviously T_H^{γ} is non-increasing with respect to γ [26]. In addition, the ranking order of prioritized aggregation values may be different with different γ values. From Fig. 1, it is clearly that the prioritized aggregation values are non-increasing with respect to γ. In our example, there are two γ values changing the ranking order of the four alternatives, $\lambda_{\mathrm{I}} \approx 16.8295$ and $\lambda_{\mathrm{II}} \approx 145.8237$. Thus five ranking orders can be obtained:

– When $0 \leq \gamma < \lambda_{\mathrm{I}}$, the ranking result is $A_3 \succ A_2 \succ A_4 \succ A_1$;
– When $\gamma = \lambda_{\mathrm{I}}$, the ranking result is $A_3 \succ A_2 \sim A_4 \succ A_1$;
– when $\lambda_{\mathrm{I}} < \gamma < \lambda_{\mathrm{II}}$, the ranking result is $A_3 \succ A_4 \succ A_2 \succ A_1$;
– When $\gamma = \lambda_{\mathrm{II}}$, the ranking result is $A_3 \sim A_4 \succ A_2 \succ A_1$;
– when $\lambda_{\mathrm{II}} < \gamma \leq 1000$, the ranking result is $A_4 \succ A_3 \succ A_2 \succ A_1$.

Remark From Fig. 1 we know that different γ values may lead to different ranking orders. Then a natural question

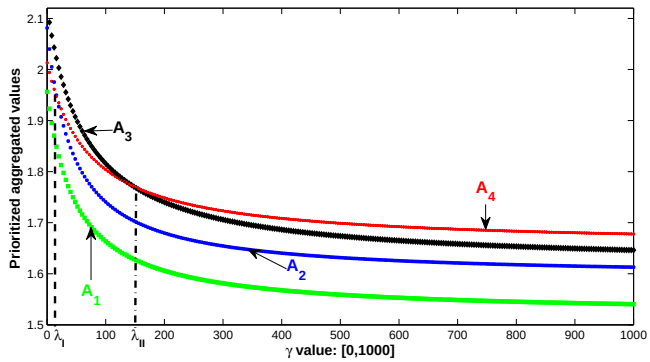

Fig. 1. Prioritized aggregated values by means of Hamacher's t-norm: $\gamma \in [0, 1000]$

that arises is how the DM selects optimal alternative with different γ values. From the theoretical point of view, we show that the strict Archimedean t-norms perform well in reducing the priority weight for each priority level. As Hamacher's family of t-norms supplies a wide class of t-norm operators ranging from the probabilistic product to the weakest t-norm, extending the product t-norm into Hamacher's family of t-norms provides a generalization of previous research. In addition, the γ value can be viewed as an index to represent strongness of the prioritization in our prioritized aggregation. The larger the γ value is, the stronger of prioritization the DM prefers. From the applicable point of view, it is not easy to specify a γ value. In fact, we can do sensitivity analysis of prioritized aggregation with respect to different γ values. When DM's preferred γ value falls into a range, we can approximately know the best alternative. If the DM does not provide his/her subjective preference, two commonly used cases are $\gamma = 0$ (Hamacher product t-norm) and $\gamma = 1$ (product t-norm). We can set $\gamma = 1$ for default.

3.5. *Comparison with previous research*

To illustrate the effectiveness and advantages of our formulation of prioritized aggregation, we shall compare our approach with previous work. Yager [39] proposed a prioritized aggregation operator according to the following three steps:

(i) To calculate the degree of satisfaction for each priority level by OWA operator as follows:

$$\mathrm{Sat}_q(\cdot) = \mathrm{OWA}_\Omega[H_q]$$

(ii) The product t-norm is used to calculate the priority weight $Z_q(\cdot)$ for priority level H_q

$$Z_q(\cdot) = \mathrm{T}_P\left(Z_{q-1}(\cdot), \mathrm{Sat}_{q-1}(\cdot)\right) = \prod_{l=0}^{q-1} \mathrm{Sat}_l(\cdot) \quad (9)$$

where $Z_0(\cdot) = \mathrm{Sat}_0(\cdot) = 1$.

(iii) To calculate the overall degree of satisfaction

(a) For strict priority order, where $N_q = 1$ and $q = 1, \cdots, N$, an averaging aggregation is used as follows:

$$V(\cdot) = \frac{\sum_{q=1}^{Q}\left[\sum_{k=1}^{N_q} Z_q(\cdot)C_{qk}(\cdot)\right]}{\sum_{q=1}^{Q} Z_q(\cdot)} = \frac{\sum_{q=1}^{Q} Z_q(\cdot)C_q(\cdot)}{\sum_{q=1}^{Q} Z_q(\cdot)} \quad (10)$$

(b) For weak priority order, a scoring aggregation is used as follows:

$$V(\cdot) = \sum_{q=1}^{Q}\left[\sum_{k=1}^{N_q} Z_q(\cdot)C_{qk}(\cdot)\right] \quad (11)$$

In the first step of both Yager's prioritized aggregation operator and our prioritized OWA operator, the OWA operator is used to obtain the degree of satisfaction for each priority level. To find out some limitations of Yager's prioritized operator, we compare our operator with Yager's operator from the following three aspects:

T-norm selection

In Yager's aggregation operator, the product t-norm is used to induce the priority weight for each priority level. As mentioned previously, we suggested that roughly speaking, any t-norm can be used to induce the priority weight for each priority level. To preserve the slight change of priority weight as well as do not want to ignore the slight change, we suggest using Hamacher parameterized t-norm. In this view, Yager's operator is one special case of our operator.

Tradeoffs of criteria in the same priority level

Let us consider a special case. If the DM does not specify the priority hierarchy, it means that the DM agrees the tradeoffs among all the criteria. In this case, only one priority level is considered, all the criteria have the same priority level, we shall use C_n instead of C_{1n} to represent the n-th criterion. Our proposed prioritized OWA operator Eq. (5) reduces to the OWA operator [32] such that

$$\begin{aligned} V(\cdot) &= \text{PRI-OWA}(H_1) \\ &= \mathrm{OWA}_\Omega\left(C_1(\cdot), \cdots, C_n(\cdot), \cdots, C_N(\cdot)\right). \end{aligned} \quad (12)$$

Whereas, Yager's prioritized aggregation operator reduces to the summation of all criteria, such that

$$V(\cdot) = \sum_{n=1}^{N} C_n(\cdot) \quad (13)$$

In this regard, Yager's prioritized aggregation operator only allows summation tradeoffs for each priority level, whereas our approach allows OWA tradeoffs. In the literature, many aggregation operators can be used to realize the tradeoffs between attributes, by allowing a positve compensation between scores, i.e. a higher degree of satisfaction of one the the attribute can compensate for a lower degree of satisfaction of another attribute to a certain extent [8]. In this study, OWA operator is used to aggregate the attributes in the same priority level, thus we shall call this type of tradeoff as OWA tradeoff.

Averaging or Scoring Aggregation

Yager [39](p.267) pointed out that under strictly ordered prioritization (only one criterion in each priority

level), the prioritized aggregation should be represented as the normalized form; under weakly ordered prioritization, the prioritized aggregation should be represented as the scoring type. The main reason is that the normalized form under weakly ordered prioritization dose not always guarantee monotonicity.

However, as the strictly ordered prioritization and weakly ordered prioritization are just two special cases of the prioritization hierarchy, hence they should have the same properties and type. To keep the general properties of aggregation operators as well as to keep the original priority changes, similar with [6,18] we shall use the scoring type of aggregation.

4. Including Benchmark into Prioritized OWA Aggregation

We now turn to a possible variation of our formulation for prioritized OWA aggregation $\mathrm{V}(\cdot) = \sum_{q=1}^{Q} Z_q(\cdot)\mathrm{Sat}_q(\cdot)$ where

$$Z_q(\cdot) = \mathrm{T}_H^{\gamma}\left(Z_{q-1}(\cdot), \mathrm{Sat}_{q-1}(\cdot)\right) = \mathrm{T}_{H\,l=0}^{\gamma\;\;q-1}\mathrm{Sat}_l(\cdot)$$
$$\mathrm{Sat}_q(\cdot) = \mathrm{OWA}_{\Omega}[H_q]$$

In this formulation, the priority weight $Z_q(\cdot)$ directly depends upon the satisfactions of the criteria in all higher priority levels. As suggested by Yager [38], as a variation of this, we can let $Z_q(\cdot)$ depend on some function of the satisfactions of the higher priority criteria. Without loss of generality, we assume that for any priority level, the DM specifies a requirement. In particular, we can let $\mathrm{E} : [0,1] \to [0,1]$ such that $\mathrm{E}(0) = 0$, $\mathrm{E}(1) = 1$, and $\mathrm{E}(x) \geq \mathrm{E}(y)$ if $x \geq y$. Using this we can express

$$\begin{aligned}\mathrm{Z}_q(\cdot) &= \mathrm{T}_H^{\gamma}(\mathrm{Z}_{q-1}(\cdot), \mathrm{E}(\mathrm{Sat}_{q-1}(\cdot))) \\ &= \mathrm{T}_H^{\gamma}\left(\mathrm{E}(\mathrm{Sat}_0(\cdot)), \mathrm{E}(\mathrm{Sat}_1(\cdot)), \cdots, \mathrm{E}(\mathrm{Sat}_{q-1}(\cdot))\right) \\ &= \mathrm{T}_{H\,l=0}^{\gamma\;\;q-1}\mathrm{E}(\mathrm{Sat}_l(\cdot))\end{aligned} \tag{14}$$

Roughly speaking, we can view $\mathrm{E}(\mathrm{Sat}_q(\cdot))$ as some kind of effective or pseudo satisfaction. Here then the score value associated with criteria can be different in its evaluation of induced priority weights and the satisfaction degree used in the aggregation. With the use of function E to transform levels of satisfaction, we are able to model the linguistically expressed DM's requirements by means of Zadeh's paradigm of computing with words. Due to this observation, DM's requirement can be viewed as *benchmark or reference level* for the degrees of satisfaction of each priority level, i.e., "***the satisfactions of the higher priority criteria are larger than or equal to the DM's requirements***" expressed as $\mathbb{E}(\mathrm{Sat}_q(\cdot) \geq G_q) = \mathrm{E}(\mathrm{Sat}_q(\cdot))$. We shall consider two possible DM's requirements such that

(i) The DM specifies a certain requirement G_q for priority level H_q,

(ii) The DM specifies an uncertain requirement $\widetilde{G}_q$ for priority level H_q.

In the following, we will deal with these two types of benchmarks respectively.

4.1. *Crisp Requirements*

Before discussing crisp requirements, we shall recall some knowledge of R-implications.

Definition An implication operator I is a mapping: $[0,1]^2 \to [0,1]$, such that [8,26]

– I is non-increasing with respect to its first argument;
– I is non-decreasing with respect to its second argument;
– $I(0,0) = I(0,1) = I(1,1) = 1$, $I(1,0) = 0$.

Basically, there are many implication operators. We shall use the R-implications. R-implications I are based on the idea that implication reflects partial ordering on proposition, i.e., $I(x,y) = 1$ if and only if $x \leq y$. In the standard semantics of t-norm based fuzzy logics, where conjunction is interpreted by a t-norm, the residuum plays the role of implication (often called R-implication). R-implications can be obtained by residuum of a continuous t-norm T [26] as follows,

$$x \to y = \sup\{z \in [0,1] | \mathrm{T}(x,z) \leq y\}, \text{ for all } x,y,z \in [0,1]. \tag{15}$$

These implications arise from the intutionistic logic formalism [8]. Typical examples of R-implication operators are

(i) Kleene-Dienes implication:

$$I(x,y) = \max(1-x, y) \tag{16}$$

(ii) Łukasiewicz implication:

$$I(x,y) = \min(1-x+y, 1) \tag{17}$$

(iii) Gödel implication:

$$I(x,y) = 1, \text{if } x \leq y; y, \text{otherwise}. \tag{18}$$

If both the degree of satisfaction and the requirement are crisp numbers, we can implement $\mathbb{E}(\mathrm{Sat}_q(\cdot) \geq G_q)$ using the strict implication operator. It is clear that this will be very sensitive to small changes of both arguments. However, we can still sustain the benchmark character if we use an R-implication operator to transform the degrees of satisfaction. As Łukasiewicz implication is the one that satisfies most of the properties pertaining to the logical implication operators [26], in this study Łukasiewicz implication was used as a technique to compute benchmark satisfaction for crisp requirements. Given DM's requirement G_q of priority level H_q, according to Łukasiewicz implication Eq. (17), we can define

$$\mathbb{E}(\mathrm{Sat}_q(\cdot) \geq G_q) = G_q \to \mathrm{Sat}_q(\cdot) = \min\{1 - G_q + \mathrm{Sat}_q(\cdot), 1\}. \tag{19}$$

It is of interest noting that in Section 3, $Z_q(\cdot)$ directly depends on the satisfaction of higher priority criteria. In this case, the DM's requirement can be modeled as $G_q = 1$. According to Eq. (19), we know that

$$\begin{aligned}\mathbb{E}(\mathrm{Sat}_q(\cdot) \geq G_q) &= \min\{1 - G_q + \mathrm{Sat}_q(\cdot), 1\} \\ &= \min\{\mathrm{Sat}_q(\cdot), 1\} = \mathrm{Sat}_q(\cdot).\end{aligned}$$

4.2. *Uncertain requirements*

Due to the vagueness or impreciseness of knowledge, it is difficult for DM(s) to estimate their requirements with precision. In many applications, fuzzy subsets [40] provide a very convenient object for the representation of uncertain information. The subjective assessments provided by DM(s) are usually conceptually vague, with uncertainty that is frequently represented in linguistic forms. To help people easily express their subjective assessments, the linguistic variables [41,42] are used to linguistically express requirements.

A fuzzy number $\widetilde{G}$ can be conveniently represented by the canonical form [17]

$$\mu_{\widetilde{G}(g)} = \begin{cases} f_{\widetilde{G}(g)}, & g_1 \le g \le g_2, \\ 1, & g_2 \le g \le g_3, \\ h_{\widetilde{G}(g)}, & g_3 \le g \le g_4, \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

where $\mu_{\widetilde{G}(g)}$ denotes the membership function of fuzzy number $\widetilde{G}$, $f_{\widetilde{G}(g)}$ is a real-valued function that is monotonically increasing, and $h_{\widetilde{G}(g)}$ is a real-valued function that is monotonically decreasing. In addition, like most applications, we assume that functions $f_{\widetilde{G}(g)}$ and $h_{\widetilde{G}(g)}$ are continuous. If $f_{\widetilde{G}(g)}$ and $h_{\widetilde{G}(g)}$ are linear functions then $\widetilde{G}$ is called a trapezoidal fuzzy number and denoted by $\widetilde{G} = (g_1, g_2, g_3, g_4)$. In particular, $\widetilde{G}$ becomes a triangular fuzzy number if $g_2 = g_3$.

Many methods can be used to compute $\mathbb{E}(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q)$. One typical method is to use the fuzzy membership function to represent the degree of preference. As pointed out by Beliakov and Warren [3], membership functions of fuzzy sets play the role similar to utility functions–the role of degrees of preference. Many researchers, including Zadeh himself, refer to membership functions as 'a kind of utility functions'. However, as empirical evidence indicates that conventional concave attribute utility function often does not provide a good description of individual preference, and usually it is difficult for DM(s) to determine their utility functions [5]. Target-oriented decision analysis lies in the philosophical root of bounded rationality as well as represents the S-shaped value function. In particular, fuzzy targets can be used to represent the uncertain requirements of DM(s). These considerations lead us to use fuzzy target-oriented decision analysis [16] to compute $\mathrm{E}(\mathrm{Sat}_q(\cdot))$ for uncertain requirements $\widetilde{G}_q$. Toward this end, we define

$$\mathbb{E}(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q) = \Pr\left(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q\right) \tag{21}$$

where $\Pr\left(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q\right)$ denotes the probability of "meeting the fuzzy benchmark $\widetilde{G}_q$". For simplicity, we shall denote $\mathrm{Sat}_q(\cdot)$ and $\widetilde{G}_q$ as X and Y respectively.

The target-oriented decision analysis [5,16] suggested that

$$\Pr(X \ge Y) = \int_{-\infty}^{\infty} P_X(x) \left[\int_{-\infty}^{x} P_Y(y)dy\right] dx \tag{22}$$

where $P_X(x)$ and $P_Y(y)$ denote the probability distributions of $\mathrm{Sat}_q(\cdot)$ and $\widetilde{G}_q$ respectively. As $\mathrm{Sat}_q(\cdot)$ is a crisp number and in our research context the bounded domain is $[0, 1]$, we can reduce as

$$\Pr\left(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q\right) = \int_0^{\mathrm{Sat}_q(\cdot)} P_Y(y)dy \tag{23}$$

To compute this, as suggested by Huynh et al. [16], a direct and simple way to define $\Pr\left(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q\right)$ is making use of Yager's method [36] for converting a possibility distribution into an associated probability distribution via the simple normalization. Particularly, the possibility distribution $\mu_{\widetilde{G}(g)}$ of the benchmark $\widetilde{G}_q$ is first converted into its associated probability distribution, denoted by $P_{\widetilde{G}_q}$, as follow

$$P_{\widetilde{G}(g)} = \frac{\mu_{\widetilde{G}(g)}}{\int_{g_1}^{g_4} \mu_{\widetilde{G}(g)} dg} \tag{24}$$

Then we can obtain the probability of "meeting the fuzzy benchmark $\widetilde{G}_q$" as follows

$$\Pr\left(\mathrm{Sat}_q(\cdot) \ge \widetilde{G}_q\right) = \frac{\int_0^{\mathrm{Sat}_q(\cdot)} \mu_{\widetilde{G}(g)} dg}{\int_{g_1}^{g_4} \mu_{\widetilde{G}(g)} dg} \tag{25}$$

4.3. *A Comparative Analysis*

To show the effectiveness and advantages of our approach, we shall use the same example (car selection) introduced in Section 3 to compare our approach with related research.

4.3.1. *Wang and Chen's Approach*

Wang and Chen [10] suggested that the weights of lower priority criteria depends on ***whether*** each alternative satisfies the requirements of all the higher priority criteria ***or not***. They proposed two benchmark achievement according to two cases.

(i) For criteria in priority level H_q, a degree of satisfaction $\mathrm{Sat}_q(\cdot)$ is calculated as follows [3]

$$\mathrm{Sat}_q(\cdot) = \mathrm{OWA}_\Omega[H_q] \tag{26}$$

(ii) Then an importance weight $Z_q(\cdot)$ for priority level H_q by means of product t-norm is calculated as follows

$$Z_q(\cdot) = \prod_{l=0}^{q-1} \mathrm{E}(\mathrm{Sat}_l(\cdot)) = Z_{q-1}(\cdot)\mathrm{E}(\mathrm{Sat}_{q-1}(\cdot)) \tag{27}$$

where $Z_0(\cdot) = \mathrm{Sat}_0(\cdot) = 1$. To obtain the benchmark achievement $\mathrm{E}(\mathrm{Sat}_q(\cdot))$, Wang and Chen considered two cases:

[3] In fact, Wang and Chen [10] applied the fuzzy linguistic quantifier [32] method to determine the OWA weighting vector. To clarify the main differences between our work and Wang&Chen, we shall use the same weight determination method proposed by O'Hagan [25].

(a) The DM wants that a good solution must have *at least* G_q degree of satisfaction such that

$$\mathrm{E}(\mathrm{Sat}_q(\cdot)) = 1, \text{ if } \mathrm{Sat}_q(\cdot) \geq G_q; 0, \text{otherwise.} \tag{28}$$

(b) The decision maker wants that a good solution must have *at least* G_q and *as high as possible* degree of satisfaction such that

$$\mathrm{E}(\mathrm{Sat}_q(\cdot)) = \mathrm{Sat}_q, \text{ if } \mathrm{Sat}_q(\cdot) \geq G_q; 0, \text{otherwise.} \tag{29}$$

(iii) To calculate the overall degree of satisfaction as follows

$$V(\cdot) = \sum_{q=1}^{Q} Z_q(\cdot)\mathrm{Sat}_q(\cdot) \tag{30}$$

4.3.2. *A comparative analysis*

In both our approach and Wang and Chen's work [10], OWA operator is used to obtain the degrees of satisfaction for each priority level. For purposes of simplicity, we shall assume that DM's attitudinal character $\Omega = 0.5$. The main difference between our approach and Wang and Chen are twofold.

(i) First, more than product t-norm, we proposed using Hamacher parameterized t-norm to induce the priority weight for each priority level. As product t-norm is one special case of Hamacher parameterized t-norm where $\gamma = 1$, and in previous section we have already discussed the t-norm selection problem, here in order to distinguish the main difference between our approach from Wang and Chen's approach, we shall just use product t-norm.

(ii) Second, instead of strict threshold method, we propose using Łukasiewicz implication to compute benchmark achievement. In addition, due to the uncertainty of DM's requirements, target-oriented decision decision analysis is used to solve the fuzzy requirement.

Let us reconsider the same example as shown in Table 2. Assume that John wants to buy a car having a requirement for the satisfaction of criterion *safety*. We also assume that John specifies his attitudinal character as $\Omega = 0.5$. Considering the uncertainty of requirement, we do comparative analysis from the following three aspects.

Benchmark: at least G_1

Assume that John specifies his requirement toward the criterion *safety* as G_1, here specify three possible values, as shown in the first column of Table 5. According to Wang and Chen's approach (Eq. (26), Eq. (27), Eq. (28) and Eq. (30)), we can obtain the aggregation values with different G_1 values, as shown in Columns 2-5 of Table 5. With the three steps of our prioritized OWA aggregation operator and the benchmark achievement Eq. (19) we can easily obtain the aggregation values as shown in Columns 6-9 of Table 5.

Table 5
Prioritized aggregated value under different crisp requirements

At least G_1	Wang and Chen				Our proposed method			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
$G_1 = 0.91$	2.0000	**2.1850**	2.1750	2.0700	2.0000	**2.1850**	2.1750	2.0700
$G_1 = 0.95$	2.0000	0.9100	**2.1750**	0.9450	2.0000	2.1340	**2.1750**	2.0644
$G_1 = 0.96$	**0.9500**	0.9100	**0.9500**	0.9450	1.9895	2.1213	**2.1628**	2.0531

Looking at the second row of Table 5 where $G_1 = 0.91$, it is clearly seen that both Wang and Chen's approach and our approach get the same result as $A_2 \succ A_3 \succ A_4 \succ A_1$. Secondly, we consider the case $G_1 = 0.96$, the fourth row of Table 5. We can see that the prioritized aggregation values are different. Taking alternative A_1 and A_3 as an example, from Table 2 we know that the degrees of satisfaction of criterion *safety* are $C_1(A_1) = C_1(A_3) = 0.95$. Using Wang and Chen's approach we know that satisfaction of criterion *safety* does not satisfy the requirement $G_1 = 0.95$ at all, i.e. the priority weights of lower priority criteria are all 0, thus A_1 and A_2 induce the same aggregation value. However, the satisfactions of lower priority criteria of alternative A_3 are higher than those alternative A_1, and 0.95 is slightly less than $G_1 = 0.96$, thus Wang and Chen's approach [10] is too strict. By using our approach, it is clearly seen that the ranking order of alternatives is $A_3 \succ A_2 \succ A_4 \succ A_1$.

The main difference of benchmark achievement in inducing priority weight between our approach and Wang and Chen's approach is illustrated in Fig. 2(a).It is clear that when the degree of satisfaction of criterion *safety* is higher than requirement G_1, we will obtain the same result with Wang and Chen. If the degree of satisfaction of criterion *safety* is less than requirement G_1, Wang and Chen's approach will be too strict.

Benchmark: at least G_1 and as high as possible

Now let us consider the second type of requirement. Column 1 of Table 6 shows the requirement values. According to Wang and Chen's approach (Eq. (26), Eq. (27), Eq. (29) and Eq. (30)), we can obtain the aggregation value with different G_1 values, as shown in Columns 2-5 of Table 6.

To model at least G_1 and as high as possible, we can use fuzzy number to represent this uncertainty, denoted as $\tilde{G}_1 = (G_1, G_1, 1, 1)$. It is in fact an interval number $[G_1, 1]$, in which a uniform probability distribution can be obtained by means of the possibility-probability conversion method in Eq. (24). And then according to Eq. (25) and our prioritized aggregation operator we can obtain the prioritized aggregation values as shown in Columns 6-10 of Table 6.

We will analysis the difference between our approach and Wang and Chen via Fig. 2(b). The ***red*** line in Fig. 2(b) shows the benchmark achievement by means of fuzzy target-oriented decision analysis under the requirement $\tilde{G}_1 = (G_1, G_1, 1, 1)$. The ***green*** line in Fig. 2(b) represents Wang and Chen's approach. According to Fig. 2(b), it is clear that when the degree of satisfaction

Table 6
Prioritized aggregated value under different fuzzy uncertain requirements

at least G_1, as high as possible	Wang and Chen's method				Our proposed method			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
$G_1 = 0.91$	1.9475	2.0703	**2.1137**	2.0081	1.4167	0.9100	**1.4944**	1.3825
$G_1 = 0.95$	1.9475	0.9100	**2.1137**	0.9450	**0.9500**	0.9100	**0.9500**	0.9450
$G_1 = 0.96$	**0.9500**	0.9100	**0.9500**	0.9450	**0.9500**	0.9100	**0.9500**	0.9450

of criterion ***safety*** is slightly less than the benchmark G_1, the induced degree of satisfaction $\mathrm{E}(\mathrm{Sat}_1(\cdot))$ will be zero, which will be too strict; when the degree of satisfaction of criterion ***safety*** is more than the benchmark G_1, the induced degree of satisfaction $\mathrm{E}(\mathrm{Sat}_1(\cdot))$ will be $\mathrm{Sat}_1(\cdot)$. Whereas our approach is more consistent than Wang and Chen's approach. In addition, our approach usually constrain the benchmark achievement into an interval range $[0, 1]$.

Benchmark: Fuzzy at least G_1

Due to the vagueness or impreciseness of knowledge, it is difficult for DMs to estimate their requirements with precision. Fuzzy min target (fuzzy at least) is the target commonly used in decision making. We can model the fuzzy min G_1 as $\tilde{G}_1 = (0, G_1, 1, 1)$. In the previous case, we considered the requirement "at least G_1 and as high as possible" via target-oriented decision analysis. The fuzzy target $(G_1, G_1, 1, 1)$ is a special case of fuzzy at least G_1. By means of the possibility-probability conversion method in Eq. (24) we can obtain the induced probability distribution function of fuzzy min target. And then according to Eq. (25) we obtain the target achievement function. Finally, according to our prioritized aggregation operator, we can easily obtain the aggregation values as shown in Columns 2-5 of Table 7, in which A_3 is always the optimal alternative.

Table 7
Prioritized aggregated value under different fuzzy min requirements

Fuzzy min G_1	Our proposed method			
	A_1	A_2	A_3	A_4
$(0, 0.91, 1, 1)$	1.9037	1.9744	**2.0626**	1.9565
$(0, 0.95, 1, 1)$	1.9000	1.9685	**2.0583**	1.9522
$(0, 0.96, 1, 1)$	1.8991	1.9675	**2.0573**	1.9513

Fig. 2(c) graphically depicts the fuzzy requirement and its associated target achievement function. It is clear from the ***red line*** in Fig. 2(c), the benchmark achievement reflects Simon's bounded rationality and the S-shaped value function. This is main reason why we utilize fuzzy target-oriented decision model to calculate fuzzy benchmark achievement.

5. Concluding Remarks

In this paper, we have concerned ourselves with multi-criteria decision making (MCDM) problems where there exists a prioritization of criteria, in which the priority weights associated with the lower priority are related to the satisfactions of the higher priority criteria. We have builded upon the work of [10,38,39] and extended it in a number of directions. First, the OWA operator is used to obtain the degree of satisfaction for each priority level. To preserve the tradeoffs between the criteria in the same priority level, the degree of satisfaction regarding each priority level is viewed a pseudo criterion. Second, we suggest that roughly speaking any t-norms can be used to model the priority relationships between the criteria in different priority levels. To preserve slight change of the priority weight, strict Archimedean t-norms perform better in inducing priority weight. As Hamacher family of t-norms provide a wide class of strict Archimedean t-norms ranging from the product to weakest t-norm [26], Hamacher t-norms are selected to induce the priority weight. Third, considering DM's requirement toward higher priority levels, a *benchmark* based approach has been proposed to induce priority weight for each priority level. In particular, Łukasiewicz implication is used as a technique to compute benchmark achievement for crisp requirements. In case of fuzzy uncertain requirements, as target-oriented decision analysis lies in the philosophical root of bounded rationality as well as represents the S-shaped value function, fuzzy target-oriented decision analysis [16] is utilized to obtain the benchmark achievement. In contrast to Wang and Chen's [10] work, our approach can catch the slight changes of DM's requirement as well as coincide with the intuition of DM.

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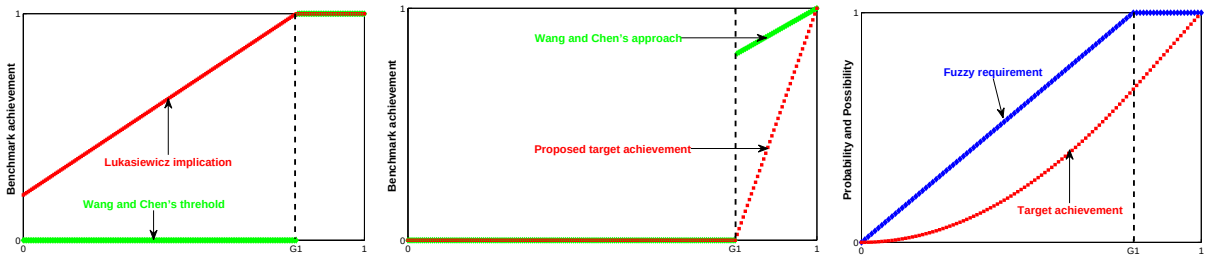

(a) At least G_1 (b) At least G_1 and as high as possible (c) Fuzzy min G_1

Fig. 2. Benchmark achievement

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