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Description	

On the Contrapositive of Countable Choice

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Abstract

We show that in elementary analysis (EL) the contrapositive of countable choice (CCC) is equivalent to double negation elimination for Σ^0_2-formulas. By also proving a recursive adaptation of this equivalence in Heyting arithmetic (HA), we give an instance of the conservativity of EL over HA with respect to recursive functions and predicates. As a complement, we prove in HA enriched with the (extended) Church thesis that every decidable predicate is recursive.

Throughout let x, y, z, z', e stand for numbers (i.e., nonnegative integers); f for everywhere defined number-number functions; and P, Q for decidable predicates of numbers. The focus is on the *contrapositive of countable choice* [5, 2, 1]:

CCC $\forall f \exists x\, P(x, f(x)) \to \exists x \forall y\, P(x, y)$.

In [1, Section 2.4, Lemma 5.5][1] it was proved that CCC follows from *double negation elimination for Σ^0_2-formulas:*

$\mathbf{\Sigma^0_2}$**-DNE** $\neg\neg\exists x\, \forall y\, P(x, y) \to \exists x \forall y\, P(x, y)$;

and that CCC implies the *law of excluded middle for Σ^0_1-formulas:*

$\mathbf{\Sigma^0_1}$**-LEM** $\exists x\, P(x) \vee \forall x\, \neg P(x)$.

In [1, Footnote 4] it was also conjectured that CCC lies strictly in between these two instances of the law of excluded middle.

The objective of the present note is to show that if one works in elementary intuitionistic analysis **EL** [4, 3.6], and restricts P to quantifier-free predicates, then CCC is actually equivalent to Σ^0_2-DNE. Now **EL** is conservative over intuitionistic first-order arithmetic **HA** ([3] and [4, 3.6.2]) whenever the function variables characteristic of **EL** are interpreted as ranging over all (total) recursive functions, and thus—by Kleene's normal form

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[1]Where Σ^0_2-DNE and Σ^0_1-LEM were called 2-Markov and 1-EM, respectively.

theorem—can be represented by numbers. Since CCC is the only place where a function variable occurs in the context of this paper, the equivalence of CCC and Σ^0_2-DNE can already be proved in **HA** provided that we limit f to recursive functions, and likewise P to recursive predicates. We give a formal proof of this result for the sake of completeness. As a complement we show, in **HA** enriched with the (extended) Church thesis, that every decidable predicate is recursive.

We first show how to arrive at the desired equivalence in **EL**. For this purpose let P stand for a quantifier-free predicate. The aforementioned implications involving CCC are valid under these circumstances too; for the reader's convenience we briefly recall, in the subsequent lemma, their proofs from [1, Section 2.4, Lemma 5.5]. It is useful to remember before that *double negation elimination for* Σ^0_1*-formulas:*

$\mathbf{\Sigma^0_1}$**-DNE** $\neg\neg\exists x\, P\,(x) \to \exists x\, P\,(x)$,

also known as *Markov's principle* (MP), follows from each of Σ^0_2-DNE and Σ^0_1-LEM.

Lemma 1 *[1, Section 2.4, Lemma 5.5]*

(a) $\mathbf{EL} + \Sigma^0_2\text{-DNE} \vdash \text{CCC}$.

(b) $\mathbf{EL} + \text{CCC} \vdash \Sigma^0_1\text{-LEM}$.

PROOF. (a) Assume that $\forall f \exists x\, P\,(x, f\,(x))$. To show $\exists x \forall y\, P\,(x, y)$, in the presence of Σ^0_2-DNE it suffices to disprove $\forall x \neg \forall y\, P\,(x, y)$. From the latter we can infer $\forall x \exists y \neg P\,(x, y)$ by Σ^0_1-DNE; whence the function $f = \lambda x.\mu y.\neg P\,(x, y)$ is everywhere defined. By assumption, for this f there is x with $P(x, f(x))$, which is in conflict with the definition of f.

(b) We first notice $\forall f \exists x\, [P\,(x) \vee \neg P(f(x))]$. (For any f, if $\neg P(f(0))$, then take $x = 0$, and if $P(f(0))$, then $x = f(0)$ is as required.) By CCC there is x with $\forall y\; [P(x) \vee \neg P(y)]$. For any such x, if $P(x)$, then $\exists x\, P(x)$, whereas if $\neg P(x)$, then $\forall y\, \neg P(y)$. Q.E.D.

It is to be noted that in **EL** the construction of a function by unbounded minimisation, as used in part (a) of the foregoing proof, can be coped with by quantifier-free choice.

Lemma 2 *In* **EL** *each of the following assertions implies the next:*

$$\exists f\, \forall x\, \neg P\,(x, f\,(x))\ ; \qquad \forall x\, \exists y\, \neg P\,(x, y)\ ; \qquad \forall x\, \neg \forall y\, P\,(x, y)\ .$$

Proposition 3 $\mathbf{EL} \vdash \Sigma^0_2\text{-DNE} \leftrightarrow \text{CCC}$.

PROOF. It suffices to show that CCC implies Σ^0_2-DNE, to which end we may use Σ^0_1-DNE. Since the consequents of CCC and Σ^0_2-DNE are identical, we only have to verify that $C \equiv \forall f \exists x\, P\,(x, f\,(x))$, the antecedent of CCC, follows from $D \equiv \neg \forall x\, \neg \forall y\, P\,(x, y)$, an equivalent of the antecedent of Σ^0_2-DNE. By Lemma 2, D implies $\forall f\, \neg \forall x\, \neg P\,(x, f\,(x))$, from which we arrive at C by means of Σ^0_1-DNE. Q.E.D.

One may have noticed in the foregoing proof (see also Lemma 2) that CCC even implies

$\neg\forall x\,\exists y\,\neg P(x,y) \to \exists x \forall y\, P(x,y)$.

This clearly entails Σ^0_2-DNE, of which it therefore is yet another equivalent in **EL**.

We next turn to **HA** and a recursive adaptation of CCC. In the rest of this paper let P, as it occurs in the axiom schemes under consideration, stand for recursive predicates. Before going into any detail we first sketch how we arrived at the equivalence we were after. To this end we relativise CCC to any collection $\mathcal{F}$ of number-number functions:

CCC$_{\mathcal{F}}$ $\forall f \in \mathcal{F} \exists x\, P(x, f(x)) \to \exists x \forall y\, P(x,y)$.

It is plain that if $\mathcal{G} \subseteq \mathcal{H}$, then CCC$_{\mathcal{G}}$ implies CCC$_{\mathcal{H}}$. Hence CCC, which is CCC$_{\mathcal{F}}$ with $\mathcal{F}$ consisting of all the number-number functions, is the weakest form. In particular, Σ^0_1-LEM follows from CCC$_{\mathcal{F}}$ for arbitrary $\mathcal{F}$.

An inspection of the proof of [1, Section 2.4, Lemma 5.5], also recalled in our Lemma 1, revealed the following:

— *Let $f_P = \lambda x.\mu y.P(x,y)$ for every recursive predicate P such that $\forall x \exists y\, P(x,y)$. If $\mathcal{F}$ contains all these f_P, then* Σ^0_2-DNE *implies* CCC$_{\mathcal{F}}$.

— *If there is a partial recursive function F such that $\mathcal{F}$ consists of the $F(\_, e)$ which are total functions with e any number, then* CCC$_{\mathcal{F}}$ *implies* Σ^0_2-DNE.

Since the collection $\mathcal{F}$ of all the recursive functions satisfies the hypotheses of both items above, we thus have achieved the desired equivalence.

Before we make this argument precise we recall some well known facts from the theory of recursive functions. Whenever one says that recursive functions can be enumerated by numbers, and thus that quantification over the former can be reduced to quantification over the latter, one tacitly invokes Kleene's normal form theorem [4, 3.4.2, 3.7.6]. This indeed provides us with a predicate T and a function U, both primitive recursive, such that the partial recursive functions are precisely the functions of type $\{e\}$ with $\{e\}(x) \simeq U(\mu z.T(e,x,z))$, and any such $\{e\}$ is a total function precisely when $\forall x \exists z\, T(e,x,z)$. Moreover, computation is deterministic in the sense that

$$T(e,x,z) \wedge T(e,x,z') \to z = z' ;$$

in particular, given e and x, if $T(e,x,z)$ for any z, then $\{e\}(x) = U(z)$ for this z. As a consequence, for every recursive f there is e with $f = \{e\}$, for which $\forall x \exists z\, T(e,x,z)$, and $f(x) = U(z)$ whenever $T(e,x,z)$.

With Kleene's normal form theorem at hand it is easy to see that CCC for recursive functions is equivalent to

CCC^0 $\forall e\, [\forall x \exists z\, T(e,x,z) \to \exists x \exists z\, [T(e,x,z) \wedge P(x, U(z))]] \to \exists x \forall y\, P(x,y)$.

Note that CCC^0 is free of function variables. If P is a recursive predicate we write

$$A \equiv \forall e\,[\forall x \exists z\, T\,(e,x,z) \to \exists x \exists z\,[T\,(e,x,z) \wedge P\,(x, U\,(z))]]$$

for the antecedent of (the outmost implication of) CCC^0, so that CCC^0 simply reads as $A \to \exists x \forall y\, P\,(x,y)$. It is useful to keep in mind that A is equivalent to

$$\forall e\,[\forall x \exists z\, T\,(e,x,z) \to \exists x \forall z\,[T\,(e,x,z) \to P\,(x, U\,(z))]]\ .$$

Parts (a) and (b) of the subsequent proposition are related to [1, Section 2.4, Lemma 5.5] as it is recalled in Lemma 1 above, whereas part (c) is related to Proposition 3.

Proposition 4

(a) $\mathbf{HA} + \Sigma^0_2\text{-DNE} \vdash \mathrm{CCC}^0$.

(b) $\mathbf{HA} + \mathrm{CCC}^0 \vdash \Sigma^0_1\text{-LEM}$.

(c) $\mathbf{HA} + \mathrm{CCC}^0 \vdash \Sigma^0_2\text{-DNE}$.

Proof. To prove (a), suppose that P is a recursive predicate for which A holds. Since $\neg P$, too, is a recursive predicate, by Kleene's normal form theorem there is e with $\{e\} \simeq \lambda x.\mu y.\neg P\,(x,y)$; for the time being this recursive function may be partial. To show $\exists x \forall y\, P\,(x,y)$, in the presence of Σ^0_2-DNE it suffices to show $\neg\forall x \neg\forall y\, P\,(x,y)$: that is, to deduce a contradiction from $\forall x \neg \forall y\, P\,(x,y)$. Assume the latter, which by Σ^0_1-DNE means that $\forall x \exists y\, \neg P\,(x,y)$. Hence $\forall x \exists z\, T\,(e,x,z)$ and thus, by A, there are x and z with $T\,(e,x,z)$ and $P\,(x, U\,(z))$, for which also $\{e\}\,(x) = U\,(z)$ and thus $\neg P\,(x, U\,(z))$, a contradiction.

To prove (b), let Q be a recursive predicate. To show $\exists x\, Q\,(x) \vee \forall x\, \neg Q\,(x)$, it suffices to verify A for $Q(x) \vee \neg Q(y)$ in place of $P\,(x,y)$. Indeed, by CCC^0 we then obtain x with $\forall y\ [Q(x) \vee \neg Q(y)]$. For this x either $Q(x)$ or $\neg Q(x)$; in the former case $\exists x\, Q(x)$, in the latter case $\forall y\, \neg Q(y)$. To verify the desired instance of A, fix e and assume that $\forall x \exists z\, T\,(e,x,z)$. In particular, for $x = 0$ there is z with $T\,(e,x,z)$. For this z either $\neg Q(U(z))$ or $Q(U(z))$. In the former case the present choice of x and z is as required. In the latter case we take $x = U(z)$, for which by assumption there is z' with $T\,(e,x,z')$.

To prove (c), let P be a recursive predicate, and assume that $\neg \forall x\, \neg \forall y\, P\,(x,y)$. To show $\exists x \forall y\, P\,(x,y)$, in the presence of CCC^0 it suffices to show A, to which end we may use Σ^0_1-DNE according to (b). To prove A fix e and assume that $\forall x \exists z\, T\,(e,x,z)$. By Σ^0_1-DNE, to show $\exists x \exists z\,[T\,(e,x,z) \wedge P\,(x, U\,(z))]$ it suffices to deduce a contradiction from $\forall x \forall z \neg\,[T\,(e,x,z) \wedge P\,(x, U\,(z))]$ as follows. For every x there is z with $T\,(e,x,z)$, for which z we must have $\neg P\,(x, U\,(z))$; whence $\forall x\, \exists y\, \neg P\,(x,y)$ and thus $\forall x\, \neg \forall y\, P\,(x,y)$, which is in conflict with our assumption. Q.E.D.

Corollary 5 $\mathbf{HA} \vdash \Sigma^0_2\text{-DNE} \leftrightarrow \mathrm{CCC}^0$.

We conclude this paper with a proof of the fact that in **HA** plus the (extended) Church thesis every decidable predicate is recursive. To be precise on this, we need to consider the following three properties of a predicate R relative to an extension **S** of **HA** :

(1) R is *externally recursive* in $\mathbf{S}$: that is, there is a closed term (or numeral) t such that $\mathbf{S} \vdash \forall x \, \exists z \; [T\,(t,x,z) \wedge (R\,(x) \leftrightarrow U\,(z) = 0)]\,;$

(2) R is *internally recursive* in $\mathbf{S}$: that is, $\mathbf{S} \vdash \exists e \, \forall x \, \exists z \; [T\,(e,x,z) \wedge (R\,(x) \leftrightarrow U\,(z) = 0)]\,;$

(3) R is *decidable* in $\mathbf{S}$: that is, $\mathbf{S} \vdash \forall x \; (R\,(x) \vee \neg R\,(x))\,.$

It is plain that (1) implies (2), and that (2) implies (3). Before we can look at possible reverse implications, we need to recall some theory from [4].

Let CT_0 and ECT_0 stands for the arithmetical form of *Church's thesis* and of the *extended Church thesis* [4, 4.3.2, 4.4.8]. As shown in [4, 4.4.10, 3.5.10], we have the following:

Lemma 6

(i) $\mathbf{HA} + \mathrm{ECT}_0 \vdash R \leftrightarrow \exists x(x \;\mathbf{r}\; R)$ *;*

(ii) $\mathbf{HA} + \mathrm{ECT}_0 \vdash R \Leftrightarrow \mathbf{HA} \vdash \exists x(x \;\mathbf{r}\; R)$ *;*

(iii) $\mathbf{HA} \vdash \exists x R(x) \Rightarrow \mathbf{HA} \vdash R\,(\underline{n})$ *for some numeral* $\underline{n}$ *.*

By Lemma 6.iii, which says that $\mathbf{HA}$ satisfies the *explicit definability property for numbers* (EDN), it is clear that if $\mathbf{S}$ is $\mathbf{HA}$, then (2) implies (1).

Proposition 7

(a) *In* $\mathbf{HA} + \mathrm{ECT}_0$*, every decidable predicate is externally recursive.*

(b) *In* $\mathbf{HA} + \mathrm{CT}_0$*, every decidable predicate is internally recursive.*

PROOF. (a) We show that

(3') $\mathbf{HA} + \mathrm{ECT}_0 \vdash \forall x(R(x) \vee \neg R(x))$

implies the existence of a numeral $\underline{m}$ for which

(1') $\mathbf{HA} + \mathrm{ECT}_0 \vdash \forall x \exists z[T(\underline{m},x,z) \wedge (U\,(z) = 0 \leftrightarrow R(x))]$.

To start with, (3') implies $\mathbf{HA} \vdash \exists u(u \;\mathbf{r}\; \forall x(R(x) \vee \neg R(x)))$ by Lemma 6.ii, and hence $\mathbf{HA} \vdash \underline{n} \;\mathbf{r}\; \forall x(R(x) \vee \neg R(x))$ for some numeral $\underline{n}$ by Lemma 6.iii. For this $\underline{n}$ we have

$$\mathbf{HA} \vdash \forall x \exists z (T(\underline{n},x,z) \wedge z \;\mathbf{r}\; (R(x) \vee \neg R(x)))$$

by the definition [4, 4.4.2] of $\mathbf{r}$, by which further

$$\mathbf{HA} \vdash \forall x \exists z[T(\underline{n},x,z) \wedge ((j_1(U\,(z)) = 0 \wedge j_2(U\,(z)) \;\mathbf{r}\; R(x)) \vee (j_1(U\,(z)) \neq 0 \wedge j_2(U\,(z)) \;\mathbf{r}\; \neg R(x)))]\ .$$

Now two invocations of $\exists$-introduction yield

$$\mathbf{HA}{+}\mathrm{ECT}_0 \vdash \forall x \exists z[T(\underline{n},x,z) \wedge ((j_1(U\,(z)) = 0 \wedge \exists v(v \;\mathbf{r}\; R(x))) \vee (j_1(U\,(z)) \neq 0 \wedge \exists v(v \;\mathbf{r}\; \neg R(x))))]\ ,$$

from which by Lemma 6.i we arrive at

$$\mathbf{HA} + \mathrm{ECT}_0 \vdash \forall x \exists z [T(\underline{n}, x, z) \wedge ((j_1(U(z)) = 0 \wedge R(x)) \vee (j_1(U(z)) \neq 0 \wedge \neg R(x)))] \ ,$$

which in turn implies (1'): simply take a numeral $\underline{m}$ such that $\{\underline{m}\} = j_1 \circ \{\underline{n}\}$.

(b) Note first that

$$\mathbf{HA} \vdash \forall x \ (R(x) \vee \neg R(x)) \leftrightarrow \forall x \, \exists y \, (R(x) \leftrightarrow y = 0) \ .$$

Now if $\mathbf{HA} + \mathrm{CT}_0 \vdash \forall x(R(x) \vee \neg R(x))$, then $\mathbf{HA} + \mathrm{CT}_0 \vdash \forall x \, \exists y(R(x) \leftrightarrow y = 0)$ and thus

$$\mathbf{HA} + \mathrm{CT}_0 \vdash \exists e \forall x \exists z [T(e, x, z) \wedge (R(x) \leftrightarrow U(z) = 0)]$$

as required. Q.E.D.

To prove part (b) already *Church's thesis for disjunctions* $\mathrm{CT}_0^{\vee}$ [4, 4.3.2] is sufficient. It is unclear, however, whether $\mathbf{HA} + \mathrm{CT}_0$ has EDN, and thus whether (a) can be proved with CT_0 in place of the stronger ECT_0.

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