Chapter VII: Gödel-Dummett Logics

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1 Introduction and History

The logics we present in this chapter, Gödel logics, can be characterized in a rough-and-ready way as follows: The language is a standard (propositional, quantified propositional, first-order) language. The logics are many-valued, and the sets of truth values considered are (closed) subsets of $[0, 1]$ which contain both 0 and 1. 1 is the 'designated value,' i.e., a formula is valid if it receives the value 1 in every interpretation. The truth functions of conjunction and disjunction are minimum and maximum, respectively, and in the first-order case quantifiers are defined by infimum and supremum over subsets of the set of truth values. The characteristic operator of Gödel logics, the Gödel conditional, is defined by $a \to b = 1$ if $a \leq b$ and $= b$ if $a > b$. Because the truth values are ordered (indeed, in many cases, densely ordered), the semantics of Gödel logics is suitable for formalizing *comparisons*. It is related in this respect to a more widely known many-valued logic, Łukasiewicz (or 'fuzzy') logic (see Chapter V– although the truth function of the Łukasiewicz conditional is defined not just using comparison, but also addition. In contrast to Łukasiewicz logic, which might be considered a logic of *absolute* or *metric comparison*, Gödel logics are logics of *relative comparison*.

There are other reasons why the study of Gödel logics is important. As noted, Gödel logics are related to other many-valued logics of recognized importance. Indeed, Gödel logic is one of the three basic t-norm based logics which have received increasing attention in the last 15 or so years (the others are Łukasiewicz and product logic; see [27]). Yet Gödel logic is also closely related to intuitionistic logic: it is the logic of linearly-ordered Heyting algebras. In the propositional case, infinite-valued Gödel logic can be axiomatized by the intuitionistic propositional calculus extended by the axiom schema $(A \to B) \vee (B \to A)$. This connection extends also to Kripke semantics for intuitionistic logic: Gödel logics can also be characterized as logics of (classes of) linearly ordered and countable intuitionistic Kripke structures with constant domains [19]. Furthermore, the infinitely valued propositional Gödel logic can be embedded into the box fragment of LTL in the same way as intuitionistic propositional logic can be embedded into S4.

We want to start here with an observation concerning implications for many-valued logics, that spotlights why Gödel logics behave well in some cases in contrast to other many-valued logics, namely that they are based on the only implication that admits both modus ponens and the deduction theorem, as can be seen from the following observation of Gaisi Takeuti.

1.1 The Gödel conditional

LEMMA 1.1.1. *Suppose we have a standard language containing a 'conditional'* $\twoheadrightarrow$ *interpreted by a truth-function into* $[0,1]$, *and some entailment relation* $\models$. *Suppose further that*

1. *a conditional evaluates to* 1 *if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if* $\mathcal{I}(A) \leq \mathcal{I}(B)$, *then* $\mathcal{I}(A \twoheadrightarrow B) = 1$;
2. *if* $\Gamma \models B$, *then* $\mathcal{I}(\Gamma) \leq \mathcal{I}(B)$;
3. *the deduction theorem holds, i.e.,* $\Gamma \cup \{A\} \models B \Leftrightarrow \Gamma \models A \twoheadrightarrow B$.

Then $\twoheadrightarrow$ *is the Gödel conditional.*

Proof. From (1), we have that $\mathcal{I}(A \twoheadrightarrow B) = 1$ if $\mathcal{I}(A) \leq \mathcal{I}(B)$. Since $\models$ is reflexive, $B \models B$. Since it is monotonic, $B, A \models B$. By the deduction theorem, $B \models A \twoheadrightarrow B$. By (2),

$$\mathcal{I}(B) \leq \mathcal{I}(A \twoheadrightarrow B).$$

From $A \twoheadrightarrow B \models A \twoheadrightarrow B$ and the deduction theorem, we get $A \twoheadrightarrow B, A \models B$. By (2),

$$\min\{\mathcal{I}(A \twoheadrightarrow B), \mathcal{I}(A)\} \leq \mathcal{I}(B).$$

Thus, if $\mathcal{I}(A) > \mathcal{I}(B), \mathcal{I}(A \twoheadrightarrow B) \leq \mathcal{I}(B)$. $\square$

A large class of many-valued logics can be developed from the theory of t-norms [27]. The class of t-norm based logics includes not only (standard) Gödel logic, but also Łukasiewicz and product logic. In these logics, the conditional is defined as the residuum of the respective t-norm, and the logics differ only in the definition of their t-norm and the respective residuum, i.e., the conditional (see Chapter VII). The truth function for the Gödel conditional is of particular interest as it can be 'deduced' from simple properties of the evaluation and the entailment relation, as shown above.

Note that all usual conditionals (Gödel, Łukasiewicz, product conditionals) satisfy condition (1). So, in some sense, the Gödel conditional is the only many-valued conditional which validates both directions of the deduction theorem for $\models$. For instance, for the Łukasiewicz conditional $\rightarrow_{Ł}$ the right-to-left direction fails: $A \rightarrow_{Ł} B \models A \rightarrow_{Ł} B$, but $A \rightarrow_{Ł} B, A \not\models B$. (With respect to $\Vdash$, the left-to-right direction of the deduction theorem fails for $\rightarrow_{Ł}$.)

One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, and already uncountably many different logics when considering propositional entailments [17] or quantification over propositions [16], there are only countably many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values (Theorem 3.5.1). For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties in general result in different sets of valid formulas.

Besides the logical and computational interest in Gödel logics, they also provide an interesting playground for various areas of more traditional mathematics, like topology, esp. Polish spaces and Order theory.

1.2 History of Gödel logics

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by G¨ödel [26] to show that there are infinitely many logics between intuitionistic and classical logic. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [23] was the first to study infinite-valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is also sometimes called Gödel-Dummett logic or Dummett's $\mathbf{LC}$. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders. The entailment relation in propositional Gödel logics was investigated in [17] and Gödel logics with quantifiers over propositions in [8].

Standard first-order Gödel logic $\mathbf{G}_{\mathbb{R}}$ – the one based on the full interval $[0, 1]$ – has been discovered and studied by several people independently. Alfred Horn [28] was probably the first: He discussed this logic under the name *logic with truth values in a linearly ordered Heyting algebra*, and gave an axiomatization and the first completeness proof. [39] called $\mathbf{G}_{\mathbb{R}}$ *intuitionistic fuzzy logic* and gave a sequent calculus axiomatization for which they proved completeness. This system incorporates the density rule

$$\frac{\Gamma \vdash A \vee (C \to p) \vee (p \to B)}{\Gamma \vdash A \vee (C \to B)}$$

(where p is any propositional variable not occurring in the lower sequent.) The rule is redundant for an axiomatization of $\mathbf{G}_{\mathbb{R}}$, as was shown by Takano [37], who gave a streamlined completeness proof of Takeuti-Titani's system without the rule. A syntactical proof of the elimination of the density rule was later given in [18]. Other proof-theoretic investigations of Gödel logics can be found in [3] and [4]. The density rule is nevertheless interesting: It forces the truth value set to be dense in itself (in the sense that, if the truth value set is not dense in itself, the rule does not preserve validity). This contrasts with the expressive power of formulas: no formula is valid only for truth value sets which are dense in themselves.

Recent developments have clarified many long standing questions, like the classification of axiomatizability, the relation to Kripke frames, status of satisfiability of the monadic class.

1.3 Syntax and semantics for propositional Gödel logics

When considering propositional Gödel logics we fix a standard propositional language $\mathscr{L}^0$ with countably many propostional variables p_i, and the connectives $\wedge$, $\vee$, $\to$ and the constant $\bot$ (for 'false'); negation is introduced as an abbreviation: we let $\neg p \equiv (p \to \bot)$. For convenience, we also define $\top \equiv \bot \to \bot$. We will sometimes use the unary connective $\triangle$, introduced in [2]. Furthermore we will use $p \prec q$ as an abbreviation for $(q \to p) \to q$.

DEFINITION 1.3.1. *Let $V \subseteq [0, 1]$ be some set of truth values which contains 0 and 1. A propositional Gödel valuation $\mathcal{I}^0$ (short valuation) based on V is a function from the* *set of propositional variables into V with $\mathcal{I}^0(\bot) = 0$. This valuation can be extended to a function mapping formulas from $\mathrm{Frm}(\mathscr{L}^0)$ into V as follows:*

$$\mathcal{I}^0(A \wedge B) = \min\{\mathcal{I}^0(A), \mathcal{I}^0(B)\}$$
$$\mathcal{I}^0(A \vee B) = \max\{\mathcal{I}^0(A), \mathcal{I}^0(B)\}$$
$$\mathcal{I}^0(\triangle A) = \begin{cases} 1 & \mathcal{I}^0(A) = 1 \\ 0 & \mathcal{I}^0(A) < 1 \end{cases}$$
$$\mathcal{I}^0(A \to B) = \begin{cases} \mathcal{I}^0(B) & \textit{if } \mathcal{I}^0(A) > \mathcal{I}^0(B) \\ 1 & \textit{if } \mathcal{I}^0(A) \leq \mathcal{I}^0(B). \end{cases}$$

A formula is called valid with respect to V if it is mapped to 1 for all valuations based on V. The set of all formulas which are valid with respect to V will be called the propositional Gödel logic based on V and will be denoted by G^0_V.

The validity of a formula A with respect to V will be denoted by

$$\models^0_V A \quad \textit{or} \quad \models_{\mathrm{G}^0_V} A.$$

REMARK 1.3.2. *The extension of the valuation $\mathcal{I}^0$ to formulas provides the following truth functions:*

$$\mathcal{I}^0(\neg A) = \begin{cases} 0 & \textit{if } \mathcal{I}^0(A) > 0 \\ 1 & \textit{otherwise} \end{cases}$$
$$\mathcal{I}^0(A \prec B) = \begin{cases} 1 & \textit{if } \mathcal{I}^0(A) < \mathcal{I}^0(B) \textit{ or } \mathcal{I}^0(A) = \mathcal{I}^0(B) = 1 \\ \mathcal{I}(B) & \textit{otherwise} \end{cases}$$

Thus, the intuition behind $A \prec B$ is that A is strictly less than B, or both are equal to 1.

1.4 Syntax and semantics for first-order Gödel logics

When considering first-order Gödel logics we fix a standard first-order language $\mathscr{L}$ with finitely or countably many predicate symbols P and finitely or countably many function symbols f for every finite arity k. In addition to the connectives of propositional Gödel logics the two quantifiers $\forall$ and $\exists$ are used.

In the first order case, where quantifiers will be interpreted as infima and suprema, we require the truth value set to be a closed subset of $[0, 1]$ (and as before $0, 1 \in V$).

DEFINITION 1.4.1 (Gödel set). *A* Gödel set *is a closed set $V \subseteq [0, 1]$ which contains 0 and 1.*

The semantics of Gödel logics, with respect to a fixed Gödel set as set of truth values and a fixed language $\mathscr{L}$ of predicate logic, is defined using the extended language $\mathscr{L}^U$, where U is the universe of the interpretation $\mathcal{I}$. $\mathscr{L}^U$ is $\mathscr{L}$ extended with constant symbols for each element of U.

DEFINITION 1.4.2 (Semantics of Gödel logic). *Let V be a Gödel set. An* interpretation *$\mathcal{I}$* into *V, or a V*-interpretation*, consists of*

1. *a nonempty set $U = U^{\mathcal{I}}$, the 'universe' of $\mathcal{I}$,*

2. *for each k-ary predicate symbol P, a function $P^{\mathcal{I}}\colon U^k \to V$,*

3. *for each k-ary function symbol f, a function $f^{\mathcal{I}}\colon U^k \to U$.*

4. *for each variable v, a value $v^{\mathcal{I}} \in U$.*

Given an interpretation $\mathcal{I}$, we can naturally define a value $t^{\mathcal{I}}$ for any term t and a truth value $\mathcal{I}(A)$ for any formula A of $\mathscr{L}^U$. For a term $t = f(u_1, \dots, u_k)$ we define $\mathcal{I}(t) = f^{\mathcal{I}}(u_1^{\mathcal{I}}, \dots, u_k^{\mathcal{I}})$. For atomic formulas $A \equiv P(t_1, \dots, t_n)$, we define $\mathcal{I}(A) = P^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}})$. For composite formulas A we extend the truth definitions from the propositonal case for the new syntactic elements by:

$$\mathcal{I}(\forall x\, A(x)) = \inf\{\mathcal{I}(A(u)) : u \in U\}$$
$$\mathcal{I}(\exists x\, A(x)) = \sup\{\mathcal{I}(A(u)) : u \in U\}$$

If $\mathcal{I}(A) = 1$, we say that $\mathcal{I}$ satisfies *A, and write $\mathcal{I} \models A$. If $\mathcal{I}(A) = 1$ for every V-interpretation $\mathcal{I}$, we say A is* valid *in $\mathbf{G}_V$ and write $\mathbf{G}_V \models A$.*
If Γ is a set of sentences, we define $\mathcal{I}(\Gamma) = \inf\{\mathcal{I}(A) : A \in \Gamma\}$.

Abusing notation slightly, we will often define interpretations simply be defining the truth values of atomic formulas in $\mathscr{L}^U$.

DEFINITION 1.4.3. *If Γ is a set of formulas (possibly infinite), we say that Γ* entails *A* in *$\mathbf{G}_V$, $\Gamma \models_V A$ iff for all $\mathcal{I}$ into V, $\mathcal{I}(\Gamma) \leq \mathcal{I}(A)$.*
Γ 1-entails *A in $\mathbf{G}_V$, $\Gamma \Vdash_V A$, iff, for all $\mathcal{I}$ into V, whenever $\mathcal{I}(B) = 1$ for all $B \in \Gamma$, then $\mathcal{I}(A) = 1$.*
We will write $\Gamma \models A$ instead of $\Gamma \models_V A$ in case it is obvious which truth value set V is meant.

DEFINITION 1.4.4. *For a Gödel set V we define the* first order Gödel logic *$\mathbf{G}_V$ as the set of all pairs (Γ, A) such that $\Gamma \models_V A$.*

One might wonder whether a different definition of the entailment relation in Gödel logic might give different results. But as the following proposition shows, the above two definitions yield the same result, allowing us to use the characterization of $\models$ or $\Vdash$ as convenient.

PROPOSITION 1.4.5. $\Pi \models_V A$ *iff* $\Pi \Vdash_V A$

Proof. See [17, Proposition 2.2] □

Note that in the presence of $\triangle$, Proposition 1.4.5 does not hold and we will use the 1-entailment. Furthermore, it is important to mention that the (1-)satisfiability in the case without $\triangle$ does *not* define the entailment, which changes when adding $\triangle$.

1.5 Axioms and deduction systems for Gödel logics

In this section we introduce deduction systems for Gödel logics, and we show soundness and completeness.

Most of the time we concentrate on Hilbert-style deduction systems, for proof theory and Gentzen style systems see Chapter IV. The only time a Gentzen style proof system will be used in this chapter is when proving the strong completeness. In this proof system the notion of sequent, written as

$$A_1, \ldots, A_n \Rightarrow B$$

is introduced which we will consider as an abbreviation for

$$A_1 \to A_2 \to \ldots \to A_n \to B,$$

and $A_1, \ldots, A_n \Rightarrow$ as an abbreviation for $A_1, \ldots, A_n \Rightarrow \bot$.

We will denote by **IL** the following complete axiom system for intuitionistic logic, where $B^{(x)}$ means that x is not free in B:

I1	$\bot \to A$	I8	$(A \to B) \to [(C \to A) \to (C \to B)]$
I2	$A \to (B \to A)$	I9	$[A \to (C \to B)] \to [C \to (A \to B)]$
I3	$(A \wedge B) \to A$	I10	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$
I4	$(A \wedge B) \to B$	I11	$(C \to A) \wedge (C \to B) \to (C \to (A \wedge B))$
I5	$A \to (B \to (A \wedge B))$	I12	$(A \to (B \to C)) \to (A \wedge B \to C)$
I6	$A \to (A \vee B)$	I13	$[A \to (A \to B)] \to (A \to B)$
I7	$B \to (A \vee B)$		
IQ1	$\dfrac{B^{(x)} \to A(x)}{B^{(x)} \to \forall x A(x)}$	IQ2	$\forall x A(x) \to A(t)$
IQ3	$A(t) \to \exists x A(x)$	IQ4	$\dfrac{A(x) \to B^{(x)}}{\exists x A(x) \to B^{(x)}}$
MP	$\dfrac{A \quad A \to B}{B}$		

The following formulas will play an important rôle when axiomatizing Gödel logics, their names can be explained as follows: QS stands for 'quantifier shift', LIN for 'linearity', ISO_0 for 'isolation axiom of 0', ISO_1 for 'isolation axiom of 1', and $\text{FIN}(n)$ for 'finite with n elements'.

QS	$\forall x(C^{(x)} \vee A(x)) \to (C^{(x)} \vee \forall x A(x))$
LIN	$(A \to B) \vee (B \to A)$
ISO_0	$\forall x \neg\neg A(x) \to \neg\neg \forall x A(x)$
ISO_1	$\forall x \triangle A(x) \to \det \exists x A(x)$
$\text{FIN}(n)$	$(\top \to p_1) \vee (p_1 \to p_2) \vee \ldots \vee (p_{n-2} \to p_{n-1}) \vee (p_{n-1} \to \bot)$

We will use aditionally, referred to as $\text{AX}\Delta$

$$
\begin{array}{ll}
\Delta 1 & \triangle A \vee \det A \\
\Delta 2 & \triangle(A \vee B) \to (\det A \vee \det B) \\
\Delta 3 & \triangle A \to A \\
\Delta 4 & \triangle A \to \det \det A \\
\Delta 5 & \triangle(A \to B) \to (\det A \to \det B) \\
\Delta 6 & \dfrac{A}{\triangle A}
\end{array}
$$

DEFINITION 1.5.1. *If $\mathcal{A}$ is an axiom system, we denote by $\mathcal{A}^0$ the propositional part of $\mathcal{A}$, i.e. all the axioms which do not contain quantifiers.*

With $\mathcal{A}\Delta$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axioms $\text{AX}\Delta$.

With $\mathcal{A}_n$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axiom $\text{FIN}(n)$.

We denote by $\mathbf{H}$ the axiom system $\mathbf{IL} + \text{QS} + \text{LIN}$.

EXAMPLE 1.5.2. $\mathbf{IL}^0$ *is* $\mathbf{IPL}$. $\mathbf{H}^0$ *is Dummett's* $\mathbf{LC}$.

For all these axiom systems the general notion of deducability can be defined:

DEFINITION 1.5.3. *If a formula/sequent Γ can be deduced from an axiom system $\mathcal{A}$ we denote this by*

$$\vdash_{\mathcal{A}} \Gamma$$

PROPOSITION 1.5.4 (Soundness). *Suppose Γ contains only closed formulas, and all axioms of $\mathcal{A}$ are valid in $\mathbf{G}_V$. Then, if $\Gamma \vdash_{\mathcal{A}} A$ then $\Gamma \models_V A$. In particular, $\mathbf{H}$ is sound for $\models_V$ for any Gödel set V; $\mathbf{H}_n$ is sound for $\models_V$ if $|V| = n$; $\mathbf{H} + \text{ISO}_0$ is sound for $\models_V$ if 0 is isolated in V; and $\mathbf{H}\triangle + \text{ISO}_1$ is sound for $\models_V$ with $\triangle$.*

1.6 Topologic and order

In the following we will recall some definitions and facts from topology and order theory which will be used later on in many places.

1.6.1 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ and all its closed subsets are Polish spaces (hence, every Gödel set is a Polish space). For a detailed exposition see [29, 31].

DEFINITION 1.6.1 (Limit point, perfect space, perfect set). *A* limit point *of a topological space is a point that is not isolated, i.e. for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$. A space is* perfect *if all its points are limit points. A set $P \subseteq \mathbb{R}$ is* perfect *if it is closed and together with the topology induced from $\mathbb{R}$ is a perfect space.*

It is obvious that all (non-trivial) closed intervals are perfect sets, as well as all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are 'everywhere dense,' i.e., contained in the closure of their inner component. There is a well-known example of a perfect set that is nowhere dense, the Cantor set:

EXAMPLE 1.6.2 (Cantor Set). *The set of all numbers in the unit interval which can be expressed in triadic notation only by digits* 0 *and* 2 *is called the* Cantor set $\mathbb{D}$.

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one for our purposes is that it is a perfect set:

PROPOSITION 1.6.3. *The Cantor set is perfect.*

It is possible to embed the Cauchy space into any perfect space, yielding the following proposition:

PROPOSITION 1.6.4 ([29], Corollary 6.3). *If* X *is a nonempty perfect Polish space, then* $|X| = 2^{\aleph_0}$. *All nonempty perfect subsets of* $[0, 1]$ *have cardinality* $2^{\aleph_0}$.

It is possible to obtain the following characterization of perfect sets (see [41]):

PROPOSITION 1.6.5 (Characterization of perfect sets in $\mathbb{R}$). *For any perfect subset of* $\mathbb{R}$ *there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.*

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

THEOREM 1.6.6 (Cantor-Bendixon). *Let* X *be a Polish space. Then* X *can be uniquely written as* $X = P \cup C$, *with* P *a perfect subset of* X *and* C *countable and open. The subset* P *is called the* perfect kernel *of* X *(denoted by* X^∞*).*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality $2^{\aleph_0}$.

2 Propositional Gödel Logics

As already mentioned Gödel introduced this family of logics on the propositional level to analyze Intuitionistic logic. This allows the approach to Gödel logics via restricting the possible accessibility relations of Kripke models of intuitionistic logic. Two somehow reasonable restrictions of the Kripke structures are the restriction to constant domains and the restriction that the Kripke worlds are linearly ordered and of order type ω. One can now ask what sentences are valid in this restricted class of Kripke models. This question has been settled by Dummett [23] for the propositional case by adding to a complete axiomatization of intuitionistic logic the axiom of linearity

$$\text{LIN} \qquad (p \to q) \vee (q \to p)$$

It is interesting to note that p and q in the linearity scheme are propositional formulas. It is *not* enough to add this axiom for atomic p and q. For an axiom scheme only necessary for atomic formulas we have to use

$$((p \to q) \to p) \vee (p \to (p \to q))$$

to obtain completeness [16].

Another interesting distinction between $\mathbf{LC}$, which is $\mathrm{G}^0_\downarrow$, and other propositional Gödel logics is the fact that while $\mathrm{G}^0_\downarrow$ and $\mathrm{G}^0_\mathbb{R}$ have the same set of tautologies, the entailment relation of the former is not compact, while the one of the latter is. The logic Dummett discussed, the logic of linearly ordered Kripke frames of order type ω, corresponds to $\mathrm{G}^0_\downarrow$. Therefore, Dummett proved only weak completeness (see Section 4.4).

One of the important properties of propositional Gödel logics is that the set of tautologies for any infinitely valued propositional Gödel logic coincides with the intersection of the sets of tautologies of all finitely valued propositional Gödel logics. Therefore, there is, with respect to the set of tautologies, only one infinite valued propositional Gödel logic, in contrast to entailment, quantified propositional logics, and first order.

2.1 Completeness of H^0 for LC

[23] proved that a formula of propositional Gödel logic is valid in any infinite truth value set if it is valid in one infinite truth value set. Moreover, all the formulas valid in these sets are axiomatized by any axiomatization of intuitionistic propositional logic extended with the linearity axiom scheme $(p \to q) \vee (q \to p)$. The proof given here is a simplified proof of the completeness of $\mathbf{H}^0$ taken from [28].

DEFINITION 2.1.1. *An algebra* $\mathbf{P} = \langle P, \cdot, +, \to, \mathbf{0}, \mathbf{1} \rangle$ *is a* Heyting algebra *if the reduct* $\langle P, \cdot, +, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice with least element* $\mathbf{0}$*, largest element* $\mathbf{1}$ *and* $x \cdot y \leq z$ *iff* $x \leq (y \to z)$.

DEFINITION 2.1.2. *An* L-algebra *is a Heyting algebra in which*

$$(x \to y) + (y \to x) = \mathbf{1}$$

is valid for all x, y.

It is obvious that if we take L-algebras as our reference models for completeness, the proof of completeness is trivial. Generally, it is not very interesting to define algebras fitting to logics like a second skin, and then proving completeness with respect to this class (Ł-algebras, ...), without giving any connection to well known algebraic structures or already accepted reference models. In our case we want to show completeness with respect to the real interval $[0, 1]$ or one of its sub-orderings. More generally we aim at completeness with respect to chains, which are special Heyting algebras:

DEFINITION 2.1.3. *A* chain *is a linearly ordered Heyting algebra.*

Chains are exactly what we are looking for as every chain (with cardinality less or equal to the continuum) is isomorphic to a sub-ordering of the $[0,1]$ interval, and vice versa. Our aim is now to show completeness of the above axiomatization with respect to chains. Furthermore we will exhibit that the length of the chains for a specific formula can be bounded by the number of propositional variables in the formula. More precisely:

THEOREM 2.1.4. *A formula α is provable in $\mathbf{H}^0 = \mathbf{LC}$ if and only if it is valid in all chains with at most $n+2$ elements, where n is the number of propositional variables in α.*

Proof. As usual we define the relation $\alpha \preceq \beta$ equivalent to $\vdash \alpha \to \beta$ and $\alpha \equiv \beta$ as $\alpha \preceq \beta$ and $\beta \preceq \alpha$. It is easy to verify that $\equiv$ is an equivalence relation. We denote $\alpha/{\equiv}$ with $|\alpha|$. It is also easy to show that with $|\alpha| + |\beta| = |\alpha \vee \beta|$, $|\alpha| \cdot |\beta| = |\alpha \wedge \beta|$, $|\alpha| \to |\beta| = |\alpha \to \beta|$ the set $\mathcal{F}/{\equiv}$ becomes a Heyting algebra, and due to the linearity axiom it is also an L-algebra. Furthermore note that $|\alpha| = \mathbf{1}$ if and only if α is provable in $\mathbf{H}^0$ ($\mathbf{1} = |p \to p|$, $|\alpha| = |p \to p|$ gives $\vdash (p \to p) \to \alpha$ which in turn gives $\vdash \alpha$).

If our aim would be completeness with respect to L-algebras the proof would be finished here, but we aim at completeness with respect to chains, therefore, we will take a close look at the structure of $\mathcal{F}/{\equiv}$ as L-algebra. Assume that a formula α is given, which is not provable, we want to give a chain where α is not valid. We already have an L-algebra where α is not valid, but how to obtain a chain?

We could use the general result from [28], Theorem 1.2, that a Heyting algebra is an L-algebra if and only if it is a subalgebra of a direct product of chains, but we will exhibit how to find explicitly a suitable chain. The idea is that the L-algebra $\mathcal{F}/{\equiv}$ describes all possible truth values for all possible orderings of the propositional variables in α. We want to make this more explicit:

DEFINITION 2.1.5. *We denote with*

$$\mathcal{C}(\bot, p_{i_1}, \ldots, p_{i_n}, \top)$$

the chain with these elements and the ordering

$$\bot \le p_{i_1} < \ldots < p_{i_n} \le \top.$$

If $\mathcal{C}$ is a chain we denote with $|\alpha|_{\mathcal{C}}$ the evaluation of the formula in the chain $\mathcal{C}$.

LEMMA 2.1.6. *The L-algebra $\mathcal{F}/{\equiv}$ is a subalgebra of the following direct product of chains*

$$X = \prod_{i=1}^{n!} \mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$$

where π_i ranges over the set of permutations of n elements. We will use $\mathcal{C}_i$ to denote $\mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$.

Proof. Define $\phi : \mathcal{F}/{\equiv} \to X$ as follows:

$$\phi(|\alpha|) = (|\alpha|_{\mathcal{C}_1}, \ldots, |\alpha|_{\mathcal{C}_{n!}}).$$

We have to show that ϕ is well defined, is a homomorphism and is injective. First assume that $\beta \in |\alpha|$ but $\phi(|\alpha|) \neq \phi(|\beta|)$, i.e.

$$(|\alpha|_{\mathcal{C}_1}, \ldots, |\alpha|_{\mathcal{C}_{n!}}) \neq (|\beta|_{\mathcal{C}_1}, \ldots, |\beta|_{\mathcal{C}_{n!}})$$

but then there must be an i such that

$$|\alpha|_{\mathcal{C}_i} \neq |\beta|_{\mathcal{C}_i}.$$

Without loss of generality, assume that $|\alpha|_{\mathcal{C}_i} < |\beta|_{\mathcal{C}_i}$. From the fact that $|\alpha| = |\beta|$ we get $\vdash \beta \to \alpha$. From this we get that $|\beta \to \alpha|_{\mathcal{C}_i} < \mathbf{1}$ and from $\vdash \beta \to \alpha$ we get that $|\beta \to \alpha|_{\mathcal{C}_i} = \mathbf{1}$, which is a contradiction. This proves the well-definedness.

To show that ϕ is a homomorphism we have to prove that

$$\begin{aligned}\phi(|\alpha| \cdot |\beta|) &= \phi(|\alpha|) \cdot \phi(|\beta|)\\ \phi(|\alpha| + |\beta|) &= \phi(|\alpha|) + \phi(|\beta|)\\ \phi(|\alpha| \to |\beta|) &= \phi(|\alpha|) \to \phi(|\beta|).\end{aligned}$$

This is a straightforward computation using $|\alpha \wedge \beta|_{\mathcal{C}} = \phi(|\alpha|_{\mathcal{C}}) \cdot \phi(|\beta|_{\mathcal{C}})$.

Finally we have to prove that ϕ is injective. Assume that $\phi(|\alpha|) = \phi(|\beta|)$ and that $|\alpha| \neq |\beta|$. From the former we obtain that $|\alpha|_{\mathcal{C}_i} = |\beta|_{\mathcal{C}_i}$ for all $1 \leq i \leq n!$, which means that

$$\mathcal{I}_{\mathcal{C}_i}(\alpha) = \mathcal{I}_{\mathcal{C}_i}(\beta) \quad \text{for all } 1 \leq i \leq n!.$$

On the other hand we know from the latter that there is an interpretation $\mathcal{I}$ such that $\mathcal{I}(\alpha) \neq \mathcal{I}(\beta)$. Without loss of generality assume that

$$\perp \leq \mathcal{I}(p_{i_1}) < \ldots < \mathcal{I}(p_{i_n}) \leq \top.$$

There is an index k such that the $\mathcal{C}_k$ is exactly the above ordering with

$$\mathcal{I}_{\mathcal{C}_k}(\alpha) \neq \mathcal{I}_{\mathcal{C}_k}(\beta),$$

this is a contradiction.

This completes the proof that $\mathcal{F}/{\equiv}$ is a subalgebra of the given direct product of chains. □

EXAMPLE 2.1.7. *For $n = 2$ the chains are $\mathcal{C}(\perp, p, q, \top)$ and $\mathcal{C}(\perp, q, p, \top)$. The product of these two chains looks as given in Figure 1, p. 142. The labels below the nodes are the products, the formulas above the nodes are representatives for the class $\alpha/{\equiv}$.*

Now the proof of Theorem 2.1.4 is trivial since, if $|\alpha| \neq \mathbf{1}$, there is a chain $\mathcal{C}_i$ where $|\alpha|_{\mathcal{C}_i} \neq \mathbf{1}$. □

This yields the following theorem:

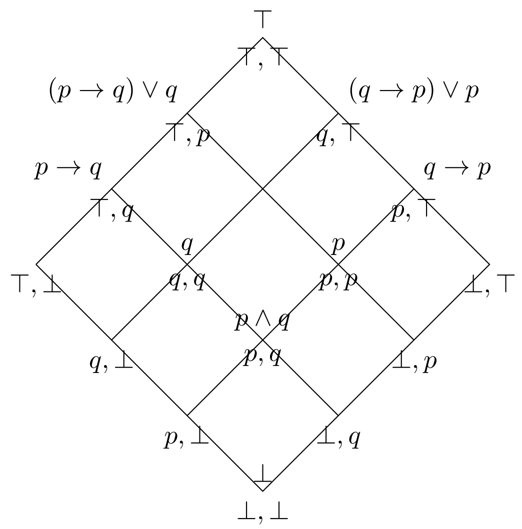

Figure 1. L-algebra of $\mathcal{C}(\bot, p, q, \top) \times \mathcal{C}(\bot, q, p, \top)$. Labels below the nodes are the elements of the direct product, formulas above the node are representatives for the class $\alpha/\equiv$.

THEOREM 2.1.8. *A propositional formula is valid in any infinite chain iff it is derivable in* $\mathbf{LC} = \mathbf{H}^0$.

Going on to finite truth value set we can give the following theorem:

THEOREM 2.1.9. *A formula is valid in any chain with at most n elements iff it is provable in* $\mathbf{LC}_n$.

Proof. Assuming that $\mathbf{H}_n^0 \nvdash \alpha$ and using the deduction theorem we can proceed as follows:

$$\begin{aligned} \mathbf{H}_n^0 &\nvdash \alpha \\ \mathbf{H}^0 + \text{FIN}(n) &\nvdash \alpha \\ \mathbf{H}^0 &\nvdash \text{FIN}(n) \to \alpha \end{aligned}$$

From this we know that there is an interpretation $\mathcal{I}$ such that

$$\mathcal{I}(\text{FIN}(n) \to \alpha) < 1$$

which is equivalent to

$$\mathcal{I}(\text{FIN}(n)) = 1 \text{ and } \mathcal{I}(\alpha) < 1.$$

The first formula ensures that the domain has at most n elements. Therefore, $\mathcal{I}$ is an interpretation with a domain with at most n elements and which evaluates α to a value less than 1. $\square$

As a simple consequence of these result the following corollaries settle the number of propositional Gödel logics and their relation:

COROLLARY 2.1.10. *The propositional Gödel logics* G_n^0 *and* $\mathrm{G}_{\mathbb{R}}^0$ *are all different, thus there are countable many different propositional Gödel logics, and*

$$\bigcap_{n\in\mathbb{N}} \mathrm{G}_n^0 = \mathrm{G}_{\mathbb{R}}^0$$

2.2 The Delta operator

For Gödel logics there is an asymmetry between 0 and 1 because 0 can be distinguished from other values (by using the negation), while 1 cannot be distinguished. The reason is that all connectives and quantifiers are continuous at 1. To overcome this assymetry the operator $\triangle$ has been introduced in [2] with the following truth function:

$$\phi(\triangle A) = \begin{cases} 1 & \text{if } \phi(A) = 1 \\ 0 & \text{o.w.} \end{cases}$$

THEOREM 2.2.1. *There is only one infinitely valued Gödel logic with* $\triangle$*, and it is axiomatized by* $\mathbf{H}^0\triangle$ *(see page 137), and this logic is the intersection of the finitely valued logics.*

It is important to note that adding $\triangle$ to the language is an actual extension, i.e., the $\triangle$ operator cannot be defined.

3 First Order Gödel Logics

After some preliminaries we discuss the relationships between different Gödel logics in Section 3.2, characterize the axiomatizable first-order Gödel logics in Sections 3.3.1, 3.3.2, and 3.3.3, followed by the characterization of those logics that are not recursively enumerable in Sections 3.3.4 and 3.3.5.

Following this complete characterization of axiomatizability we explicate the relation between (linear) Kripke frames based logics and Gödel logics in Section 3.4, and briefly discuss the very surprising result on the number of different first-order Gödel logics in Section 3.5.

All the result in the following sections are from [4, 15, 17–20].

3.1 Preliminaries

We will be concerned below with the relationships between Gödel logics, here considered as entailment relations. Note that $\mathbf{G}_V \models A$ iff $(\emptyset, A) \in \mathbf{G}_V$, so in particular, showing that $\mathbf{G}_V \subseteq \mathbf{G}_W$ also shows that every valid formula of $\mathbf{G}_V$ is also valid in $\mathbf{G}_W$. On the other hand, to show that $\mathbf{G}_V \not\subseteq \mathbf{G}_W$ it suffices to show that for some A, $\mathbf{G}_V \models A$ but $\mathbf{G}_W \not\models A$.

REMARK 3.1.1. *The case that a formula* A *evaluates to* 1 *under a certain interpretation* $\mathcal{I}$ *depends only on the* relative ordering *of the truth values of the atomic formulas (in* $\mathscr{L}^{\mathcal{I}}$*), and not directly on the set* V *or on the specific values of the atomic formulas.*

If $V \subseteq W$ are both Gödel sets, and $\mathcal{I}$ is a V-interpretation, then $\mathcal{I}$ can be seen also as a W-interpretation, and the values generated during the computation of $\mathcal{I}(A)$ do not depend on whether we view $\mathcal{I}$ as a V-interpretation or a W-interpretation. Consequently, if $V \subseteq W$, there are more interpretations into W than into V. Hence, if $\Gamma \models_W A$, then also $\Gamma \models_V A$ and $\mathbf{G}_W \subseteq \mathbf{G}_V$.

This can be generalized to embeddings between Gödel sets other than inclusion. First, we make precise which formulas are involved in the computation of the truth-value of a formula A in an interpretation $\mathcal{I}$:

DEFINITION 3.1.2. *The only subformula of an atomic formula A in $\mathscr{L}^U$ is A itself. The subformulas of $A \star B$ for $\star \in \{\rightarrow, \wedge, \vee\}$ are the subformulas of A and of B, together with $A \star B$ itself. The subformulas of $\forall x\, A(x)$ and $\exists x\, A(x)$ with respect to a universe U are all subformulas of all $A(u)$ for $u \in U$, together with $\forall x\, A(x)$ (or, $\exists x\, A(x)$, respectively) itself.*

The set of truth-values of subformulas of A under a given interpretation $\mathcal{I}$ is denoted by

$$\mathrm{Val}(\mathcal{I}, A) = \{\mathcal{I}(B) : B \text{ subformula of } A \text{ w.r.t. } U^{\mathcal{I}}\} \cup \{0, 1\}$$

If Γ is a set of formulas, then $\mathrm{Val}(\mathcal{I}, \Gamma) = \bigcup\{\mathrm{Val}(\mathcal{I}, A) : A \in \Gamma\}$.

LEMMA 3.1.3. *Let $\mathcal{I}$ be a V-interpretation, and let $h\colon \mathrm{Val}(\mathcal{I}, \Gamma) \to W$ be a mapping satisfying the following properties:*

1. *$h(0) = 0$, $h(1) = 1$;*
2. *h is strictly monotonic, i.e., if $a < b$, then $h(a) < h(b)$;*
3. *for every $X \subseteq \mathrm{Val}(\mathcal{I}, \Gamma)$, $h(\inf X) = \inf h(X)$ and $h(\sup X) = \sup h(X)$ (provided $\inf X, \sup X \in \mathrm{Val}(\mathcal{I}, \Gamma)$).*

Then the W-interpretation $\mathcal{I}_h$ with universe $U^{\mathcal{I}}$, $f^{\mathcal{I}_h} = f^{\mathcal{I}}$, and for atomic $B \in \mathscr{L}^{\mathcal{I}}$,

$$\mathcal{I}_h(B) = \begin{cases} h(\mathcal{I}(B)) & \text{if } \mathcal{I}(B) \in \operatorname{dom} h \\ 1 & \text{otherwise} \end{cases}$$

satisfies $\mathcal{I}_h(A) = h(\mathcal{I}(A))$ for all $A \in \Gamma$.

Proof. By induction on the complexity of A. If $A \equiv \bot$, the claim follows from (1). If A is atomic, it follows from the definition of $\mathcal{I}_h$. For the propositional connectives the claim follows from the strict monotonicity of h (2). For the quantifiers, it follows from property (3). $\square$

PROPOSITION 3.1.4 (Downward Löwenheim-Skolem). *For any interpretation $\mathcal{I}$ with $U^{\mathcal{I}}$ infinite, there is an interpretation $\mathcal{I}' \prec \mathcal{I}$ with a countable universe $U^{\mathcal{I}'}$.*

LEMMA 3.1.5. *Let $\mathcal{I}$ be an interpretation into V, $w \in [0,1]$, and let $\mathcal{I}_w$ be defined by*

$$\mathcal{I}_w(B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(B) < w \\ 1 & \text{otherwise} \end{cases}$$

for atomic formulas B in $\mathscr{L}^{\mathcal{I}}$. Then $\mathcal{I}_w$ is an interpretation into V. If $w \notin \mathrm{Val}(\mathcal{I}, A)$, then $\mathcal{I}_w(A) = \mathcal{I}(A)$ if $\mathcal{I}(A) < w$, and $\mathcal{I}_w(A) = 1$ otherwise.

Proof. By induction on the complexity of formulas A in $\mathscr{L}^{\mathcal{I}}$. The condition that $w \notin \mathrm{Val}(\mathcal{I}, A)$ is needed to prove the case of $A \equiv \exists x\, B(x)$, since if $\mathcal{I}(\exists x\, B(x)) = w$ and $\mathcal{I}(B(d)) < w$ for all d, we would have $\mathcal{I}_w(\exists x\, B(x)) = w$ and not $= 1$. □

Using this lemma we can obtain the following observation.

LEMMA 3.1.6. *If for all interpretations $\mathcal{I}(A) = 1$ iff $\mathcal{I}(B) = 1$, then already $A \leftrightarrow B$ is valid.*[1]

Proof. If $A \leftrightarrow B$ is not valid, then there is a real number w strictly between the valuations of A and B, such that either w is not a truth value, or w does not occur in the set of valuations of all sub-formulas of A and B. Assuming w.l.o.g. that $\mathcal{I}(A) > \mathcal{I}(B)$ we obtain that $I_w(A) = 1$ while $I_w(B) = I(B) < 1$. □

The following lemma was originally proved in [34], where it was used to extend the proof of recursive axiomatizability of the 'standard' Gödel logic $\mathbf{G}_{\mathbb{R}}$ to Gödel logics with a truth value set containing a perfect set in the general case. The following simpler proof is inspired by [20]:

LEMMA 3.1.7. *Suppose that $M \subseteq [0,1]$ is countable and $P \subseteq [0,1]$ is perfect. Then there is a strictly monotone continuous map $h\colon M \to P$ (i.e., infima and suprema already existing in M are preserved). Furthermore, if $\inf M \in M$, then one can choose h such that $h(\inf M) = \inf P$.*

Proof. Let σ be the mapping which scales and shifts M into $[0,1]$, i.e. the mapping $x \mapsto (x - \inf M)/(\sup M - \inf M)$ (assuming that M contains more than one point). Let w be an injective monotone map from $\sigma(M)$ into 2^ω, i.e. $w(m)$ is a fixed binary representation of m. For dyadic rational numbers (i.e. those with different binary representations) we fix one possible.

Let i be the natural bijection from 2^ω (the set of infinite $\{0,1\}$-sequences, ordered lexicographically) onto $\mathbb{D}$, the Cantor set. i is an order preserving homeomorphism. Since P is perfect, we can find a continuous strictly monotone map c from the Cantor set $\mathbb{D} \subseteq [0,1]$ into P, and c can be chosen so that $c(0) = \inf P$. Now $h = c \circ i \circ w \circ \sigma$ is also a strictly monotone map from M into P, and $h(\inf M) = \inf P$, if $\inf M \in M$. Since c is continuous, existing infima and suprema are preserved. □

COROLLARY 3.1.8. *A Gödel set V is uncountable iff it contains a non-trivial dense linear subordering.*

[1] Vincenzo Mara: oral communication

Proof. If: Every countable non-trivial dense linear order has order type η, $\mathbf{1}+\eta$, $\eta+\mathbf{1}$, or $\mathbf{1}+\eta+\mathbf{1}$ [36, Corollary 2.9], where η is the order type of $\mathbb{Q}$. The completion of any ordering of order type η has order type λ, the order type of $\mathbb{R}$ [36, Theorem 2.30], thus the truth value set must be uncountable. Only if: By Theorem 1.6.6, V^∞ is non-empty. Take $M = \mathbb{Q} \cap [0,1]$ and $P = V^\infty$ in Lemma 3.1.7. The image of M under h is a non-trivial dense linear subordering in V. □

THEOREM 3.1.9. *Suppose V is a truth value set with non-empty perfect kernel P, and let $W = V \cup [\inf P, 1]$. Then $\Gamma \models_V A$ iff $\Gamma \models_W A$, i.e., $\mathbf{G}_V = \mathbf{G}_W$.*

Proof. As $V \subseteq W$ we have $\mathbf{G}_W \subseteq \mathbf{G}_V$ (cf. Remark 3.1.1). Now assume that $\mathcal{I}$ is a W-interpretation which shows that $\Gamma \models_W A$ does *not* hold, i.e., $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. By Proposition 3.1.4, we may assume that $U^{\mathcal{I}}$ is countable. The set $\mathrm{Val}(\mathcal{I}, \Gamma \cup A)$ has cardinality at most $\aleph_0$, thus there is a $w \in [0,1]$ such that $w \notin \mathrm{Val}(\mathcal{I}, \Gamma \cup A)$ and $\mathcal{I}(A) < w < 1$. By Lemma 3.1.5, $\mathcal{I}_w(A) < w < 1$. Now consider $M = \mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$: these are all the truth values from $W = V \cup [\inf P, 1]$ required to compute $\mathcal{I}_w(A)$ and $\mathcal{I}_w(B)$ for all $B \in \Gamma$. We have to find some way to map them to V so that the induced interpretation is a counterexample to $\Gamma \models_V A$.

Let $M_0 = M \cap [0, \inf P)$ and $M_1 = (M \cap [\inf P, w]) \cup \{\inf P\}$. By Lemma 3.1.7 there is a strictly monotone continuous (i.e. preserving all existing infima and suprema) map h from M_1 into P. Furthermore, we can choose h such that $h(\inf M_1) = \inf P$.

We define a function g from $\mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$ to V as follows:

$$g(x) = \begin{cases} x & 0 \le x \le \inf P \\ h(x) & \inf P \le x \le w \\ 1 & x = 1 \end{cases}$$

Note that there is no $x \in \mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$ with $w < x < 1$. This function has the following properties: $g(0) = 0$, $g(1) = 1$, g is strictly monotonic and preserves existing infima and suprema. Using Lemma 3.1.3 we obtain that $\mathcal{I}_g$ is a V-interpretation with $\mathcal{I}_g(C) = g(\mathcal{I}_w(C))$ for all $C \in \Gamma \cup A$, thus also $\mathcal{I}_g(\Gamma) > \mathcal{I}_g(A)$. □

3.2 Relationships between Gödel logics

We now establish some results regarding the relationships between various first-order Gödel logics. For this, it is useful to consider several ‘prototypical’ Gödel sets.

$$\begin{aligned} &V_{\mathbb{R}} = [0,1] \qquad V_0 = \{0\} \cup [1/2, 1] \\ &V_{\downarrow} = \{1/k : k \ge 1\} \cup \{0\} \\ &V_{\uparrow} = \{1 - 1/k : k \ge 1\} \cup \{1\} \\ &V_n = \{1 - 1/k : 1 \le k \le m-1\} \cup \{1\} \end{aligned}$$

The corresponding Gödel logics are $\mathbf{G}_{\mathbb{R}}$, $\mathbf{G}_0$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and $\mathbf{G}_n$. $\mathbf{G}_{\mathbb{R}}$ is the *standard* Gödel logic.

The logic $\mathbf{G}_{\downarrow}$ also turns out to be closely related to some temporal logics [1, 11]. $\mathbf{G}_{\uparrow}$ is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 3.2.4.

PROPOSITION 3.2.1. *Intuitionistic predicate logic* **IL** *is contained in all first-order Gödel logics.*

Proof. The axioms and rules of **IL** are sound for the Gödel truth functions. □

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically valid formula when working in any of the Gödel logics.

PROPOSITION 3.2.2. $\mathbf{G}_{\mathbb{R}} = \bigcap_V \mathbf{G}_V$*, where* V *ranges over all Gödel sets.*

Proof. If $\Gamma \models_V A$ for every Gödel set V, then it does so in particular for $V = [0,1]$. Conversely, if $\Gamma \not\models_V A$ for a Gödel set V, there is a V-interpretation $\mathcal{I}$ with $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. Since $\mathcal{I}$ is also a $[0,1]$-interpretation, $\Gamma \not\models_{\mathbb{R}} A$. □

PROPOSITION 3.2.3. *The following strict containment relationships hold:*

1. $\mathbf{G}_n \supsetneq \mathbf{G}_{n+1}$,
2. $\mathbf{G}_n \supsetneq \mathbf{G}_{\uparrow} \supsetneq \mathbf{G}_{\mathbb{R}}$,
3. $\mathbf{G}_n \supsetneq \mathbf{G}_{\downarrow} \supsetneq \mathbf{G}_{\mathbb{R}}$,
4. $\mathbf{G}_0 \supsetneq \mathbf{G}_{\mathbb{R}}$.

Proof. The only non-trivial part is proving that the containments are strict. For this note that

$$\textsc{fin}(n) \equiv (\top \to A_1) \vee \ldots \vee (A_{n-1} \to \bot)$$

is valid in $\mathbf{G}_n$ but not in $\mathbf{G}_{n+1}$. Furthermore, let

$$C_{\uparrow} = \exists x (A(x) \to \forall y\, A(y)) \text{ and}$$
$$C_{\downarrow} = \exists x (\exists y\, A(y) \to A(x)).$$

$C_{\downarrow}$ is valid in all $\mathbf{G}_n$ and in $\mathbf{G}_{\uparrow}$ and $\mathbf{G}_{\downarrow}$; $C_{\uparrow}$ is valid in all $\mathbf{G}_n$ and in $\mathbf{G}_{\uparrow}$, but not in $\mathbf{G}_{\downarrow}$; neither is valid in $\mathbf{G}_0$ or $\mathbf{G}_{\mathbb{R}}$ [11, Corollary 2.9].

$\mathbf{G}_0 \models \textsc{iso}_0$ but $\mathbf{G}_{\mathbb{R}} \not\models \textsc{iso}_0$. □

The formulas $C_{\uparrow}$ and $C_{\downarrow}$ are of some importance in the study of first-order infinite-valued Gödel logics. $C_{\uparrow}$ expresses the fact that the infimum of any subset of the set of truth values is contained in the subset (every infimum is a minimum), and $C_{\downarrow}$ states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

$$(\forall x\, A(x) \wedge B) \leftrightarrow \forall x (A(x) \wedge B) \tag{1}$$
$$(\exists x\, A(x) \wedge B) \leftrightarrow \exists x (A(x) \wedge B) \tag{2}$$
$$(\forall x\, A(x) \vee B) \to \forall x (A(x) \vee B) \tag{3}$$
$$(\exists x\, A(x) \vee B) \leftrightarrow \exists x (A(x) \vee B) \tag{4}$$
$$(B \to \forall x\, A(x)) \leftrightarrow \forall x (B \to A(x)) \tag{5}$$
$$(B \to \exists x\, A(x)) \leftarrow \exists x (B \to A(x)) \tag{6}$$
$$(\forall x\, A(x) \to B) \leftarrow \exists x (A(x) \to B) \tag{7}$$
$$(\exists x\, A(x) \to B) \leftrightarrow \forall x (A(x) \to B) \tag{8}$$

The remaining three are:

$$(\forall x\, A(x) \vee B) \leftarrow \forall x(A(x) \vee B) \qquad (S_1)$$

$$(B \to \exists x\, A(x)) \to \exists x(B \to A(x)) \qquad (S_2)$$

$$(\forall x\, A(x) \to B) \to \exists x(A(x) \to B) \qquad (S_3)$$

Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply and are implied by $C_\downarrow$ and $C_\uparrow$, respectively (take $\exists y\, A(y)$ and $\forall y\, A(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in $\mathbf{G}_\uparrow$, invalid and valid in $\mathbf{G}_\downarrow$, and both invalid in $\mathbf{G}_\mathbb{R}$.

Note that since we defined $\neg A \equiv A \to \bot$, the quantifier shifts for $\to$ (7, 8, S_3) include the various directions of De Morgan's laws as special cases. Specifically, the only direction of De Morgan's laws which is not valid in all Gödel logics is the one corresponding to (S_3), i.e., $\neg\forall x\, A(x) \to \exists x \neg A(x)$. This formula is equivalent to $\textsc{iso}_0$. For, $\mathbf{G}_V \models \forall x \neg\neg A(x) \leftrightarrow \neg\exists\neg A(x)$ by (8). We get $\textsc{iso}_0$ using $\neg\exists x\neg A(x) \to \neg\neg\forall x\, A(x)$, which is an instance of (S_3). The other direction is given in Lemma 3.3.6.

We now also know that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

THEOREM 3.2.4 ([15], Theorem 23).

$$\mathbf{G}_\uparrow = \bigcap_{n\geq 2} \mathbf{G}_n$$

Proof. By Proposition 3.2.3, $\mathbf{G}_\uparrow \subseteq \bigcap_{n\geq 2} \mathbf{G}_n$. We now prove the reverse inclusion. Suppose $\Gamma \not\models_{V_\uparrow} A$, i.e., there is a $V_\uparrow$-interpretation $\mathcal{I}$ such that $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. Let $\mathcal{I}(A) = 1 - 1/k$, and pick w somewhere between $1 - 1/k$ and $1 - 1/(k+1)$. Then the interpretation $\mathcal{I}_w$ given by Lemma 3.1.5 is so that $\mathcal{I}(\Gamma) = 1$ and $\mathcal{I}(A) = 1 - 1/k$. Since there are only finitely many truth values below w in $V_\uparrow$, $\mathcal{I}_w$ is also a $\mathbf{G}_{k+1}$ interpretation which shows that $\Gamma \not\models_{V_{k+1}} A$. Hence, $(\Gamma, A) \notin \bigcap_{n\geq 2} \mathbf{G}_n$. □

COROLLARY 3.2.5. $\mathbf{G}_n \supsetneq \bigcap_n \mathbf{G}_n = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R} = \bigcap_V \mathbf{G}_V$

Note that also $\mathbf{G}_\uparrow \supsetneq \mathbf{G}_0 \supsetneq \mathbf{G}_\mathbb{R}$ by the above, and that neither $\mathbf{G}_0 \subseteq \mathbf{G}_\downarrow$ nor $\mathbf{G}_\downarrow \subseteq \mathbf{G}_0$ (counterexamples are $\textsc{iso}_0$ or $\neg\forall x\, A(x) \to \exists\neg A(x)$, and $C_\downarrow$, respectively).

LEMMA 3.2.6. *If all infima in the truth value set are minima or A contains no quantifiers, and A evaluates to some $v < 1$ in $\mathcal{I}$, then A also evaluates to v in $\mathcal{I}_v$ where*

$$\mathcal{I}_v(P) = \begin{cases} 1 & \text{if } \mathcal{I}(P) > v \\ \mathcal{I}(P) & \text{otherwise} \end{cases}$$

for P atomic sub-formula of A.

Proof. We prove by induction on the complexity of formulas that any sub-formula F of A with $\mathcal{I}(F) \leq v$ has $\mathcal{I}'(F) = \mathcal{I}(F)$. This is clear for atomic sub-formulas. We distinguish cases according to the logical form of F:

$F \equiv D \wedge E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \leq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) \geq \mathcal{I}(E)$, so

$\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) > v$ and $\mathcal{I}(E) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \vee E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \geq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then, again without loss of generality, $\mathcal{I}(F) = \mathcal{I}(D) > v$, by induction hypothesis $\mathcal{I}'(D) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \to E$. Since $v < 1$, we must have $\mathcal{I}(D) > \mathcal{I}(E) = \mathcal{I}(F)$. By induction hypothesis, $\mathcal{I}'(D) \geq \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) \geq \mathcal{I}(E) = \mathcal{I}(F) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = \mathcal{I}'(F) = 1$.

$F \equiv \exists x D(x)$. First assume that $\mathcal{I}(F) \leq v$. Since $D(c)$ evaluates to a value less or equal to v in $\mathcal{I}$ and, by induction hypothesis, in $\mathcal{I}'$ also the supremum of these values is less or equal to v in $\mathcal{I}'$, thus $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then there is a c such that $\mathcal{I}(D(c)) > v$, by induction hypothesis $\mathcal{I}'(D(c)) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv \forall x D(x)$. This is the crucial part. First assume that $\mathcal{I}(F) < v$. Then there is a witness c such that $\mathcal{I}(F) \leq \mathcal{I}(D(c)) < v$ and, by induction hypothesis, also $\mathcal{I}'(D(c)) < v$ and therefore, $\mathcal{I}'(F) = \mathcal{I}(F)$. For $\mathcal{I}(F) > v$ it is obvious that $\mathcal{I}'(F) = \mathcal{I}(F) = 1$. Finally assume that $\mathcal{I}(F) = v$. If this infimum would be proper, i.e. no minimum, then the value of all witnesses under $\mathcal{I}'$ would be 1, but the value of F under $\mathcal{I}'$ would be v, which would contradict the definition of the semantic of the $\forall$ quantifier. Since all infima are minima, there is a witness c such that $\mathcal{I}(D(c)) = v$ and therefore, also $\mathcal{I}'(D(c)) = v$ and thus $\mathcal{I}'(F) = \mathcal{I}(F)$. $\square$

As we will see later, the axioms FIN(n) axiomatize exactly the finite-valued Gödel logics. In these logics the quantifier shift axiom QS is not necessary. Furthermore, all quantifier shift rules are valid in the finite valued logics. Since $\mathbf{G}_{\uparrow}$ is the intersection of all the finite ones, all quantifier shift rules are valid in $\mathbf{G}_{\uparrow}$. Moreover, any infinite-valued Gödel logic other than $\mathbf{G}_{\uparrow}$ is defined by some V which either contains an infimum which is not a minimum, or a supremum (other than 1) which is not a maximum. Hence, in V either $C_{\uparrow}$ or $C_{\downarrow}$ will be invalid, and therewith either S_3 or S_2. We have:

COROLLARY 3.2.7. *In $\mathbf{G}_V$ all quantifier shift rules are valid iff there is a strictly monotone and continuous embedding from V to $V_{\uparrow}$, i.e., V is either finite or order isomorphic to $V_{\uparrow}$.*

This means that it is in general not possible to transform formulas to equivalent prenex formulas in the usual way. Moreover, in general there is not even a recursive procedure for mapping formulas to equivalent, or even just validity-equivalent formulas in prenex form, since for some V, $\mathbf{G}_V$ is not r.e. whereas the corresponding prenex fragment is r.e., $V = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup [0.5, 1]$ is such an example.

3.3 Axiomatizability results

3.3.1 Axiomatizable case 1: 0 is contained in the perfect kernel

If V is uncountable, and 0 is contained in V^{∞}, then $\mathbf{G}_V$ is axiomatizable. Indeed, Theorem 3.1.9 showed that all such logics $\mathbf{G}_V$ coincide. Thus, it is only necessary to establish completeness of the axioms system $\mathbf{H}$ with respect to $\mathbf{G}_{\mathbb{R}}$. This result has been shown by several researchers over the years. We give here a generalization of the proof of [37]. Alternative proofs can be found in [27, 28, 39]. The proof of [28], however, does not give strong completeness, while the proof of [39] is specific to the Gödel set $[0,1]$. Our proof is self-contained and applies to Gödel logics directly, making an extension of the result easier.

THEOREM 3.3.1 ([37], [15] Theorem 37, Strong completeness of Gödel logic). *If* $\Gamma \models_{\mathbb{R}} A$*, then* $\Gamma \vdash_{\mathbf{H}} A$*.*

Proof. Assume that $\Gamma \nvdash A$, we construct an interpretation $\mathcal{I}$ in which $\mathcal{I}(A) = 1$ for all $B \in \Gamma$ and $\mathcal{I}(A) < 1$. Let $y_1, y_2, \ldots$ be a sequence of free variables which do not occur in $\Gamma \cup \triangle$, let $\mathcal{T}$ be the set of all terms in the language of $\Gamma \cup \Delta$ together with the new variables $y_1, y_2, \ldots$, and let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be an enumeration of the formulas in this language in which y_i does not appear in $F_1, \ldots, F_i$ and in which each formula appears infinitely often.

If Δ is a set of formulas, we write $\Gamma \Rightarrow \Delta$ if for some $A_1, \ldots, A_n \in \Gamma$, and some $B_1, \ldots, B_m \in \Delta$, $\vdash_{\mathbf{H}} (A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$ (and $\not\Rightarrow$ if this is not the case). We define a sequence of sets of formulas Γ_n, Δ_n such that $\Gamma_n \not\Rightarrow \Delta_n$ by induction. First, $\Gamma_0 = \Gamma$ and $\Delta_0 = \{A\}$. By the assumption of the theorem, $\Gamma_0 \not\Rightarrow \Delta_0$.

If $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n \cup \{F_n\}$ and $\Delta_{n+1} = \Delta_n$. In this case, $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$, since otherwise we would have $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$ and $\Gamma_n \cup \{F_n\} \Rightarrow \Delta_n$. But then, we'd have that $\Gamma_n \Rightarrow \Delta_n$, which contradicts the induction hypothesis (note that $\vdash_{\mathbf{H}} (A \to B \vee F) \to ((A \wedge F \to B) \to (A \to B)))$.

If $\Gamma_n \not\Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{F_n, B(y_n)\}$ if $F_n \equiv \forall x\, B(x)$, and $\Delta_{n+1} = \Delta_n \cup \{F_n\}$ otherwise. In the latter case, it is obvious that $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$. In the former, observe that by $I10$ and QS, if $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x\, B(x), B(y_n)\}$ then also $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x\, B(x)\}$ (note that y_n does not occur in Γ_n or Δ_n).

Let $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^* = \bigcup_{i=0}^{\infty} \Delta_i$. We have:

1. $\Gamma^* \not\Rightarrow \Delta^*$, for otherwise there would be a k so that $\Gamma_k \Rightarrow \Delta_k$.

2. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$ (by construction).

3. $\Gamma^* = \mathcal{F} \setminus \Delta^*$, since each F_n is either in Γ_{n+1} or Δ_{n+1}, and if for some n, $F_n \in \Gamma^* \cap \Delta^*$, there would be a k so that $F_n \in \Gamma_k \cap \Delta_k$, which is impossible since $\Gamma_k \not\Rightarrow \Delta_k$.

4. If $\Gamma^* \Rightarrow B_1 \vee \ldots \vee B_n$, then $B_i \in \Gamma^*$ for some i. For suppose not, then for $i = 1, \ldots, n$, $B_i \notin \Gamma^*$, and hence, by (3), $B_i \in \Delta^*$. But then $\Gamma^* \Rightarrow \Delta^*$, contradicting (1).

5. If $B(t) \in \Gamma^*$ for every $t \in \mathcal{T}$, then $\forall x\, B(x) \in \Gamma^*$. Otherwise, by (3), $\forall x\, B(x) \in \Delta^*$ and so there is some n so that $\forall x\, B(x) = F_n$ and Δ_{n+1} contains $\forall x\, B(x)$ and $B(y_n)$. But, again by (3), then $B(y_n) \notin \Gamma^*$.

6. Γ^* is closed under provable implication, since if $\Gamma^* \Rightarrow A$, then $A \notin \Delta^*$ and so, again by (3), $A \in \Gamma^*$. In particular, if $\vdash_{\mathbf{H}} A$, then $A \in \Gamma^*$.

Define relations $\preceq$ and $\equiv$ on $\mathcal{F}$ by

$$B \preceq C \Leftrightarrow B \to C \in \Gamma^* \quad \text{and} \quad B \equiv C \Leftrightarrow B \preceq C \wedge C \preceq B.$$

Then $\preceq$ is reflexive and transitive, since for every B, $\vdash_{\mathbf{H}} B \to B$ and so $B \to B \in \Gamma^*$, and if $B \to C \in \Gamma^*$ and $C \to D \in \Gamma^*$ then $B \to D \in \Gamma^*$, since $B \to C, C \to D \Rightarrow B \to D$ (recall (6) above). Hence, $\equiv$ is an equivalence relation on $\mathcal{F}$. For every B in $\mathcal{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathcal{F}/\!\equiv$ the set of all equivalence classes. Next we define the relation $\leq$ on $\mathcal{F}/\!\equiv$ by

$$|B| \leq |C| \Leftrightarrow B \preceq C \Leftrightarrow B \to C \in \Gamma^*.$$

Obviously, $\leq$ is independent of the choice of representatives A, B.

LEMMA 3.3.2. *$\langle \mathcal{F}/\!\equiv, \leq \rangle$ is a countably linearly ordered structure with distinct maximal element $|\top|$ and minimal element $|\bot|$.*

Proof. Since $\mathcal{F}$ is countably infinite, $\mathcal{F}/\!\equiv$ is countable. For every B and C, $\vdash_{\mathbf{H}} (B \to C) \vee (C \to B)$ by LIN, and so either $B \to C \in \Gamma^*$ or $C \to B \in \Gamma^*$ (by (4)), hence $\leq$ is linear. For every B, $\vdash_{\mathbf{H}} B \to \top$ and $\vdash_{\mathbf{H}} \bot \to B$, and so $B \to \top \in \Gamma^*$ and $\bot \to B \in \Gamma^*$, hence $|\top|$ and $|\bot|$ are the maximal and minimal elements, respectively. Pick any A in Δ^*. Since $\top \to \bot \Rightarrow A$, and $A \notin \Gamma^*$, $\top \to \bot \notin \Gamma^*$, so $|\top| \neq |\bot|$. $\square$

We abbreviate $|\top|$ by $\mathbf{1}$ and $|\bot|$ by $\mathbf{0}$.

LEMMA 3.3.3. *The following properties hold in $\langle \mathcal{F}/\!\equiv, \leq \rangle$:*

1. $|B| = \mathbf{1} \Leftrightarrow B \in \Gamma^*$.
2. $|B \wedge C| = \min\{|B|, |C|\}$.
3. $|B \vee C| = \max\{|B|, |C|\}$.
4. $|B \to C| = \mathbf{1}$ *if* $|B| \leq |C|$, $|B \to C| = |C|$ *otherwise.*
5. $|\neg B| = \mathbf{1}$ *if* $|B| = \mathbf{0}$*;* $|\neg B| = \mathbf{0}$ *otherwise.*
6. $|\exists x B(x)| = \sup\{|B(t)| : t \in \mathcal{T}\}$.
7. $|\forall x B(x)| = \inf\{|B(t)| : t \in \mathcal{T}\}$.

Proof. (1) If $|B| = \mathbf{1}$, then $\top \to B \in \Gamma^*$, and hence $B \in \Gamma^*$. And if $B \in \Gamma^*$, then $\top \to B \in \Gamma^*$ since $B \Rightarrow \top \to B$. So $|\top| \leq |B|$. It follows that $|\top| = |B|$ as also $|B| \leq |\top|$.

(2) From $\Rightarrow B \wedge C \to B$, $\Rightarrow B \wedge C \to C$ and $D \to B, D \to C \Rightarrow D \to B \wedge C$ for every D, it follows that $|B \wedge C| = \inf\{|B|, |C|\}$, from which (2) follows since $\leq$ is linear. (3) is proved analogously.

(4) If $|B| \leq |C|$, then $B \to C \in \Gamma^*$, and since $\top \in \Gamma^*$ as well, $|B \to C| = \mathbf{1}$. Now suppose that $|B| \not\leq |C|$. From $B \wedge (B \to C) \Rightarrow C$ it follows that $\min\{|B|, |B \to C|\} \leq |C|$. Because $|B| \not\leq |C|$, $\min\{|B|, |B \to C|\} \neq |B|$, hence $|B \to C| \leq |C|$. On the other hand, $\vdash C \to (B \to C)$, so $|C| \leq |B \to C|$.

(5) If $|B| = \mathbf{0}$, $\neg B = B \to \bot \in \Gamma^*$, and hence $|\neg B| = \mathbf{1}$ by (1). Otherwise, $|B| \not\leq |\bot|$, and so by (4), $|\neg B| = |B \to \bot| = \mathbf{0}$.

(6) Since $\vdash_{\mathbf{H}} B(t) \to \exists x\, B(x)$, $|B(t)| \leq |\exists x\, B(x)|$ for every $t \in \mathcal{T}$. On the other hand, for every D without x free,

$$\begin{array}{lll} & |B(t)| \leq |D| & \text{for every } t \in \mathcal{T} \\ \Leftrightarrow & B(t) \to D \in \Gamma^* & \text{for every } t \in \mathcal{T} \\ \Rightarrow & \forall x(B(x) \to D) \in \Gamma^* & \text{by property (5) of } \Gamma^* \\ \Rightarrow & \exists x\, B(x) \to D \in \Gamma^* & \text{since } \forall x(B(x) \to D) \Rightarrow \exists x\, B(x) \to D \\ \Leftrightarrow & |\exists x\, B(x)| \leq |D|. & \end{array}$$

(7) is proved analogously. □

$\langle \mathcal{F}/\equiv, \leq \rangle$ is countable, let $\mathbf{0} = a_0, \mathbf{1} = a_1, a_2, \ldots$ be an enumeration. Define $h(\mathbf{0}) = 0$, $h(\mathbf{1}) = 1$, and define $h(a_n)$ inductively for $n > 1$: Let $a_n^- = \max\{a_i : i < n \text{ and } a_i < a_n\}$ and $a_n^+ = \min\{a_i : i < n \text{ and } a_i > a_n\}$, and define $h(a_n) = (h(a_n^-) + h(a_n^+))/2$ (thus, $a_2^- = \mathbf{0}$ and $a_2^+ = \mathbf{1}$ as $\mathbf{0} = a_0 < a_2 < a_1 = \mathbf{1}$, hence $h(a_2) = \frac{1}{2}$). Then $h\colon \langle \mathcal{F}/\equiv, \leq \rangle \to \mathbb{Q} \cap [0,1]$ is a strictly monotone map which preserves infs and sups. By Lemma 3.1.7 there exists a $\mathbf{G}$-embedding h' from $\mathbb{Q} \cap [0,1]$ into $\langle [0,1], \leq \rangle$ which is also strictly monotone and preserves infs and sups. Put $\mathcal{I}(B) = h'(h(|B|))$ for every atomic $B \in \mathcal{F}$ and we obtain a $V_{\mathbb{R}}$-interpretation.

Note that for every $B, \mathcal{I}(B) = 1$ iff $|B| = \mathbf{1}$ iff $B \in \Gamma^*$. Hence, we have $\mathcal{I}(B) = 1$ for all $B \in \Gamma$ while if $A \notin \Gamma^*$, then $\mathcal{I}(A) < 1$, so $\Gamma \not\models A$. Thus we have proven that on the assumption that if $\Gamma \nvdash A$, then $\Gamma \not\models A$ □

This completeness proof can be adapted to hypersequent calculi for Gödel logics (Chapter IV, [4, 21]), even including the $\triangle$ projection operator [14].

As already mentioned we obtain from this completeness proof together with the soundness theorem (Theorem 1.5.4) and Theorem 3.1.9 the characterization of recursive axiomatizability:

THEOREM 3.3.4 ([15], Theorem 40). *Let V be a Gödel set with 0 contained in the perfect kernel of V. Suppose that Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}} A$.*

COROLLARY 3.3.5 (Deduction theorem for Gödel logics). *Suppose that Γ is a set of formulas, and A is a closed formula. Then*

$$\Gamma, A \vdash_{\mathbf{H}} B \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{H}} A \to B.$$

Proof. Use the soundness and completeness theorems (Theorem 1.5.4 and 3.3.4, resp.) and a straight-forward semantic deduction. Another proof would be by induction on the length of the proof. See [27, Theorem 2.2.18]. □

3.3.2 Axiomatizable case 2: 0 is isolated

In the case where 0 is isolated in V, and thus also not contained in the perfect kernel, we will transform a counter example in $\mathbf{G}_{\mathbb{R}}$ for $\Gamma, \Pi \models A$, where Π is a set of sentences stating that every infimum is a minimum, into a counterexample in $\mathbf{G}_V$ to $\Gamma \models A$.

LEMMA 3.3.6. *Let $x, \bar{y}$ be the free variables in A.*

$$\vdash_{\mathbf{H}_0} \forall \bar{y}(\neg \forall x\, A(x, \bar{y}) \to \exists x\, \neg A(x, \bar{y}))$$

Proof. It is easy to see that in all Gödel logics the following weak form of the law of excluded middle is valid: $\neg\neg A(x) \vee \neg A(x)$. By quantification we obtain $\forall x \neg\neg A(x) \vee \exists x \neg A(x)$ and, by $\textsc{iso}_0$, $\neg\neg\forall x\, A(x) \vee \exists \neg A(x)$. Using the intuitionistically valid schema $(\neg A \vee B) \to (A \to B)$ we can prove $\neg \forall x\, A(x) \to \exists x \neg A(x)$. A final quantification of the free variables concludes the proof. □

THEOREM 3.3.7 ([15], Theorem 43). *Let V be an uncountable Gödel set where 0 is isolated. Suppose Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}_0} A$.*

Proof. If: Follows from soundness (Theorem 1.5.4) and the observation that $\textsc{iso}_0$ is valid for any V where 0 is isolated.

Only if: We already know from Theorem 3.1.9 that the entailment relations of V and $V \cup [\inf P, 1]$ coincide, where P is the perfect kernel of V. So we may assume without loss of generality that V already is of this form, i.e., that $w = \inf P$ and $V \cap [w, 1] = [w, 1]$. Let $V' = [0, 1]$. Define

$$\Pi = \{\forall \bar{y}(\neg \forall x\, A(x, \bar{y}) \to \exists x \neg A(x, \bar{y})) : A(x, \bar{y}) \text{ has } x, \bar{y} \text{ free}\}$$

where $A(x, \bar{y})$ ranges over all formulas with free variables x and $\bar{y}$. We consider the entailment relation in V'. Either $\Pi, \Gamma \models_{V'} A$ or $\Pi, \Gamma \not\models_{V'} A$. In the former case we know from the strong completeness of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ that there are finite subsets Π' and Γ' of Π and Γ, respectively, such that $\Pi', \Gamma' \vdash_{\mathbf{H}} A$. Since all the sentences in Π are provable in $\mathbf{H}_0$ (see Lemma 3.3.6) we obtain that $\Gamma' \vdash_{\mathbf{H}_0} A$. In the latter case there is an interpretation $\mathcal{I}'$ such that $\mathcal{I}'(\Pi \cup \Gamma) > \mathcal{I}'(A)$.

It is obvious from the structure of the formulas in Π that their truth value will always be either 0 or 1. Combined with the above we know that for all $B \in \Pi$, $\mathcal{I}'(B) = 1$. Next we define a function $h(x)$ which maps values from $\mathrm{Val}(\mathcal{I}', \Gamma \cup \Pi \cup \{A\})$ into V:

$$h(x) = \begin{cases} 0 & x = 0 \\ w + x/(1-w) & x > 0 \end{cases}$$

We see that h satisfies conditions (1) and (2) of Lemma 3.1.3, but we cannot use this Lemma directly, as not all existing infima and suprema are necessarily preserved.

Consider as in Lemma 3.1.3 the interpretation $\mathcal{I}_h(B) = h(\mathcal{I}'(B))$ for atomic subformulas of $\Gamma \cup \Pi \cup \{A\}$. We want to show that the identity $\mathcal{I}_h(B) = h(\mathcal{I}'(B))$ extends to all subformulas of $\Gamma \cup \Pi \cup \{A\}$. For propositional connectives and the existentially quantified formulas this is obvious. The important case is $\forall x\, A(x)$. First assume that $\mathcal{I}'(\forall x\, A(x)) > 0$. Then it is obvious that $\mathcal{I}_h(\forall x\, A(x)) = h(\mathcal{I}'(\forall x\, A(x)))$. In the case where $\mathcal{I}'(\forall x\, A(x)) = 0$ we observe that $A(x)$ contains a free variable and therefore $\neg\forall x\, A(x) \to \exists x \neg A(x) \in \Pi$, thus $\mathcal{I}'(\neg\forall x\, A(x) \to \exists x \neg A(x)) = 1$. This implies that there is a witness u such that $\mathcal{I}'(A(u)) = 0$. Using the induction hypothesis we know that $\mathcal{I}_h(A(u)) = 0$, too. We obtain that $\mathcal{I}_h(\forall x\, A(x)) = 0$, concluding the proof.

Thus we have shown that $\mathcal{I}_h$ is a counterexample to $\Gamma \models_V A$. □

3.3.3 Axiomatizable case 3: Finite Gödel sets

In this section we show that entailment over finite truth value sets are axiomatized by $\mathbf{H}_n$.

THEOREM 3.3.8 ([15], Theorem 45). *Suppose* Γ *contains only closed formulas. Then* $\Gamma \models_{V_n} A$ *iff* $\Gamma \vdash_{\mathbf{H}_n} A$.

Proof. If: By Theorem 1.5.4, since every instance of $\textsc{fin}(n)$ is valid in $\mathbf{G}_n$.

Only if: Suppose $\Gamma \nvdash_{\mathbf{H}_n} A$, and consider the set Π of closed formulas of the form

$$\forall \bar{x}_1 \ldots \bar{x}_{n-1}((A_0(\bar{x}_0) \to A_1(\bar{x}_1)) \vee \cdots \vee (A_{n-1}(\bar{x}_{n-1}) \to A_n(\bar{x}_n)))),$$

where $A_0, \ldots, A_n$ ranges over all sequences (with repetitions) of length $n+1$ where each A_i is $P(\bar{x})$ for some predicate symbol P occurring in Γ or A. Each formula in Π follows from an instance of $\textsc{fin}(n)$ by generalization. Hence, $\Gamma, \Pi \nvdash_{\mathbf{H}} A$. From the (strong) completeness (Theorem 3.3.4) of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ we know there is an interpretation $\mathcal{I}_{\mathbb{R}}$ (into $[0,1]$) such that $\mathcal{I}_{\mathbb{R}}(B) = 1$ for all $B \in \Gamma \cup \Pi$ and $\mathcal{I}_{\mathbb{R}}(A) < 1$.

For sake of brevity let $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Delta)$ for a set of formulas Δ be the set of all truth values of atomic subformulas of formulas in Δ, i.e., $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Delta) = \{\mathcal{I}_{\mathbb{R}}(P(\bar{u})) :$ $\bar{u}$ constants from $\mathscr{L}^{\mathcal{I}}\}$. We claim that $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ contains at most n elements. To see this, assume that it contains more than n elements. Then there exist atomic subformulas (w.r.t. $\mathcal{I}$) $B_0, \ldots, B_n$ of A or of formulas in Γ such that $\mathcal{I}_{\mathbb{R}}(B_i) > \mathcal{I}_{\mathbb{R}}(B_{i+1})$ for $i = 0, \ldots, n-1$. Thus, $\mathcal{I}_{\mathbb{R}}((B_0 \to B_1) \vee \ldots \vee (B_{n-1} \to B_n)) < 1$. But this formula is an instance of a formula in Π, and so we have a contradiction with $\mathcal{I}_{\mathbb{R}}(B) = 1$.

Now let $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\}) = \{0, v_1, \ldots, v_k, 1\}$ be sorted in increasing order, and let $h(0) = 0$, $h(1) = 1$, and $h(v_i) = 1 - 1/(i+1)$. Note that any truth value occurring in $\mathrm{Val}(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ must be one of the elements of $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$. This is easily seen by induction on the complexity of subformulas of $\Gamma \cup \{A\}$ w.r.t. $\mathcal{I}_{\mathbb{R}}$, as the inf and sup of any subset of the finite set $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ is a member of the finite set. By Lemma 3.1.3, $\mathcal{I}_h$ is a V_n-interpretation with $\mathcal{I}_h(B) = h(\mathcal{I}_{\mathbb{R}}) = 1$ for all $B \in \Gamma$ and $\mathcal{I}_h(A) = h(\mathcal{I}_{\mathbb{R}}) < 1$. □

3.3.4 Not recursively enumerable case 1: Countable Gödel sets

In this section we show that the first-order Gödel logics where the set of truth values does not contain a dense subset are not r.e. We establish this result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot's Theorem).

DEFINITION 3.3.9. *A formula is called* crisp *if all its atomic subformulas occur either negated or double-negated in it.*

LEMMA 3.3.10. *If A and B are crisp and classically equivalent, then also $\mathbf{G}_V \models A \leftrightarrow B$, for any Gödel set V. Specifically, if $A(x)$ and $B^{(x)}$ are crisp, then*

$$\mathbf{G}_V \models (\forall x\, A(x) \to B^{(x)}) \leftrightarrow \exists x(A(x) \to B^{(x)}) \quad \textit{and}$$
$$\mathbf{G}_V \models (B^{(x)} \to \exists x\, A(x)) \leftrightarrow \exists x(B^{(x)} \to A(x)).$$

Proof. Given an interpretation $\mathcal{I}$, define $\mathcal{I}'(C) = 1$ if $\mathcal{I}(C) > 0$ and $= 0$ if $\mathcal{I}(C) = 0$ for atomic C. It is easily seen that if A, B are crisp, then $\mathcal{I}(A) = \mathcal{I}'(A)$ and $\mathcal{I}(B) = \mathcal{I}'(B)$. But $\mathcal{I}'$ is a classical interpretation, so by assumption $\mathcal{I}'(A) = \mathcal{I}'(B)$. □

THEOREM 3.3.11 ([15], Theorem 36). *If V is countably infinite, then the set of validities of $\mathbf{G}_V$ is not r.e.*

Proof. By Theorem 3.1.8, V is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence A there is a sentence A^g s.t. A^g is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let $Q_1, \ldots, Q_n$ be all the predicate symbols in A. We use the abbreviations $x \in y \equiv \neg\neg L(x, y)$ and $x \prec y \equiv (P(y) \to P(x)) \to P(y)$. Note that for any interpretation $\mathcal{I}$, $\mathcal{I}(x \in y)$ is either 0 or 1, and as long as $\mathcal{I}(P(x)) < 1$ for all x (in particular, if $\mathcal{I}(\exists z\, P(z)) < 1$), we have $\mathcal{I}(x \prec y) = 1$ iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$. Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \quad \forall i \big[\forall x, y \forall j \forall k \exists z\, D \vee \forall x \neg(x \in s(i))\big] \end{array} \right\} \to (A' \vee \exists u\, P(u))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$D \equiv \begin{array}{c} (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \to \\ \to (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x(x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i. If the antecendent is true, then the true standard axioms S force the domain to be a model of the reduct of PA to the language without $+$ and $\times$, which could be either a standard model (isomorphic to $\mathbb{N}$) or a non-standard model ($\mathbb{N}$ followed by copies of $\mathbb{Z}$). P orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

The idea is that for any two elements in a level $\leq i$ there is an element in a not-empty level $j \geq i$ which lies strictly between those two elements in the ordering given by $\prec$. If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied in an interpretation $\mathcal{I}$ only for finitely many levels if V does not contain a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{\mathcal{I}(P(d)) : \mathcal{I} \models d \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. □

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset, i.e., no countably infinite Gödel logic, is r.e.

3.3.5 Not recursively enumerable case 2: 0 not isolated but not in the perfect kernel

In the preceding sections, we gave axiomatizations for the logics based on those uncountably infinite Gödel sets V where 0 is either isolated or in the perfect kernel of V. It remains to determine whether logics based on uncountable Gödel sets where 0 is neither isolated nor in the perfect kernel are axiomatizable. The answer in this case is negative. If 0 is not isolated in V, 0 has a countably infinite neighborhood. Furthermore, any sequence $(a_n)_{n\in\mathbb{N}} \to 0$ is so that, for sufficiently large n, $V \cap [0, a_n]$ is countable and hence, by (the proof of) Theorem 3.1.8, contains no densely ordered subset. This fact is the basis for the following non-axiomatizability proof, which is a variation on the proof of Theorem 3.3.11.

THEOREM 3.3.12 ([15], Theorem 48). *If V is uncountable, 0 is not isolated in V, but not in the perfect kernel of V, then the set of validities of $\mathbf{G}_V$ is not r.e.*

Proof. We show that for every sentence A there is a sentence A^h s.t. A^h is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

The definition of A^h mirrors the definition of A^g in the proof of Theorem 3.3.11, except that the construction there is carried out infinitely many times for $V \cap [0, a_n]$, where $(a_n)_{n\in\mathbb{N}}$ is a strictly descending sequence, $0 < a_n < 1$ for all n, which converges to 0. Let P be a binary and L be a ternary predicate symbol not occurring in A and let $R_1, \ldots, R_n$ be all the predicate symbols in A. We use the abbreviations $x \in_n y \equiv \neg\neg L(x, y, n)$ and $x \prec_n y \equiv (P(y, n) \to P(x, n)) \to P(y, n)$. As before, for a fixed n, provided $\mathcal{I}(\exists x\, P(x, n)) < 1$, $\mathcal{I}(x \prec_n y) = 1$ iff $\mathcal{I}(P(x, n)) < \mathcal{I}(P(y, n))$, and $\mathcal{I}(x \in_n y)$ is always either 0 or 1. We also need a unary predicate symbol $Q(n)$ to give us the descending sequence $(a_n)_{n\in\mathbb{N}}$: Note that $\mathcal{I}(\neg\forall n\, Q(n)) = 1$ iff $\inf\{\mathcal{I}(Q(d)) : d \in U_{\mathcal{I}}\} = 0$ and $\mathcal{I}(\forall n\, \neg\neg Q(n)) = 1$ iff $0 \notin \{\mathcal{I}(Q(d)) : d \in U_{\mathcal{I}}\}$.

Let $A^h \equiv$

$$\left\{\begin{array}{l} S \wedge \forall n((Q(n) \to Q(s(n))) \to Q(n)) \wedge \\ \quad \neg\forall n\, Q(n) \wedge \forall n\, \neg\neg Q(n) \wedge \\ \quad \forall n \forall x((Q(n) \to P(x, n)) \to Q(n)) \wedge \\ \quad \forall n \exists x \exists y(x \in_n 0 \wedge y \in_n 0 \wedge x \prec_n y) \wedge \\ \qquad \forall n \forall i\big[\forall x, y \forall j \forall k \exists z\, E \vee \forall x \neg(x \in_n s(i))\big] \end{array}\right\} \to (A' \vee \exists n \exists u\, P(u, n) \vee \exists n\, Q(n))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$E \equiv \begin{array}{c} (j \leq i \wedge x \in_n j \wedge k \leq i \wedge y \in_n k \wedge x \prec_n y) \to \\ \to (z \in_n s(i) \wedge x \prec_n z \wedge z \prec_n y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(n) \equiv \forall i \exists x(x \in_n i)$.

The idea here is that an interpretation $\mathcal{I}$ will define a sequence $(a_n)_{n\in\mathbb{N}} \to 0$ by $a_n = \mathcal{I}(Q(n))$ where $a_n > a_{n+1}$, and $0 < a_n < 1$ for all n. Let $L_n^i = \{x : \mathcal{I}(x \in_n i)\}$

be the i-th n-level. $P(x,n)$ orders the set $\bigcup_i L_n^i = \{x : \mathcal{I}(\exists i\, x \in_n i) = 1\}$ in a subordering of $V \cap [0,a_n]$: $x \prec_n y$ iff $\mathcal{I}(x \prec_n y) = 1$. Again we force that whenever $x, y \in L_n^i$ with $x \prec_n y$, there is a $z \in L_n^{i+1}$ with $x \prec_n z \prec_n y$, or, if no possible such z exists, $L_n^{i+1} = \emptyset$. Let $r(n)$ be the least i so that L_n^i is empty, or ∞ otherwise. If $r(n) = \infty$ then there is a densely ordered subset of $V \cap [0,a_n]$. So if 0 is not in the perfect kernel, for some sufficiently large L, $r(n) < \infty$ for all $n > L$. $\mathcal{I}(R(n)) = 1$ iff $r(n) = \infty$ hence $\{n : \mathcal{I}(R(n)) = 1\}$ is finite whenever the interpretations of P, L, and Q are as intended.

Now if A is classically false in some finite structure $\mathcal{I}$, we can again choose a $\mathbf{G}_V$-interpretation $\mathcal{I}^h$ so that there are as many n with $\mathcal{I}^h(R(n)) = 1$ as there are elements in the domain of $\mathcal{I}$, and the predicates of A behave on $\{n : \mathcal{I}(R(n)) = 1\}$ just as they do on $\mathcal{I}$.

For instance, we can define $\mathcal{I}^h$ as follows. We may assume that the domain of $\mathcal{I} = \{0,\ldots,m\}$. Let $U^{\mathcal{I}^h} = \mathbb{N}$, and $\mathcal{I}^h(B) = \mathcal{I}(B)$ for B an atomic subformula of A in the language $\mathscr{L}^{\mathcal{I}}$. Pick a strictly monotone descending sequence $(a_n)_{n\in\mathbb{N}}$ in V with $\lim a_n = 0$ so that $a_0, \ldots, a_{m+1} \in V^\infty$, $a_0 < 1$, $a_{m+1} = \inf V^\infty$, and let $\mathcal{I}^h(Q(n)) = a_n$. This guarantees that $\mathcal{I}^h(\forall n((Q(n) \to Q(s(n))) \to Q(n))) = 1$ (because $a_n > a_{n+1}$), $\mathcal{I}^h(\neg\forall n\, Q(n)) = 1$ (because $\inf a_n = 0$), $\mathcal{I}^h(\forall n \neg\neg Q(n)) = 1$ (because $a_n > 0$), and $\mathcal{I}^h(\exists n\, Q(n)) < 1$ (because $a_0 < 1$). Then $V \cap [0,a_n]$ is uncountable if $n \le m$, and countable if $n > m$. For $n \le m$, let $D_n \subseteq V \cap [0,a_n)$ be countable and densely ordered, and let $j_n\colon \mathbb{N} \to D_n$ be bijective.

For $n > m$, let $j_n(0) = a_{n+1}$, and $j_n(i) = 0$ for $i > 0$. Define $\mathcal{I}^h(P(i,n)) = j_n(i)$. Then, since $j_n(i) < a_n$ for all i, $\mathcal{I}^h(\forall n \forall x((Q(n) \to P(x,n)) \to Q(n))) = 1$, and, since $j_n(i) < a_0 < 1$, $\mathcal{I}^h(\exists n \exists u\, P(u,n)) < 1$. Finally, let $\mathcal{I}^h(L(x,y,n)) = 1$ for all x, $y \in \mathbb{N}$ if $n \le m$ (i.e., $L_n^i = \mathbb{N}$), and if $n > m$ let $\mathcal{I}^h(L(0,0,n)) = \mathcal{I}^h(L(1,0,n)) = 1$ and $\mathcal{I}^h(L(x,y,n)) = 0$ if $x > 1$ and $y \in \mathbb{N}$, and if $x \in \mathbb{N}$ and $y > 0$ (i.e., $L_n^0 = \{0,1\}$, $L_n^i = \emptyset$ for $i > m$). This makes the rest of the antecedent of A^h true and ensures that $\mathcal{I}^h(R(n)) = \mathcal{I}^h(\forall i \exists x(x \in_n i)) = 1$ if $n \le m$ and $= 0$ otherwise. Hence $\mathcal{I}^h(A') = 0$ and $\mathcal{I}^h \not\models A^h$.

On the other hand, if $\mathcal{I} \not\models A^h$, then the value of the consequent is < 1. Then as required, for all x, n, $\mathcal{I}(P(x,n)) < 1$ and $\mathcal{I}(Q(n)) < 1$. Since the antecedent, as before, must be $= 1$, this means that $x \prec_n y$ expresses a strict ordering of the elements of L_n^i and $\mathcal{I}((Q(n) \to Q(s(n))) \to Q(n)) = 1$ for all n guarantees that $\mathcal{I}(Q(s(n))) = a_{n+1} < a_n = \mathcal{I}(Q(n))$. The other conditions are likewise seen to hold as intended, so that we can extract a finite countermodel for A based on the interpretation of the predicate symbols of A on $\{n : \mathcal{I}(R(n)) = 1\}$, which must be finite. □

3.4 Relation to Kripke frames

For propositional logic the truth value sets on which Gödel logics are based can be considered as linear Heyting algebras (or pseudo-Boolean algebras). By taking the prime filters of a Heyting algebra as the Kripke frame it is easy to see that the induced logics coincide (cf. [24, 32]). This direct method does not work for first order logics as the structure of the prime filters does not coincide with the possible evaluations in the first order case.

[19] showed that the class of logics defined by countable linear Kripke frames on constant domains and the class of all Gödel logics coincide. More precisely, for every countable Kripke frame we will *construct* a truth value set such that the logic induced by the Kripke frame and the one induced by the truth value set coincide, and vice versa (Theorems 3.4.1 and 3.4.6). As corollaries a complete characterisation of axiomatisability of logics based on countable linear Kripke frames with constant domains (Corollaries 3.4.7 and 3.4.8) have been obtained. Furthermore, we obtain that there are only countable many different logics based on countable linear Kripke frames with constant domains (Corollary 3.4.9). This is especially surprising for at least two reasons: Due to a result obtained in [17] there are uncountably many different propositional quantified Gödel logics, and thus also uncountably many propositional quantified logics based on countable linear Kripke frames. Furthermore, the number of all intermediate (predicate) logics extending the basic linear logic with constant domains is uncountable.

In the following we will state the central results and give proof ideas and sketches for these results.

THEOREM 3.4.1 ([19], Theorem 18). *For every countable linear Kripke frame K there is a Gödel set V_K such that* $\mathbf{L}(K) = \mathbf{G}_{V_K}$.

Proof. Let $K = (W, \preceq)$ be a countable linear Kripke frame. The construction of V_K will be in three steps: First, we will enlarge K by doubling all limit worlds; then we will apply the Horn monomorphism (Lemma 3.1.7) to embed the enlarged Kripke frame to $\mathbb{Q} \cap [0, 1]$; finally, V_K will be the completion of the range of this embedding.

Let W^* be a disjoint copy of W whose elements can be accessed by the bijection ${}^*: W \to W^*$. Elements of W^* serve as names for points which we may have to add. We extend $\preceq$ to a total order $\preceq^*$ on $W \cup W^*$ by putting w^* as the direct successor of w for each $w \in W$, see Fig. 2.

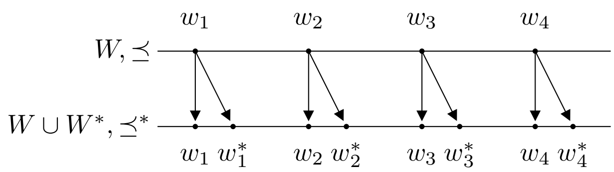

Figure 2. Extending $(W, \preceq)$ to $(W \cup W^*, \preceq^*)$.

Formally we define $\preceq^*$ as follows:

$$\preceq^* \quad := \quad \preceq \cup \{(v^*, w^*) : v \preceq w\} \cup \{(v, w^*) : v \preceq w\} \cup \{(v^*, w) : v \prec w\} \quad .$$

Let $\mathrm{Lim}(W)$ denote the set of limit worlds in W:

$$w \in \mathrm{Lim}(W) \quad \text{iff} \quad (\forall w' \succ w)(\exists w'' \succ w)(w'' \prec w') \ .$$

Observe that a maximal element of K, if it exists, would be in $\mathrm{Lim}(W)$. We define

$$W' \quad := \quad W \cup \{w^* : w \in \mathrm{Lim}(W)\}$$

and we define $\preceq'$ as the restriction of $\preceq^*$ to W':

$$\preceq' \quad := \quad \preceq^* \cap (W' \times W') \ .$$

Let $K' := (W', \preceq')$.

Next, we apply the Horn monomorphism from the proof of Lemma 3.1.7 to the converse of K', i.e. to $(W', \succeq')$. We obtain an embedding σ from $(W', \succeq')$ to $(\mathbb{Q} \cap [0,1], \leq)$ which satisfies the following form of continuity: for any subsets X and Y of W', if

$$\{w \in W' : \exists x \in X w \preceq' x\} \cap \{w \in W' : \exists y \in Y y \preceq' w\} = \emptyset$$

and

$$\{w \in W' : \exists x \in X w \preceq' x\} \cup \{w \in W' : \exists y \in Y y \preceq' w\} = W'$$

then $\sup \sigma(Y) = \inf \sigma(X)$.

To finish our construction, let V_K be the closure of $\sigma(W')$:

$$V_K \quad := \quad \overline{\sigma(W')} \ .$$

It remains to show that the logics $\mathbf{L}(K)$ and $\mathbf{G}_{V_K}$ coincide, which can be shown using notions and lemmas from [19]. $\square$

The following example considers the logic of the Kripke frame with set of worlds $\mathbb{Q}$. Takano [38] has shown that this logic is axiomatised by any complete axiom system for first-order intuitionistic logic (see e.g. [40]) plus the axiom scheme of linearity $(A \to B) \vee (B \to A)$ and the axiom scheme of constant domain (or quantifier shift) $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$, where x must not occur free in A. This axiomatisation is the same as the one for the standard first-order Gödel logic, i.e. the one based on the full interval $[0,1]$ (cf. [28]). Hence, we can expect that our construction derives a related Gödel set from the Kripke frame $\mathbb{Q}$.

EXAMPLE 3.4.2 (The logic $\mathbf{L}(\mathbb{Q})$). *Let $K_{\mathbb{Q}} = (\mathbb{Q}, \leq)$ be the Kripke frame of $\mathbb{Q}$. We want to describe the Gödel set $V_{\mathbb{Q}}$ corresponding to $K_{\mathbb{Q}}$ which is obtained by the construction given in the proof of the previous Theorem. $V_{\mathbb{Q}}$ will be isomorphic to the set of upsets of $\mathbb{Q}$.*

Note that for every element $q \in \mathbb{Q}$ there are two designated upsets in $\mathrm{Up}(K_{\mathbb{Q}})$*, $q^{\uparrow}$ and $q^{\uparrow} \setminus \{q\}$. Between these two upsets there is no other upset in* $\mathrm{Up}(K_{\mathbb{Q}})$*. Thus, $q^{\uparrow}$ and $q^{\uparrow} \setminus \{q\}$ under the isomorphism between* $\mathrm{Up}(K_{\mathbb{Q}})$ *and $V_{\mathbb{Q}}$ determine an open interval of* $[0,1]$ *which will never contain a point during our construction. Hence, doing this for all elements of $\mathbb{Q}$, countably many disjoint open intervals are generated which are densely ordered, which is achieved by a set isomorphic to the Cantor set.*

To be more precise: For every $q \in \mathbb{Q}$ the upset $q^{\uparrow} \setminus \{q\}$ is of type β. Thus, our construction from the last proof duplicates all the rational number, i.e. $\mathbb{Q}' = \mathbb{Q} \cup \{q^* : q \in \mathbb{Q}\}$ and $\leq' = \leq^*$. Now fix a particular enumeration of $\mathbb{Q} = \{q_1, q_2, \dots\}$ and consider the following enumeration induced on $\mathbb{Q}' = \{q_1, q_1^*, q_2, q_2^*, \dots\}$. The images of the pairs q_1, q_1^*, q_2, q_2^*, etc., under the Horn function h determine a sequence of disjoint open intervals of $[0,1]$ *which are removed from* $[0,1]$*. This obviously mimics Cantors middle third construction of repeatedly removing the middle thirds of line segments of*

$[0,1]$. *Hence the image of* $\mathbb{Q}'$ *under the Horn function* h *for this enumeration is a set isomorphic to the set of boundary points of the Cantor set, and the completion of* $h(\mathbb{Q}')$ *is a set isomorphic to the Cantor set.*

Now, the Gödel logic $\mathbf{G}_{\mathbb{C}_{[0,1]}}$ *generated by the Cantor set* $\mathbb{C}_{[0,1]}$ *is equal to the Gödel logic of the full interval,* $\mathbf{G}_{[0,1]}$ *(Theorem 3.3.4). To obtain an idea for this, first observe that obviously* $\mathbf{G}_{[0,1]} \subseteq \mathbf{G}_{\mathbb{C}_{[0,1]}}$. *Furthermore, for each* $\varphi \notin \mathbf{G}_{[0,1]}$ *we can find a valuation based on a countable model which makes* φ *false; hence the occurring truth values form a countable set (not necessarily closed!) which can be embedded into* $\mathbb{C}_{[0,1]}$ *such that existing infima and suprema are preserved. This gives rise to an interpretation based on* $\mathbb{C}_{[0,1]}$ *which also makes* φ *false. Hence, also* $\varphi \notin \mathbf{G}_{\mathbb{C}_{[0,1]}}$.

Going from Gödel sets to Kripke frames is not as complicated as the other direction. First we consider countable Gödel sets. For the general case of uncountable Gödel sets we will use Example 3.4.2 and a splitting lemma (Lemma 3.4.5) which divides uncountable Gödel sets into a countable part and a part containing a perfect set.

LEMMA 3.4.3. *For every countable Gödel set* V *there is a countable linear Kripke frame* K_V *such that* $\mathbf{G}_V = \mathbf{L}(K_V)$.

Proof. Since V is countable and closed, it can be viewed as a complete and completely distributive linear lattice. Every element of V is either an isolated point, or it is the limit of some isolated points. Thus every element of V is the join of a set of completely join-irreducible elements and V is isomorphic to a complete linear ring of sets (see [35] for definitions of join-irreducibility and this result). Furthermore, a lattice is isomorphic to a complete ring of sets if and only if it is isomorphic to the lattice of order ideals of some partial order P (see e.g. [22] for the definition of order ideals and this result). It is an easy exercise to show now that the logicss $\mathbf{G}_V$ and $\mathbf{L}(P)$ coincide. ☐

REMARK 3.4.4. *It is worth explicitly describing the construction of the Kripke frame underlying the previous proof. This is useful for finding Kripke frames for concretely given Gödel sets such that the logics defined by the Kripke frames are the same as the logics defined by the Gödel sets.*

Let V *be a countable Gödel set. By removing proper suprema from* V *we obtain a corresponding Kripke frame* K_V*: Let* $\mathrm{Sups}(V)$ *be the set of all suprema of* V*,*

$$\mathrm{Sups}(V) := \{p \in V : \exists (p_n) \subset V \text{ strictly increasing to } p\} \cup \{0\}.$$

We define the set of worlds as $W_V := V \setminus \mathrm{Sups}(V)$. *Then the Kripke frame* $K_V := (W_V, \geq)$ *defines the same logic as the Gödel set* V. *This construction works because a supremum of V will reoccur in* $\mathrm{Up}(K_V)$ *as the upset of all elements smaller than that supremum.*

For the treatment of general, i.e. uncountable, Gödel sets we need the following splitting Lemma which allows to split Kripke frames into parts and consider the logics of these parts only.

LEMMA 3.4.5. *Let V_1 and V_2 be Gödel sets and $K_1 = (W_1, \preceq_1)$ and $K_2 = (W_2, \preceq_2)$ be Kripke frames such that $(V_i, \leq)$ and $(\mathrm{Up}(K_i), \subseteq)$ are isomorphic. Assume $W_1 \cap W_2 = \emptyset$. Let $\alpha \in (0, 1)$, define*

$$V := \alpha V_1 \cup ((1 - \alpha)V_2 + \alpha)$$

and $K := (W_2 \cup W_1, \preceq)$ with

$$\preceq := \preceq_2 \cup \preceq_1 \cup \{(w_2, w_1) : w_2 \in W_2, w_1 \in W_1\} \quad ,$$

see Fig. 3. Then $(V, \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic, too.

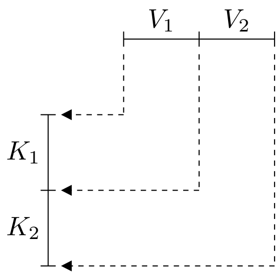

Figure 3. The relation of V_1, V_2 to K_1, K_2.

Proof. Let f_i be the isomorphism from V_i to $\mathrm{Up}(K_i)$. We define $f\colon V \to \mathrm{Up}(K)$ as follows: If $v \in [0, \alpha] \cap V$ then $f(v) = f_1(v/\alpha)$. If $v \in [\alpha, 1] \cap V$ then $f(v) = W_1 \cup f_2((v - \alpha)/(1 - \alpha))$.

First observe that f is well defined: the only critical point is at α where we have two ways to compute $f(\alpha)$:

$$f(\alpha) = f_1(\alpha/\alpha) = f_1(1) = W_1$$

and

$$f(\alpha) = W_1 \cup f_2((\alpha - \alpha)/(1 - \alpha)) = W_1 \cup f_2(0) = W_1 \cup \emptyset = W_1 \quad .$$

It is easy to verify that f is a $(V, \leq)$–$(\mathrm{Up}(K), \subseteq)$ isomorphism: f being bijective is reduced to f_1 and f_2 being bijective, and it is also immediate from the construction that f is a $\leq$–$\subseteq$ homomorphism. $\square$

THEOREM 3.4.6 ([19], Theorem 25). *For every Gödel set V there is a countable linear Kripke frame K_V such that $\mathbf{G}_V = \mathbf{L}(K_V)$.*

Proof. In Corollary 1.6.6 the Cantor-Bendixon representation of V gives a countable set C and a perfect set P such that $V = C \cup P$ and $C \cap P = \emptyset$. If V is countable, then P is empty and the Gödel logic induced by V can already be represented using Lemma 3.4.3. So assume that V is not countable, which means P is not empty. Let

$\alpha := \inf P$, $V'' := V \cup [\alpha, 1]$ and $V' := (V \cap [0, \alpha]) \cup \mathbb{C}_{[\alpha,1]}$, where $\mathbb{C}_I$ is the Cantor middle-third set on the interval I, which is a perfect set. Using Theorem 3.1.9 we obtain that $\mathbf{G}_V = \mathbf{G}_{V''} = \mathbf{G}_{V'}$. Hence, it is enough to consider V'.

In the case that $\alpha = 0$ we have $V' = \mathbb{C}_{[0,1]}$, in which case the Gödel logic based on V' is the same as the $\mathbf{L}(\mathbb{Q})$, see Example 3.4.2.

Otherwise let $V_1 := (1/\alpha)(V \cap [0, \alpha])$ and $V_2 := \mathbb{C}_{[0,1]}$. Then we can write V' as

$$V' = \alpha V_1 \cup ((1 - \alpha)V_2 + \alpha).$$

By construction of α, V_1 is countable and due to V being closed V_1 is also closed. Hence, by the proof of Lemma 3.4.3 we can find a countable linear Kripke frame K_1 such that $(V_1, \leq)$ and $(\mathrm{Up}(K_1), \subseteq)$ are isomorphic. Due to Example 3.4.2 we know that $(V_2, \leq)$ and $(\mathrm{Up}(K_{\mathbb{Q}}), \subseteq)$ are isomorphic. Applying Lemma 3.4.5 we obtain a countable Kripke frame K such that $(V', \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic. Finally, it is easy to see that the induced logics agree:

$$\mathbf{G}_V = \mathbf{G}_{V'} = \mathbf{L}(\mathrm{Up}(K)) = \mathbf{L}(K) \ .$$

□

It is worth pointing out some structural consequences which can be inferred from our constructions. Let K be a countable linear Kripke frame and let V_K be the corresponding Gödel set. K having a top element is equivalent to 0 being isolated in V_K, and K having a bottom element is equivalent to 1 being isolated in V_K. Let K *ends with* $\mathbb{Q}$ denote that there is an embedding σ of $\mathbb{Q}$ into K such that $\forall k \in K\ \exists q \in \mathbb{Q}\ k \preceq \sigma(q)$. In this case we have that $\mathbf{L}(K) = \mathbf{L}(\mathbb{Q})$. To see this observe that, as in Example 3.4.2, the condition 'K ends with $\mathbb{Q}$' implies that V_K contains a Cantor set which contains 0. But then Theorem 3.1.9 shows that the induced Gödel logic $\mathbf{G}_{V_K}$ is the same as the Gödel logic of the full unit interval, hence

$$\mathbf{L}(K) = \mathbf{G}_{V_K} = \mathbf{G}_{[0,1]} = \mathbf{L}(\mathbb{Q}) \ .$$

It is interesting to note that Theorem 3.4.6 cannot be deduced from the Löwenheim-Skolem Theorems in [33] and Lemma 3.4.3. Rather, the results presented in the present paper indicate that the Löwenheim-Skolem Theorem in [33, Theorem 4.8], which deals with reducing the cardinality of the pseudo-Boolean algebra, cannot be strengthened in the form that it is reduced to the cardinality of the universe (assuming it is infinite), i.e. in terms of [33, Theorem 4.8], $\lambda' = 2^\lambda$ cannot be replaced by λ in general. To see this observe that the pseudo-Boolean algebra $[0, 1]$ cannot be replaced by any countable pseudo-Boolean algebra: the Gödel logic of the former is axiomatisable (see above), where the Gödel logic of any countable truth value set is not axiomatisable (see Theorem 3.3.11).

As consequences of previous results on axiomatizability we obtain

COROLLARY 3.4.7. *Let K be a countable linear Kripke frame. The intermediate predicate logic defined by K on constant domains is axiomatisable if and only if K is finite, or if $\mathbb{Q}$ can be embedded into K, and either K has a top element or ends with a copy of $\mathbb{Q}$.*

COROLLARY 3.4.8. *Let K be a countable linear Kripke frame. If K is either not finite and $\mathbb{Q}$ cannot be embedded into K (i.e., K is scattered), or $\mathbb{Q}$ can be embedded into K, but K does not end with $\mathbb{Q}$ and K does not have a top element, then the intermediate predicate logic defined by K on constant domains is not recursively enumerable.*

COROLLARY 3.4.9. *The set of intermediate predicate logics defined by countable linear Kripke frames on constant domains is countable.*

Another surprising aspect from the point of view of the last corollary is that while there are uncountably many different countable linear orderings (which can be taken as Kripke frames), the class of logics defined by them on constant domains only contains countably many elements. Furthermore, the last result is contrasted by the fact that the number of all intermediate logics extending the basic linear logic with constant domains is uncountable.

In a similar way one can show that the logics of scattered Kripke frames with constant domains are not recursively enumerable.

3.5 Number of different Gödel logics

Following the last remark we will now consider the number of different logics. A reasonable argumentation for a lower bound on it would be as follows: If we have a basic logic with extensions in which each of countable many principles can be either true or false, then we would expect uncountably many different logics. As an example let us consider the class of all intermediate predicate logics, i.e. all those logics which are between Intuitionistic Logic and Classical Logic (cf. [33]). Here, we have a common basic logic, Intuitionistic Logic, and extensions of it by different principles. And in fact there are uncountably many intermediate predicate logics. Another example is the class of modal logics which has K as its common basic logic.

Considering Gödel logics, there is a common basic logic, the logic of the full interval, which is included in all other Gödel logics. On the side of logics defined by linear Kripke frames on constant domains this corresponds to the logic determined by a set of worlds of order-type $\mathbb{Q}$. There are still countably many extension principles but, surprisingly, in total only countably many different logics. This has been proven recently by formulating and solving a variant of a Fraïssé Conjecture [25] on the structure of countable linear orderings w.r.t. continuous embeddability.

The ordering relation in this article is smc-embeddability, which is a strictly monotone and continuous embedding of one countable closed subset of the reals into another.

THEOREM 3.5.1 ([20], Corollary 40). *The set of Gödel logics*

(a) is countable

(b) is a (lightface) Σ^1_2 set

(c) is a subset of Gödel's constructible universe L.

Proof. (a) First note that the set of countable Gödel logics (i.e. those with countable truth value set), ordered by $\supseteq$, is a wqo. To see this, assume that $\langle \mathbf{G}_n : n \in \omega \rangle$ is a sequence of countable Gödel logics. Take the sequence of countable Gödel sets $\langle V_n : n \in \omega\rangle$ generating these logics and define the respective Q-labeled countable closed linear ordering (cclo) (also denoted with V_n) with $Q = \{0, 1\}$, $0 <_Q 1$ and $V_n(0) = V_n(1) = 1$, and $V_n(x) = 0$ otherwise. Using results from [20] one shows that original sequence of Gödel logics $\langle \mathbf{G}_n : n \in \omega\rangle$ is good, i.e., it is neither an infinite anti-chain or infinitely decreasing chain with respect to embeddability.

As each countable Gödel logic is a subset of a fixed countable set (the set of all formulas), the family of countable Gödel logics cannot contain a copy of ω_1. So by [20, Lemma 39], the family of countable Gödel logics must be countable.

According to Theorem 3.1.9 any uncountable Gödel logic, i.e. Gödel logic determined by an uncountable Gödel set, such that 0 is not included in the prefect kernel P of the Gödel set is completely determined by the countable part $V \cap [0, \inf P]$. So the total number of Gödel logics is at most two times the number of countable Gödel logics plus 1 for the logic based on the full interval, i.e. countable.

(b) First, note that the set

$$\{(v, \varphi, v(\varphi)) : M^v = \mathbb{N}\}$$

is a Borel set, since we can show by induction on the quantifier complexity of φ that the sets $\{(v, q) : M^v = \mathbb{N}, v(\varphi) \geq q\}$ are Borel sets (even of finite rank).

Next, a set G of formulas is a Gödel logic iff

> There *exists* a closed set $V \subseteq [0, 1]$ (say, coded as the complement of a sequence of finite intervals) such that:
>
> - For every $\varphi \in G$, *for every* v with $M^v = \mathbb{N}$, $v(\varphi) = 1$, and
> - For every $\varphi \notin G$, there exists v with $M^v = \mathbb{N}$, $v(\varphi) < 1$.

(We can restrict our attention to valuations v with $v^M = \mathbb{N}$ because of Proposition 3.1.4.)

Counting quantifiers we see that this is a Σ^1_2 property.

(c) follows from (a) and (b) by the Mansfield-Solovay theorem (see [30], [31, 8G.1 and 8G.2]). $\square$

4 Further topics

In the following we shortly mention some further topics and observations and refer the interested reader to the cited references.

4.1 The Delta operator

With respect to the set of valid formulas the $\triangle$ operator extends the recursively enumerable infinitely valued Gödel logics.

THEOREM 4.1.1 ([15]). *A Gödel logic with $\triangle$ is axiomatizable iff the truth value set is one of:*

- $\{0, 1\} \subset V^\infty$
  axiomatization: $\mathbf{H\Delta}$

- $0 \in V^\infty$, 1 *isolated*
  axiomatization: $\mathbf{H}\Delta + \textsc{iso}_0$
- 0 *isolated,* $1 \in V^\infty$
  axiomatization: $\mathbf{H}\Delta + \textsc{iso}_1$
- 0 *and* 1 *isolated and* $V^\infty \neq \emptyset$ *(thus* V *is uncountable)*
  axiomatization: $\mathbf{H}\Delta + \textsc{iso}_0 + \textsc{iso}_1$
- V *finite*
  axiomatization: $\mathbf{H}\Delta + \textsc{fin}(n)$

Another observation is that $G^{\Delta}_{\uparrow}$ is not anymore the intersection of the finitely valued Gödel logics with Delta.

4.2 The Takeuti-Titani rule and quantified propositional logics

As already mentioned and discussed in the introduction, Takeuti and Titani in [39] introduced the following rule

$$\frac{C \vee (A \to x) \vee (x \to B)}{C \vee A \to B}$$

where the variable x does not occur in the conclusion. Recall that this rule can be semantically and syntactically eliminated from proofs, see [18, 37]. This shows that using dependent rules certain semantic properties can forced.

Using this rule we can introduce quantified propositional Gödel logics on $[0, 1]$.

DEFINITION 4.2.1 (Propositional quantified Gödel logic over V). *The language is the propositional language with quantifiers binding propositional variables. The propositional quantifiers range over all truth values.*

THEOREM 4.2.2 ([16]). *The quantified propositional Gödel logic on* $[0, 1]$ *admits quantifier elimination and is axiomatized by the quantified propositional variant of* $\mathbf{H}$ *together with the Takeuti-Titani rule.*

Note that in this case the Takeuti-Titani rule cannot be eliminated, but can be replaced by the equivalent formulation as axiom.

By coding open and closed intervals in the language one obtains the following result:

PROPOSITION 4.2.3 ([16]). *There are uncountable (*$\aleph_1$*) many quantified propositional Gödel logics.*

Other quantified propositional logics that admit quantifier elimination are $G_\uparrow$ and $G_\downarrow$, but in these cases a syntactical extension of the language by an unary operator is necessary [9, 12].

Considering the intersection of quantified propositional Gödel logics a varied image is shown:

- the intersection of all finitely valued q.p. without $\triangle$ is the q.p. logic over $V_\uparrow$

- the intersection of all finitely valued q.p. with $\triangle$ is not a Gödel logic (with $\triangle$ and propositional quantifiers finiteness can be expressed)
- the intersection of all q.p. without $\triangle$ is not a Gödel logic.

We conclude this further topics with an observation of rarely considered logics where first order and quantified propositional quantifiers are mixed.

First note that in all first order Gödel logics the following two implications are valid:

$$\begin{array}{lcl} \forall x A(x) \vee B & \to & \exists x(A(x) \to C \vee C \to B) \\ A \to \exists x B(x) & \to & \exists x(A \to C \vee C \to B(x)) \end{array}$$

Thus, in the quantified propositional Gödel logic over $[0,1]$ the following equivalences are valid (where the first quantifier can be a propositional or first order quantifier)

$$\begin{array}{lcl} (\forall x A(x) \to B) & \leftrightarrow & \forall p \exists x(A(x) \to p \vee p \to B) \\ (A \to \exists x B(x)) & \leftrightarrow & \forall p \exists x(A \to p \vee p \to B(x) \end{array}$$

Combining that with the fact that all other quantifier shifts are valid we obtain

THEOREM 4.2.4. *In the Gödel logic over* $[0,1]$ *with first order and quantified propositional quantifiers all formulas can be transformed into equivalent prenex formulas.*

4.3 Fragments of Gödel logics

THEOREM 4.3.1. *[13] The bottom-less fragment, the prenex fragment, and the existential fragment for infinitely valued Gödel logics are recursively enumerable if and only if the truth value set is uncountable. The resulting sets of valid formulas conincide.*

4.4 Entailment

The compactness of the underlying entailment relation is of central importance for the deductive properties of the logic under consideration. If the entailment is not compact, no effective representation of the entailment can be constructed [17].

DEFINITION 4.4.1 (Compactness). G^0_V *is* compact *if, whenever* $\Pi \models_V A$ *there is a finite* $\Pi' \subset \Pi$ *such that* $\Pi' \models_V A$.

It is important to mention that if we consider entailment relations or compactness, the underlying truth value set has to be closed under infima.

In the case of propositional tautologies, all logics of infinite truth value sets are the same (Theorem 2.1.8). The case for the entailment relation is similar with dense linear subset taking the position of the infinite subset.

It is an easy but fundamental result that $\mathrm{Taut}(V) = \mathrm{G}^0_V$ and $\mathrm{Ent}(V)$, the set of valid entailment relations, depend only on the order type of V. This central property of Gödel logics is dependent on the specific definition of the Gödel implication, other definitions of implication might not allow this kind of equivalence (see Lemma 3.1.3).

THEOREM 4.4.2 ([17], Proposition 3.2). *If* V *is finite then* G_V *is compact.*

Proof. We are discussing the entailment $\Pi \Vdash A$. Let $\Pi = \{B_1, B_2, \ldots\}$, and let $X = \{p_0, p_1, \ldots\}$, be an enumeration of variables occurring in Π, A such that all variables in B_i occur before the variables in B_{i+1}. We show that either $\{B_1, \ldots, B_k\} \Vdash A$ for some $k \in \mathbb{N}$ or $\Pi \nVdash A$.

Let T be the complete semantic tree on X, i.e. $T = V^{<\omega}$. An element of T of length k is a valuation of $p_0, \ldots, p_{k-1}$. Since V is finite, T is finitary. Let T' be the subtree of T defined by: $v \in T'$ if for every initial segment v' of v and every k such that all the variables in $A, B_1, \ldots, B_k$ are among $p_0, \ldots, p_{\ell(v')}$,

$$v'(\{B_1, \ldots, B_k\}) = \min\{v(B_1), \ldots, v(B_k)\} > v'(A).$$

In other words, branches in T' terminate at nodes v', where

$$v'(\{B_1, \ldots, B_k\}) \leq v(A).$$

Now if T' is finite, there is a k such that $B_1, \ldots, B_k \Vdash_V A$. Otherwise, since T' is finitary, it contains an infinite branch. Let v be the limit of the partial valuations in that branch. Obviously, since V is finite, $v(\Pi) > v(A)$ and so $\Pi \nVdash_V A$. □

THEOREM 4.4.3 ([17], Theorem 3.4). *If V is uncountable, then G_V is compact.*

Proof. Let W be a densely ordered, countable subset of V. Such a subset exists according to Proposition 1.6.4. Let X be a set of variables. A *chain on X* is an arrangement of X in a linear order. Formally, a chain C on X is a sequence of pairs $\langle p_i, o_i \rangle$ where $o_i \in \{<, =, >\}$ where p_i appears exactly once. A valuation $\mathcal{I}$ respects C if $\mathcal{I}(p_i) = \mathcal{I}(p_{i+1})$ if o_i is $=$, $\mathcal{I}(p_i) > \mathcal{I}(p_{i+1})$ if o_i is $>$, and $\mathcal{I}(p_i) < \mathcal{I}(p_{i+1})$ if o_i is $<$. If X is finite, there are only finitely many chains on X.

We consider the entailment relation $\Pi \Vdash A$ and construct a tree in stages as follows: The initial node is labeled by $0 < 1$ and an empty valuation. Stage $n + 1$: A node N constructed in stage n is labeled by a chain on the variables $p_1, \ldots, p_n$ and a valuation $\mathcal{I}_N$ of $p_1, \ldots, p_n$ respecting the chain. N receives successor nodes, one for each possibility of extending the chain by inserting p_{n+1}. The labels of each successor node N' are the corresponding extended chain and an extension of $\mathcal{I}_N$ which respects the extended chain. The value $\mathcal{I}_{N'}(p_{n+1})$ is chosen inside W, i.e. the endpoints of W may not be chosen as values. Since W is densely ordered, this ensures that such a choice can be made at every stage.

We call a branch of T *closed at node N* (constructed at stage n) if for some finite $\Pi' \subseteq \Pi$ such that $\mathrm{var}(\Pi') \cup \mathrm{var}(A) \subseteq \{p_1, \ldots, p_n\}$ it holds that $\mathcal{I}_N(\Pi') \leq \mathcal{I}_N(A)$. T is *closed* if it is closed on every branch. In that case, for some finite $\Pi' \subseteq \Pi$, we have $\Pi' \Vdash A$.

If T is not closed, it contains an infinite branch. Let $\mathcal{I}$ be the limit of the $\mathcal{I}_N$ of nodes N on the infinite branch. It holds that $\mathcal{I}(B) > \mathcal{I}(A)$ for all $B \in \Pi$, for otherwise the branch would be closed at the first stage where all the variables in A were assigned values. Let $w = \mathcal{I}(A)$. By Lemma 3.2.6, $\mathcal{I}_w(A) = \mathcal{I}(A)$ and $\mathcal{I}_w(\Pi) = \inf\{\mathcal{I}_w(B) : B \in \Pi\} = 1$, and so $\Pi \nVdash A$, a contradiction. □

THEOREM 4.4.4 ([17], Theorem 3.5). *If V is countably infinite, then G_V is not compact.*

Proof. Note that if V is countable, it cannot contain a densely ordered subset, since truth-value sets for entailment have to be closed (under infima). We define a sequence of formulas Γ_k as follows:

$$\begin{aligned}
\Gamma_k' &= \{p_{0/2^k} \prec p_{1/2^k} \prec \ldots \prec p_{(2^k-1)/2^k} \prec p_{2^k/2^k}\} \\
\Gamma_k'' &= \{p_{0/2^k} \to q, \ldots, p_{2^k/2^k} \to q\} \\
\Gamma_k &= \Gamma_k' \cup \Gamma_k'' \\
\Gamma &= \bigcup_{k \in \omega} \Gamma_k
\end{aligned}$$

Intuitively, Γ_k' expresses that the $p_r = p_{i/2^k}$ are linearly ordered and $\bigcup_{k\in\omega} \Gamma_k'$ expresses that the variables p_r are densely ordered. Since V does not contain a densely ordered subset, we have

$$\Gamma \Vdash_V q.$$

In fact the only $\mathcal{I}$ such that $\mathcal{I}(\Gamma) = 1$ is $\mathcal{I}(p_r) = 1$ for all r, and $\mathcal{I}(q) = 1$. Now assume a finite $\Gamma' \subset \Gamma$ such that

$$\Gamma' \Vdash_V q.$$

There is a $\Gamma_k \supseteq \Gamma'$. Since V is infinite we can choose at least $2^k + 2$ different truth values $v_0 < \cdots < v_{2^k+1} < 1$. Define the valuation $\mathcal{I}$ as

$$\begin{aligned}
\mathcal{I}(p_{i/2^k}) &= v_i \\
\mathcal{I}(q) &= v_{2^k+1}.
\end{aligned}$$

Then we have $\mathcal{I}(\Gamma_k) = \mathcal{I}(\Gamma') = 1$, but $\mathcal{I}(q) < 1$ and therefore, $\Gamma' \nVdash_V q$. □

Thus, we have succeeded in characterizing the compact propositional Gödel logics. They are all those where the set of truth values V is either finite or contains a nontrivial densely ordered subset.

Although the number of *first-order Gödel logics* has been settled to countable (see Theorem 3.5.1), it is possible to prove that there are uncountably ($2^{\aleph_0}$) many entailment relations:

PROPOSITION 4.4.5. *The number of different entailment relations of first-order Gödel logics is* $2^{\aleph_0}$.

Sketch. It is possible to express ordinals and their orderings with entailment relations. Oberving that there are uncountably many such orderings concludes the proof. □

4.5 Satisfiability

In the case of Gödel logics, the connection between satisfiability and entailment one is used to from classical logic breaks down. It is not the case that $\models A$ iff $\{\neg A\}$ is unsatisfiable. For instance, $B \vee \neg B$ is not a tautology, but can also never take the value 0, hence $\mathcal{I}(\neg(B \vee \neg B)) = 0$ for all $\mathcal{I}$, i.e., $\neg(B \vee \neg B)$ is unsatisfiable. So entailment cannot be defined in terms of satisfiability in the same way as in classical logic. Yet, satisfiability can be defined in terms of entailment: Γ is satisfiable iff $\Gamma \not\models \perp$. Hence also for Gödel logics, establishing soundness and strong completeness for entailment yields the familiar versions of soundness and completeness in terms of satisfiability: a set of formulas Γ is satisfiable iff it is consistent.

In the following we concentrate on satisfiability as many practical problems are connected to it: large ontologies are always checked for consistency, i.e., satisfiability. For the applicability of Gödel logics it is essential that satisfiability in many cases means *classical* satisfiability, and therefore usual automated theorem provers may be used for consistency checks.

THEOREM 4.5.1 ([6]). *In the following cases satisfiability in Gödel logics is classical satisfiability:*

- *in the propositional case,*
- 0 *is isolated in the truth value set,*
- *prenex fragment for any truth value set.*
- *existential fragment for any truth value set.*

Proof. For the first two cases let Q be any formula of G_V. If Q is satisfiable in classical logic then Q is satisfiable in G_V. For the converse direction, consider any interpretation $\mathcal{I}_{\mathbf{G}}$ of G_V such that $\mathcal{I}_{\mathbf{G}}(Q) = 1$. An interpretation $\mathcal{I}_{\mathbf{CL}}$ of classical logic such that $\mathcal{I}_{\mathbf{CL}}(Q) = 1$ is defined as follows: for any atomic formula A

$$\mathcal{I}_{\mathbf{CL}}(A) = \begin{cases} 1 & \text{if } \mathcal{I}_{\mathbf{G}}(A) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for each formula P, $(*)$ $\mathcal{I}_{\mathbf{G}}(P) = 0$ if and only if $\mathcal{I}_{\mathbf{CL}}(P) = 0$ and $\mathcal{I}_{\mathbf{G}}(P) > 0$ if and only if $\mathcal{I}_{\mathbf{CL}}(P) = 1$. The proof proceeds by induction on the complexity of P and all cases go though for *all* Gödel logics except when P has the form $\forall x P_1(x)$; in this case, being 0 an isolated point in V, $\mathcal{I}_{\mathbf{G}}(P) = 0$ if and only if there is an element u in the domain of $\mathcal{I}_{\mathbf{G}}$ such that $\mathcal{I}_{\mathbf{G}}(P_1(u)) = 0$; by induction hypothesis $\mathcal{I}_{\mathbf{CL}}(P_1(u)) = 0$ and hence $\mathcal{I}_{\mathbf{CL}}(\forall x P_1(x)) = 0$.

Considering the prenex case, let $Q = \mathcal{Q}\bar{x}P$ be any prenex formula, were $\mathcal{Q}\bar{x}$ is the formula prefix and P does not contain quantifiers. Assume that $\mathcal{I}_{\mathbf{G}}(Q) = 1$. As above we can prove $(*)$ for P. $\mathcal{I}_{\mathbf{CL}}(Q) = 1$ easily follows by induction on the number n of quantifiers in $\mathcal{Q}\bar{x}$. $\square$

4.5.1 Recursively enumerability of (un)satisfiability

This is an area with many open questions. The only known results are

THEOREM 4.5.2.

- *for finitely valued logics, unsatisfiability is r.e.*
- *for the prenex fragment with* $\triangle$ *over the truth value set* $[0, 1]$, *unsatisfiability is r.e. [10].*

- *for the class* FO^1_{mon} *consisting of all formulas in the first-order language with* $\triangle$ *of the form*

$$\bigvee_{i=1}^{n} (\exists x A_1^i(x) \wedge \ldots \wedge \exists x A_{n_i}^i(x) \wedge \forall x B_1^i(x) \wedge \ldots \wedge \forall x B_{m_i}^i(x)),$$

  satisfiability is decidable [7]

It can be easily shown that with respect to satisfiability there are at least countably many logics, but the upper limit is not known.

We conclude this part with two observations, the first concerning the Löwenheim-Skolem theorem. Consider

$$\neg(x = y) \to \neg\triangle(P(x) \leftrightarrow P(y)).$$

which implies that the upward Löwenheim-Skolem theorem does not hold for infinitely valued Gödel logics with $\triangle$.

To show the contrast between validity and satisfiability, we emphasize the following theorem:

THEOREM 4.5.3. *The prenex fragment of the monadic class (with at least 2 predicate symbols) for infinitely valued Gödel logics is*

- *undecidable (with the possible exceptions of* $\mathrm{G}_\uparrow$*) with respect to validity (the construction in [5] is easily adaptable to the prenex case)*
- *decidable with respect to satisfiability (see Theorem 4.5.1)*

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