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A Categorical Description of Relativization

By Kazuto Yoshimura

A thesis submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
Graduate Program in Information Science

Written under the direction of
Hajime Ishihara

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1 Introduction

The aim of this thesis is to give a foundation for computable analysis which does not depend on a particular effectivity concept.

The main purpose in computable analysis is investigations of computational structures appear in analysis, geometry, topology, or any other fields of mathematics. Although many researchers developed foundations for computable analysis, most of those are based on particular effectivity concepts, such as computability, polynomial time computability or limit computability, and are also based on choices of special kind of spaces, such as computable topological space [8], effective uniform neighborhood system [2] or effective equilogical space [1].

Our goal is to reformulate fundamental results from computable analysis without a particular choice of an effectivity concept or of a special kind of space. To do that, we give a description of "relativization to oracles" on a pure category theoretical setting, based on the approach from [6]. Using the description, at the end of this thesis, a corresponding result to the equivalence between oracle co-r.e. closedness and topological closedness will be shown categorically.

In what follows, we give some backgrounds and motivations of our work.

Backgrounds TTE, *type-2 theory of effectivity*, a foundation for computable analysis developed mainly by K. Weihrauch, has been broadly embraced by many researchers (see [9]).

The main effectivity concept in TTE is type-2 computability. Although it is defined only on Cantor space, the space of binary sequences of countably infinite length, even on an abstract space, relative computability can be derived by a representation [2] . So-called *continuous computation* or *arbitrary precision computation* is formally described by relative computability.

An abstract space equipped with a representation is called a represented space. TTE provides us a theory on systematic structures, such as computable topological space or computable metric space, to construct a represented space with many desirable properties [8].

A central idea of TTE might be expressed as "each topological notion is the relativization of a computational notion". As a fragment of our reasoning for the idea, it is well-known that for every subset of a given computable topological space, oracle co-r.e. closedness coincides with topological closedness. This sort of equivalences between a relativized computational notion and a topological notion are thought as necessary fundamental results, and are used in practical ways [3].

The problem on which we focus in this thesis is non-axiomatized style of TTE. Caused by its arbitrary choice of an effectivity concept and of a special kind of space, when we prove even a fundamental result, it is unclear what the critical assumption to obtain the result. This kind of problems is not unique on TTE, but, as we explained at the beginning of this introduction, is common among many other foundations for computable analysis.

[2] A representation is a partial surjection from Cantor space to the concerned space.

Hence there is a need for a foundation for computable analysis which does not depend on any particular effectivity concept and on any special kind of space.

Our Approach and Its Concept Avoiding a choice of a special kind of computational structure as our basis, we use categories. A category is an algebraic structure consists of the class of *objects*, the class of *morphisms* and some other arguments. Typically, each object is a kind of space (e.g. topological space or represented space), and each morphism is a function between two objects with special properties (continuous functions and relatively computable functions are possible choices respectively for topological spaces and for represented spaces).

Recently, a categorical foundation for general topology, known as *a functional approach to general topology*, was introduced by [6]. On this foundation, many topological notions, such as closedness, openness, density or compactness, are captured in a pure categorical way. However, generality of the categorical approach might allow us to capture other notions such as computational notions as well.

Actually we try to describe "relativization to oracles" on a setting basically from [6] and to reformulate a fundamental result from computable analysis, the equivalence between oracle co-r.e. closedness and topological closedness, using the description.

What is important is that only a category equipped with an additional structure but nothing else is supposed to be given in our approach. Thus a particular choice of an effectivity concept nor of a special kind of space is no longer in need.

Summary of Main Works Let us explain, firstly, our settings. In the following, as a typical but a simple example, the category Cp, whose objects are subsets of Cantor space and whose morphisms are computable total functions, will be used on our explanation.

We work on a (large and well-powered) category $\mathbb{E}$ equipped with a proper factorization system $(\mathscr{S}, \mathscr{T})$, a pair of two classes of morphisms. The class $\mathscr{S}$ is supposed to be stable under pullback and our category $\mathbb{E}$ is supposed to have $\mathscr{T}$-intersection (cf. Section 3.3.1). One can think of $\mathbb{E}$ as a broad generalization of the category of topological spaces. A subclass of $\mathscr{T}$ is called a fundamental class on $\mathbb{E}$ when it contains all isomorphisms, is closed under composition and is stable under pullback (cf. Section 3.1). A fundamental class can be thought of as defining a topology-like structure on our category $\mathbb{E}$. This notion is basically from [6]. On Cp, if $\mathscr{S}$ and $\mathscr{T}$ are suitably defined for it, one can define a fundamental class $\mathscr{B}_{0,\mathrm{Cp}}$ which identify the notion of co-r.e. closedness.

Our primal work is a categorical abstraction of the notion of oracle. We call an object with a certain property an imaginary. In the case of Cp, the set of all imaginaries coincides with the set of all oracles. As the next work, we define two closure operators $\mathscr{I}$ and $\mathscr{L}$ for fundamental classes. On the one hand, the action of $\mathscr{I}$ is an abstraction of "relativization to oracles". In the case of Cp, it turns out that $\mathscr{I}\mathscr{B}_{0,\mathrm{Cp}}$ identifies the notion of oracle co-r.e. closedness. On the other hand, the action of $\mathscr{L}$ is an abstraction of "generation of topology". In the case of Cp, it turns out that $\mathscr{L}\mathscr{B}_{0,\mathrm{Cp}}$ identifies the notion of topological closedness.

Two theorems will be shown in this thesis as our main works. Both of them are on a comparison of $\mathscr{I}\mathscr{F}$ and $\mathscr{I}\mathscr{F}$ where $\mathscr{F}$ is a given fundamental class on $\mathbb{E}$.

The first main theorem, Theorem 3.70 from Section 3.3.4, is stated as follows. For a given fundamental class $\mathscr{F}$ on $\mathbb{E}$, the inclusion $\mathscr{I}\mathscr{F} \subseteq \mathscr{L}\mathscr{F}$ holds if and only if all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact. Therefore this is a complete characterization of the concerned inclusion. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted respectively as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, the concerned inclusion corresponds to the fact that oracle co-r.e. closedness implies topological closedness.

The second main theorem is concerned with a slightly complicated situation. We have to prepare another category $\mathbb{E}^*$ with a certain structure and its equipped factorization system $(\mathscr{S}^*, \mathscr{T}^*)$. Suppose that we are given two fundamental classes $\mathscr{F}$ and $\mathscr{F}^*$ respectively on $\mathbb{E}$ and $\mathbb{E}^*$. Assume also $\mathbb{E}$ is suitably related to $\mathbb{E}^*$ with respect to $\mathscr{F}, \mathscr{F}^*$ and a functor $G : \mathbb{E} \to \mathbb{E}^*$. In this situation, we define another class of morphisms ${}^*\!\mathscr{I}\mathscr{F}$. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted respectively as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, the class ${}^*\!\mathscr{I}\mathscr{F}$ identifies what is called r.e. closedness.

The second main theorem, Theorem 5.24 from Section 5.2.2, is stated as follows. The equality $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ holds if the following three conditions are fulfilled: (i) all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact; (ii) all objects of $\mathbb{E}$ are $\mathscr{I}\mathscr{F}$-full; (iii) ${}^*\!\mathscr{I}\mathscr{F}$ is included in $\mathscr{I}\mathscr{F}$. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, respectively, the concerned equality, of course, corresponds to the fact that oracle co-r.e. closedness coincides with topological closedness. Actually the three conditions (i)-(iii) are certainly fulfilled in Cp.

The category Cp is, as we have already mentioned, a typical and a simple example. However, it is quite narrow in a sense. As a broader category, the category $\mathtt{Rep}_{\mathrm{op}}$, whose objects are represented topological spaces with an open representation and whose morphisms are relatively computable functions, will be constructed. All effective topological spaces can be regarded as objects of this category $\mathtt{Rep}_{\mathrm{op}}$ with respect to standard representation, and similarly, all effective metric spaces can be regarded as objects of this category $\mathtt{Rep}_{\mathrm{op}}$ with respect to Cauchy representation.

At the end of this thesis, $\mathtt{Rep}_{\mathrm{op}}$ will also be applied to Theorem 5.24, and as a result, it turns out that oracle co-r.e. closedness coincides with topological closedness on each object of $\mathtt{Rep}_{\mathrm{op}}$, a represented topological space with open representation.

2 Preliminairies

In this section, we introduce some basic notions and give a quick review on basics of category theory.

Firstly, we work on a fixed sufficiently strong set theoretical foundation e.g. ZF=Zermelo-Fraenkel's set theory. So our language has equality $=$ and membership relation $\in$. We don't enumerate our axioms, but, at least, guarantee that all notions and notations which will be introduced below are certainly well-defined. In what follows, we use the term "set" in the same sense with "variable".

2.1 Set Theoretic Notations

On a sufficiently strong set theoretical foundation, one can perform most of our usual mathematical implementations. We introduce some elemental notions and notations below.

Set theoretic operations If $a \in x$, as usual, a is called an element of x. Assume that for every two sets x and y, one has $x = y$ if and only if $x \subseteq y$ and $y \subseteq x$ where $x \subseteq y$ is an abbreviation of ${}^{\forall}a \in x,\ a \in x$. This property is called extensionality of sets.

Now we introduce some notations. Let a, b, x, y and σ be sets and let P be an arbitrary formula. All terms defined as the left side of each of the following equations is supposed to be well-defined as new sets.

$$
\begin{aligned}
\{a, b\} &= \{c : c = a \vee c = b\}, \quad \{a\} \ = \ \{a, a\} \\
(a, b) &= \{\{a\}, \{a, b\}\} \\
x \times y &= \{c : {}^{\exists}a \in x,\ {}^{\exists}b \in y \text{ s.t. } c = (a, b)\} \\
\mathrm{Pow}(x) &= \{u : u \subseteq x\} \\
\bigcup \sigma &= \{a : {}^{\exists}x \in \sigma \text{ s.t. } a \in x\} \qquad\quad ,\ x \cup y \ := \ \bigcup\{x, y\} \\
\bigcap \sigma &= \{a : {}^{\forall}x \in \sigma,\ a \in x\} \text{ if } \sigma \neq \emptyset\ ,\ x \cap y \ := \ \bigcap\{x, y\} \\
\{a \in x : P(a)\} &= \{a : a \in x \wedge P(a)\} \\
x - y &= \{a \in x : a \notin y\}
\end{aligned}
$$

Here $\emptyset$ is the empty set what is unique existence identified by the property ${}^{\forall}a,\ a \notin \emptyset$. We call any set of the form:

$$
\begin{array}{rcl}
\{a,b\} & , & \text{pair set of } a \text{ and } b; \\
\{a\} & , & \text{singleton of } a; \\
(a,b) & , & \text{ordered pair of } a \text{ and } b; \\
x \times y & , & \text{Cartesian product of } x \text{ and } y; \\
\mathrm{Pow}(x) & , & \text{power set of } y; \\
\bigcup \sigma & , & \text{union of } \sigma; \\
x \cup y & , & \text{binary union of } x \text{ and } y; \\
\bigcap \sigma & , & \text{intersection of } \sigma; \\
x \cap y & , & \text{binary intersection of } x \text{ and } y; \\
\{a \in x : P(a)\} & , & \text{a subset of } x; \\
x - y & , & \text{relative complement of } y \text{ in } x:
\end{array}
$$

For two ordered pairs (a, b) and (a', b'), it can be easily checked that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$. We abbreviate as $(a, b, c) = ((a, b), c)$, $(a, b, c, d) = (((a, b), c), d)$, ...etc. A set u is a subset of x if and only if $u \subseteq x$.

Function Now we define the notion of function and give some related definitions.

Definition 2.1. For each $F \in \mathrm{Pow}(x \times y)$, the triple $f = ((x, y), F)$ is said to be a function from x to y, written as $f : x \to y$, if the following condition holds:

$$
{}^{\forall}a \in x,\ {}^{\exists!}b \in y \text{ s.t. } (a, b) \in F
$$

If f is a function, x is called domain of f and y is called codomain of f.

Let f be a function from x to y and P be an arbitrary formula. For each $a \in x$, intending the situation that corresponding unique element of y satisfies the property P, we abbreviate as follows.

$$
P(f(a)) \iff {}^{\forall}b \in y\ \big[(a, b) \in F \Longrightarrow P(b)\big]
$$

Each $f(a)$ is called the value of f at a, or f of a. For every two functions f and g, both from x to y, one can see that $f = g$ if and only if $f(a) = g(a)$ for every $a \in x$. This property is called extensionality of functions.

We also abbreviate as $\{f(a) : P(a)\} = \{b \in y : {}^{\exists}a \in x \text{ s.t. } P(a) \wedge b = f(a)\}$. Using this abbreviation, we introduce some notations as follows.

$$
\begin{array}{rcl}
f\big[u\big] & = & \{f(a) : a \in u\} \\
\mathrm{range}(f) & = & f\big[x\big] \\
f^{-1}\big[v\big] & = & \{a \in x : f(a) \in v\}
\end{array}
$$

where $u \subseteq x$ and $v \subseteq y$. We denote by y^x the set of all functions from x to y. Explicitly, one may define:

$$
y^x = \{(x, y, F) : F \in \mathrm{Pow}(x \times y),\ (x, y, F) : x \to y\}
$$

When defining a new function, "maps to" denotation is frequently used. For example, projection functions π_1, π_2 for a Cartesian product $x \times y$ can be defined as follows.

$$\begin{aligned} \pi_1 : x \times y &\rightarrow x \\ (a,b) &\mapsto a \\ \pi_2 : x \times y &\rightarrow y \\ (a,b) &\mapsto b \end{aligned}$$

Of course this is meant to be $\pi_1(a,b) = a, \pi_2(a,b) = b$ at each $(a,b) \in x \times y$. As another example, if $u \subseteq x$, we usually denote by ι identical embedding of u into x which is defined as follows.

$$\begin{aligned} \iota : u &\rightarrow x \\ a &\mapsto a \end{aligned}$$

Of course this is meant to be $\iota(a) = a$ at each $a \in x$.

Next we define some properties for functions.

Definition 2.2. A function $f : x \rightarrow y$ is said to be:

injective	if	$^{\forall}a, a' \in x \ \big[f(a) = f(a') \Longrightarrow a = a'\big]$;
surjective	if	$y = \mathrm{range}(f)$;
bijective	if	f is injective and surjective.

An injective (resp. surjective, bijective) function is called an injection (resp. surjection, bijection).

Let $f : x \rightarrow y$ and $g : y \rightarrow z$ be two functions. We define composition $g \circ f$ of them as follows:

$$g \circ f \quad := \quad (x, z, \{(a,c) \in x \times z : c = g(f(a))\})$$

One can easily see that $g \circ f$ is a function from x to z and $g \circ f(a) = g(f(a))$ for each $a \in x$. It is also easy to show that composition preserves injectivity, surjectivity and bijectivity i.e. $g \circ f$ is injective (resp. surjective, bijective) whenever both f and g are injective (resp. surjective, bijective).

Classes and Families Let P be a formula. We occasionally write $a \in P$, or $a \in \{a : P(a)\}$, instead of $P(a)$. In that case, P, also $\{a : P(a)\}$, is called a class. A class P is said to be a set if the following condition holds.

$$^{\exists}x \text{ s.t. } x = P$$

Here $x = P$ is an abbreviation of $^{\forall}a \ \big[a \in x \Longleftrightarrow a \in P\big]$. Of course such x is at most unique. One can introduce similar notions and notations for classes just as we've done for sets. For example, if P is a class, we define:

$$\bigcap P \quad = \quad \{a : {}^{\forall}x \in P, \ a \in x\}$$

Note that $\bigcap P$ is always a set whenever there is a set $x_0 \in P$. Namely, one has the following equality.

$$\{a \in x_0 : {}^{\forall}x \in P,\ a \in x\} \quad = \quad \bigcap P$$

As another example, if P and Q are two classes, we define:

$$\begin{aligned} P \subseteq Q &\iff {}^{\forall}a \in P,\ a \in Q \\ (a,b) \in P \times Q &\iff a \in P \wedge b \in Q \end{aligned}$$

Also one can call a class $F \subseteq P \times Q$ a correspondence (the corresponding notion of function) if the following condition holds.

$${}^{\forall}a \in P,\ {}^{\exists!}b \in Q \text{ s.t. } (a,b) \in F$$

If F is a correspondence, we use the notation $F(a)$ defined in a same habit to the case of function. A correspondence F is, occasionally, called a family. In that case, it will be denoted by $\{F_i\}_{i \in P}$, and its value at $i \in P$ will be denoted by F_i instead of $F(i)$.

Natural Number There are several alternative ways to define what is natural numbers. Peano system is one of them.

Definition 2.3. Let N be a set with an element $0 \in N$ and let $s : N \to N$ be a function. The triple $(0, s, N)$ is said to be a Peano system if the following three conditions hold:

(Pi) for each $i \in N$, $s(i) \neq 0$;

(Pii) s is injective;

(Piii) for each $u \subseteq N$, if $0 \in u$ and $s(i) \in u$ (${}^{\forall}i \in u$), then $u = N$.

On a sufficiently strong set theoretical foundation, it can be assumed that there is a set x with the following property $(*)$:

$$0 \in x \wedge {}^{\forall}i \in x, i+1 \in x$$

where 0 is an alternative denotation of $\emptyset$ and $i+1$ is an abbreviation of $i \cup \{i\}$. Then one can define:

$$\omega = \bigcap\{x : x \text{ satisfies } (*)\}$$

This ω is the smallest set with property $(*)$. It is easy to see that $(0, +1, \omega)$ forms a Peano system.

On a Peano system, one can define order $\leq$, addition $+$, multiplication $\times$, ...etc, and imitate our usual settings in the theory of natural numbers. But we don't refer to the detail of implementations. Note that each $k \in \omega$ is being of the form $\{0, \cdots, k-1\}$.

Sequence For each function $f : k \to y$ $(k \in \omega)$, we occasionally write $\{f_i\}_{i<k}$, or $f_0, \cdots, f_{k-1}$, and call it a finite sequence of length k on y. Each value at $i < k$ will be denoted by f_i instead of $f(i)$ in that case. We define $\{f_0, \cdots, f_{k-1}\} = \{f_i : i < k\}$.

For each function $f : \omega \to y$, again, we occasionally write $\{f_i\}_{i\in\omega}$ and call it a sequence on y. Each value at $i \in \omega$ will be denoted by f_i instead of $f(i)$ in that case.

Finite and Countable Sets For two sets $x$and y, we write $x \approx y$ if and only if there is a bijection from x to y. A set x is said to be finite if there exists $k \in \omega$ such that $x \approx k$. Also it is said to be countably infinite if $x \approx \omega$. A set is said to be countable if it is finite or countably infinite.

2.2 Category

In the following, we give a definition of category and some related notions. Particularly, several simple examples of universal construction (e.g. product, equalizer, ...etc) and some kinds of morphism will be introduced.

Definition of Category This paragraph is devoted to the definition of category and introduction of abbreviations or expressions.

Definition 2.4. Let $\mathrm{ob}(\mathbb{E})$ and $\mathrm{mor}(\mathbb{E})$ be two sets and let:

$$\begin{aligned} \mathrm{dom} &: \mathrm{mor}(\mathbb{E}) \to \mathrm{ob}(\mathbb{E}) \\ \mathrm{cod} &: \mathrm{mor}(\mathbb{E}) \to \mathrm{ob}(\mathbb{E}) \\ \mathrm{id} &: \mathrm{ob}(\mathbb{E}) \ \ \to \mathrm{mor}(\mathbb{E}) \\ - \circ - &: \mathrm{mor}(\mathbb{E})|_{\mathrm{dom,cod}} \to \mathrm{mor}(\mathbb{E}) \end{aligned}$$

where $\mathrm{mor}(\mathbb{E})|_{\mathrm{dom,cod}} = \{(f, g) \in \mathrm{mor}(\mathbb{E}) \times \mathrm{mor}(\mathbb{E}) : \mathrm{dom}(f) = \mathrm{cod}(g)\}$. Suppose that $x = \mathrm{dom}(\mathrm{id}(x)) = \mathrm{cod}(\mathrm{id}(x))$ for every $x \in \mathrm{ob}(\mathbb{E})$. We say that $\mathbb{E} = (\mathrm{ob}(\mathbb{E}), \mathrm{mor}(\mathbb{E}), \mathrm{dom}, \mathrm{cod}, \mathrm{id}, - \circ -)$ is a category if the follwing two conditions hold:

(C0) $^\forall f, g, h \in \mathrm{mor}(\mathbb{E}),\ f \circ (g \circ h) = (f \circ g) \circ h$;

(C1) $^\forall f \in \mathrm{mor}(\mathbb{E}),\ f = f \circ \mathrm{id}(\mathrm{dom}(f)) = \mathrm{id}(\mathrm{cod}(f)) \circ f$.

Let $\mathbb{E}$ be also a category. We usually assume that $\mathbb{E}$ is being of the form:

$$\mathbb{E} = (\mathrm{ob}(\mathbb{E}), \mathrm{mor}(\mathbb{E}), \mathrm{dom}, \mathrm{cod}, \mathrm{id}, - \circ -)$$

And we write:

$x \in \mathbb{E}$	instead of	$x \in \mathrm{ob}(\mathbb{E})$;
f in $\mathbb{E}$	instead of	$f \in \mathrm{mor}(\mathbb{E})$;
id_x	instead of	$\mathrm{id}(x)$;
$\mathrm{dom} f$	instead of	$\mathrm{dom}(f)$;
$\mathrm{cod} f$	instead of	$\mathrm{cod}(f)$;
fg	instead of	$f \circ g$;
$\begin{cases} f : x \to y \\ x \xrightarrow{f} y \end{cases}$	instead of	$x = \mathrm{dom}(f) \wedge y = \mathrm{cod}(f)$.

Obvious combinations of above notations will be used. For example, $x \xrightarrow{f} y$ in $\mathbb{E}$ means $x, y \in \mathbb{E} \wedge f$ in $\mathbb{E} \wedge x \xrightarrow{f} y$. If $x \xrightarrow{f} y$ in $\mathbb{E}$, then f is said to be a morphism from x to y, x is called domain of f and y is called codomain of f. Each element of $\mathrm{ob}(\mathbb{E})$ is called an object of $\mathbb{E}$. For two objects $x, y \in \mathbb{E}$, we denote by $\mathbb{E}(x, y)$ the set of all morphisms from x to y and call it hom-set between them. Explicitly, one may define:

$$\mathbb{E}(x, y) \quad = \quad \{f \text{ in } \mathbb{E} : x \xrightarrow{f} y\}$$

Example 2.5. Let x be a set and $\leq \subseteq x \times x$. The pair $(x, \leq)$ is said to be a pre-ordered system if $\leq$ is reflective and transitive, namely if: $a \leq a$; $a \leq b$ and $b \leq c$ implies $a \leq c$: for each $a, b, c \in x$. Each pre-ordered system can be regarded as a category in the following sense. First, its objects are elements of x. And for every two objects $a, b \in (x, \leq)$, the hom-set $(x, \leq)(a, b)$ is a singleton if $a \leq b$ and the empty set otherwise. It can easily be checked that $(x, \leq)$ certainly forms a category.

Example 2.6. We call 2^ω, the set of all functions from ω to $2 = \{0, 1\}$, Cantor set. Each function $f : u \to v$ with $u, v \subseteq \omega$ is said to be computable if there is a type-2 Turing machine which always outputs $f(p)$ with input $p \in u$. See [9] for detail on type-2 computability. We define a new category as follows:

Cp
    object : subsets of 2^ω
    morphism : computable functions

Explicitly:

$$\begin{aligned} \mathrm{ob}(\mathrm{Cp}) &= \mathrm{Pow}(2^\omega) \\ \mathrm{Cp}(u, v) &= \{f \in v^u : f \text{ is computable}\} \ ({}^\forall u, v \in \mathrm{Cp}) \end{aligned}$$

Composition of morphisms is supposed to be given by usual composition of functions.

Small Sets and Large Sets We assume that there is a set V^* with the following properties:

- ${}^\forall x \in V^*,\ {}^\forall y \in x,\ y \in V^*$;
- ${}^\forall x, y \in V^*,\ \{x, y\} \in V^*$;
- ${}^\forall x \in V^*,\ \mathrm{Pow}(x),\ \bigcup x \in V^*$;
- $\omega \in V^*$;
- for each surjective function $f : x \to y$ with $x \in V^*$ and $y \subseteq V^*$, $y \in V^*$.

We fix such a set V^* and call it universe. Each set is said to be small if it belongs to V^*, and to be large if it is a subset of V^*. A category $\mathbb{E}$ is said to be large if both $\mathrm{ob}(\mathbb{E})$ and $\mathrm{mor}(\mathbb{E})$ are large. We show some examples of large categories below.

Example 2.7. Cp is a large category.

Example 2.8. We define a new category as follows:

$\mathtt{Set}$
 object : small sets
 morphism : functions

Explicitly:

$$\begin{aligned} \mathrm{ob}(\mathtt{Set}) &= V^* \\ \mathtt{Set}(x,y) &= y^x \quad ({}^\forall x, y \in \mathtt{Set}) \end{aligned}$$

Composition of morphisms is given by usual composition of functions. Of course $\mathtt{Set}$ is a large category.

Example 2.9. Let x be a set and let $\tau \subseteq \mathrm{Pow}(x)$. We say that τ is a topology on x if the following three conditions hold:

(Oi) $\emptyset, x \in \tau$;

(Oii) ${}^\forall u, v \in \tau,\ u \cap v \in \tau$;

(Oiii) ${}^\forall \sigma \subseteq \tau,\ \bigcup \sigma \in \tau$.

If τ is a topology on x, the pair (x, τ) is called a topological space and x is called its underlying set. In that case, each $u \subseteq x$ is said to be open (resp. closed) if $u \in \tau$ (resp. $x - u \in \tau$). For instance, we define as follows.

$$\begin{aligned} 2^* &= \bigcup_{i \in \omega} 2^i \\ \big[w\big] &= \{p \in 2^\omega : w \sqsubseteq p\} \\ \tau_{2^\omega} &= \{\bigcup_{w \in W} \big[w\big] : W \subseteq 2^*\} \end{aligned}$$

Here $w \sqsubseteq p$ is an abbreviation of ${}^\forall i < |w|,\ p(i) = w(i)$ and $|w|$ is the unique $k \in \omega$ such that $w \in 2^k$. Then this τ_{2^ω} is a topology on Cantor space. It's called Cantor topology. We abbreviate as $2^\omega = (2^\omega, \tau_{2^\omega})$ and call it Cantor space.

A topological space is said to be small if it is a small set, or equivalently, if its underlying set is small. Let $x = (x, \tau_x)$ and $y = (y, \tau_y)$ be two topological spaces. Each function $f : x \to y$ is said to be continuous with respect to τ_x and τ_y, or to be a (τ_x, τ_y)-continuous function, if $f^{-1}\big[u\big] \in \tau_x$ for every $u \in \tau_y$. We define a new category as follows:

$\mathtt{Top}$
 object : small topological spaces
 morphism : continuous functions

Precisely, each morphism of `Top` is supposed to being of the form $((x, \tau_x), (y, \tau_y), f)$ where both (x, τ_x) and (y, τ_y) are topological spaces and f is a (τ_x, τ_y)-continuous function from x to y. Composition of morphisms in `Top` can be defined using composition of functions in an obvious way [3] . Of course `Top` is a large category.

Simple Examples of Universal Construction In the following, we give some simple examples of what is called universal construction, such as terminal object, product, equalizer and pullback. As a preparation, we define a kind of morphism, isomorphism, below.

Let $\mathbb{E}$ be an arbitrarily fixed category. A morphism f in $\mathbb{E}$ is said to be an isomorphism if it has a left-right inverse i.e. there exists a morphism g in $\mathbb{E}$ such that $fg = \mathrm{id}$ and $gf = \mathrm{id}$ [4] . It is easy to see that left-right inverse of a morphism is at most unique, and we denote it by f^{-1} for an isomorphism f. We denote by Iso the class of all isomorphisms in $\mathbb{E}$. Each two objects $x, y \in \mathbb{E}$ are said to be isomorphic to each other, written as $x \cong y$, if there is an isomorphism from x to y. Note that Iso is closed under composition.

As the first example, we give a definition of terminal object below.

Definition 2.10. Each $x \in \mathbb{E}$ is said to be a terminal object if for each $y \in \mathbb{E}$, there is exactly one morphism from y to x.

One can see that terminal objects are essentially unique. Namely, if both x and y are terminal objects of $\mathbb{E}$, there exists unique morphism from x to y and it is an isomorphism. We usually denote by 1 an arbitrarily fixed terminal object, of course, if it exists. For each $x \in \mathbb{E}$, the unique morphism from x to 1 will be written by $!_x$.

Example 2.11. In `Set`, each singleton $\{*\}$ is a terminal object.

Example 2.12. In `Top`, each topological space of the form $(\{*\}, \mathrm{Pow}(\{*\}))$ is a terminal object.

Example 2.13. Each $* \in 2^\omega$ is said to be computable if the constant function $c_* : 2^\omega \to 2^\omega$ defined by $p \mapsto *$ $({}^\forall p \in 2^\omega)$ is computable. In Cp, each singleton $\{*\} \subseteq 2^\omega$ with being computable of its unique element $*$ is a terminal object.

Next, we introduce product.

Definition 2.14. Let $x, y \in \mathbb{E}$. A pair of morphisms $x \xleftarrow{\pi_1} x \times y \xrightarrow{\pi_2} y$ in $\mathbb{E}$ is called a (binary) product of x and y if the following condition holds:

- for each pair of morphisms $x \xleftarrow{f_1} \cdot \xrightarrow{f_2} y$, there exists unique morphism j which makes the following diagram commute:

$$\begin{tikzcd} & \cdot \arrow[dl, "f_1"'] \arrow[d, "j"] \arrow[dr, "f_2"] & \\ x & x \times y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"] & y \end{tikzcd}$$

[3] We usually omit these detailed constructions as long as it can be guessed from the context.

[4] Of course these are abbreviations and should be written as "$fg = \mathrm{id}_{\mathrm{dom} g}$ and $gf = \mathrm{id}_{\mathrm{dom} f}$" precisely. We usually abbreviate likewise whenever there seems to be no confusion.

Such unique j is usually denoted by $\langle f_1, f_2 \rangle$.

In that case, $x \times y$ is called vertex, π_1 is called first projection and π_2 is called second projection.

For emphasis, we occasionally use the term "categorical product" instead of "product". We say that $\mathbb{E}$ has binary product if there is a product for every two objects.

If we denote as $x \times y$ without a special notice, it is meant to be the vertex of a product of x and y with implicitly equipped first and second projections π_1, π_2. Our categorical discussions may not be changed by the choice of a product $x \times y$ since the following statement hold: if $x \xleftarrow{\pi_1} z \xrightarrow{\pi_2} y$ and $x \xleftarrow{\pi_1'} z' \xrightarrow{\pi_2'} y$ are two products of x and y, $\mathrm{can} = \langle \pi_1', \pi_2' \rangle$ is an isomorphism.

$$\begin{tikzcd} & z \arrow[r, "\pi_2"] \arrow[dl, "\pi_1"'] & y \\ x & z' \arrow[l, "\pi_1'"] \arrow[u, "\mathrm{can}"] \arrow[ur, "\pi_2'"'] & \end{tikzcd}$$

The above can is occasionally called canonical isomorphism. This is so called essential uniqueness of binary product. If $\mathbb{E}$ has binary product, for every two morphisms f, g in $\mathbb{E}$, we define a new morphism $f \times g$ in the following commutative diagram.

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \\ \cdot \times \cdot \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[r, "f \times g"] & \cdot \times \cdot \arrow[u, "\pi_1'"'] \arrow[d, "\pi_2'"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

Example 2.15. $\mathtt{Set}$ has binary product. Let $x, y \in \mathtt{Set}$. The Cartesian product $x \times y$ is a categorical product of x and y in $\mathtt{Set}$. One can suppose the usual first projection function π_1 and the second projection function π_2 as implicitly equipped projections.

Example 2.16. $\mathtt{Top}$ has binary product. Let x be a set and let $\sigma \subseteq \mathrm{Pow}(x)$. We say that σ is a base on x if the following two conditions hold:

(Bi) $x = \bigcup \sigma$;

(Bii) ${}^\forall u, v \in \sigma$, ${}^\exists w \in \sigma$ s.t. $w \subseteq u \cap v$.

For each base σ on x, the following set is a topology on x.

$$\tau(\sigma) = \{\bigcup \sigma' : \sigma' \subseteq \sigma\}$$

For instance, if we define:

$$\sigma_{2^\omega} = \{\big[w\big] : w \in 2^*\}$$

then this σ_{2^ω} is a base on Cantor space, and the equality $\tau_{2^\omega} = \tau(\sigma_{2^\omega})$ can easily be checked.

Now let $x = (x, \tau_x), y = (y, \tau_y) \in \texttt{Top}$. We define:

$$[\tau_x, \tau_y] \quad = \quad \{u \times v : u \in \tau_x,\ v \in \tau_y\}$$

This $[\tau_x, \tau_y]$ is a base on the Cartesian product $x \times y$. We call $\tau_x \times \tau_y = \tau([\tau_x, \tau_y])$ product topology of τ_x and τ_y. Then $x \times y = (x \times y, \tau_x \times \tau_y)$ is a categorical product of x and y in $\texttt{Top}$. One can suppose the usual first projection function π_1 and second projection function π_2 as implicitly equipped projections. It is easy to see that both π_1 and π_2 are continuous.

Example 2.17. Cp has binary product. For each $p, q \in 2^\omega$, we define:

$$\begin{array}{rcl} \langle p, q\rangle : \omega & \to & 2 \\ 2i & \mapsto & p(i) \\ 2i+1 & \mapsto & q(i) \end{array}$$

And for each $u, v \in$ Cp, we define:

$$\big[u, v\big] \quad = \quad \{\langle p, q\rangle : p \in u,\ q \in v\}$$

This $\big[u, v\big]$ is a categorical product of u and v in Cp. We can define its projections as follows:

$$\begin{array}{rcl} \pi_1 : \big[u, v\big] & \to & u \\ \langle p, q\rangle & \mapsto & p \\ \pi_2 : \big[u, v\big] & \to & v \\ \langle p, q\rangle & \mapsto & q \end{array}$$

Both π_1 and π_2 are computable, and thus, are morphisms in Cp.

A definition of equalizer can be given as follows.

Definition 2.18. Let $x \underset{g}{\overset{f}{\rightrightarrows}} y$ be a pair of parallel morphisms in $\mathbb{E}$. A morphism $\cdot \xrightarrow{t} x$ is said to be an equalizer of f, g if the following two conditions hold:

- $ft = gt$;
- if $fh = gh$, there exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "t"] & x \\ \cdot \arrow[u, "j"] \arrow[ur, "h"'] & \end{tikzcd}$$

We say that $\mathbb{E}$ has equalizer if there is an equalizer for every pair of parallel morphisms. Similar to the case of binary product, an equalizer has essential uniqueness.

Example 2.19. `Set` has equalizer. Let $x \overset{f}{\underset{g}{\rightrightarrows}} y$ be a pair of parallel morphisms in `Set`. Define:

$$x|_{f,g} \quad = \quad \{a \in x : f(a) = g(a)\} \ (\subseteq \ x)$$

Let us denote by ι the identical embedding of $x|_{f,g}$ into x. This ι is an equalizer of f and g.

Example 2.20. `Top` has equalizer. Let $x = (x, \tau) \in$ `Top`. For each $m \subseteq x$, we define:

$$\tau|_m \quad = \quad \{u \cap m : u \in \tau\}$$

This $\tau|_m$ is a topology on m. We call $x|_m = (m, \tau|_m)$ restriction of x to m, or a subspace of x. Let $x \overset{f}{\underset{g}{\rightrightarrows}} y$ be a pair of parallel morphisms in `Top`. Let us denote by ι the identical embedding of $x|_{f,g}$ into x. This ι is continuous with respect to $\tau|_{x|_{f,g}}$ and τ, and is an equalizer of f and g in `Top`.

Example 2.21. Cp has equalizer. One can construct an equalizer of a pair of parallel morphisms in a same habit to the case of `Set`.

Finally, we give a definition of pullback.

Definition 2.22. Let $x \xrightarrow{f} z \xleftarrow{g} y$ be a pair of morphisms in $\mathbb{E}$ with shared codomain. A pair of morphisms $x \xleftarrow{g'} \cdot \xrightarrow{f'} y$ is said to be a pullback of f and g if the following two conditions hold:

- the following diagram commutes i.e. $fg' = gf'$;

$$\begin{tikzcd}
\cdot \arrow[r, "f'"] \arrow[d, "g'"'] & y \arrow[d, "g"] \\
x \arrow[r, "f"'] & z
\end{tikzcd}$$

- for any commutative diagram shown as the left one below, there exists unique morphism j which makes the right one below commute.

$$\begin{tikzcd}
\cdot \arrow[r, "f^*"] \arrow[d, "g^*"'] & y \arrow[d, "g"] \\
x \arrow[r, "f"'] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[dr, "j"] \arrow[drr, bend left, "f^*"] \arrow[ddr, bend right, "g^*"'] & & \\
& \cdot \arrow[r, "f'"] \arrow[d, "g'"'] & y \arrow[d, "g"] \\
& x \arrow[r, "f"'] & z
\end{tikzcd}$$

We say that $\mathbb{E}$ has pullback if there is a pullback for every pair of morphisms with shared codomain. Similar to the case of binary product, a pullback has essential uniqueness. We can construct a pullback in a uniform way when we already have binary product and equalizer.

Fact 2.23. If $\mathbb{E}$ has binary product and equalizer, it also has pullback.

> **Proof.** Let $x \xrightarrow{f} z \xleftarrow{g} y$ be a pair of morphisms in $\mathbb{E}$ with shared codomain. Suppose that $x \xleftarrow{\pi_1} x \times y \xrightarrow{\pi_2} y$ is a product and that t is an equalizer of $f\pi_1$ and $g\pi_2$. It is easy to see that $x \xleftarrow{\pi_1 t} \cdot \xrightarrow{\pi_2 t} y$ is a pullback of f and g. $\square$

Not only in the above case, but we can construct many kinds of "limit" from a terminal object, binary product and equalizer. We introduce an additional expression.

Definition 2.24. $\mathbb{E}$ is said to be finitely complete if it has a terminal object, binary product and equalizer.

Example 2.25. `Set`, `Top` and Cp is finitely complete.

Mono and Epi We define two kinds of morphism below. Let $\mathbb{E}$ be an arbitrarily fixed category.

Definition 2.26. Each m in $\mathbb{E}$ is said to be a monomorphism if it is left-cancellable i.e. $mf = mg$ implies $f = g$. Each e in $\mathbb{E}$ is said to be an epimorphism if it is right-cancellable i.e. $fe = ge$ implies $f = g$.

For example, each equalizer is monic.

For a given category $\mathbb{E}$, we denote by $\mathbb{E}^{\mathrm{op}}$ its dual category which is defined as follows.

$$
\begin{array}{rcl}
\mathrm{ob}(\mathbb{E}^{\mathrm{op}}) & = & \mathrm{ob}(\mathbb{E}) \\
\mathrm{mor}(\mathbb{E}^{\mathrm{op}}) & = & \mathrm{mor}(\mathbb{E}) \\
\mathbb{E}^{\mathrm{op}}(x, y) & = & \mathbb{E}(y, x) \quad ({}^{\forall} x, y \in \mathbb{E}^{\mathrm{op}})
\end{array}
$$

One may see that a morphism in $\mathbb{E}$ is monic if and only if it is epic in $\mathbb{E}^{\mathrm{op}}$. And, "dually", a morphism in $\mathbb{E}$ is epic if and only if it is monic in $\mathbb{E}^{\mathrm{op}}$. In this sense, epicity is called dual of monicity.

We denote by Mono (resp. Epi) the class of all monomorphisms (resp. epimorphisms). It is easy to see that Mono contains all isomorphisms, is closed under composition and is stable under pullback. Here we mean by "stable under pullback" a stability property stated as follows: for any pullback diagram shown below, we have $m' \in$ Mono whenever $m \in$ Mono.

$$
\begin{tikzcd}
\cdot \arrow[r, "f'"] \arrow[d, "m'"'] & \cdot \arrow[d, "m"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
$$

Dually, Epi contains all isomorphisms and is closed under composition. Furthermore, Epi has a stability property which is dual of "stable under pullback", but we don't indicate precise statement.

Example 2.27. In `Set`, monicity coincides with injectivity, and similarly, epicity coincides with surjectivity. Same statement also holds in `Top`.

Example 2.28. In Cp, monicity coincides with injectivity, but epicity does not coincide with surjectivity (cf. Example 3.47).

In any category, an isomorphism is monic and epic. However the converse direction doesn't holds in general.

Definition 2.29. A category is said to be balanced if every morphism is an isomorphism whenever it is monic and epic.

Example 2.30. `Set` is balanced, but neither of `Top` or Cp is.

We give a definition of intersection. A sink to $x \in \mathbb{E}$ is a family of morphisms $\{f_i\}_{i\in I}$ with $x = \mathrm{cod} f_i$ ($^\forall i \in I$). Suppose that $\mathcal{M} \subseteq$ Mono contains all isomorphisms and is closed under composition. A $\mathcal{M}$-sink to x is a sink $\{m_i\}_{i\in I}$ with $m_i \in \mathcal{M}$ ($^\forall i \in I$).

Definition 2.31. Let $\{m_i\}_{i\in I}$ be a $\mathcal{M}$-sink to $x \in \mathbb{E}$. A $\mathcal{M}$-intersection of $\{m_i\}_{i\in I}$ is a morphism $(\cdot \xrightarrow{m} x) \in \mathcal{M}$ with the following property:

- there exists (necessarily unique) morphism j_i such that $m = m_i j_i$ for every $i \in I$;

$$\begin{tikzcd}
& \cdot \arrow[dr, "m_i"] & \\
\cdot \arrow[ur, "j_i"] \arrow[rr, "m"'] & & x
\end{tikzcd}$$

- for a morphism $\cdot \xrightarrow{f} x$ in $\mathbb{E}$, if there exists (necessarily unique) morphism j_i' such that $f = m_i j_i'$ for every $i \in I$, then there exists (necessarily unique) morphism j such that $f = mj$.

$$\begin{tikzcd}
& \cdot \arrow[dr, "m_i"] & \\
\cdot \arrow[ur, "j_i"] \arrow[rr, "m"] & & x \\
& \cdot \arrow[uu, "j_i'"] \arrow[ul, "j"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We say that $\mathbb{E}$ has $\mathcal{M}$-intersection if there is a $\mathcal{M}$-intersection for every $\mathcal{M}$-sink.

Example 2.32. Let $\{m_i\}_{i\in I}$ be a Mono-sink to $x \in$ `Set`. If we define:

$$u = \bigcap_{i\in I} \mathrm{range}(m_i) \ (\subseteq x)$$

then the identical embedding of u into x is a Mono-intersection of $\{m_i\}_{i\in I}$.

2.3 Functor

In most of, or possibly all, theories on a kind of algebraic structure (e.g. monoid, group, vector space, ...etc), the notion of homomorphism plays an important roll. A homomorphism is often described as a correspondence which preserves the concerned structure. In category theory, such a homomorphism is called a functor.

Functor The notion of functor can be introduced as follows.

Definition 2.33. Let $\mathbb{D}$ and $\mathbb{E}$ be two categories and $F : \mathrm{mor}(\mathbb{D}) \to \mathrm{mor}(\mathbb{E})$ be a function. We say that F is a functor from $\mathbb{D}$ to $\mathbb{E}$, written as $F : \mathbb{D} \to \mathbb{E}$, if the following conditions hold:

- for every f, g in $\mathbb{D}$ at being compositionable as fg, one has $F(fg) = F(f) \circ F(g)$;
- for every $x \in \mathbb{D}$, there exists $y \in \mathbb{E}$ such that $F(\mathrm{id}_x) = \mathrm{id}_y$.

For each functor $F : \mathbb{D} \to \mathbb{E}$, usually, $F(f)$ will be abbreviated as Ff where f in $\mathbb{D}$. And we extends its correspondence by defining $Fx = \mathrm{dom}F(\mathrm{id}_x)$ where $x \in \mathbb{D}$. It is easy to see that $Fx \xrightarrow{Ff} Fy$ for each $x \xrightarrow{f} y$ in $\mathbb{D}$. We call $\mathbb{D}$ (resp. $\mathbb{E}$) domain (resp. codomain) of F.

Example 2.34. Let $x = (x, \leq)$ and $y = (y, \leq)$ be two pre-ordered system and let us regard them as two categories in the usual way (cf. Example 2.5). A function $f : x \to y$ is said to be monotonically increasing if $a \leq b$ implies $f(a) \leq f(b)$ for every $a, b \in x$. One can see that the notion of monotonically increasing function coincides with the notion of functor in this case.

We give definitions of various kinds of functor.

Definition 2.35. Let $F : \mathbb{D} \to \mathbb{E}$ be a functor. F is said to be:

- faithful if it is injective on each hom-set
  i.e. for each pair of parallel morphisms $\cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot$ in $\mathbb{D}$, $Ff = Fg$ implies $f = g$;
- full if it is surjective on each hom-set
  i.e. for each $x, y \in \mathbb{D}$ and $Fx \xrightarrow{f} Fy$ in $\mathbb{E}$, there exists $x \xrightarrow{g} y$ in $\mathbb{D}$ such that $f = Fg$;
- injective on objects if it is injective as a function $F : \mathrm{ob}(\mathbb{D}) \to \mathrm{ob}(\mathrm{E})$;
- surjective on objects if it is surjective as a function $F : \mathrm{ob}(\mathbb{D}) \to \mathrm{ob}(\mathrm{E})$;
- an embedding if it is faithful and is injective on objects.

And $\mathbb{D}$ is said to be embeddable into $\mathbb{E}$ if there is an embedding from $\mathbb{D}$ into $\mathbb{E}$.

As a related notion, we define the notion of subcategory.

Definition 2.36. Let $\mathbb{D}$ and $\mathbb{E}$ be two categories with $\mathrm{ob}(\mathbb{D}) \subseteq \mathrm{ob}(\mathrm{E})$ and with $\mathrm{mor}(\mathbb{D}) \subseteq \mathrm{mor}(\mathbb{E})$. We say that $\mathbb{D}$ is a subcategory of $\mathbb{E}$ if the identical embedding of $\mathrm{mor}(\mathbb{D})$ into $\mathrm{mor}(\mathbb{E})$ forms a functor.

We give some examples of functors.

Example 2.37. We define a new functor as follows:

$$
\begin{array}{lll}
U : \texttt{Top} \to \texttt{Set} & & \\
\quad \text{object} & : & x = (x, \tau) \mapsto x \\
\quad \text{morphism} & : & f = ((x, \tau_x), (y, \tau_y), f) \mapsto f
\end{array}
$$

This U is well-defined and is faithful.

Example 2.38. We define a new functor as follows:

$$
\begin{array}{lll}
U : \mathrm{Cp} \to \texttt{Set} & & \\
\quad \text{object} & : & u \mapsto u \\
\quad \text{morphism} & : & f \mapsto f
\end{array}
$$

This U is well-defined. So Cp is a subcategory of `Set`.

Example 2.39. We define a new functor as follows:

$$
\begin{array}{lll}
U : \mathrm{Cp} \to \texttt{Top} & & \\
\quad \text{object} & : & u \mapsto (u, \tau_{2^\omega}|_u) \\
\quad \text{morphism} & : & f \mapsto f
\end{array}
$$

This U is well-defined [5] and is an embedding. So Cp is embeddable into `Top`.

Definition 2.40. Let $F : \mathbb{D} \to \mathbb{E}$ be a functor. We say that F preserves binary product if for every $x, y \in \mathbb{D}$ with a product $x \times y$, $F(x \times y)$ is a product of Fx and Fy. We also say that, in according situations, F preserves terminal object, equalizer, pullback, ...etc.

Being careful on our proof of Fact 2.23, one may see that if $\mathbb{D}$ is finitely complete, F preserves pullback whenever it preserves binary product and equalizer. Let us say that F preserves finite limit if it preserves terminal object, binary product and equalizer.

Example 2.41. All of $U : \texttt{Top} \to \texttt{Set}$, $U : \mathrm{Cp} \to \texttt{Set}$ and $U : \mathrm{Cp} \to \texttt{Top}$ from Example 2.37, Example 2.38 and Example 2.39, respectively, preserves finite limit.

[5] This is equivalent to say that every computable functions are continuous. To see this, we have to pick up one of alternative but formal definitions of type-2 computability. See [9] or [7] for instance.

3 Fundamental Class

In this section, we introduce the notion of fundamental class. A fundamental class can be thought as defining a topology-like structure on a given category. This notion has various examples, but we postpone constructing them to the next section and focus on only few examples here. Instead, we develop general methods to analyze a given fundamental class. Our main theorem in this section is Theorem 3.70.

In section 3.1, we introduce some basic notions in need and define our first main structure, pre-effectiveness. This is a general abstraction of the category of topological spaces. In section 3.2, we define a closure operater [6] , denoted by $\mathscr{I}$, for fundamental classes. The action of that closure operator $\mathscr{I}$ abstract what is called "relativization" in computability theory. For example, the relativization of the notion of co-r.e. closedness can be obtained as oracle co-r.e. closedness (cf. Example 3.34). We describe such relativization in a pure categorical way. In section 3.3, our second main structure, effectiveness, will be introduced. And, again, we define a closure operater, denoted by $\mathscr{L}$, for fundamental classes. The action of that closure operator $\mathscr{L}$ is an abstraction of "generation of topology". For example, the limit completion of the notion of co-r.e. closedness can be obtained as usual topological closedness (cf. Example 3.53). We characterize the situation, as Proposition 3.56, that the action of $\mathscr{L}$ dominates the action of $\mathscr{I}$ for a given fundamental class. In Section 3.3.2 and Section 3.3.4, a proper assumption of Proposition 3.56 will be sharpen by a pure categorical discussion and by a set theoretical discussion, respectively. Particulary in Section 3.3.4, our main theorem in this section, Theorem 3.70, will be shown with a wide scope of application.

In what follows, we restrict our considerations to large categories. Without a special notice, the term "category" means "large category".

3.1 Bottom Fundamental Class

The most of this subsection is devoted to preparations. In section 3.1.1, we introduce some classes of morphisms. In section 3.1.2, we give a definition of factorization system and prove some useful statements. In section 3.1.1, our first main structure, pre-effectiveness, and the notion of fundamental class will be defined.

3.1.1 Classes of Morphisms

In Section 2.2, we defined three kinds of morphism, isomorphism, monomorphism and epimorphism. We introduce further notions for morphism below. In the following, let us denote by $\mathbb{E}$ an arbitrarily fixed category.

Definition 3.1. Each t in $\mathbb{E}$ is said to be:

[6] A pre-ordered system $(x, \leq)$ is called a partially ordered system if $\leq$ is anti-symmetric i.e. $a \leq b \wedge b \leq a$ implies $a = b$ for every $a, b \in x$. A function c on a partially ordered system $(x, \leq)$ is said to be a closure operator provided that it is extensive, monotone (monotonically increasing) and idempotent. Namely, c is a closure operator if and only if: $a \leq ca$; $a \leq a'$ implies $ca \leq ca'$; $cca \leq ca$: for every $a, a' \in X$. Each $a \in x$ is said to be c-closed if $a = ca$ holds.

- an element if its domain is a terminal object;
- a split monomorphism if it has a left-inverse i.e. $^{\exists}g$ in $\mathbb{E}$ s.t. $gt = \mathrm{id}$;
- a regular monomorphism if it is an equalizer of a pair of parallel morphisms;
- an extremal monomorphism if in any factorization $t = gh$ with $h \in \mathrm{Epi}$, one has $h \in \mathrm{Iso}$.

We denote by Elm (resp. SplitMono, RegMono, ExtMono) the class of all elements (resp. split monomorphisms, regular monomorphisms, extremal monomorphisms). It is almost trivial that: each element is a split monomorphism; each split monomorphism is a monomorphism; each regular monomorphism is a monomorphism.

We establish some points.

Lemma 3.2. The following statements hold:

(i) $\mathrm{Iso} \subseteq \mathrm{SplitMono}$;

(ii) $\mathrm{Iso} = \mathrm{SplitMono} \cap \mathrm{Epi}$;

(iii) $\mathrm{Elm} \subseteq \mathrm{SplitMono} \subseteq \mathrm{RegMono} \subseteq \mathrm{ExtMono}$;

(iv) $\mathrm{Iso} = \mathrm{ExtMono} \cap \mathrm{Epi}$;

(v) RegMono is stable under pullback i.e. in any pullback diagram:

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "t'"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

one has $t' \in \mathrm{RegMono}$ whenever $t \in \mathrm{RegMono}$.

Proof. (i): Trivial.
(ii): $\mathrm{Iso} \subseteq \mathrm{SplitMono} \cap \mathrm{Epi}$ follows from (i). Let $t \in \mathrm{SplitMono} \cap \mathrm{Epi}$. By the definition of split monomorphism, there is a left-inverse g of t. Then $tgt = t$ and this implies $tg = \mathrm{id}$ since $t \in \mathrm{Epi}$. Hence $t \in \mathrm{Iso}$. We conclude the desired equality.
(iii): The first inclusion is trivial. Note that if $t \in \mathrm{SplitMono}$ and g is one of its left-inverse, then t is an equalizer of id and tg. Now the second inclusion follows. Finally, let $t \in \mathrm{RegMono}$ and let $t = gh$ with $h \in \mathrm{Epi}$. By the definition of regular monomorphism, there is a pair of parallel morphisms f_1, f_2 such that t is an equalizer of them. Since h is epic, we have $f_1 g = f_2 g$. So, by universality of equalizer, we obtain unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "t"] & \cdot \\ \cdot \arrow[u, "j"] \arrow[ur, "g"'] & \end{tikzcd}$$

Hence $t = gh = tjh$ and this implies $\mathrm{id} = jh$ since each regular monomorphism is monic. Then $h \in \mathrm{SplitMono} \cap \mathrm{Epi} = \mathrm{Iso}$ by (ii). Thus $t \in \mathrm{ExtMono}$. The third inclusion holds.
(iv): $\mathrm{Iso} \subseteq \mathrm{ExtMono} \cap \mathrm{Epi}$ follows from (i) and (iii). The other inclusion is trivial.
(v): Trivial. □

We can, for instance, introduce dual notion for split monomorphism. We say that a morphism is a split epimorphism if it has a right-inverse, and denote by SplitEpi the class of all split epimorphisms. Then one has dual statement of (ii) of Lemma 3.2. Namely, $\mathrm{Iso} = \mathrm{SplitEpi} \cap \mathrm{Mono}$. Introductions of other dual notions, regular epimorphism and extremal epimorphism, will also be given when they are in need.

3.1.2 Factorization System

In what follows, we give a quick review on the theory of factorization system. As we shall see, factorization systems are quite useful and play an important roll in the rest of this section.

The Definition and Basic Properties Let $\mathbb{E}$ be an arbitrarily fixed category. For each f, g in $\mathbb{E}$, we say that f is orthogonal to g, written as $f \perp g$, if the following condition holds: for any commutative diagram shown as the left one below, there exists unique morphism j which makes the right one below commutes.

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] \arrow[ur, "j"] & \cdot \end{tikzcd}$$

The definition of factorization system is given as follows.

Definition 3.3. Let $\mathscr{S}$ and $\mathscr{T}$ be two classes of morphisms in $\mathbb{E}$. We say that $(\mathscr{S}, \mathscr{T})$ is a factorization system on $\mathbb{E}$ if the following three conditions hold:

(F0) both $\mathscr{S}$ and $\mathscr{T}$ are closed under composition with isomorphisms
i.e. for every $s \in \mathscr{S}$ and $h \in \mathrm{Iso}$, one has $sh, hs \in \mathscr{S}$, and similar with $\mathscr{T}$;

(F1) each f in $\mathbb{E}$ has $(\mathscr{S}, \mathscr{T})$-factorization
i.e. there is a pair of morphisms $s \in \mathscr{S}$ and $t \in \mathscr{T}$ such that $f = ts$;

(F2) every $s \in \mathscr{S}$ is orthogonal to every $t \in \mathscr{T}$.

A factorization system $(\mathscr{S}, \mathscr{T})$ is said to be proper if $\mathscr{S} \subseteq \mathrm{Epi}$ and $\mathscr{T} \subseteq \mathrm{Mono}$.

We check some famous statements for a factorization system. All proofs which we give below can be found in [4]. Assume that $(\mathscr{S}, \mathscr{T})$ is a factorization system on $\mathbb{E}$.

Lemma 3.4. The following statements hold:

(i) $(\mathscr{S}, \mathscr{T})$-factorization of each morphism is essentially unique
i.e. for each f in $\mathbb{E}$, if $f = ts = t's'$ are two $(\mathscr{S}, \mathscr{T})$-factorization of f, there exists unique isomorphism can which makes the following diagram commutes;

$$\begin{tikzcd}
& \cdot \arrow[r, "t'"] & \cdot \\
\cdot \arrow[ur, "s'"] \arrow[r, "s"'] & \cdot \arrow[u, "\mathrm{can}"'] \arrow[ur, "t"'] &
\end{tikzcd}$$

(ii) in any commutative diagram shown below with $s \in \mathscr{S}$ and $t \in \mathscr{T}$, one has $t \in \mathrm{Iso}$ and $f \in \mathscr{S}$;

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "j"] \arrow[r, "\mathrm{id}"'] & \cdot
\end{tikzcd}$$

(iii) $\mathscr{S} = \{f \text{ in } \mathbb{E} : {}^{\forall}t \in \mathscr{T},\ f \perp t\}$, $\mathscr{T} = \{f \text{ in } \mathbb{E} : {}^{\forall}s \in \mathscr{S},\ s \perp f\}$;

(iv) $\mathrm{Iso} = \mathscr{S} \cap \mathscr{T}$;

(v) both $\mathscr{S}$ and $\mathscr{T}$ are closed under composition.

Proof. (i): Let f in $\mathbb{E}$ and $f = ts = t's'$ be two $(\mathscr{S}, \mathscr{T})$-factorizations of f. Since $s' \perp t$ and $s \perp t'$, we have the following two commutative diagrams.

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "s'"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "\exists! j"] \arrow[r, "t'"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "s'"] \arrow[d, "s"'] & \cdot \arrow[d, "t'"] \\
\cdot \arrow[ur, "\exists! k"] \arrow[r, "t"'] & \cdot
\end{tikzcd}$$

And then we have:

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "s"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "jk"] \arrow[r, "t"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "s'"] \arrow[d, "s'"'] & \cdot \arrow[d, "t'"] \\
\cdot \arrow[ur, "kj"] \arrow[r, "t'"'] & \cdot
\end{tikzcd}$$

By the uniqueness stated in (F2), we obtain $jk = \mathrm{id}$, $kj = \mathrm{id}$. Thus $j, k \in \mathrm{Iso}$. This is the essential uniqueness of $(\mathscr{S}, \mathscr{T})$-factorization.
(ii): If the left one below commutes, then the right one below also commutes since $(jt)s = j(ts) = jf = s$ and $t(jt) = (tj)t = t$.

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "j"] \arrow[r, "\mathrm{id}"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "s"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "jt"] \arrow[r, "t"'] & \cdot
\end{tikzcd}$$

By the uniqueness stated in (F2), we obtain $jt = \mathrm{id}$ and thus $t \in \mathrm{Iso}$. $f \in \mathscr{S}$ is follows from (F0).
(iii): Let f in $\mathbb{E}$ be satisfying: for any $t \in \mathscr{T}$, $f \perp t$: and let $f = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. Since $f \perp t$, there exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "\mathrm{id}"'] \arrow[ur, "j"] & \cdot \end{tikzcd}$$

Then $f \in \mathscr{S}$ follows from (ii). Hence, the first equality holds and so $\mathscr{S}$ is characterized by $\mathscr{T}$ via orthogonality. Dually one may obtain the second equality.
(iv): Let $f \in \mathscr{S} \cap \mathscr{T}$ and $f = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. Since $f \perp t$, there exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "\mathrm{id}"'] \arrow[ur, "j"] & \cdot \end{tikzcd}$$

Then $t \in \mathrm{Iso}$ follows from (ii). Dually, one can see $s \in \mathrm{Iso}$. Therefore, $f \in \mathrm{Iso}$ and thus $\mathscr{S} \cap \mathscr{T} \subseteq \mathrm{Iso}$. The other inclusion follows from (iii).
(v): Let $s_0, s_1 \in \mathscr{S}$ at being compositionable as $s_0 s_1$. For any commutative diagram shown below with $t \in \mathscr{T}$:

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "s_0 s_1"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

we have first the left one below, and then the right one below.

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "s_1"'] & \cdot \arrow[dd, "t"] \\ \cdot \arrow[ur, "\exists! k_1"] \arrow[d, "s_0"'] & \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "s_1"'] & \cdot \arrow[dd, "t"] \\ \cdot \arrow[ur, "k_1"] \arrow[d, "s_0"'] & \\ \cdot \arrow[r, "v"'] \arrow[uur, "\exists! k_0"'] & \cdot \end{tikzcd}$$

That k_0 is the desired morphism. The expected uniqueness of k_0 can easily be obtained. Hence $\mathscr{S}$ is closed under composition. Similar for $\mathscr{T}$. □

As dual notion for extremal monomorphism, we can introduce the notion of extremal epimorphism as follows. Each f in $\mathbb{E}$ is said to be an extremal epimorphism if in any factorization $f = gh$ with $g \in \mathrm{Mono}$, one has $g \in \mathrm{Iso}$. We denote by ExtEpi the class of all extremal epimorphisms in $\mathbb{E}$.

Lemma 3.5. If $(\mathscr{S}, \mathscr{T})$ is proper, the following three statements and their duals hold:

(i) $\mathrm{ExtMono} \subseteq \mathscr{T}$:

(ii) $\mathscr{T}$ is stable under pullback;

(iii) $\mathscr{T}$ is stable under arbitrary intersection;

(iv) $fg \in \mathscr{T}$ implies $g \in \mathscr{T}$.

Proof. (i): Let $f \in \mathrm{ExtMono}$ and $f = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. If $\mathscr{S} \subseteq \mathrm{Epi}$, then by the definition of extremal epimorphism, we have $s \in \mathrm{Iso}$ and thus $f \in \mathscr{T}$.
(ii): Let the left one below be pullback with $t \in \mathscr{T}$ and let the right one below be commutative with $s \in \mathscr{S}$.

$$
\begin{tikzcd}
\cdot \arrow[r, "f'"] \arrow[d, "t'"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "s"'] & \cdot \arrow[d, "t'"] \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd}
$$

We obtain first the left one below by $s \perp t$, and then the right one below by universality of pullback.

$$
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "s"'] & \cdot \arrow[r, "f'"] \arrow[d, "t'"] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "v"'] \arrow[urr, "\exists! j"'] & \cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "s"'] & \cdot \arrow[r, "f'"] \arrow[d, "t'"] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "v"'] \arrow[ur, "\exists! k"] \arrow[urr, "j"'] & \cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
$$

That k is the desired unique morphism. This shows that $t' \in \mathscr{T}$ and thus $\mathscr{T}$ is stable under pullback.
(iii): Similar with (ii).
(iv): Suppose that $fg \in \mathscr{T}$ and $g = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. Since $s \perp fg$, there exists unique morphism which makes the following diagram commute.

$$
\begin{tikzcd}
\cdot \arrow[r, "\mathrm{id}"] \arrow[d, "s"'] & \cdot \arrow[d, "fg"] \\
\cdot \arrow[r, "ft"'] \arrow[ur, "j"] & \cdot
\end{tikzcd}
$$

By properness of $(\mathscr{S}, \mathscr{T})$, we obtain $s \in \mathrm{SplitMono} \cap \mathrm{Epi} = \mathrm{Iso}$. Hence $g \in \mathscr{T}$ follows. $\square$

Image and Inverse Image Using the notion of factorization system, we introduce image and inverse image of subobjects. Particularly, the two correspondences among subobjects given by image and inverse image, respectively, form two functors (monotonically increasing functions, see Example 2.34).

First we give a formulation of subobjects. Let $\mathbb{E}$ be an arbitrary category. Suppose that a set $\mathcal{M} \subseteq \mathrm{Mono}$ contains all isomorphisms and is closed under composition. For each $x \in \mathbb{E}$ and $(\cdot \xrightarrow{t} x), (\cdot \xrightarrow{u} x) \in \mathcal{M}$, we write $t \leq u$ (resp. $t \cong u$) if and only if there is a (necessarily unique) morphism j such that $t = uj$ (resp. $t \leq u$ and $u \leq t$). This "$\cong$" forms an equivalence relation on Mono [7] Each equivalence class $[t] = [t]_{\cong}$ is called a $\mathcal{M}$-subobject of x and we denote by $\mathcal{M}(x)$ the set of all $\mathcal{M}$-subobjects of x. We think $\mathcal{M}(x)$ is ordered by the partial order induced from $\leq$ [8] , and occasionally, regard it as a category in a usual way (cf. Example 2.5).

Now we give the definition of image and inverse image. Let $\mathbb{E}$ be a category having pullback and let $(\mathscr{S}, \mathscr{T})$ be its proper factorization system.

Definition 3.6. Let $f : x \to y$ in $\mathbb{E}$. For each $(\cdot \xrightarrow{t} x) \in \mathscr{T}$, we call $f\big[t\big]$ an image of t in a $(\mathscr{S}, \mathscr{T})$-factorization $ft = f\big[t\big] \circ s$ of ft.

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "t"'] & \cdot \arrow[d, "f[t]"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

For each $(\cdot \xrightarrow{u} y) \in \mathscr{T}$, we call $f^{-1}\big[u\big]$ an inverse image of u in a pullback diagram below:

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "f^{-1}[u]"'] & \cdot \arrow[d, "u"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

Lemma 3.7. For any $f : x \to y$ in $\mathbb{E}$, $(\cdot \xrightarrow{t} x), (\cdot \xrightarrow{t'} y) \in \mathscr{T}$, one has the following equivalence.

$$f\big[t\big] \leq t' \iff t \leq f^{-1}\big[t'\big]$$

Corollary 1. For each $f : x \to y$ in $\mathbb{E}$, we can regard: $f\big[-\big]$ as a functor from $\mathscr{T}(x)$ to $\mathscr{T}(y)$; $f^{-1}\big[-\big]$ as a functor from $\mathscr{T}(y)$ to $\mathscr{T}(x)$.

Corollary 2. Let $f : x \to y$ in $\mathbb{E}$. If $f \in \mathscr{T}$, then $t \cong f^{-1}\big[f\big[t\big]\big]$ for each $t \in \mathscr{T}$ and so $f^{-1}\big[-\big] \circ f\big[-\big]$ gives identical correspondence. Also, if $\mathscr{S}$ is stable under pullback and $f \in \mathscr{S}$, then $t \cong f\big[f^{-1}\big[t\big]\big]$ for each $t \in \mathscr{T}$ and so $f\big[-\big] \circ f^{-1}\big[-\big]$ gives identical correspondence.

---

[7] Let x be a set. Each $\equiv \subseteq x \times x$ is called an equivalence relation if it is reflective, symmetric and transitive, namely, if for every $a, b, c \in x$, one has $a \equiv a$; $a \equiv b$ implies $b \equiv a$; $a \equiv b \wedge b \equiv c$ implies $a \equiv c$. If $\equiv$ is an equivalence relation, for each $a \in x$, we define $[a]_{\equiv} = \{b \in x : a \equiv b\}$ and call it equivalence class of a.

[8] Explicitly, we define $[t] \leq [t']$ by $t \leq t'$.

Corollary 3. Each f in $\mathbb{E}$ belongs to $\mathscr{S}$ if and only if $f\big[h\big] \in \mathrm{Iso}$ for every $h \in \mathrm{Iso}$.

Note that for each $\mathscr{T}$-sink $\{t_i\}_{i \in I}$ to x, a morphism $(\cdot \xrightarrow{t} x) \in \mathscr{T}$ is an intersection of $\{t_i\}_{i \in I}$ if and only if for every $(\cdot \xrightarrow{t'} x) \in \mathscr{T}$, $t' \leq t$ coinsides with $t' \leq t_i$ $({}^{\forall} i \in I)$. Hence by Lemma 3.7, the following lemma also holds.

Lemma 3.8. Let $x \xrightarrow{f} y$ in $\mathbb{E}$ and let $\{t_i\}_{i \in I}$ be a $\mathscr{T}$-sink to y. The following holds if there is an intersection $\bigwedge_{i \in I} t_i$ of $\{t_i\}_{i \in I}$.

$$\bigwedge_{i \in I} f^{-1}\big[t_i\big] \;\; \cong \;\; f^{-1}\big[\bigwedge_{i \in I} t_i\big]$$

3.1.3 Pre-Effectiveness

We define the notion of pre-effectiveness below what is our first main structure.

Definition 3.9. Let $\mathbb{E}$ be a finitely complete category and let $(\mathscr{S}, \mathscr{T})$ be a proper factorization system on $\mathbb{E}$. We say that $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a pre-effectiveness if the following condition holds:

(E0) $\mathscr{S}$ is stable under pullback.

Example 3.10. $\mathtt{Set} = (\mathtt{Set}, \mathrm{Epi}, \mathrm{Mono})$ is a pre-effectiveness. More generally, for a topos $\mathbb{E}$, one can prove that $\mathbb{E} = (\mathbb{E}, \mathrm{Epi}, \mathrm{Mono})$ is a pre-effectiveness. This is a consequence of topos theory. See [5] for detail.

Example 3.11. Let us denote by $\mathrm{RefEpi}_{\mathtt{Top}}$ the class of all surjective morphisms in $\mathtt{Top}$ and by $\mathrm{RefMono}_{\mathtt{Top}}$ the class of all injective morphisms in $\mathtt{Top}$. We also define a subclass $\mathrm{Emb}_{\mathtt{Top}}$ of $\mathrm{RefMono}_{\mathtt{Top}}$ as follows.

$t \in \mathrm{Emb}_{\mathtt{Top}} \iff$ in any commutative diagram in $\mathtt{Set}$ shown below with f in $\mathtt{Top}$

$$\begin{tikzcd} x \arrow[r, "t"] & y \\ x' \arrow[u, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

g must be a morphism in $\mathtt{Top}$

Then $\mathtt{Top} = (\mathtt{Top}, \mathrm{RefEpi}_{\mathtt{Top}}, \mathrm{Emb}_{\mathtt{Top}})$ is a pre-effectiveness.

Example 3.12. In a similar habit with the case of $\mathtt{Top}$, let us define $\mathrm{RefEpi}_{\mathrm{Cp}}$, $\mathrm{RefMono}_{\mathrm{Cp}}$ and $\mathrm{Emb}_{\mathrm{Cp}}$. Then $\mathrm{Cp} = (\mathrm{Cp}, \mathrm{RefEpi}_{\mathrm{Cp}}, \mathrm{Emb}_{\mathrm{Cp}})$ is, again, a pre-effectiveness. Actually, we can generalize these constructions. See Proposition 4.11.

Assume that $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a pre-effectiveness. A subclass $\mathscr{F} \subseteq \mathscr{T}$ is said to be a fundamental class on $\mathbb{E}$ if $\mathscr{F}$ contains all isomorphisms, is closed under composition and is stable under pullback.

Example 3.13. We define a subclass $\mathrm{ClsEmb}_{\mathtt{Top}}$ of $\mathrm{Emb}_{\mathtt{Top}}$ as follows:

$$t \in \mathrm{ClsEmb}_{\mathtt{Top}} \iff \mathrm{range}(t) \text{ is closed in } x$$

where $(\cdot \xrightarrow{t} x) \in \mathrm{Emb}_{\mathtt{Top}}$. This $\mathrm{ClsEmb}_{\mathtt{Top}}$ is a fundamental class on $\mathtt{Top}$.

Example 3.14. Similar with the case of $\mathtt{Top}$, we define a subclass $\mathrm{ClsEmb}_{\mathrm{Cp}}$ of $\mathrm{Emb}_{\mathrm{Cp}}$ as follows:

$$t \in \mathrm{ClsEmb}_{\mathrm{Cp}} \iff \mathrm{range}(t) \text{ is closed with respect to } \tau_{2^\omega}|_u$$

where $(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}}$. This $\mathrm{ClsEmb}_{\mathrm{Cp}}$ is well-defined as a fundamental class on Cp.

Let us denote by $\left[\mathscr{T}\right]$ the class of all fundamental classes on $\mathbb{E}$ [9] . This $\left[\mathscr{T}\right]$ is partially ordered by inclusion $\subseteq$ and is closed under $\bigcap$. For a given subset $\mathfrak{T} \subseteq \mathscr{T}$, we denote by $\mathscr{B}\mathfrak{T}$ the smallest fundamental class containing $\mathfrak{T}$. Explicitly, one may define:

$$\mathscr{B}\mathfrak{T} \;=\; \bigcap\{\mathscr{F} \in \left[\mathscr{T}\right] : \mathfrak{T} \subseteq \mathscr{F}\}$$

or equivalently:

$$\begin{aligned} \mathfrak{T}' &= \mathfrak{T} \cup \mathrm{Iso} \\ \overline{\mathfrak{T}'} &= \{t \in \mathscr{T} : {}^\exists f \text{ in } \mathbb{E}, {}^\exists t' \in \mathfrak{T}' \text{ s.t. } t \cong f^{-1}\left[t'\right]\} \\ \mathscr{B}\mathfrak{T} &= \{t_0 \cdots t_k : k \in \mathbb{N},\ t_0, \cdots, t_k \in \overline{\mathfrak{T}'}\} \end{aligned}$$

Particulary, we abbreviate as $\mathscr{B}_0 = \mathscr{B}\mathrm{RegMono}$ and call this $\mathscr{B}_0$ bottom fundamental class on $\mathbb{E}$. This is well-defined since $\mathrm{RegMono} \subseteq \mathrm{ExtMono} \subseteq \mathscr{T}$ by (iii) of Lemma 3.2 and (i) of Lemma 3.5. Recall the fact that RegMono contains all isomorphisms and is stable under pullback (cf. Lemma 3.2). Hence $\mathrm{RegMono} = \overline{\mathrm{RegMono}'}$ in the above notation. So each $t \in \mathscr{T}$ belongs to $\mathscr{B}_0$ if and only if it can be represented as the composition of some regular monomorphisms i.e. there exists $t_0, \cdots, t_k \in \mathrm{RegMono}$ such that $t = t_0 \cdots t_k$.

Fact 3.15. For a fundamental class $\mathscr{F}$ on $\mathbb{E}$, we define:

$$f \in \mathscr{F}\text{-Cls} \iff {}^\forall t \in \mathscr{F}, f\left[t\right] \in \mathscr{F}$$

where f in $\mathbb{E}$. The following statements hold:

(i) $\mathscr{F} = \mathscr{F}\text{-Cls} \cap \mathscr{T}$;

(ii) for any $fg \in \mathscr{F}\text{-Cls}$ with $g \in \mathscr{S}$, we have $f \in \mathscr{F}\text{-Cls}$:

Fact 3.15 shows a relationship between our terminologies and that of [6]. Particularly, it shows that $\mathscr{F}$-Cls is a "$(\mathscr{S}, \mathscr{T})$-closed" class of morphisms. By this fact, we can borrow many useful results and terminologies from [6].

Example 3.16. $\mathscr{B}_{0,\mathtt{Set}} = \mathrm{Mono}_{\mathtt{Set}}$.

[9] As we promised at the beginning of this section, our observations are restricted to large categories. So each fundamental class on $\mathbb{E}$ is actually a set, and so is $\left[\mathscr{T}\right]$.

Example 3.17. $\mathscr{B}_{0,\mathtt{Top}} = \mathrm{Emb}_{\mathtt{Top}}$.

Example 3.18. For any $v \subseteq u \subseteq 2^\omega$, v is said to be co-r.e. closed in u provided that: there is a type-2 Turing machine which halts with input $p \in u$ if and only if $p \notin v$: or equivalently: there is a computable function $\chi : u \to 2^\omega$ in Cp with the following property.

$$v = \chi^{-1}\big[\{0^\omega\}\big] = \{p \in u : \chi(p) = 0^\omega\}$$

Here $0^\omega \in 2^\omega$ is defined by $i \mapsto 0$ $({}^\forall i \in \omega)$. This 0^ω is, of course, computable. In Cp, regarded as a pre-effectiveness, one can see that:

$$\begin{aligned} \mathscr{B}_{0,\mathrm{Cp}} &= \{(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}} : {}^\exists\chi : u \to 2^\omega \text{ in Cp s.t. } t \cong \chi^{-1}\big[\top\big]\} \\ &= \{(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}} : \mathrm{range}(t) \text{ is co-r.e. closed in } u\} \end{aligned}$$

where $\top : \{\mathtt{0}^\omega\} \to \Sigma^\omega$ defined by $\mathtt{0}^\omega \mapsto \mathtt{0}^\omega$.

3.2 Imaginary Fundamental Class

In this subsection, we develop a method for extending fundamental classes. Particularly, a closure operator for fundamental classes will be constructed. We give our first observation below.

Suppose that $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a pre-effectiveness and $\mathscr{F}$ is a fundamental class on $\mathbb{E}$. Recall Corollary 2 of Lemma 3.7. Let α and x be two objects of $\mathbb{E}$. If $(\alpha \times x \xrightarrow{\pi_2} x) \in \mathscr{S}$, then $\pi_2\big[-\big] \circ \pi_2^{-1}\big[-\big]$ gives identical correspondence and thus $\pi^{-1}\big[-\big]$ is injective. Since a fundamental class is stable under pullback, one can say that $\pi_2^{-1}\big[-\big]$ embeds $\mathscr{F}(x)$ into $\mathscr{F}(\alpha \times x)$. So it could be expected that $\alpha \times x$ has more $\mathscr{F}$-subobjects than x.

Based on the above idea, we introduce some notions in Section 3.2.1. In Section 3.2.2, our closure operator is formulated and its several desirable properties will be observed. Section 3.2.3 is devoted to an additional consideration.

3.2.1 Extending Fundamental classes

Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness.

Definition 3.19. Each $\alpha \in \mathbb{E}$ is said to be non-empty if $(\alpha \xrightarrow{!_\alpha} 1) \in \mathscr{S}$.

Of course, a terminal object is non-empty. If $\alpha \in \mathbb{E}$ is non-empty, then for any $x \in \mathbb{E}$, $(\alpha \times x \xrightarrow{\pi_2} x) \in \mathscr{S}$ always holds since the following diagram is pullback and since $\mathscr{S}$ is stable under pullback.

$$\begin{tikzcd} \alpha \times x \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & x \arrow[d, "!_x"] \\ \alpha \arrow[r, "!_\alpha"'] & 1 \end{tikzcd}$$

So each second projection $\alpha \times x \xrightarrow{\pi_2} x$ satisfies the assumption of our first observation which we gave at the begging of this subsection. We denote by $\mathrm{Ne}(\mathbb{E})$ the class of all non-empty objects of $\mathbb{E}$. Then $\mathrm{Ne}(\mathbb{E})$ contains terminal objects and is closed under binary product since $!_{\alpha\times\beta} = (\alpha \times \beta \xrightarrow{\pi_2} \beta \xrightarrow{!_\beta} 1) \in \mathscr{S}$ for each $\alpha, \beta \in \mathrm{Ne}(\mathbb{E})$. Occasionally, this $\mathrm{Ne}(\mathbb{E})$ will be regarded as a pre-ordered system defining $\alpha \leq \beta$ by $\mathbb{E}(\alpha, \beta) \neq \emptyset$ for each $\alpha, \beta \in \mathrm{Ne}(\mathbb{E})$.

Example 3.20. Let us consider on Cp as a pre-effectiveness. One may see that $\mathrm{ob}(\mathrm{Cp}) - \mathrm{Ne}(\mathrm{Cp}) = \{\emptyset\}$. So each object of Cp is non-empty if and only if it is non-empty in the usual sense. Same statement also holds in `Set` and `Top`. Actually, we can generalizes this observation as Proposition 3.42.

We define a new function as follows:

$$\begin{aligned} \mathscr{I} : \mathrm{Ne}(\mathbb{E})^{\mathrm{op}} \times \big[\mathscr{T}\big] &\to \big[\mathscr{T}\big] \\ (\alpha, \mathscr{F}) &\mapsto \{t \in \mathscr{T} : \alpha \times t \in \mathscr{F}\} \end{aligned}$$

where $\alpha \times t = \mathrm{id}_\alpha \times t$. Note that $\alpha \times t \cong \pi_2^{-1}\big[t\big]$ if we denote by π_2 the second projection from $\alpha \times \mathrm{cod}t$ to $\mathrm{cod}t$. As an abbreviation, we write $\mathscr{I}_\alpha\mathscr{F}$ instead of $\mathscr{I}(\alpha, \mathscr{F})$. Some points can easily be checked.

Lemma 3.21. Suppose that we are given a concatenation of two squares as shown below.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

If both the whole square and the lower square are pullback, so is the upper square.

Lemma 3.22. The following statements hold:

(i) $\mathscr{I}$ is well-defined as a function. Particularly, it is extensive;

(ii) $\mathscr{I}$ is monotone (monotonically increasing);

(iii) $\mathscr{I}_1$ behaves identically (i.e. $\mathscr{I}_1\mathscr{F} = \mathscr{F}$) for each fundamental class $\mathscr{F}$.

Proof. (i): Trivial.
(ii): Let $\alpha, \beta \in \mathrm{Ne}(\mathbb{E})$ with $\alpha \leq \beta$ i.e. $\mathbb{E}(\alpha, \beta) \neq \emptyset$ and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. We show that $\mathscr{I}_\beta\mathscr{F} \leq \mathscr{I}_\alpha\mathscr{F}$. Suppose that $t \in \mathscr{I}_\beta\mathscr{F}$ and $\alpha \xrightarrow{f} \beta$. We have the following commutative diagram.

$$\begin{tikzcd} \alpha \times \cdot \arrow[rr] \arrow[dr, "f\times\mathrm{id}"] \arrow[dd, "\alpha\times t"'] & & \cdot \arrow[dd, "t"] \\ & \beta \times \cdot \arrow[ur] \arrow[dd, "\beta\times t"] & \\ \alpha \times \cdot \arrow[rr] \arrow[dr, "f\times\mathrm{id}"'] & & \cdot \\ & \beta \times \cdot \arrow[ur] & \end{tikzcd}$$

All unnamed morphisms are second projections. Hence by Lemma 3.21, we obtain $\alpha \times t \in \mathscr{F}$ from $\beta \times t \in \mathscr{F}$. This implies $t \in \mathscr{I}_\alpha \mathscr{F}$.
(iii): Trivial. $\square$

Definition 3.23. Each non-empty class $R \subseteq \mathrm{Ne}(\mathbb{E})$ is said to be a rule if for every two elements of R, any product of them belongs to R again.

For instance, $\mathrm{Ne}(\mathbb{E})$ itself is a rule. We define a technical notion below.

Definition 3.24. Let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. Each $x \in \mathbb{E}$ is said to be $\mathscr{F}$-compact if $(x \times y \xrightarrow{\pi_2} y) \in \mathscr{F}$-Cls for each $y \in \mathbb{E}$ (cf. Fact 3.15).

Example 3.25. Let $x = (x, \tau)$ be a topological space. Each $\sigma \subseteq \tau$ is called a cover if $x = \bigcup \sigma$. A subcover of a cover σ is a subset $\sigma' \subseteq \sigma$ which is a cover again. Then x is said to be Heine-Borel compact if for every cover σ, there exists a finite subcover, a subcover at being a finite set, of it. It is a well-known fact that Heine-Borel compactness coincides with $\mathrm{ClsEmb}_{\mathsf{Top}}$-compactness. See [6] for detail.

For each rule R, we define $\mathscr{I}_R : \mathscr{F} \mapsto \bigcup_{\alpha \in R} \mathscr{I}_\alpha \mathscr{F}$ $\;({}^\forall \mathscr{F} \in [\mathscr{T}])$. As an abbreviation, we usually write $\mathscr{I}_R \mathfrak{T}$ instead of $\mathscr{I}_R \mathscr{B} \mathfrak{T}$ for each $\mathfrak{T} \subseteq \mathscr{T}$.

Proposition 3.26. Let R be a rule. The following statements hold:

(i) $\mathscr{I}_R$ is a closure operator on $[\mathscr{T}]$;

(ii) Each fundamental class $\mathscr{F}$ on $\mathbb{E}$ is $\mathscr{I}_R$-closed (i.e. $\mathscr{F} = \mathscr{I}_R \mathscr{F}$) if every $\alpha \in R$ is $\mathscr{F}$-compact.

Proof. (i): First we show that $\mathscr{I}_R$ is well-defined as a function from $[\mathscr{T}]$ to $[\mathscr{T}]$. Let $\mathscr{F} \in [\mathscr{T}]$.

$\mathrm{Iso} \subseteq \mathscr{I}_R \mathscr{F}$:
One has $\mathrm{Iso} \subseteq \mathscr{F} \subseteq \mathscr{I}_{\alpha_0} \mathscr{F} \subseteq \bigcup_{\alpha \in R} \mathscr{I}_\alpha \mathscr{F} = \mathscr{I}_R \mathscr{F}$ by (i) of Lemma 3.22 where α_0 is an element of R.

$\mathscr{I}_R \mathscr{F}$ is closed under composition:
Let $t, t' \in \mathscr{I}_R \mathscr{F}$ at being compositionable as tt'. There exists $\alpha, \beta \in R$ s.t. $t \in \mathscr{I}_\alpha \mathscr{F}$ and $t' \in \mathscr{I}_\beta \mathscr{F}$, respectively. Since R is closed under binary products, we obtain $t \in \mathscr{I}_\alpha \mathscr{F} \subseteq \mathscr{I}_{\alpha \times \beta} \mathscr{F}$ by (ii) of Lemma 3.22, and similarly $t' \in \mathscr{I}_\beta \mathscr{F} \subseteq \mathscr{I}_{\alpha \times \beta} \mathscr{F}$. Hence $tt' \in \mathscr{I}_{\alpha \times \beta} \mathscr{F} \subseteq \mathscr{I}_R \mathscr{F}$.

$\mathscr{I}_R \mathscr{F}$ is stable under pullback:
Trivial.

Next we show that $\mathscr{I}_R \mathscr{F}$ is a closure operator. But extensivity and monotonicity is trivial since each of $\mathscr{I}_\alpha \mathscr{F}$ is extensive and monotone where $\alpha \in R$. To see its idempotency, let $t \in \mathscr{I}_R \mathscr{I}_R \mathscr{F}$. There exists $\alpha, \beta \in R$ such that $(\alpha \times \beta) \times t \cong \alpha \times (\beta \times t) \in \mathscr{F}$. Hence $t \in \mathscr{I}_{\alpha \times \beta} \mathscr{F} \subseteq \mathscr{I}_R \mathscr{F}$.

(ii): Let $\alpha \in R$ and $(\cdot \xrightarrow{t} x) \in \mathscr{I}_\alpha \mathscr{F}$. We denote by π_2 the second projection from $\alpha \times x$ to x. Since $\pi_2^{-1}\big[t\big] \cong \alpha \times t \in \mathscr{F}$ and $\pi_2 \in \mathscr{S}$, we obtain:

$$t \cong \pi_2\big[\pi_2^{-1}\big[t\big]\big] \cong \pi_2\big[\alpha \times t\big] \in \mathscr{F}$$

by the definition of $\mathscr{F}$-compactness and by $\alpha \times t \in \mathscr{F}$. Therefore $\mathscr{F} = \mathscr{I}_R \mathscr{F}$. $\square$

3.2.2 Imaginary Operator

We introduce the notion of imaginary and define a closure operator, denoted by $\mathscr{I}$, for fundamental classes. One may see that the action of $\mathscr{I}$ abstract what is called "relativization" in computability theory.

Definition 3.27. Let $\mathbb{E}$ be an arbitrary category. Each $\alpha \in \mathbb{E}$ is said to be a habobject if $\mathbb{E}(x, \alpha)$ has at most one element for any $x \in \mathbb{E}$.

Of course a terminal object is a habobject. For a habobject $\alpha \in \mathbb{E}$ and an arbitrary $x \in \mathbb{E}$, we denote by $x \xrightarrow{!} \alpha$ the unique morphism from x to α if it exists. If $\mathbb{E}$ has a terminal object 1, each $\alpha \in \mathbb{E}$ is a habobject if and only if $\alpha \xrightarrow{!_\alpha} 1$ is monic. Assume that $\mathbb{E}$ is finitely complete. In that case, for a habobject $\alpha \in \mathbb{E}$ and an arbitrary $x \in \mathbb{E}$, $(\alpha \times x \xrightarrow{\pi_2} x) \in \mathrm{Mono}$ always holds since the following diagram is pullback and since Mono is stable under pullback.

$$\begin{tikzcd}
\alpha \times x \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & x \arrow[d, "!_x"] \\
\alpha \arrow[r, "!_\alpha"'] & 1
\end{tikzcd}$$

So the class of all habobjects is closed under binary product since $!_{\alpha\times\beta} = (\alpha \times \beta \xrightarrow{\pi_2} \beta \xrightarrow{!_\beta} 1) \in \mathrm{Mono}$ for each two habobjects α, β. These observations are analogous to the case of non-empty objects.

Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness.

Definition 3.28. Each $\alpha \in \mathbb{E}$ is said to be an imaginary if it is a non-empty habobject.

We denote by $\mathfrak{Im}$ the class of all imaginaries of $\mathbb{E}$. Of course $\mathfrak{Im}$ is a rule. As an abbreviation, Deg will be used to denote the class of all equivalence classes of imaginaries by $\cong$. This Deg is called degree structure of imaginaries. Occasionally, we regard Deg as a partially ordered system where $[\alpha] \le [\beta]$ is defined by "$\beta \xrightarrow{!} \alpha$ exists".

Example 3.29. In $\mathtt{Set}$, the degree structure of imaginaries $\mathrm{Deg}_{\mathtt{Set}}$ has exactly one object, so equivalence class of terminal objects. Same with $\mathtt{Top}$.

Example 3.30. Let us consider on Cp. One may see that $\mathfrak{Im}_{\mathrm{Cp}}$ is the class of all singleton objects of Cp (see also Proposition 3.42). So each imaginary $\{*\} \in \mathrm{Cp}$ can be thought as corresponding to an "oracle" $* \in 2^\omega$. Furthermore, $\mathrm{Deg}_{\mathrm{Cp}}$ is exactly same structure with what is called Turing degree structure.

Lemma 3.31. Let $\alpha \in \mathbb{E}$ be a habobject. For each $x \in \mathbb{E}$, the second projection $\pi_2 : \alpha \times x \to x$ is isomorphic if and only if $x \xrightarrow{!} \alpha$ exists.

> **Proof.** "only if" is trivial. Assume that $x \xrightarrow{!} \alpha$ exists. Then $\pi_2 \circ \langle !, \mathrm{id} \rangle = \mathrm{id}$ and thus $\pi_2 \in \mathrm{SplitEpi} \cap \mathrm{Mono} = \mathrm{Iso}$. This is the assertion of "if". $\square$

Lemma 3.32. Let $\alpha \in \mathbb{E}$ be an imaginary. The following statements hold:

(i) $\pi_2^{-1}\big[-\big] : \mathscr{T}(x) \to \mathscr{T}(\alpha \times x)$ gives a bijective correspondence and $\pi_2\big[-\big]$ is its inverse for each $x \in \mathbb{E}$;

(ii) α is $\mathscr{I}_\alpha\mathscr{F}$-compact for each fundamental class $\mathscr{F}$ on $\mathbb{E}$.

> **Proof.** (i): Since α is an imaginary, $\pi_2 \in \mathscr{S}$ and thus $\pi_2^{-1}\big[-\big]$ gives an injective correspondence on $\mathscr{T}$-subobjects. So now we have to show its surjectivity. Let $(\cdot \xrightarrow{t'} \alpha \times x) \in \mathscr{T}$ and $\pi_2 t' = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. By the universality of pullback, there exists unique morphism j which makes the following diagram commute.
>
> $$\begin{tikzcd} \cdot \arrow[rr, "\mathrm{id}"] \arrow[dd, "t'"'] \arrow[dr, "j"] & & \cdot \arrow[dd, "\pi_2 t'"' near start] \arrow[dr, "s"] & \\ & \cdot \arrow[rr, "\pi_2'"'] \arrow[dl, "\pi_2^{-1}[t]"] & & \cdot \arrow[dl, "t"] \\ \alpha \times x \arrow[rr, "\pi_2"'] & & x & \end{tikzcd}$$
>
> Note that $j \in \mathscr{T}$ by (iv) of Lemma 3.5. Since π_2 is monic, $\cdot \xleftarrow{t'} \cdot \xrightarrow{\mathrm{id}} \cdot$ is a pullback of $\alpha \times e \xrightarrow{\pi_2} e \xleftarrow{\pi_2 t'} \cdot$. Hence by Lemma 3.21, the following square is also pullback.
>
> $$\begin{tikzcd} \cdot \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & \cdot \arrow[d, "s"] \\ \alpha \times \cdot \arrow[r, "\pi_2'"'] & \cdot \end{tikzcd}$$
>
> So we obtain $j \in \mathscr{S} \cap \mathscr{T} = \mathrm{Iso}$ since $\mathscr{S}$ is stable under pullback. Therefore, $t' \cong \pi_2^{-1}\big[t\big] \cong \pi_2^{-1}\big[\pi_2\big[t'\big]\big]$. We proved that $\pi_2^{-1}\big[-\big]$ is bijective, and particularly, $\pi_2\big[-\big]$ is its inverse.
> (ii): Note that if we denote by π_2' the second projection from $\alpha \times (\alpha \times x)$ to $\alpha \times x$, this π_2' is isomorphic by Lemma 3.31. Suppose that $(\cdot \xrightarrow{t} \alpha \times x) \in \mathscr{T}$. Then one observes $\alpha \times t \in \mathscr{F}$ if and only if $t \in \mathscr{F}$, and thus, $t \in \mathscr{I}_\alpha\mathscr{F}$ if and only if $t \in \mathscr{F}$. Note that $t \cong \pi_2^{-1}\big[\pi_2\big[t\big]\big] \cong \alpha \times \pi_2\big[t\big]$ by (i) where $\alpha \times x \xrightarrow{\pi_2} x$. So we obtain $\pi_2\big[t\big] \in \mathscr{I}_\alpha\mathscr{F}$ whenever $t \in \mathscr{I}_\alpha\mathscr{F}$. This shows that α is $\mathscr{I}_\alpha\mathscr{F}$-compact. $\square$

As an abbreviation, we write $\mathscr{I}$ instead of $\mathscr{I}_{\mathfrak{Im}}$.

Proposition 3.33. The following statements hold:

(i) $\mathscr{I}$ is a closure operator on $\left[\mathscr{T}\right]$;

(ii) a fundamental class $\mathscr{F}$ on $\mathbb{E}$ is $\mathscr{I}$-closed (i.e. $\mathscr{F} = \mathscr{I}\mathscr{F}$) if and only if all imaginaries are $\mathscr{F}$-compact.

> **Proof.** (i): Follows from (i) of Proposition 3.26.
> (ii): "if" has been already shown as (ii) of Proposition 3.26. Suppose that $\mathscr{F}$ is $\mathscr{I}$-closed and let α be an imaginary. Then $\mathscr{F} = \mathscr{I}_\alpha\mathscr{F}$ and α is $\mathscr{F}$-compact by (ii) of Lemma 3.32. This is the assertion of "only if ". $\square$

We usually write $\mathscr{I}_0$ instead of $\mathscr{I}\mathscr{B}_0$ and call this $\mathscr{I}_0$ imaginary fundamental class on $\mathbb{E}$.

Example 3.34. Let us consider on Cp. Let $\alpha \in 2^\omega$. For any $v \subseteq u \subseteq 2^\omega$, v is said to be α-co-r.e. closed in u provided that: there exists a type-2 Turing machine (having two input tapes) which halts with input α and $p \in u$ if and only if $p \notin v$. One can see that:

$$t \in \mathscr{I}_{\{\alpha\}}\mathscr{B}_{0,\mathrm{Cp}} \iff \text{range}(t) \text{ is } \alpha\text{-co-r.e. closed in } u$$

where $(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}}$. Also v is said to be oracle co-r.e. closed in u if there is an $\alpha \in 2^\omega$ such that v is α-co-r.e. closed in u. Oracle co-r.e. closedness is said to be "relativization" of co-r.e. closedness. One can see that:

$$t \in \mathscr{I}_{0,\mathrm{Cp}} \iff \text{range}(t) \text{ is oracle co-r.e. closed in } u$$

where $(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}}$. Now one can say that the action of $\mathscr{I}$ abstract the notion of relativization.

3.2.3 The Strongest Extension Ability

In the following, we discuss about extension ability of rules. Particularly, one may see that the class of all imaginaries has the strongest extension ability under a certain condition.

Definition 3.35. Let $\mathbb{E}$ be an arbitrary category. A class $D \subseteq \mathrm{ob}(\mathbb{E})$ is called a generating class of $\mathbb{E}$ if for each pair of parallel morphisms $x \underset{f_1}{\overset{f_0}{\rightrightarrows}} y$ in $\mathbb{E}$ at being distinct (i.e. $f_0 \neq f_1$), there exists $\alpha \xrightarrow{k} x$ such that $\alpha \in D$ and $f_0 k \neq f_1 k$.

Definition 3.36. A pre-effectiveness $\mathbb{E}$ is said to be:

- imaginary extensional if $\mathfrak{Im}$, the class of all imaginaries, is a generating class of $\mathbb{E}$;

- extensional if $\{1\}$ is a generating class of $\mathbb{E}$.

Definition 3.37. A pre-effectiveness $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is said to be strictly (imaginary) extensional if $\mathbb{E}$ is (imaginary) extensional and the following condition holds:

- if $fg \in \mathrm{Mono}$, then $f \in \mathrm{Mono}$ whenever $g \in \mathscr{S}$.

Note that a stability condition for Mono appears in the above definition is trivially holds when our category $\mathbb{E}$ is balanced (cf. Definition 2.29).

Example 3.38. `Set` is strictly extensional.

Example 3.39. Both of `Top` and Cp is strictly imaginary extensional (see also (iii) of Lemma 4.12).

Let $(\cdot \xrightarrow{t} x) \in \mathscr{T}$ where $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a pre-effectiveness. We call t an imaginary element of x if its domain is an imaginary.

Lemma 3.40. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a strictly imaginary extensional pre-effectiveness. For every $(x \xrightarrow{s} y) \in \mathscr{S}$, if x is an imaginary, so is y.

Proof. The following diagram commutes.

$$\begin{tikzcd} y \arrow[r, "!_y"] & 1 \\ x \arrow[u, "s"] \arrow[ur, "!_x"'] & \end{tikzcd}$$

Suppose that x is an imaginary. Then $!_x \in \mathscr{S} \cap \mathrm{Mono}$. Hence $!_y \in \mathscr{S} \cap \mathrm{Mono}$ holds by dual of (iv) of Lemma 3.5 and by the definition of strict imaginary extensionality. This shows that y is an imaginary. $\square$

Lemma 3.41. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a strictly imaginary extensional pre-effectiveness. Assume that we are given a pair of parallel morphisms $\alpha \underset{g}{\overset{f}{\rightrightarrows}} x$ with being an imaginary of the shared domain α, and also $(\mathscr{S}, \mathscr{T})$-factorizations as $f = ts$ and $g = t's'$. Then one has $f = g$ if and only if $t \cong t'$.

Proof. Firstly, "only if" follows immediately from essential uniqueness of $(\mathscr{S}, \mathscr{T})$-factorization. Suppose that $t \cong t'$. There exists unique morphism can which makes the following diagram commute and is an isomorphism.

$$\begin{tikzcd} \cdot \arrow[rr, "\mathrm{can}"] \arrow[dr, "t'"'] & & \cdot \arrow[dl, "t"] \\ & x & \end{tikzcd}$$

Note that both dom$t$ and domt' are imaginaries by Lemma 3.40. Hence whole of the following diagram commutes.

$$\begin{tikzcd} & \cdot \arrow[r, "t"] & x \\ \alpha \arrow[ur, "s"] \arrow[r, "s'"'] & \cdot \arrow[u, "\mathrm{can}"] \arrow[ur, "t'"'] & \end{tikzcd}$$

This implies $f = ts = t(\mathrm{can} \circ \mathrm{can}^{-1})s = t's' = g$. ☐

Proposition 3.42. In a strictly imaginary extensional pre-effectiveness, each object is non-empty if and only if it has an imaginary element. Furthermore, each non-empty object is an imaginary if and only if it has essentially unique imaginary element.

Proof. Let $\mathbb{E}$ be a strictly imaginary extensional pre-effectiveness.
For the first assertion: Let $\alpha \in \mathbb{E}$ be a non-empty object. If α is not an imaginary, of course it is not a habobject, and hence there is a pair of parallel morphisms $x \underset{g}{\overset{f}{\rightrightarrows}} \alpha$ with being distinct i.e. $f \neq g$. Since the class of all imaginaries forms a generating class of $\mathbb{E}$, there exists $\beta \xrightarrow{k} x$ with being imaginary of its domain β such that $fk \neq gk$. Let $fk = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. By Lemma 3.40, dom$t$ is an imaginary and thus t is an imaginary element of α. This shows "only if" of the first assertion. Next, suppose that α has an imaginary element $\beta \xrightarrow{t} \alpha$. The following diagram commutes.

$$\begin{tikzcd} \alpha \arrow[r, "!_\alpha"] & 1 \\ \beta \arrow[u, "t"] \arrow[ur, "!_\beta"'] & \end{tikzcd}$$

Hence by dual of (iv) of Lemma 3.5, we obtain $!_\alpha \in \mathscr{S}$ and this implies that α is non-empty. This is "if" of the first assertion.
For the second assertion: Let $\alpha \in \mathbb{E}$ be non-empty. If α is not an imaginary, we can set up f, g and k just as same with the above discussion. Now let $fk = ts$ and $gk = t's'$ be two $(\mathscr{S}, \mathscr{T})$-factorizations of fk and gk, respectively. By Lemma 3.41, we obtain $t \ncong t'$. This means $\alpha \in \mathrm{Ne}(\mathbb{E}) - \mathfrak{Im}$ has at least two essentially distinct imaginary elements. Hence "if" of the second assertion follows. Finally, suppose that α is an imaginary and let $\beta \xrightarrow{t} \alpha$ and $\gamma \xrightarrow{t'} \alpha$ be its two imaginary elements. The following diagram is pullback.

$$\begin{tikzcd} \beta \times \gamma \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \gamma \arrow[d, "t'"] \\ \beta \arrow[r, "t"'] & \alpha \end{tikzcd}$$

Commutativity follows from the fact that α is a habobject, and universality follows from that of a product $\beta \times \gamma$. Since $t, t' \in \mathscr{T}$, one has $\pi_1, \pi_2 \in \mathscr{S} \cap \mathscr{T} = \mathrm{Iso}$. This implies $t \cong t'$. Hence α has essentially unique imaginary element. This is "only if" of the second assertion. $\square$

Proposition 3.43. Suppose that $\mathbb{E}$ is an imaginary extensional pre-effectiveness. Then $\mathfrak{Im}$ has the strongest extension ability over all rules. Namely, for any rule R, one has $\mathscr{I}_R \leq \mathscr{I}$ (i.e. $\mathscr{I}_R\mathscr{F} \leq \mathscr{I}\mathscr{F}$ for every fundamental class $\mathscr{F}$ on $\mathbb{E}$).

Proof. Follows from (ii) of Lemma 3.22. $\square$

3.2.4 Dense morphisms

We define a technical notion which plays an important roll in Section 3.3. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness.

Definition 3.44. Let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. Each f in $\mathbb{E}$ is said to be $\mathscr{F}$-dense if in any factorization $f = th$, one has $t \in \mathrm{Iso}$ whenever $t \in \mathscr{F}$.

We denote by $\mathscr{F}$-Dense the class of all $\mathscr{F}$-dense morphisms. One may see easily that $\mathscr{S} \subseteq \mathscr{F}$-Dense. The following characterization of $\mathscr{F}$-Dense is quite useful.

Lemma 3.45. Let $\mathscr{F}$ be a fundamental class. Then $\mathscr{F}\text{-Dense} = \{f \text{ in } \mathbb{E} : {}^\forall t \in \mathscr{F},\ f \perp t\}$.

Example 3.46. Let $x = (x, \tau)$ be a topological space. Each $u \subseteq x$ is said to be dense in x if $u \cap v \neq \emptyset$ for every non-empty $v \in \tau$. For instance, the set of all computable elements is dense in 2^ω. It is easy to see that:

$$f \in \mathrm{ClsEmb}_{\mathtt{Top}}\text{-Dense} \iff \mathrm{range}(f) \text{ is dense in } x:$$

where $\cdot \xrightarrow{f} x$ in $\mathtt{Top}$.

Example 3.47. Similar characterization with the case of $\mathtt{Top}$ can be obtained even in Cp. For each $\cdot \xrightarrow{f} u$ in Cp:

$$f \in \mathrm{ClsEmb}_{\mathrm{Cp}}\text{-Dense} \iff \mathrm{range}(f) \text{ is dense in } u = (u, \tau_{2^\omega}|_u).$$

Furthermore, one can see that $\mathscr{B}_{0,\mathrm{Cp}}$-density coincides with $\mathrm{ClsEmb}_{\mathrm{Cp}}$-density. At first sight, it may looks slightly strange that although $\mathscr{B}_{0,\mathrm{Cp}}$ is characterized by a computational notion, co-r.e. closedness, $\mathscr{B}_{0,\mathrm{Cp}}$-Dense is characterized by a topological notion, topological density. However this phenomenon will be observed again in a greatly generalized situation (cf. Lemma 3.58, Proposition 3.69).

Proposition 3.48. The following identification holds: $\mathscr{B}_0\text{-Dense} = \mathrm{Epi}$.

Proof. $\mathscr{B}_0\text{-Dense} \subseteq \mathrm{Epi}$: Let $f \in \mathscr{B}_0$-Dense and let $g_0 f = g_1 f$. We have an equalizer t of g_0, g_1 and there exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "t"] & \cdot \\ \cdot \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd}$$

Since $t \in \mathrm{RegMono} \subseteq \mathscr{B}_0$, we obtain $t \in \mathrm{Iso}$. This implies $g_0 = g_1$ and hence $f \in \mathrm{Epi}$.
$\mathrm{Epi} \subseteq \mathscr{B}_0$-Dense: Let $f \in \mathrm{Epi}$ and $f = th$ with $t \in \mathscr{B}_0$. Then there exists $t_0, \cdots, t_k \in \mathrm{RegMono}$ such that $t = t_0 \cdots t_k$. In the following, we use mathematical induction for k. As the base case, suppose that $k = 0$. Namely, t is an equalizer of a pair of parallel morphisms g_0, g_1. Then commutativity of the following diagram implies $g_0 = g_1$.

$$\begin{tikzcd} \cdot \arrow[r, "t"] & \cdot \\ \cdot \arrow[u, "h"] \arrow[ur, "f"'] & \end{tikzcd}$$

So we obtain $t \in \mathrm{Iso}$. Next, suppose that $k = k' + 1$ and $t_i \in \mathrm{Iso}$ for each $i \leq k'$. Then defining $f' = t_{k'}^{-1} \cdots t_0^{-1} f \in \mathrm{Epi}$, we can resolve this as a base case and $t_k \in \mathrm{Iso}$ will be obtained. Therefore, $t \in \mathrm{Iso}$ follows. This shows that $f \in \mathscr{B}_0$-Dense. □

Example 3.49. We have already mentioned to the point that epicity does not coincide with surjectivity in Cp (cf. Example 2.28). By Proposition 3.48 and by Example 3.47, each $\cdot \xrightarrow{f} u$ in Cp is epic if and only if $\mathrm{range}(f)$ is dense in $u = (u, \tau_{2^\omega}|_u)$.

We show a technical lemma.

Lemma 3.50. Let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. Then $\mathscr{I}_\alpha\mathscr{F}$-Dense is closed under $\alpha \times -$ where α is an imaginary.

Proof. Let $(x \xrightarrow{f} y) \in \mathscr{I}_\alpha\mathscr{F}$-Dense and $\alpha \times f = th$ with $t \in \mathscr{I}_\alpha\mathscr{F}$. By $\pi_2' \perp \pi_2\big[t\big]$, there exists unique morphism j which makes the following diagram commute:

$$\begin{tikzcd} \alpha \times x \arrow[rr, "\alpha\times f"] \arrow[dr, "h"] \arrow[dd, "\pi_2'"'] & & \alpha \times y \arrow[dd, "\pi_2"] \\ & \cdot \arrow[ur, "t"'] \arrow[dd, "s"] & \\ x \arrow[rr, "f"] \arrow[dr, "j"'] & & y \\ & \cdot \arrow[ur, "\pi_2[t]"'] & \end{tikzcd}$$

Note that $\pi_2\big[t\big] \in \mathscr{I}_\alpha\mathscr{F}$ holds by (ii) of Lemma 3.32. Since $f \in \mathscr{I}_\alpha\mathscr{F}$-Dense, we obtain $\pi_2\big[t\big] \in \mathrm{Iso}$ and thus $t \cong \pi_2^{-1}\big[\pi_2\big[t\big]\big] \in \mathrm{Iso}$ follows from (i) of Lemma 3.32. □

3.3 Limit Fundamental Class

Here we define another closure operator $\mathscr{L}$ for fundamental classes. Its action abstract "generation of topology". To do that, we also define our second main structure, effectiveness.

In Section 3.3.1, as a central statement, Proposition 3.56 will be proved. Section 3.3.2 is devoted to a consideration on a proper assumption of Proposition 3.56. It will be done by a pure categorical discussion, and as a result, Proposition 3.59 will be obtained. Section 3.3.2 is also devoted to a consideration on the same assumption of Proposition 3.56. It will be done by a set theoretical discussion, and as a result, Theorem 3.70, our first main theorem in this thesis, will be obtained. Theorem 3.70 provides us a complete characterization of the following situation: the action of $\mathscr{L}$ dominates that of $\mathscr{I}$ for a given fundamental class $\mathscr{F}$ i.e. $\mathscr{I}\mathscr{F} \leq \mathscr{L}\mathscr{F}$.

3.3.1 Effectiveness

Definition 3.51. A pre-effectiveness $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is said to be an effectiveness if the following condition holds:

(E1) $\mathbb{E}$ has $\mathscr{T}$-intersection.

Example 3.52. Cp is an effectiveness.

In an effectiveness $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$, we denote by $\mathscr{L}\mathscr{F}$ the smallest intersection closed fundamental class containing a given fundamental class $\mathscr{F}$, and call it limit completion of $\mathscr{F}$. Explicitly, one may define:

$$\mathscr{L}\mathscr{F} = \bigcap\{\mathscr{F}' \in \left[\mathscr{T}\right] : \mathscr{F} \subseteq \mathscr{F}',\ \mathscr{F}' \text{ is closed under } \mathscr{T}\text{-intersection}\}$$

Particularly, we abbreviate as $\mathscr{L}_0 = \mathscr{L}\mathscr{B}_0$ and call this $\mathscr{L}_0$ limit fundamental class on $\mathbb{E}$. Obviously, we can regard $\mathscr{L}$ as a closure operator on $\left[\mathscr{T}\right]$. We usually write $\mathscr{L}\mathfrak{T}$ instead of $\mathscr{L}\mathscr{B}\mathfrak{T}$ for each $\mathfrak{T} \subseteq \mathscr{T}$.

Example 3.53. Let us consider on Cp. We remark to the point that for each $u \in \mathrm{Cp}$ and $w \in 2^*$, $u \cap [w]$ is always co-r.e. closed in u and that each co-r.e. closed set in u is closed in $u = (u, \tau_{2^\omega})$. Recall Example 3.18. Now one can see that each $(\cdot \xrightarrow{t} u) \in \mathrm{Emb}_{\mathrm{Cp}}$ belongs to $\mathscr{L}_{0,\mathrm{Cp}}$ if and only if range(t) is closed in $u = (u, \tau_{2^\omega}|_u)$ (see also Example 3.72 and Lemma 3.73).

Let $\mathscr{F}$ be a $\mathscr{L}$-closed fundamental class on $\mathbb{E}$. For each $t \in \mathscr{T}$, we define:

$$\mathrm{cl}_{\mathscr{F}}(t) \quad = \quad \bigwedge\{t' \in \mathscr{F} : t \leq t'\}$$

Of course $\mathrm{cl}_{\mathscr{F}}(t) \in \mathscr{F}$ always holds.

Lemma 3.54. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be an effectiveness and let $\mathscr{F}$ be a $\mathscr{L}$-closed fundamental class on $\mathbb{E}$. Then ($\mathscr{F}$-Dense, $\mathscr{F}$) is a (possibly non-proper) factorization system on $\mathbb{E}$.

Proof. (F0): Trivial.
(F1): Let f in $\mathbb{E}$ and $f = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization. We can find a (necessarily unique) morphism h such that $f = \mathrm{cl}_{\mathscr{F}}(t) \circ h$. It is sufficient to see that $h \in \mathscr{F}$-Dense. Suppose that $h = t'h'$ with $t' \in \mathscr{F}$. Then $t \leq \mathrm{cl}_{\mathscr{F}}(t) \circ t' \in \mathscr{F}$ and thus, by the definition of intersection, there exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} & \cdot \arrow[dr, "\mathrm{cl}_{\mathscr{F}}(t) \circ t'"] & \\ \cdot \arrow[ur, "j"] \arrow[rr, "\mathrm{cl}_{\mathscr{F}}(t)"'] & & \cdot \end{tikzcd}$$

We obtain:

$$\begin{aligned} \mathrm{cl}_{\mathscr{F}}(t) \circ t'j = \mathrm{cl}_{\mathscr{F}}(t) &\implies t'j = \mathrm{id} \\ &\implies t' \in \mathrm{SplitEpi} \cap \mathrm{Mono} = \mathrm{Iso} \end{aligned}$$

Hence $h \in \mathscr{F}$-Dense.
(F2): Follows from Lemma 3.45. $\square$

Lemma 3.55. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. If $tt' \in \mathscr{F}$, one has $t' \in \mathscr{F}$ whenever $t \in \mathscr{F}$.

Proof. Let $t, tt' \in \mathscr{F}$. Note that $t' \in \mathscr{T}$ follows from (iv) of Lemma 3.5. By Corollary 2 of Lemma 3.7, we have $t' \cong t^{-1}\big[t\big[t'\big]\big] \cong t^{-1}\big[tt'\big]$. Hence $t' \in \mathscr{F}$ since $\mathscr{F}$, a fundamental class, is stable under pullback. $\square$

Proposition 3.56. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be an effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. If $\mathscr{F}$-Dense $= \mathscr{L}\mathscr{F}$-Dense, the following statements are equivalent:

(i) $\mathscr{I}\mathscr{F} \leq \mathscr{L}\mathscr{F}$;

(ii) all imaginaries are $\mathscr{L}\mathscr{F}$-compact;

(iii) $\mathscr{F}$-Dense is closed under $\alpha \times -$ for every imaginary α.

Proof. (iii)⇒(ii): Assume (iii). Let α be an imaginary and $(\cdot \xrightarrow{t} \alpha \times x) \in \mathscr{L}\mathscr{F}$. Suppose that $\pi_2\big[t\big] = gh$ is a $(\mathscr{L}\mathscr{F}\text{-Dense}, \mathscr{L}\mathscr{F})$-factorization where $\alpha \times x \xrightarrow{\pi_2} x$. Note that the following diagram commutes.

$$\begin{tikzcd} \alpha \times \cdot \arrow[rr, "\alpha \times \pi_2[t]"] \arrow[dr, "\alpha \times h"'] \arrow[dd, "\pi_2''"'] & & \alpha \times x \arrow[dd, "\pi_2"] \\ & \alpha \times \cdot \arrow[ur, "\alpha \times g"'] \arrow[dd, "\pi_2'"] & \\ \cdot \arrow[rr, "\pi_2[t]"] \arrow[dr, "h"'] & & x \\ & \cdot \arrow[ur, "g"'] & \end{tikzcd}$$

By (i) of Lemma 3.32, $\alpha\times\pi_2\big[t\big]\cong\pi_2^{-1}\big[\pi_2\big[t\big]\big]\cong t\in\mathscr{L}\mathscr{F}$. And $\alpha\times g\cong\pi_2^{-1}\big[g\big]\in\mathscr{L}\mathscr{F}$. So we obtain $\alpha\times h\in\mathscr{L}\mathscr{F}$ by Lemma 3.55. Hence, by Lemma 3.54, one can see that:

$$\begin{aligned} h\in\mathscr{L}\mathscr{F}\text{-Dense}=\mathscr{F}\text{-Dense} &\implies \alpha\times h\in\mathscr{F}\text{-Dense}=\mathscr{L}\mathscr{F}\text{-Dense}\\ &\implies \alpha\times h\in\mathscr{L}\mathscr{F}\text{-Dense}\cap\mathscr{L}\mathscr{F}=\mathrm{Iso}\end{aligned}$$

Since $\pi_2',\pi_2''\in\mathscr{S}$ and $h\in\mathscr{T}$ by (iv) of Lemma 3.5, $h\cong\pi_2'\big[\alpha\times h\big]\in\mathrm{Iso}$ follows from Corollary 3 of Lemma 3.7. Hence $\pi_2\big[t\big]\cong g\in\mathscr{L}\mathscr{F}$. This shows that α is $\mathscr{L}\mathscr{F}$-compact.
(ii)⇒(i): Assume (ii). Let $(x,\leq)$ be a partially ordered system and let c c' be two closure operators on it. It is easy to see that for every $a\in x$, one has $ca\leq c'a$ whenever $c'a$ is c-closed. So now we should show that $\mathscr{L}\mathscr{F}$ is $\mathscr{I}$-closed since both of $\mathscr{I}$ and $\mathscr{L}$ is closure operator on $\big[\mathscr{T}\big]$. But this is an immediate consequence of (ii) of Proposition 3.33.
(i)⇒(iii): Assume (i). One has:

$$\begin{aligned} \mathscr{F}\text{-Dense}=\mathscr{L}\mathscr{F}\text{-Dense} &\subseteq \mathscr{I}\mathscr{F}\text{-Dense}\\ &\subseteq \mathscr{I}_\alpha\mathscr{F}\text{-Dense}\subseteq\mathscr{F}\text{-Dense}\end{aligned}$$

where α is an arbitrary imaginary. Here $\mathscr{F}$-Dense $=\mathscr{I}_\alpha\mathscr{F}$-Dense is closed under $\alpha\times-$ by Lemma 3.50. So the desired statement holds.

□

3.3.2 A Sharpen by A Pure Categorical Discussion

Here we scrutinize the proper assumption of Proposition 3.56, $\mathscr{F}$-Dense $=\mathscr{L}\mathscr{F}$-Dense, by a pure categorical discussion. Let $\mathbb{E}=(\mathbb{E},\mathscr{S},\mathscr{T})$ be an effectiveness.

Lemma 3.57. The following statements hold on $\mathbb{E}$:

(i) for each $h\in\mathrm{Epi}$ and each $t\in\mathrm{ExtMono}$, $h\perp t$;

(ii) ExtMono is closed under composition;

(iii) $\mathbb{E}$ has ExtMono-intersections;

(iv) $(\mathrm{Epi},\mathrm{ExtMono})$ is a proper factorization system on $\mathbb{E}$;

(v) $\mathrm{ExtMono}=\mathscr{L}\mathrm{ExtMono}$:

Proof. (i): Let the following square be commutative with $h\in\mathrm{Epi}$ and $t\in\mathrm{ExtMono}$.

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "h"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

Let us enumerate as $\{t_i\}_{i\in I}$ all morphisms in $\mathscr{L}$ExtMono such that $g = t_i k_i$ and $t = t_i l_i$ with existing (necessarily unique) morphisms k_i, l_i.

$$\begin{tikzcd}
\cdot \arrow[rr, "g'"] \arrow[dd, "h"'] & & \cdot \arrow[dl, "l_i"'] \arrow[dd, "t"] \\
& \cdot \arrow[dr, "t_i"] & \\
\cdot \arrow[ur, "k_i"] \arrow[rr, "g"'] & & \cdot
\end{tikzcd}$$

By the universality of intersection, we have $g = (\bigwedge_{i\in I} t_i) \circ k$ and $t = (\bigwedge_{i\in I} t_i) \circ l$ with the existing unique morphisms k, l. Since $\bigwedge_{i\in I} t_i \in \mathscr{L}$ExtMono, there is an index $j \in I$ with $t_j = \bigwedge_{i\in I} t_i$, $k_j = k$ and $l_j = l$. We show that $l_j \in$ Epi. For any commutative $\cdot \xrightarrow{l_j} \cdot \underset{f_1}{\overset{f_0}{\rightrightarrows}} \cdot$, we have:

$$\begin{aligned}
f_0 l_j = f_1 l_j \quad &\Longrightarrow \quad f_0 l_j g' = f_1 l_j g' \\
&\Longleftrightarrow \quad f_0 k_j h = f_1 k_j h \\
&\Longleftrightarrow \quad f_0 k_j = f_1 k_j
\end{aligned}$$

since $t_j \in$ Mono and $h \in$ Epi. Hence for an equalizer u of f_0 and f_1, there exists unique morphisms k_u, l_u such that $k_j = u k_u$ and $l_j = u l_u$. Since u is regular monic, and thus is extremal monic, $t_j u \in \mathscr{L}$ExtMono. So there is an index $j' \in I$ with $t_{j'} = t_j u$, $k_{j'} = k_u$ and $l_{j'} = l_u$. Then by the definition of intersection, the unique morphism p exists with $t_j = t_{j'} p = t_j u p$. This implies $u \in \mathrm{SplitEpi} \cap \mathrm{Mono} = \mathrm{Iso}$. So $f_0 = f_1$ and then $l_j \in$ Epi follows. Furthermore, one can obtain $l_j \in \mathrm{Epi} \cap \mathrm{ExtMono} = \mathrm{Iso}$. Therefore, $d = l_j^{-1} k_j$ is the desired unique morphism for $h \perp t$ what makes the following diagram commute.

$$\begin{tikzcd}
\cdot \arrow[r, "g'"] \arrow[d, "h"'] & \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "d"] \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

(ii): Let $(\cdot \xrightarrow{t'} \cdot \xrightarrow{t} \cdot) \in$ ExtMono and $tt' = gh$ with $h \in$ Epi. Then we obtain first the left one below by $h \perp t$ and then the right one below by $h \perp t'$, respectively from (i).

$$\begin{tikzcd}
\cdot \arrow[r, "\mathrm{id}"] \arrow[dd, "h"'] & \cdot \arrow[d, "t'"] \\
& \cdot \arrow[d, "t"] \\
\cdot \arrow[ur, "\exists! j"'] \arrow[r, "g"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\mathrm{id}"] \arrow[dd, "h"'] & \cdot \arrow[d, "t'"] \\
& \cdot \arrow[d, "t"] \\
\cdot \arrow[uur, "\exists! k"] \arrow[ur, "j"'] \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

Hence $kh = \mathrm{id}$ and this implies $h \in \mathrm{SplitMono} \cap \mathrm{Epi} = \mathrm{Iso}$. This shows that $tt' \in$ ExtMono.

(iii): Let $\{t_i\}_{i\in I}$ be a family of extremal monomorphisms with shared codomain and $t = \bigwedge_{i\in I} t_i = gh$ with $h \in \mathrm{Epi}$. By the definition of intersection, there exists unique morphism p_i such that $t = t_i p_i$. Using (i), we obtain the unique morphism d_i which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "p_i"] \arrow[d, "h"'] & \cdot \arrow[d, "t_i"] \\ \cdot \arrow[r, "g"'] \arrow[ur, "d_i"] & \cdot \end{tikzcd}$$

Hence by the universality of intersection, there exists unique morphism k such that $g = tk$. Then we have:

$$\begin{aligned} g = tk \quad &\Longrightarrow \quad t = gh = tkh \\ &\Longleftrightarrow \quad \mathrm{id} = kh \iff h \in \mathrm{SplitMono} \cap \mathrm{Epi} = \mathrm{Iso} \end{aligned}$$

This shows that $t = \bigwedge_{i\in I} t_i \in \mathrm{ExtMono}$.
(iv): It is sufficient to see that each f in $\mathbb{E}$ has an $(\mathrm{Epi}, \mathrm{ExtMono})$-factorization. Let us enumerate as $\{t_i\}_{i\in I}$ all extremal monomorphisms such that $f = t_i h_i$ with existing (necessarily unique) morphism h_i. By (iii), $\bigwedge_{i\in I} t_i \in \mathrm{ExtMono}$ and thus there is an index j with $t_j = \bigwedge_{i\in I} t_i$. We have to show $h_j \in \mathrm{Epi}$, but we can solve this with just similar process when we showed "$l_j \in \mathrm{Epi}$" in the proof of (i).
(v): Obvious. □

Lemma 3.58. One has the following two identifications of classes of morphisms: $\mathrm{ExtMono} = \mathscr{L}_0$; $\mathrm{Epi} = \mathscr{B}_0\text{-Dense} = \mathscr{L}_0\text{-Dense}$.

Proof. First identification. Note that $\mathscr{B}_0 \le \mathrm{ExtMono}$ and ExtMono is a $\mathscr{L}$-closed fundamental class (see (v) of Lemma 3.57). Hence $\mathscr{L}_0 \le \mathrm{ExtMono}$. Let $t \in \mathrm{ExtMono}$. Let us denote by j the unique morphism with $t = \mathrm{cl}_{\mathscr{L}_0}(t) \circ j$. It is easy to see that $j \in \mathscr{B}_0\text{-Dense}$. Hence $j \in \mathrm{Epi} \cap \mathrm{ExtMono} = \mathrm{Iso}$ by Proposition 3.48 and thus $t \cong \mathrm{cl}_{\mathscr{L}_0}(t) \in \mathscr{L}_0$.
Second identification. The first equality follows from (iv) of Lemma 3.57 and (iii) of Lemma 3.4. The second equality follows from Proposition 3.48. □

Proposition 3.59. The following statements are equivalent:

(i) $\mathscr{I}_0 \le \mathscr{L}_0$;

(ii) all imaginaries are $\mathscr{L}_0$-compact;

(iii) $\mathscr{B}_0$-Dense is closed under $\alpha \times -$ for every imaginary α.

3.3.3 Ordinals and Cardinals

Ordinal numbers and cardinal numbers are elemental tools in set theories. In the following, we give some definitions and introduce some notations.

A partially ordered system $(x, \leq)$ is said to be a well-ordered system if for an arbitrary formula P:

$$^\forall a \in x \left[\left[^\forall b \in x\langle a\rangle,\ P(b)\right] \Longrightarrow P(a)\right] \Longrightarrow {}^\forall a \in x,\ P(a)$$

where $x\langle a\rangle = \{b \in x : b < a\}$. This is equivalent to say $(x, \leq)$ is a well-ordered system if every non-empty subset of x has minimum. Here minimum of $u \subseteq x$ is an element $a_0 \in u$ with the property: $^\forall a \in u,\ a_0 \leq a$.

For given two sets x and y, we define $x \leq y$ by $x \in y \vee x = y$. When we concern with ordinal numbers, without a special notice, $\leq$ will be used in this sense.

Definition 3.60. A set η is said to be an ordinal, or an ordinal number, if the following three conditions hold:

(ONi) $^\forall\theta \in \eta,\ {}^\forall\theta' \in \theta,\ \theta' \in \eta$;

(ONii) $^\forall\theta, \theta' \in \eta,\ \theta \in \theta' \vee \theta = \theta' \vee \theta' \in \theta$;

(ONiii) $^\forall u \in \eta\left[u \neq \emptyset \Longrightarrow {}^\exists\theta_0 \in u \text{ s.t. } {}^\forall\theta \in u,\ \neg\left[\theta < \theta_0\right]\right]$;

For example, both 0 $(= \emptyset)$ and ω are ordinals. Particularly, 0 is the smallest one. If η is an ordinal, $(\eta, \leq)$ forms a well-ordered system. In ZF set theory, condition (ONiii) is always fulfilled. Let us denote by Ord the class of all ordinals. Each subclass $A \subseteq \mathrm{Ord}$ is said to be bounded if there is an ordinal η such that $A \subseteq \eta$. And, in that case, A is said to be bounded in η. Of course a bounded subclass of Ord forms a set. One can see the following lemma.

Lemma 3.61. Let η be an ordinal. The following statements hold:

(i) every $\theta \in \eta$ is an ordinal again;

(ii) if η' is another ordinal, $\eta \leq \eta'$ or $\eta' \leq \eta$;

(iii) $\eta + 1$ $(= \eta \cup \{\eta\})$ is an ordinal again;

(iv) for a bounded subclass $A \subseteq \mathrm{Ord}$, the union $\bigcup A$ is an ordinal again;

(v) for a non-empty subclass $A \subseteq \mathrm{Ord}$, the intersection $\bigcap A$ is an ordinal again;

(vi) Ord is well-ordered by $\leq$ in an extended sense, namely, for an arbitrary formula, the following holds.

$$^\forall\eta \in \mathrm{Ord}\ \left[\left[^\forall\theta < \eta,\ P(\theta)\right] \Longrightarrow P(\eta)\right] \Longrightarrow {}^\forall\eta \in \mathrm{Ord},\ P(\eta)$$

For a non-empty subclass $\{\eta : \eta \in \mathrm{Ord}, P(\eta)\} \subseteq \mathrm{Ord}$, we write $\mu\eta\big[P(\eta)\big]$ instead of $\bigcap\{\eta : \eta \in \mathrm{Ord}, P(\eta)\}$. The scheme of formulae appeared in (v) of the above lemma is called induction scheme on ordinals.

We call any ordinal of the form $\eta + 1$ a successor ordinal. An ordinal which is not a successor is called a limit ordinal. By induction on ordinals, we obtain the following theorem.

Theorem 3.62. Let $F : V \to V$ be a correspondence where $V = \{x : x = x\}$. There exists unique correspondence $G : \mathrm{Ord} \to V$ such that for every ordinal η, $F(G\langle\eta\rangle) = G(\eta)$ where $G\langle\eta\rangle = \{(\theta, G(\theta)) : \theta < \eta\}$ [10] .

Example 3.63. For each set x, we define as follows [11].

$$\begin{aligned} {}^\sharp x &= \mu\eta\big[\eta \approx x\big] \\ x^+ &= \mu\eta\big[{}^\sharp x < {}^\sharp\eta\big] \end{aligned}$$

For the definition of $\approx$, see Section 2.1. Using the above Theorem 3.62, we define a new correspondence $\omega : \mathrm{Ord} \to V$ as follows.

$$\omega_\eta = \begin{cases} \omega & \eta = 0 \\ \omega_\theta^+ & \eta = \theta + 1 \\ \bigcup_{\theta<\eta} \omega_\theta & \eta \ : \ \text{limit} \end{cases}$$

We give a definition of cardinal number.

Definition 3.64. Any ordinal of the form ${}^\sharp x$ is called a cardinal number, or cardinality of x.

The cardinality of our universe V^* (cf. Section 2.2) will, usually, be denoted by κ. A set x is said to be have small cardinality if ${}^\sharp x < \kappa$. In ZF set theory, it can be seen that each small set has small cardinality.

We define a technical notion.

Definition 3.65. An ordinal η is said to be regular if for every $\theta < \eta$ and $f : \theta \to \eta$, the set $\{f(\theta') : \theta' < \theta\}$ is bounded in η.

Obviously, each regular ordinal is a cardinal.

The following lemma plays an important roll in Section 3.3.4.

Lemma 3.66. Each cardinal of the form x^+ is regular.

---

[10] It is not trivial that $G\langle\eta\rangle$ forms a set. We need the axiom of collection which is an axiom of ZF set theory. But we, here, merely assume that $G\langle\eta\rangle$ is certainly a set.

[11] In ZF set theory, these notations are well-defined for every set x when we admit the axiom of choice. But we, here, merely assume that they are certainly well-defined.

3.3.4 A Sharpen by A Set Theoretical Discussion

Here we scrutinize the proper assumption of Proposition 3.56, $\mathscr{F}$-Dense $= \mathscr{L}\mathscr{F}$-Dense, by a set theoretical discussion. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be an effectiveness. For each $\mathfrak{T} \subseteq \mathscr{T}$ which contains all isomorphisms and is closed under composition, we define:

$$\begin{aligned} \overline{\mathfrak{T}} &= \{t \in \mathscr{T} : \text{there is a } \mathfrak{T}\text{-sink } \{t_i\}_{i\in I} \text{ such that } t \cong \bigwedge_{i\in I} t_i\} \\ \mathfrak{T}^\circ &= \{t_0 \cdots t_k : k \in \mathbb{N},\ t_0, \cdots, t_k \in \mathfrak{T}\} \end{aligned}$$

Also, for each fundamental class $\mathscr{F}$ on $\mathbb{E}$, we define:

$$\begin{aligned} \mathscr{L}^0\mathscr{F} &= \mathscr{F} \\ \mathscr{L}^{\eta+1}\mathscr{F} &= \overline{\mathscr{L}^\eta\mathscr{F}}^\circ \\ \mathscr{L}^\eta\mathscr{F} &= \bigcup_{\theta<\eta} \mathscr{L}^\theta\mathscr{F} \quad \text{if } \eta \text{ is a limit ordinal} \end{aligned}$$

where η is an arbitrary ordinal.

We define a technical notion.

Definition 3.67. We say that a pre-effectiveness $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is well-powered if $\mathscr{T}(x)$ is a set with small cardinality for every $x \in \mathbb{E}$.

We establish some points.

Lemma 3.68. Let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. The following statements hold:

(i) $\mathscr{L}^\eta\mathscr{F} \subseteq \mathscr{L}^{\eta'}\mathscr{F}$ for every two ordinals η and η' with $\eta \le \eta'$;

(ii) $\mathscr{L}^\eta\mathscr{F}$ is a fundamental class on $\mathbb{E}$ for every ordinal η;

(iii) $\mathscr{L}^\eta\mathscr{F} \le \mathscr{L}\mathscr{F}$ for every ordinal η;

(iv) if $\mathbb{E}$ is well-powered, then there is an ordinal η such that $\mathscr{L}\mathscr{F} = \mathscr{L}^\eta\mathscr{F}$.

Proof. (i): Trivial.
(ii): Let η be an ordinal. Firstly, $\mathscr{L}^0\mathscr{F} = \mathscr{F}$ is a fundamental class. Next, suppose that $\eta = \eta' + 1$ and that $\mathscr{L}^{\eta'}\mathscr{F}$ is a fundamental class. We have $\mathscr{L}^{\eta'}\mathscr{F} \subseteq \mathscr{L}^\eta\mathscr{F}$ by (i) and hence Iso $\subseteq \mathscr{L}^\eta\mathscr{F}$ follows. For each $(\cdot \xrightarrow{t} x) \in \overline{\mathscr{L}^{\eta'}\mathscr{F}}$, there is a $\mathscr{L}^{\eta'}\mathscr{F}$-sink $\{t_i\}_{i\in I}$ such that $t \cong \bigwedge_{i\in I} t_i$. If $(\cdot \xrightarrow{f} x)$ in $\mathbb{E}$, by Lemma 3.8, one has:

$$f^{-1}[t] \;\cong\; f^{-1}[\bigwedge_{i\in I} t_i] \;\cong\; \bigwedge_{i\in I} f^{-1}[t_i]$$

Since $\mathscr{L}^{\eta'}\mathscr{F}$ is a fundamental class, each $f^{-1}[t_i]$ belongs to $L^{\eta'}\mathscr{F}$. Thus $f^{-1}[t] \in \overline{\mathscr{L}^{\eta'}\mathscr{F}}$ and this shows that $\overline{\mathscr{L}^{\eta'}\mathscr{F}}$ is stable under pullback. It is easy to see that

$\mathscr{L}^{\eta}\mathscr{F} = \overline{\mathscr{L}^{\eta'}\mathscr{F}}^{\circ}$ is also stable under pullback and is closed under composition. Therefore $\mathscr{L}^{\eta}\mathscr{F}$ is a fundamental class in this case. Finally, suppose that η is a limit ordinal and that $\mathscr{L}^{\theta}\mathscr{F}$ is a fundamental class for every $\theta < \eta$. It is almost trivial that Iso $\subseteq \mathscr{L}^{\eta}\mathscr{F}$ and that $\mathscr{L}^{\eta}\mathscr{F}$ is stable under pullback. Let $t, t' \in \mathscr{L}^{\eta}\mathscr{F}$ with being compositionable as tt'. There are two ordinal θ and θ' such that $t \in \mathscr{L}^{\theta}\mathscr{F}$ and $t' \in \mathscr{L}^{\theta'}\mathscr{F}$, respectively. If $\theta^* = \max\{\theta, \theta'\} < \eta$, then $t, t' \in \mathscr{L}^{\theta^*}\mathscr{F}$ by (i) and this implies $tt' \in \mathscr{L}^{\theta^*}\mathscr{F} \subseteq \mathscr{L}^{\eta}\mathscr{F}$, again, by (i). Therefore $\mathscr{L}^{\eta}\mathscr{F}$ is a fundamental class even in this case.
(iii): Trivial.
(iv): Suppose that $\mathbb{E}$ is well-powered. Note that $\sharp\mathscr{T}(x) \leq \kappa < \kappa^+$ where we denote by κ the cardinality of our universe. We show that $\mathscr{L}^{\kappa^+}\mathscr{F}$ is $\mathscr{L}$-closed. Let $\{t_i\}_{i\in I}$ be a $\mathscr{L}^{\kappa^+}\mathscr{F}$-sink to x and let $t \cong \bigwedge_{i\in I} t_i$. Without loss of generality, we can assume that the correspondence $i \mapsto [t_i]$ ($^\forall i \in I$) is injective and then I is a set with $\sharp I \leq \sharp\mathscr{T}(x) < \kappa^+$. Note that $\mathscr{L}^{\kappa^+}\mathscr{F} = \bigcup_{\theta<\kappa^+} \mathscr{L}^{\theta}\mathscr{F}$ since κ^+, an infinite cardinal, is a limit ordinal. By Lemma 3.66, κ^+ is regular and hence the following set is bounded in κ^+.

$$\{\mu\theta\big[t_i \in \mathscr{L}^{\theta}\mathscr{F}\big] : i \in I\}$$

So if $\eta < \kappa^+$ is sufficiently large, we can regard $\{t_i\}_{i\in I}$ as a family on $\mathscr{L}^{\eta}\mathscr{F}$. Hence we can conclude $t \in \mathscr{L}^{\eta+1}\mathscr{F} \leq \mathscr{L}^{\kappa^+}\mathscr{F}$. This shows that $\mathscr{L}^{\kappa^+}\mathscr{F}$ is $\mathscr{L}$-closed. Since $\mathscr{L}^{\kappa^+}\mathscr{F} \leq \mathscr{L}\mathscr{F}$ by (iii), $\mathscr{L}\mathscr{F} = \mathscr{L}^{\kappa^+}\mathscr{F}$ follows by the minimality of $\mathscr{L}\mathscr{F}$. □

Proposition 3.69. Suppose that $\mathbb{E}$ is a well-powered effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. Then $\mathscr{F}$-Dense $= \mathscr{L}\mathscr{F}$-Dense.

Proof. By (ii) and (iv) of Lemma 3.68, it is sufficient to see that $\mathscr{F}$-Dense $\subseteq \mathscr{L}^{\eta}\mathscr{F}$-Dense where η is an arbitrary ordinal. First, suppose that $\eta = \eta' + 1$ and $\mathscr{F}$-Dense $\subseteq \mathscr{L}^{\eta'}\mathscr{F}$-Dense. Let $f \in \mathscr{F}$-Dense and $f = th$ with $t \in \mathscr{L}^{\eta}\mathscr{F}$. Then, there exists $\mathscr{L}^{\eta'}\mathscr{F}$-sinks $\{t_{0i}\}_{i\in I_0}, \cdots, \{t_{ki}\}_{i\in I_k}$ such that:

$$\begin{array}{rcll} t & \cong & t_0 \cdots t_k & \\ t_j & = & \displaystyle\bigwedge_{i\in I_j} t_{ji} & \qquad (j \leq k) \end{array}$$

We use a mathematical induction for k. In the case of $k = 0$, each t_{0i} must be isomorphic and thus $t = t_0$ is also isomorphic. If $k = k' + 1$ and t_j is isomorphic for each $j \leq k$, we can apply a same discussion with the base case to $f' = t_k \circ h$ where $f' = t_{k'}^{-1} t_{k'-1}^{-1} \cdots t_0^{-1} f \in \mathscr{F}$-Dense. So t_k is again isomorphic even in this case. We conclude that t is isomorphic and consequently $f \in \mathscr{L}^{\eta}\mathscr{F}$-Dense. Next, we should discuss about the case of η is a limit ordinal. But this case is trivial. □

Theorem 3.70. Suppose that $\mathbb{E}$ is a well-powered effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. The following statements are equivalent:

(i) $\mathscr{I}\mathscr{F} \leq \mathscr{L}\mathscr{F}$;

(ii) all imaginaries are $\mathscr{L}\mathscr{F}$-compact;

(iii) $\mathscr{F}$-Dense is closed under $\alpha \times -$ for every imaginary α.

> **Proof.** Recall Proposition 3.56. We should check its proper assumption, $\mathscr{F}$-Dense $=$ $\mathscr{L}\mathscr{F}$-Dense. But it has already shown as Proposition 3.69. $\square$

The following is devoted to an additional consideration which is useful to analyze concrete examples of fundamental classes.

Definition 3.71. Each fundamental class $\mathscr{F}$ on $\mathbb{E}$ is said to be initial if the following condition holds.

- for every $t, t' \in \mathscr{T}$ at being compositionable as tt', if $t' \in \mathscr{F}$, then t' has a representation in the form $t' \cong t^{-1}\big[t^*\big]$ where $t^* \in \mathscr{F}$.

Example 3.72. $\mathscr{B}_{0,\mathrm{Cp}}$ is initial.

Lemma 3.73. Let $\mathscr{F}$ be an initial fundamental class on $\mathbb{E}$. One has the following equality.

$$\mathscr{L}\mathscr{F} = \mathscr{L}^1\mathscr{F}$$

> **Proof.** By (ii) and (iii) of Lemma 3.68, it is sufficient to see that $\overline{\mathscr{F}}$ is closed under composition. Let $t, t' \in \overline{\mathscr{F}}$ at being compositionable as tt'. One can find two $\mathscr{F}$-sinks $\{t_i\}_{i\in I}$ and $\{t'_i\}_{i\in I'}$ such that $t \cong \bigwedge_{i\in I} t_i$ and that $t' \cong \bigwedge_{i\in I'} t'_i$. Also there exists a $\mathscr{F}$-sink $\{t^*_i\}_{i\in I'}$ such that $t'_i \cong t^{-1}\big[t^*_i\big]$ for each $i \in I'$. Note that $t \wedge t^*_i = \bigwedge\{t, t_i\} \cong t \circ (t^{-1}\big[t^*_i\big])$. We obtain:
>
> $$\begin{aligned} tt' &\cong t \circ \bigwedge_{i\in I'} t'_i \;\cong\; t \circ \bigwedge_{i\in I'} t^{-1}\big[t^*_i\big] \\ &\cong t \circ t^{-1}\big[\bigwedge_{i\in I'} t^*_i\big] \\ &\cong t \wedge (\bigwedge_{i\in I'} t^*_i) \\ &\cong (\bigwedge_{i\in I} t_i) \wedge (\bigwedge_{i\in I'} t^*_i) \;\cong\; \bigwedge_{i\in I, j\in I'} (t_i \wedge t^*_j) \end{aligned}$$
>
> Since $t_i \wedge t^*_j \cong t_i \circ t_i^{-1}\big[t^*_j\big]$ belongs to $\mathscr{F}$, this shows that tt' belongs to $\overline{\mathscr{F}}$. $\square$

4 Concrete System

In category theory, concrete categories are quite useful. A concrete category is a category equipped with a faithful functor to the base category which is usually called forgetful functor. Using this mechanical system, the structure of the base category can be "reflected" to the concerned category. As a result, compare to when we regard the concerned category as a mere plain category, one may obtain more rich structure.

Our two main structures, pre-effectiveness and effectiveness, were introduced in the previous section. In this section, we define concrete versions of them, regular pre-effectiveness and regular effectiveness, respectively. Basically, these are intended to provides a generalized construction of which we did in Example 3.11 and Example 3.12.

Those structures may have some additional properties. For example, a regular effectiveness over the category of sets always have imaginary extensionality, the property about which we discussed in Section 3.2.3.

It turns out that all topological categories over the category of sets, in the sense of [4], are regular effectiveness. Since topological categories over the category of sets have many desirable properties, this fact come out important in a situation.

In Section 4.2.2, a proper assumption of Proposition 3.56 will again be considered. As a result, Theorem 4.24 will be proved, and as an its application, we show that "oracle co-r.e. closedness implies topological closedness" on each object of Cp, a subset of Cantor space.

4.1 Relativization

In this section, we define concrete versions of our main structures, pre-effectiveness and effectiveness.

4.1.1 Concrete Category and Concrete Functor

Concrete category We give a definition of concrete category.

Definition 4.1. Let $\mathbb{B}$ and $\mathbb{E}$ be two categories and $U : \mathbb{E} \to \mathbb{B}$ be a faithful functor. The pair $\mathbb{E} = (\mathbb{E}, U)$ is called a concrete category over $\mathbb{B}$.

Let $\mathbb{E}$ be a concrete category over $\mathbb{B}$. We usually assume that $\mathbb{E}$ is being of the form $\mathbb{E} = (\mathbb{E}, |-|)$. This $|-|$ is occasionally called as equipped forgetful functor of $\mathbb{E}$. For each $x \xrightarrow{f} y$ in $\mathbb{E}$, we write $|x| \xrightarrow{f} |y|$ instead of $|f|$ and call it underlying morphism of f. And for each $x, y \in \mathbb{E}$ and $|x| \xrightarrow{g} |y|$ in $\mathbb{B}$, g is said to be a morphism in $\mathbb{E}$, written as $x \xrightarrow{g} y$ in $\mathbb{E}$, if there exists (necessarily unique) morphism from x to y in $\mathbb{E}$ whose underlying morphism is g.

We define two kinds of morphism.

Definition 4.2. Let $\mathbb{E}$ be a concrete category over $\mathbb{B}$. A morphism in $\mathbb{E}$ is said to be reflected monomorphism (resp. reflected epimorphism) if its underlying morphism is monic (resp. epic) in $\mathbb{B}$.

We denote by $\mathrm{RefMono}_{\mathbb{E}}$ (resp. $\mathrm{RefEpi}_{\mathbb{E}}$) the class of all reflected monomorphisms (resp. reflected epimorphisms) in $\mathbb{E}$. Obviously $\mathrm{RefMono}_{\mathbb{E}} \subseteq \mathrm{Mono}_{\mathbb{E}}$ and $\mathrm{RefEpi}_{\mathbb{E}} \subseteq \mathrm{Epi}_{\mathbb{E}}$.

Example 4.3. Recall $U : \mathtt{Top} \to \mathtt{Set}$ from Example 2.37. We can regard $\mathtt{Top}$ as a concrete category over $\mathtt{Set}$ with respect to that U.

Example 4.4. Recall $U : \mathrm{Cp} \to \mathtt{Set}$ from Example 2.38. We can regard Cp as a concrete category over $\mathtt{Set}$ with respect to that U.

Concrete Limit We introduce the concrete version of the notion of limit.

Definition 4.5. Let $\mathbb{E}$ be a concrete category over $\mathbb{B}$. We say that $\mathbb{E}$ has a concrete binary product if $\mathbb{E}$ has binary product and the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ preserves binary product.

We also use the prefix "concrete" for each other kind of "limit" e.g. terminal object, equalizer, pullback,...etc. Also we will say as $\mathbb{E}$ is finitely concretely complete, or $\mathbb{E}$ has finite concrete product, in according situations.

Example 4.6. $\mathtt{Top}$ as a concrete category over $\mathtt{Set}$ is finitely concretely complete. This can easily be checked (cf. Example 2.41). Similar with Cp. See also (i) of Lemma 4.14.

Concrete Functor We define the notion of concrete functor.

Definition 4.7. Let $\mathbb{E}$ and $\mathbb{E}'$ be two concrete category over $\mathbb{B}$. A functor $F : \mathbb{E} \to \mathbb{E}'$ is said to be concrete if the following triangle commutes.

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "F"] \arrow[dr, "|-|"'] & & \mathbb{E}' \arrow[dl, "|-|"] \\ & \mathbb{B} & \end{tikzcd}$$

A concrete functor must be faithful.

Example 4.8. Recall $U : \mathrm{Cp} \to \mathtt{Top}$ from 2.39. That U is concrete since it doesn't change the underlying set. So U is faithful.

4.1.2 Regular Effectiveness

We define a new category as follows:

$\mathtt{Cat}$
- object : large categories
- morphism : functors

The category 1 who has only one object $* \in V^*$ and has only one morphism, so id_*, is a terminal object of $\mathtt{Cat}$. Let $\mathbb{E}$ be a category. Suppose that $\mathcal{M} \subseteq \mathrm{Mono}_{\mathbb{E}}$ contains all isomorphisms and is closed under composition. We define a new category as follows:

$\mathcal{M}(\mathbb{E})$

object : $[m]$ where $m \in \mathcal{M}$

morphism : Let $[(\cdot \xrightarrow{m} x)], [(\cdot \xrightarrow{m'}) y] \in \mathcal{M}$.
A morphism $x \xrightarrow{f} y$ will be regarded as a morphism from $[m]$ to $[m']$ if there exists (necessarily unique) morphism j which makes the following diagram commute

$$\begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "m"'] & \cdot \arrow[d, "m'"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

We also define a new functor as follows:

$U : \mathcal{M}(\mathbb{E}) \to \mathbb{E}$

object : $[m] \mapsto \mathrm{cod} m$

morphism : $f \mapsto f$

Note that for every $x \in \mathbb{E}$, the following diagram is pullback.

$$\begin{tikzcd} \mathcal{M}(x) \arrow[r, "\iota"] \arrow[d, "!"'] & \mathcal{M}(\mathbb{E}) \arrow[d, "U"] \\ 1 \arrow[r, "c_x"'] & \mathbb{E} \end{tikzcd}$$

where ι is a trivial embedding, 1 is a terminal object of `Cat` with unique object $*$ and c_x is defined by $* \mapsto x$.

Let $\mathbb{E}$ be a concrete category over $\mathbb{B}$. A morphism f in $\mathbb{E}$ is said to be initial if the following condition hold:

in any commutative diagram in $\mathbb{B}$ shown below with g in $\mathbb{E}$

$$\begin{tikzcd} |\cdot| \arrow[r, "f"] & |\cdot| \\ |\cdot| \arrow[u, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

h must be a morphism of $\mathbb{E}$.

We denote by Init the class of all initial morphisms. Now let $\mathbb{B} = (\mathbb{B}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness. We define two classes of morphisms as follows:

$$\begin{aligned} \mathrm{Ext}^{\mathscr{S}} &= \{(\cdot \xrightarrow{f} \cdot) \text{ in } \mathbb{E} : |\cdot| \xrightarrow{f} |\cdot| \in \mathscr{S}\} \\ \mathrm{Ext}^{\mathscr{T}} &= \{(\cdot \xrightarrow{f} \cdot) \text{ in } \mathbb{E} : |\cdot| \xrightarrow{f} |\cdot| \in \mathscr{T}\} \end{aligned}$$

As an abbreviation, let us denote $\mathrm{Emb}^{\mathscr{T}}$ instead of $\mathrm{Ext}^{\mathscr{T}} \cap \mathrm{Init}$. We define the following functor:

$$
\begin{array}{l}
|-| : \mathrm{Emb}^{\mathscr{T}}(\mathbb{E}) \to \mathscr{T}(\mathbb{E}) \\
\quad \text{object} \quad : \quad [(\cdot \xrightarrow{u} \cdot)] \mapsto [(|\cdot| \xrightarrow{u} |\cdot|)] \\
\quad \text{morphism} \quad : \quad (\cdot \xrightarrow{f} \cdot) \mapsto (|\cdot| \xrightarrow{f} |\cdot|)
\end{array}
$$

Now we introduce the notion of regular pre-effectiveness.

Definition 4.9. $\mathbb{E}$ is said to be a regular pre-effectiveness over $\mathbb{B}$ if the following two conditions hold:

(Ri) $\mathbb{E}$ has finite concrete products;

(Rii) $\mathbb{E}$ extends $\mathscr{T}$-subobjects i.e. for each $x \in \mathbb{E}$ and $(\cdot \xrightarrow{t} |x|) \in \mathscr{T}$, there exists $(\cdot \xrightarrow{u} x) \in \mathrm{Emb}^{\mathscr{T}}$ such that $t \cong u$ in $\mathbb{B}$.

A regular pre-effectiveness is called a regular effectiveness if its base category is an effectiveness.

Concerning with (Rii), the following lemma is important.

Lemma 4.10. The following conditions are equivalent:

(i) $\mathbb{E}$ extends $\mathscr{T}$-subobjects;

(ii) ${}^{\forall}x \in \mathbb{E}$, ${}^{\forall}[t] \in \mathscr{T}(|x|)$, ${}^{\exists!}[u] \in \mathrm{Emb}^{\mathscr{T}}$ s.t. $[t] = |[u]|$;

(iii) the following square is pullback in `Cat`.

$$
\begin{tikzcd}
\mathrm{Emb}^{\mathscr{T}}(\mathbb{E}) \arrow[r, "U"] \arrow[d, "|-|"'] & \mathbb{E} \arrow[d, "|-|"] \\
\mathscr{T}(\mathbb{B}) \arrow[r, "U"'] & \mathbb{B}
\end{tikzcd}
$$

Proof. (i) $\Rightarrow$ (ii): Assume (i). Let $x \in \mathbb{E}$ and $[t] \in \mathscr{T}(|x|)$. We can find $(\cdot \xrightarrow{u} x), (\cdot \xrightarrow{u'} x) \in \mathrm{Emb}^{\mathscr{T}}$ such that $u \cong t \cong u'$ in $\mathbb{B}$. So there exist two isomorphisms h in $\mathbb{B}$ which makes the following diagram commute in $\mathbb{B}$.

$$
\begin{tikzcd}
|\cdot| \arrow[rr, shift left, "h"] \arrow[dr, "u"'] & & |\cdot| \arrow[ll, shift left, "h^{-1}"] \arrow[dl, "u'"] \\
& |x| &
\end{tikzcd}
$$

By the definition of $\mathrm{Emb}^{\mathscr{T}}$ and by the fact that the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, the following diagram also commutes in $\mathbb{E}$.

$$
\begin{tikzcd}
\cdot \arrow[rr, shift left, "h"] \arrow[dr, "u"'] & & \cdot \arrow[ll, shift left, "h^{-1}"] \arrow[dl, "u'"] \\
& x &
\end{tikzcd}
$$

Hence h is isomorphic, also ,in $\mathbb{E}$. This shows that $u \cong u'$, and thus $[u] = [u']$.
(ii) $\Rightarrow$ (iii): Assume (ii). Let the following diagram be commutative:

$$\begin{tikzcd}
\mathbb{D} \arrow[rrd, bend left, "G"] \arrow[rdd, bend right, "F"'] & & \\
& \mathrm{Emb}^{\mathscr{T}}(\mathbb{E}) \arrow[r, "U"] \arrow[d, "|-|"'] & \mathbb{E} \arrow[d, "|-|"] \\
& \mathscr{T}(\mathbb{B}) \arrow[r, "U"'] & \mathbb{B}
\end{tikzcd}$$

and let $x \in \mathbb{D}$. Ofcourse $|Gx| = UFx$. Then there exists unique $[u] \in \mathrm{Emb}^{\mathscr{T}}(y)$ such that $Fx = |[u]|$. We denote by K this unique correspondence $x \mapsto [u]$. We show that this correspondence can be extended as a functor. Let $x_0 \xrightarrow{f} x_1$ in $\mathbb{D}$ and $Kx_i = [u_i]$ where $i = 0, 1$. Then the underlying morphism of Gf is UFf, and there exists unique morphism j which makes the following diagram commute in $\mathbb{B}$.

$$\begin{tikzcd}
|\cdot| \arrow[r, "u_0"] \arrow[d, "j"'] & |Gx_0| \arrow[d, "Gh=UFf"] \\
|\cdot| \arrow[r, "u_1"'] & |Gx_1|
\end{tikzcd}$$

By the definition of embedding, this j is a morphism in $\mathbb{E}$. Since the equipped forgetful functor is faithful, the following diagram commutes in $\mathbb{E}$.

$$\begin{tikzcd}
\cdot \arrow[r, "u_0"] \arrow[d, "j"'] & Gx_0 \arrow[d, "Gh"] \\
\cdot \arrow[r, "u_1"'] & Gx_1
\end{tikzcd}$$

This shows that $[u_0] \xrightarrow{Gh} [u_1]$ in $\mathrm{Emb}(\mathbb{C})$. Define $Kh = Gh$. Now K is a functor which makes the following diagram commute.

$$\begin{tikzcd}
\mathbb{D} \arrow[rrd, bend left, "G"] \arrow[rd, "K"] \arrow[rdd, bend right, "F"'] & & \\
& \mathrm{Emb}^{\mathscr{T}}(\mathbb{E}) \arrow[r, "U"] \arrow[d, "|-|"'] & \mathbb{E} \arrow[d, "|-|"] \\
& \mathscr{T}(\mathbb{B}) \arrow[r, "U"'] & \mathbb{B}
\end{tikzcd}$$

The expected uniqueness of K immediately follows from its construction.
(iii) $\Rightarrow$ (i): Let $x \in \mathbb{E}$ and $(\cdot \xrightarrow{t} |x|) \in \mathscr{T}$. Then there exists unique fuctor K which makes the following diagram commute.

$$\begin{tikzcd}
1 \arrow[rrd, bend left, "c_x"] \arrow[rd, "K"] \arrow[rdd, bend right, "c_{[t]}"'] & & \\
& \mathrm{Emb}^{\mathscr{T}}(\mathbb{E}) \arrow[r] \arrow[d, "|-|"'] & \mathbb{E} \arrow[d, "|-|"] \\
& \mathscr{T}(\mathbb{B}) \arrow[r] & \mathbb{B}
\end{tikzcd}$$

where 1 is a terminal object of `Cat` with unique object $*$ and x, $[t]$ are defined by $* \mapsto x$, $* \mapsto [t]$, respectivly. One can see that $(\cdot \xrightarrow{u} x) \in \mathrm{Emb}^{\mathscr{T}}$ and $u \cong t$ in $\mathbb{B}$ if we define $K* = [u]$. $\square$

Corollary 1. If $\mathbb{E}$ extends $\mathscr{T}$-subobjects, one has $\mathrm{Emb}^{\mathscr{T}}(x) \cong \mathscr{T}(|x|)$ for every $x \in \mathbb{E}$.

Corollary 2. If $\mathbb{E}$ extends $\mathscr{T}$-subobjects, $\mathbb{E}$ is well-powered whenever $\mathbb{B}$ is well-powered.

Proposition 4.11. If $\mathbb{E}$ is a regular (pre-)effectiveness over $\mathbb{B}$, then $\mathbb{E} = (\mathbb{E}, \mathrm{Ext}^{\mathscr{S}}, \mathrm{Emb}^{\mathscr{T}})$ is a (pre-)effectiveness. Particulary, $\mathbb{E}$ is finitely concretely complete.

Proof. Assume that $\mathbb{E}$ be a regular pre-effectiveness over $\mathbb{B}$.
$\mathbb{E}$ is finitely (concretely) complete: It is sufficient to see that $\mathbb{E}$ has concrete equalizer. Let $x \underset{g}{\overset{f}{\rightrightarrows}} y$ be a pair of parallel morphisms in $\mathbb{E}$. One can find an equalizer t of f and g in $\mathbb{B}$.

$$\cdot \xrightarrow{t} |x| \underset{g}{\overset{f}{\rightrightarrows}} |y|$$

Since $t \in \mathrm{RegMono} \subseteq \mathscr{T}$, there exists a morphism $(\cdot \xrightarrow{u} x) \in \mathrm{Emb}^{\mathscr{T}}$ such that $t \cong u$ in $\mathbb{B}$. Obviously u is, again, an equalizer of f and g in $\mathbb{B}$. We show that u is an equalizer of f and g also in $\mathbb{E}$. Let $fh = gh$. There exists unique morphism j in $\mathbb{B}$ which makes the following diagram commute.

$$\begin{tikzcd} |\cdot| \arrow[r, "u"] & |x| \\ |\cdot| \arrow[u, "j"] \arrow[ur, "h"'] & \end{tikzcd}$$

By the definition of $\mathrm{Emb}^{\mathscr{T}}$ and by the fact that the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, the following diagram also commutes in $\mathbb{E}$.

$$\begin{tikzcd} \cdot \arrow[r, "u"] & x \\ \cdot \arrow[u, "j"] \arrow[ur, "h"'] & \end{tikzcd}$$

The expected uniqueness of j in $\mathbb{E}$ follows, again, from the fact that the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful.
$(\mathrm{Ext}^{\mathscr{S}}, \mathrm{Emb}^{\mathscr{T}})$ forms a proper factorization system: Firstly, (F0) is trivial. Next we check (F1). Let $(x \xrightarrow{f} y)$ in $\mathbb{E}$. We obtain a $(\mathscr{S}, \mathscr{T})$-factorization of $|x| \xrightarrow{f} |y|$ as $f = ts$. There exists a morphism $(\cdot \xrightarrow{u} y) \in \mathrm{Emb}^{\mathscr{T}}$ such that $t \cong u$ in $\mathbb{B}$. Suppose taht can is the unique isomorphism in $\mathbb{B}$ which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[rr, "\mathrm{can}"] \arrow[dr, "t"'] & & |\cdot| \arrow[dl, "u"] \\ & |y| & \end{tikzcd}$$

Note that $\mathrm{can} \circ s \in \mathscr{S}$. So the left one below forms a $(\mathscr{S}, \mathscr{T})$-factorization of f in $\mathbb{B}$. Also, by the definition of $\mathrm{Emb}^{\mathscr{T}}$ and by the fact that the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, the right one below forms an $(\mathrm{Ext}^{\mathscr{S}}, \mathrm{Emb}^{\mathscr{T}})$-factorization of f in $\mathbb{E}$.

$$\begin{tikzcd} {|x|} \arrow[rr, "f"] \arrow[dr, "\mathrm{can}\circ s"'] & & {|y|} \\ & {|\cdot|} \arrow[ur, "u"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "\mathrm{can}\circ s"'] & & y \\ & \cdot \arrow[ur, "u"'] & \end{tikzcd}$$

Finally, we check (F2). Let the following diagram in $\mathbb{E}$ be commutative, and suppose that $s \in \mathrm{Ext}^{\mathscr{T}}$ and $u \in \mathrm{Emb}^{\mathscr{T}}$.

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "s"'] & \cdot \arrow[d, "u"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

There exists unique morphism j in $\mathbb{B}$ which makes the left one below commutes. By the definition of $\mathrm{Emb}^{\mathscr{T}}$ and by the fact that the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, the right one below also commutes.

$$\begin{tikzcd} {|\cdot|} \arrow[r, "f"] \arrow[d, "s"'] & {|\cdot|} \arrow[d, "u"] \\ {|\cdot|} \arrow[r, "g"'] \arrow[ur, "j"] & {|\cdot|} \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "s"'] & \cdot \arrow[d, "u"] \\ \cdot \arrow[r, "g"'] \arrow[ur, "j"] & \cdot \end{tikzcd}$$

This shows that $s \perp u$ and thus (F2) holds.
$\mathrm{Ext}^{\mathscr{S}}$ is stable under pullback: Note that any pullback diagram in $\mathbb{E}$ is concrete. Now it's trivial.
Finally, suppose that $\mathbb{E}$ is a regular effectiveness over $\mathbb{B}$. So $\mathbb{B}$ is an effectiveness. By Corollary 1 of Lemma 4.10, $\mathbb{E}$ has $\mathrm{Emb}^{\mathscr{T}}$-intersection. Hence $\mathbb{E} = (\mathbb{E}, \mathrm{Ext}^{\mathscr{T}}, \mathrm{Emb}^{\mathscr{T}})$ is an effectiveness in this case. □

Lemma 4.12. Let $\mathbb{E}$ be a regular pre-effectiveness over $\mathbb{B}$. If $\mathbb{B}$ is strictly imaginary extensional, the following statements hold:

(i) the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ preserves monomorphism, and hence $\mathrm{Mono}_{\mathbb{E}} = \mathrm{RefMono}_{\mathbb{E}}$;

(ii) each $\alpha \in \mathbb{E}$ is an imaginary (resp. habobject) if and only if $|\alpha|$ is an imaginary (resp. habobject) of $\mathbb{B}$;

(iii) $\mathbb{E}$ is strictly imaginary extensional.

Proof. (i): Let $(x \xrightarrow{m} y) \in \mathrm{Mono}_{\mathbb{E}}$. Suppose that $mg_1 = mg_2$ where $a \overset{g_1}{\underset{g_2}{\rightrightarrows}} |x| \xrightarrow{m} |y|$ in $\mathbb{B}$ and that $g_1 \neq g_2$. There is a morphism $\alpha \xrightarrow{k} a$ in $\mathbb{B}$ with being an imaginary of its domain α such that $g_1k \neq g_2k$. Now let $g_1k = t_1s_1$ and $g_2k = t_2s_2$ be two $(\mathscr{S}, \mathscr{T})$-factorizations of g_1k and g_2k, respectively. By the definition of regular pre-effectiveness, one can find two morphisms $(\beta_1 \xrightarrow{l_1} x), (\beta_2 \xrightarrow{l_2} x) \in \mathrm{Emb}^{\mathscr{T}}$ such that $t_1 \cong l_1$, $t_2 \cong l_2$ in $\mathbb{B}$. Let the following two diagrams be commutative.

$$\begin{tikzcd} \cdot \arrow[rr, "\mathrm{can}_1"] \arrow[dr, "t"'] & & {|\beta_1|} \arrow[dl, "l_1"] \\ & {|x|} & \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[rr, "\mathrm{can}_2"] \arrow[dr, "t"'] & & {|\beta_2|} \arrow[dl, "l_2"] \\ & {|x|} & \end{tikzcd}$$

Note that both $|\beta_1|$ and $|\beta_2|$ are imaginaries by Lemma 3.40 and that $|\beta_1 \times \beta_2|$ is a product of $|\beta_1|$ and $|\beta_2|$. Thus $|\beta_1 \times \beta_2| \xrightarrow{\pi_2} |\beta_2| \in \mathrm{Mono}_{\mathbb{B}}$. Let us denote the unique morphism by j which makes the following diagram commute.

$$\begin{tikzcd} & \alpha \arrow[dl, "\mathrm{can}_1 s_1"'] \arrow[d, "j"] \arrow[dr, "\mathrm{can}_2 s_2"] & \\ {|\beta_1|} & {|\beta_1 \times \beta_2|} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & {|\beta_2|} \end{tikzcd}$$

It is easy to see that the following diagram is pullback.

$$\begin{tikzcd} \alpha \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & \alpha \arrow[d, "\mathrm{can}_2 s_2"] \\ {|\beta_1 \times \beta_2|} \arrow[r, "\pi_2"'] & {|\beta_2|} \end{tikzcd}$$

Hence $j \in \mathscr{S} \subseteq \mathrm{Epi}_{\mathbb{B}}$. Now we obtain:

$$\begin{aligned} mg_1 = mg_2 \text{ in } \mathbb{B} &\iff mg_1k = mg_2k \text{ in } \mathbb{B} \\ &\iff mt_1s_1 = mt_2s_2 \text{ in } \mathbb{B} \\ &\iff ml_1\mathrm{can}_1s_1 = ml_2\mathrm{can}_2s_2 \text{ in } \mathbb{B} \\ &\iff ml_1\pi_1j = ml_2\pi_2j \text{ in } \mathbb{B} \\ &\iff ml_1\pi_1 = ml_2\pi_2 \text{ in } \mathbb{B} \end{aligned}$$

Since the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, one has $ml_1\pi_1 = ml_2\pi_2$ in $\mathbb{E}$. Therefore $l_1\pi_1 = l_2\pi_2$ in $\mathbb{E}$. This implies:

$$\begin{aligned} l_1\pi_1 = l_2\pi_2 \text{ in } \mathbb{E} &\implies l_1\pi_1j = l_2\pi_2j \text{ in } \mathbb{B} \\ &\implies g_1k = g_2k \end{aligned}$$

Contradiction. We conclude $g_1 = g_2$ and thus $m \in \mathrm{Mono}_{\mathbb{B}}$.

(ii): Let $\alpha \in \mathbb{E}$ be a habobject. Then $(\alpha \xrightarrow{!} 1) \in \mathrm{Mono}_{\mathbb{E}} = \mathrm{RefMono}_{\mathbb{E}}$. So we obtain $(|\alpha| \xrightarrow{!} |1|) \in \mathrm{Mono}_{\mathbb{B}}$ by (i). Conversely, for every $\alpha \in \mathbb{E}$, if $|\alpha|$ is a habobject, then $(|\alpha| \xrightarrow{!} |1|) \in \mathrm{Mono}_{\mathbb{B}}$ and thus $(\alpha \xrightarrow{!} 1) \in \mathrm{RefMono}_{\mathbb{E}} = \mathrm{Mono}_{\mathbb{E}}$. Hence α is a habobject if and only if $|\alpha|$ is a habobject. Similary, one may see that α is an imaginary if and only if $|\alpha|$ is an imaginary.

(iii): Let $x \underset{g}{\overset{f}{\rightrightarrows}} y$ be a pair of parallel morphisms in $\mathbb{E}$ with being distinct i.e. $f \neq g$. Since the equipped forgetful functor $|-| : \mathbb{E} \to \mathbb{B}$ is faithful, $f \neq g$ in $\mathbb{B}$. Thus there is a morphism $\alpha \xrightarrow{k} |x|$ with being an imaginary of its domain α such that $fk \neq gk$ in $\mathbb{B}$. Let $k = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization of k. Then $ft \neq gt$ follows. One can find $(\beta \xrightarrow{l} x) \in \mathrm{Emb}^{\mathscr{T}}$ such that $t \cong l$. Note that β is an imaginary by Lemma 3.40 and by (ii). One may obtain $fl \neq gl$ in $\mathbb{E}$. This shows that $\mathbb{E}$ is imaginary extensional. Finally, we check its strictness. Let $fg \in \mathrm{Mono}_{\mathbb{E}}$ with $g \in \mathrm{Ext}^{\mathscr{S}}$. By (i), if we consider on the underlying morphisms of them, $fg \in \mathrm{Mono}_{\mathbb{B}}$ and $g \in \mathscr{S}$. Since $\mathbb{B}$ is strictly imaginaly extensional, one has $f \in \mathrm{Mono}_{\mathbb{B}}$ and this implies $f \in \mathrm{Mono}_{\mathbb{E}}$. Hence $\mathbb{E}$ is strictly imaginary extensional. □

Corollary 1. Let $\mathbb{D}$ and $\mathbb{E}$ be two regular pre-effectiveness over $\mathbb{B}$. If $F : \mathbb{D} \to \mathbb{E}$ is a concrete functor, each $\alpha \in \mathbb{D}$ is an imaginary if and only if $F\alpha$ is an imaginary.

Let $\mathbb{E}$ be a cateogry. As dual notion of sink, we define sourse. So a sourse from $x \in \mathbb{E}$ is a family $\{f_i\}_{i \in I}$ of morphisms in $\mathbb{E}$ with being the shared domain of x i.e. $x = \mathrm{dom} f_i$ for every $i \in I$. Next let $\mathbb{E}$ be a concrete category over the base category $\mathbb{B}$. Suppose that we are given a family $\{f_i\}_{i \in I}$ and a family $\{x_i\}_{i \in I}$ such that $\{f_i\}_{i \in I}$ is a sourse from $a \in \mathbb{B}$ and $a \xrightarrow{f_i} |x_i|$ for every $i \in I$. We call the pair $\{(a \xrightarrow{f_i} |x_i|)\}_{i \in I} = (\{f_i\}_{i \in I}, \{x_i\}_{i \in I})$ an $\mathbb{E}$-structured sourse from a. Furthermore, each sourse $\{g_i\}_{\{}i \in I\}$ from $x \in \mathbb{E}$ is said to be an initial lifting of $\{(a \xrightarrow{f_i} |x_i|)\}_{i \in I}$ if $a = |x|$ and each g_i is initial.

Definition 4.13. Let $\mathbb{E}$ be a concrete category over $\mathbb{B}$. We say that $\mathbb{E}$ is a topological category over $\mathbb{B}$ if every $\mathbb{E}$-structured sourse has unique initial lifting.

Topological categories have many desirable properties. We list some of them.

Lemma 4.14. Let $\mathbb{E}$ be a topological category over `Set`. The following statements hold:

(i) $\mathbb{E}$ is a regular effectiveness over `Set`;

(ii) $\mathrm{Emb}_{\mathbb{E}}^{\mathrm{Mono}_{\mathtt{Set}}} = \mathrm{ExtMono}_{\mathbb{E}} = \mathrm{RegMono}_{\mathbb{E}}$.

Example 4.15. `Top` is topological categories.

Example 4.16. We construct a topological category. Let x be a set. Each $\sigma \subseteq \mathrm{Pow}(x)$ is said to be a semi-topology on x if the following two conditions hold:

(STi) $\emptyset, x \in \sigma$;

(STii) $^\forall\sigma' \subseteq \sigma,\ \bigcup\sigma' \in \sigma.$

In that case the pair (x, σ) will be called a semi-topological space. A topological space (x, τ), obviously, is a semi-topological space. In a same habit with the case of topological space, we can define smallness for a semi-topological space and continuity for a function between two semi-topological spaces. We define a new category as follows:

`STop`		
object	:	small semi-topological spaces
morphism	:	continuous functions

We also define a new functor as follows:

$\mid - \mid$: `STop` $\to$ `Set`		
object	:	$(x, \sigma) \mapsto x$
morphism	:	$f \mapsto f$

`STop` is a topological category over `Set` with respect to the above forgetful functor. A proof can be given in a similar way with the case of `Top`. We omit here. This `STop` will play an important roll in Section 4.2.2. Note that, similar to the case of `Top`, each terminal object of `STop` is being of the form $(\{*\}, \mathrm{Pow}(\{*\}))$. It is easy to see that each imaginary is a terminal object in `STop` (see (ii) of Lemma 4.12).
We also define the following functor:

U^* : `Top` $\to$ `STop`		
object	:	$x \mapsto x$
morphism	:	$f \mapsto f$

4.1.3 Representation Operator

Let $\mathbb{B} = (\mathbb{B}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness. We define a new category as follows:

$\mathtt{Reg}_{\mathbb{B}}$		
object	:	regular pre-effectivenesses
morphism	:	$\mathbb{D} \xrightarrow{F} \mathbb{E}$ with the following property:
		(i) F is concrete;
		(ii) F preserves binary products;
		(iii) F preserves embeddings i.e. $F\big[\mathrm{Emb}^{\mathscr{T}}_{\mathbb{D}}\big] \subseteq \mathrm{Emb}^{\mathscr{T}}_{\mathbb{D}}$.

Recall here that as we promised at the beggining of this section, our observations are restricted to large categories. Hence $\mathrm{ob}(\mathtt{Reg}_{\mathbb{B}})$ forms certainly a set, and thus $\mathtt{Reg}_{\mathbb{B}}$ is well-defined as a category.

Lemma 4.17. Let $F \in \mathtt{Reg}_{\mathbb{B}}(\mathbb{D}, \mathbb{E})$. For each $(\cdot \xrightarrow{v} y) \in \mathrm{Emb}^{\mathscr{T}}_{\mathbb{E}}$, there exists $(\cdot \xrightarrow{u} x) \in \mathrm{Emb}^{\mathscr{T}}_{\mathbb{D}}$ such that $y = Fx$ and $v \cong Fu$.

We also define a new function as follows:

$$\begin{aligned}\mathscr{R} = \mathscr{R}_{\mathbb{B}} : \mathrm{mor}(\mathtt{Reg}_{\mathbb{B}}) &\to \mathrm{mor}(\mathtt{Reg}_{\mathbb{B}}) \\ (\mathbb{D} \xrightarrow{F} \mathbb{E}) &\mapsto \mathscr{R}F = (\mathscr{R}F \xrightarrow{U} \mathbb{E})\end{aligned}$$

where:

$\mathscr{R}F$

object : (x, y, δ) where $x \in \mathbb{D}$ and $(Fx \xrightarrow{\delta} y) \in \mathrm{Ext}^{\mathscr{S}}_{\mathbb{E}}$

morphism : Let $\delta = (x, y, \delta)$, $\delta' = (x', y', \delta') \in \mathscr{R}F$.
Each $y \xrightarrow{f} y'$ in $\mathbb{E}$ will be regarded as a morphism from δ to δ' if there exists a morphism $(x \xrightarrow{g} x')$ in $\mathbb{D}$ such that the following diagram commutes:

$$\begin{tikzcd} Fx \arrow[r, "\delta"] \arrow[d, "g"'] & y \arrow[d, "f"] \\ Fx' \arrow[r, "\delta'"'] & y' \end{tikzcd}$$

$U : \mathscr{R}F \to \mathbb{E}$

object : $(x, y, \delta) \mapsto \delta$

morphism : $f \mapsto f$

As equipped forgetful functor of $\mathscr{R}F$, we use $|-| \circ U$ where $|-|$ is the equipped forgetful functor of $\mathbb{E}$. Let us denote as $(Fx \xrightarrow{\delta} y)$ instead of (x, y, δ) for each object of $\mathscr{R}F$.

We establish a point.

Fact 4.18. $\mathscr{R}$ is well-defined.

Proof. Let $F \in \mathtt{Reg}_{\mathbb{B}}(\mathbb{D}, \mathbb{E})$ and $\mathscr{R}F = (\mathscr{R}F \xrightarrow{U} \mathbb{E})$. We first show that $\mathscr{R}F$ is a regular pre-effectiveness over $\mathbb{B}$.

$\mathscr{R}F$ has a terminal object: $(1_{\mathbb{D}} \xrightarrow{!} 1_{\mathbb{E}})$ is a terminal object of $\mathscr{R}F$.

$\mathscr{R}F$ has binary product: Let $\delta_1 = (Fx_1 \xrightarrow{\delta_1} y_1)$, $\delta_2 = (Fx_2 \xrightarrow{\delta_2} y_2) \in \mathscr{R}F$. There exists unique morphism $\delta_1 \times \delta_2$ in $\mathbb{E}$ which makes the following diagram commute.

$$\begin{tikzcd} Fx_1 \arrow[r, "\delta"] & y_1 \\ F(x_1 \times x_2) \arrow[u, "F\pi'_1"] \arrow[r, "\delta_1 \times \delta_2"] \arrow[d, "F\pi'_2"'] & y_1 \times y_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ Fx_2 \arrow[r, "\delta_2"'] & y_2 \end{tikzcd}$$

Let us define $\delta_1 \times \delta_2 = (F(x_1 \times x_2) \xrightarrow{\delta_1 \times \delta_2} y_1 \times y_2)$. One can easily see that $\delta_1 \xleftarrow{\pi_1} \delta_1 \times \delta_2 \xrightarrow{\pi_2} \delta_2$ is a product of δ_1 and δ_2 in $\mathscr{R}F$.

$\mathscr{R}F$ extends $\mathscr{T}$-subobject: Let $\delta = (Fx \xrightarrow{\delta} y) \in \mathscr{R}F$ and let $(\cdot \xrightarrow{t} |\delta|) \in \mathscr{T}$. Note that $|\delta| = |y|$. One can find a morphism $(y^* \xrightarrow{u} y) \in \mathrm{Emb}_{\mathbb{E}}^{\mathscr{T}}$ such that $t \cong$ in $\mathbb{B}$. Let the following diagram be pullback.

$$\begin{tikzcd}
x' \arrow[r, "\delta'"] \arrow[d, "u'"'] & y^* \arrow[d, "u"] \\
x \arrow[r, "\delta"'] & y
\end{tikzcd}$$

Since $(\mathrm{Ext}_{\mathbb{E}}^{\mathscr{T}}, \mathrm{Emb}_{\mathbb{E}}^{\mathscr{T}})$ is a factorization system of $\mathbb{E}$, ofcourse $\mathrm{Emb}_{\mathbb{E}}^{\mathscr{T}}$ is stable under pullback and thus $u \in \mathrm{Emb}_{\mathbb{E}}^{\mathscr{T}}$. By the definition of $\mathtt{Reg}_{\mathbb{B}}$, one can find again a morphism $(x^* \xrightarrow{v} x) \in \mathrm{Emb}_{\mathbb{D}}^{\mathscr{T}}$ such that $u' \cong Fv$. Let the following diagram be commutative.

$$\begin{tikzcd}
Fx^* \arrow[rr, "\mathrm{can}"] \arrow[dr, "Fv"'] & & x' \arrow[dl, "u'"] \\
& x &
\end{tikzcd}$$

We define $\delta^* = \delta'\mathrm{can}$ and denote $\delta^* = (Fx^* \xrightarrow{\delta^*} y^*)$. One can easily see that $\delta^* \xrightarrow{u} \delta \in \mathrm{Emb}_{\mathscr{R}F}^{\mathscr{T}}$. This shows that $\mathscr{R}F$ extends $\mathscr{T}$-subobjects.
Finally, we have to show that U in $\mathtt{Reg}_{\mathbb{B}}$. But this is trivial from the above constructions. $\square$

Example 4.19. Regard $\mathtt{Set}$ as a regular effectiveness over $\mathtt{Set}$ itself. Recall $U : \mathrm{Cp} \to \mathtt{Set}$ from Example 2.38. Let us define $\mathscr{R}(U) = (\mathtt{Rep} \xrightarrow{U} \mathtt{Set})$.
We give a fundamental class on $\mathtt{Rep}$. Define $\Omega = (2^\omega, 2, \delta_\Omega)$ where δ_Ω is defined as:

$$\begin{aligned}
\delta_\Omega : 2^\omega &\to 2 \\
p &\mapsto \begin{cases} 0 & \text{if } p = 0^\omega \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

Define also $\top : 1_\Omega \to \Omega$, where $1_\Omega = (\{0^\omega\}, \{*\}, !)$, by $* \mapsto 0$. Let us define a subclass Π_1^0 of $\mathrm{Emb}_{\mathtt{Rep}}^{\mathrm{Mono}_{\mathtt{Set}}}$ as follows: for every $(\cdot \xrightarrow{t} \delta) \in \mathrm{Emb}_{\mathtt{Rep}}^{\mathrm{Mono}_{\mathtt{Set}}}$, t belongs to $\Pi_{1,\mathtt{Rep}}^0$ if and only if there exists a (necessaliry unique) morphism $\delta \xrightarrow{\chi} \Omega$ such that $t \cong \chi^{-1}\big[\top\big]$. One can easily see that $t \in \Pi_{1,\mathtt{Rep}}^0$ if and only if $\delta^{-1}\big[\mathrm{range}(t)\big]$ is co-r.e. closed in u where $\delta = (u, x, \delta)$. This $\Pi_{1,\mathtt{Rep}}^0$ turns out immediately to be initial (cf. Section 4.2.2).

Let $\mathbb{D} \xrightarrow{F} \mathbb{E}$ in $\mathtt{Reg}_{\mathbb{B}}$ and suppose that we have a subclass $\mathcal{S} \subseteq \mathrm{Ext}^{\mathscr{S}}$ which contains all isomorphisms, is closed under composition and is stable under pullback. Being careful on our proof of Fact 4.18, one can see that the following category is well-defined as a regular pre-effectiveness over $\mathbb{B}$:

$\mathscr{R}F|_{\mathcal{S}}$

object : $(x, y, \delta) \in \mathscr{R}F$ with $\delta \in \mathcal{S}$

morphism : same with $\mathscr{R}F$

We use this additional construction in the next Section 4.2.

Example 4.20. Each $(x, \tau_x) \xrightarrow{f} (y, \tau_y) \in \texttt{Top}$ is said to be open if $f[u] \in \tau_y$ for every $u \in \tau_x$. We denote by Opn the class of all open maps. Opn is a subclass of $\mathrm{Quot}_{\texttt{Top}}$ and thus of $\mathrm{Ext}_{\texttt{Top}}^{\mathrm{Epi}_{\texttt{Set}}}$. Also it contains all isomorphisms, is closed under composition and is stable under pullback (see [6] for details). Let us denote $\texttt{Rep}_{\mathrm{op}}$ instead of $\mathscr{R}U|_{\mathrm{Opn}}$ where $U : \mathrm{Cp} \to \texttt{Top}$ from Example 2.39.
We give a fundamental class on $\texttt{Rep}_{\mathrm{op}}$. Define $\Omega = (2^\omega, 2_\Omega, \delta_\Omega)$ where $2_\Omega = (2, \{\emptyset, \{1\}, 2\})$. And δ_Ω, $\top$ and $\Pi^0_{1,\texttt{Rep}_{\mathrm{op}}}$ is supposed to be similary defined with the case of $\texttt{Rep}$. One can easily see that $t \in \Pi^0_{1,\texttt{Rep}_{\mathrm{op}}}$ if and only if $\delta^{-1}[\mathrm{range}(t)]$ is co-r.e. closed in u where $\delta = (u, x, \delta)$. This $\Pi^0_{1,\texttt{Rep}_{\mathrm{op}}}$ turns out immediately to be initial (cf. Section 4.2.2).

4.2 Regular Effectivenesses Over The Category of Sets

In Section 4.1, we introduced the notion of regular effectiveness. Particulary, one can find some addisional properties and structures in a regular effectiveness over the category of sets.

4.2.1 Simultaneous Functor and Compactness

We define a new category as follows:

$\texttt{PreEff}$

object : pre-effectivenesses

morphism : Let $\mathbb{D} = (\mathbb{D}, \mathscr{S}_{\mathbb{D}}, \mathscr{T}_{\mathbb{D}})$ and $\mathbb{E} = (\mathbb{E}, \mathscr{S}_{\mathbb{E}}, \mathscr{T}_{\mathbb{E}})$ be two pre-effectivenesses.
A functor $F : \mathbb{D} \to \mathbb{E}$ will be regarded as a morphism of $\texttt{PreEff}$
if it preserves finite limits, $F[\mathscr{S}_{\mathbb{D}}] \subseteq \mathscr{S}_{\mathbb{E}}$ and $F[\mathscr{T}_{\mathbb{D}}] \subseteq \mathscr{T}_{\mathbb{E}}$

Let $\mathbb{B} = (\mathbb{B}, \mathscr{S}, \mathscr{T})$ be a pre-effectiveness. One can see that the following functor is well-defined.

$U : \texttt{Reg}_{\mathbb{B}} \to \texttt{PreEff}$

object : $\mathbb{E} \mapsto \mathbb{E} = (\mathbb{E}, \mathrm{Ext}^{\mathscr{T}}, \mathrm{Emb}^{\mathscr{T}})$

morphism : $F \mapsto F$

We define a technical notion as follows.

Definition 4.21. Let $\mathbb{D} = (\mathbb{D}, \mathscr{S}_{\mathbb{D}}, \mathscr{T}_{\mathbb{D}})$ and $\mathbb{E} = (\mathbb{E}, \mathscr{S}_{\mathbb{E}}, \mathscr{T}_{\mathbb{E}})$ be two pre-effectivenesses. And let $\mathscr{F}, \mathscr{F}'$ be two fundamental classes on $\mathbb{D}$ and $\mathbb{E}$, respectively. Each $F \in \texttt{PreEff}(\mathbb{D}, \mathbb{E})$ is said to be $(\mathscr{F}, \mathscr{F}')$-simultaneous if for every f in $\mathbb{D}$, $f \in \mathscr{F}$ coincides with $Ff \in \mathscr{F}'$.

Proposition 4.22. A simultaneous functor reflects compactness i.e. if $F : \mathbb{D} \to \mathbb{E}$ is $(\mathscr{F}, \mathscr{F}')$-simultaneous, each $x \in \mathbb{D}$ is $\mathscr{F}$-compact whenever Fx is $\mathscr{F}'$-compact.

Proof. Let $x \in \mathbb{D}$ and suppose that Fx is $\mathscr{F}'$-compact. Let $y \in \mathbb{D}$. We denote by π_2 the second projection from $x \times y$ to y. For each $(\cdot \xrightarrow{u} x \times y) \in \mathscr{F}$, if $\pi_2 u = u's'$ is an $(\mathrm{Ext}^{\mathscr{T}}_{\mathbb{D}}, \mathrm{Emb}^{\mathscr{T}}_{\mathbb{D}})$-factorization of $\pi_2 u$, then $F\pi_2 Fu = Fu'Fs'$ is an $(\mathrm{Ext}^{\mathscr{T}}_{\mathbb{E}}, \mathrm{Emb}^{\mathscr{T}}_{\mathbb{E}})$-factorization of $F\pi_2 Fu$ by the definition of $\mathtt{PreEff}$. Since F is $(\mathscr{F}, \mathscr{F}')$-simultaneous, one has $Fu \in \mathscr{F}'$. Then $Fu' \in \mathscr{F}'$ follows from the fact that Fx is $\mathscr{F}'$-compact. Since, again, F is $(\mathscr{F}, \mathscr{F}')$-simultaneous, this implies $u' \in \mathscr{F}$. Therefore x is $\mathscr{F}$-compact. $\square$

4.2.2 A Sharpen by Concrete Systems

Note that for every $x \in \mathtt{Set}$, there exists unique morphism $! : \emptyset \to x$, and furthermore, $!$ is monic.

Let $\mathbb{E}$ be a regular effectiveness over $\mathtt{Set}$ and suppose that $\mathscr{F}$ is a $\mathscr{L}$-closed fundamental class on $\mathbb{E}$. We say that $\mathscr{F}$ is semi-topological if the following condition holds:

- for every $x \in \mathbb{E}$ and $(\cdot \xrightarrow{u} x) \in \mathrm{Emb}^{\mathrm{Mono}_{\mathtt{Set}}}$, if $|\cdot|$ is the empty set, $u \in \mathscr{F}$.

Let $\mathscr{F}$ be semi-topological. For each $x \in \mathbb{E}$, we define a semi-topology $\sigma_{\mathscr{F},x}$ on $|x|$ as follows: for every $u \subseteq x$, $u \in \sigma_{\mathscr{F},x}$ if and only if each $(\cdot \to x) \in \mathrm{Emb}^{\mathrm{Mono}_{\mathtt{Set}}}$ belongs to $\mathscr{F}$ whenever $\mathrm{range}(v) = x - u$. We define a new functor as follows:

$$\begin{array}{l} \overline{\mathscr{F}}^{\mathtt{STop}} : \mathbb{E} \to \mathtt{STop} \\ \quad \begin{array}{lcl} \text{object} & : & x \mapsto (|x|, \sigma_{\mathscr{F},x}) \\ \text{morphism} & : & f \mapsto f \end{array} \end{array}$$

This is obviously well-defined as a concrete functor.

Example 4.23. Recall $U^* : \mathtt{Top} \to \mathtt{STop}$ from Example 4.16. Obviously $U^* = \overline{\mathrm{ClsEmb}_{\mathtt{Top}}}^{\mathtt{STop}}$.

Theorem 4.24. Let $\mathbb{E}$ be a regular effectiveness over $\mathtt{Set}$ and let $\mathscr{F}$ be a semi-topological fundamental class on $\mathbb{E}$. One has $\mathscr{I}\mathscr{F} \leq \mathscr{L}\mathscr{F}$ whenever $\overline{\mathscr{L}\mathscr{F}}^{\mathtt{STop}}$ preserves embeddings and finite product.

Proof. If $\overline{\mathscr{L}\mathscr{F}}^{\mathtt{STop}}$ preserves embeddings and finite product, it forms a $(\mathscr{F}, \mathrm{ClsEmb}_{\mathtt{STop}})$-simultaneous functor. Hence by Corollary 1 of Lemma 4.12 and by Proposition 4.22, it follows that every imaginary $\alpha \in \mathbb{E}$ is $\mathscr{F}$-compact. $\square$

Example 4.25. One can apply Theorem 4.24 to $\mathscr{B}_{0,\mathrm{Cp}}$. Since $\mathscr{L}_{0,\mathrm{Cp}}$ identify topological closedness (cf. Example 3.53), we can factorize $\overline{\mathscr{L}_{0,\mathrm{Cp}}}^{\mathtt{STop}}$ as U^*U where $U^* : \mathtt{Top} \to \mathtt{STop}$ is from Example 4.16 and where $U : \mathrm{Cp} \to \mathtt{Top}$ is from Example 2.39. Therefore, oracle co-r.e. closedness implies topological closedness on each object of Cp.

Example 4.26. One can apply Theorem 4.24 to $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. Since $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}} \leq \mathscr{L}\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ is trivial, this provides us no new information, but it is valuable from another perspective. Let $(\cdot \xrightarrow{t} \delta) \in \mathrm{Emb}^{\mathrm{Mono}_{\mathtt{Set}}}$, $\delta = (u, x, \delta)$ and $x = (x, \tau_x)$. We show the following equivalence.

$(*)$ t belongs to $\mathscr{L}\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ if and only if range(t) is closed in x.

Proof for "if" of $(*)$. Note that the following set is a base on x and it generate τ_x.

$$\sigma_\delta \quad = \quad \{\delta\big[[w] \cap u\big] : w \in 2^*\}$$

Since $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ is initial (cf. Example 4.20), to see "if" of $(*)$, we should show that t belongs to $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ whenever $x - \mathrm{range}(t)$ belongs to σ_δ. Suppose that $x - \mathrm{range}(t) \in \sigma_\delta$. Let us enumerate as $\{(w_i, w_i')\}_{i\in\omega}$ all pairs of finite binary sequences $w, w' \in 2^*$ such that $\delta\big[[w] \cap u\big] = \delta\big[[w'] \cap u\big]$. Define an imaginary $\{p\}$ of Cp as follows.

$$p \quad = \quad \iota(w_0)\iota(w_0')\iota(w_1)\iota(w_1')\iota(w_2)\iota(w_2')\cdots$$

where $\iota(w) = 0a_00a_10\cdots 0a_k0$ for every $w = a_0a_1\cdots a_k \in 2^*$. It follows that $\alpha \times t \in \Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ where $\alpha = (\{p\}, 1_{\mathtt{Top}}, !)$. Thus $t \in \mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$.
Proof for "only if" of $(*)$. Since, again, $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ is initial, it is sufficient to see that range(t) is closed in x whenever $t \in \mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. Suppose particulary that $t \in \mathscr{I}_{\alpha^*}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ and let $\alpha' = (\alpha', 1_{\mathtt{Top}}, !)$. Without loss of generality, α' can be assumed to be an imaginary of Cp. So then $\delta^{-1}\big[\mathrm{range}(t)\big]$ is oracle co-r.e. closed in u. This implies that $\delta^{-1}\big[\mathrm{range}(t)\big]$ is topologically closed in u (cf. Example 4.25). As δ is an open map, it follows that range(t) is closed in x.
Now $(*)$ holds. So we have shown that topological closedness is captured by $\mathscr{L}\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. Finally, we notice that $\overline{\mathscr{L}\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}}^{\mathtt{STop}} = U^*U$ where $U^* : \mathtt{Top} \to \mathtt{STop}$ is from Example 4.16 and where $U : \mathtt{Rep}_{\mathrm{op}} \to \mathtt{Top}$. Hence $\overline{\mathscr{L}\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}}^{\mathtt{STop}}$ fulfills the condition appear in Theorem 4.24.

5 Internalization

Internalizability is an important concept in category theory. Existences of exponentials coincides with the internalizability of hom-classes (see Section 5.1.1). Existence of a classifier coincides with the internalizability of classes of subobjects (see Section 5.1.2).

We discuss about such internalizabilities of "imaginary structures" in this section. In Section 5.1.1, one may see that the usual exponentials is sufficient to internalize imaginary morphisms, what correspond to oracle computable functions in computable analysis. In Section 5.1.2, it turns out that the usual classifier is not sufficient to internalize imaginary subobjects, and we define the notion of strong classifier with an additional requirements.

Using those developed methods, we analyze the situation that the action of $\mathscr{I}$ is equivalent to that of $\mathscr{L}$ for a given fundamental class $\mathscr{F}$, so the desired inequality $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$. Our second main theorem, Theorem 5.24, will be shown at the end of this section, Section 5.2.2.

As an application of Theorem 5.24, finally, we show that "oracle co-r.e. closedness coincides with topological closedness" on each object of $\mathtt{Rep}_{\mathrm{op}}$, a represented topological space with an open representation.

5.1 Coding

Here we define exponential and classifier. Their existences guarantee that our category has sufficiently rich structure so that hom-classes and classes of subobjects can be internalized. We also discuss about internalizability of "imaginary structures".

5.1.1 Exponential

First we introduce the notion of imaginary morphism. Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ be a strictly imaginary extensional pre-effectiveness. Let $x, y \in \mathbb{E}$ and let α be an imaginary. Suppose that $x \xleftarrow{\pi_1} x \times \alpha \xrightarrow{!} \alpha$ is a product. For each $x \times \alpha \xrightarrow{f} y$ in $\mathbb{E}$, let us call $f = (f, \alpha, \pi_1)$ an imaginary morphism from x to y, written as $f : x \xrightarrow{\mathfrak{Im}} y$ [12]. We occasionally write $f = (f, \alpha)$ when π is regarded as an implicit argument. For each $(\cdot \xrightarrow{t} x) \in \mathscr{T}$, we write $f[t]$ instead of $f[\pi_1^{-1}[t]]$. Particularly, if a is an imaginary element of x, we write $f(a)$ instead of $f[a]$. Also, for each $(\cdot \xrightarrow{t'} y) \in \mathscr{T}$, we write $f^{-1}[t']$ instead of $\pi_1[f^{-1}[t']]$. We can see the following result what is an extended result of Lemma 3.7.

Lemma 5.1. Let $f = (f, \alpha) : x \xrightarrow{\mathfrak{Im}} y$ and let $(\cdot \xrightarrow{t} x), (\cdot \xrightarrow{t'} y) \in \mathscr{T}$. One has the following equivalence.

$$f[t] \leq t' \iff t \leq f^{-1}[t']$$

[12]Each morphism $x \xrightarrow{f} y$ in $\mathbb{E}$ can be regarded as an imaginary morphism of the form $f = (f, 1, \mathrm{id})$.

Proof. Note that $x \times \alpha \xrightarrow{\pi_1} x$ gives a bijective correspondence by (i) of Lemma 3.32. Hence, we obtain:

$$\begin{array}{rcl} f[t] \le t' & \Longleftrightarrow & f[\pi_1^{-1}[t]] \le t' \\ & \Longleftrightarrow & \pi_1^{-1}[t] \le f^{-1}[t'] \\ & \Longleftrightarrow & t \le \pi_1[f^{-1}[t']] \\ & \Longleftrightarrow & t \le f^{-1}[t'] \end{array}$$

This shows the desired equivalence. □

One can also see the following lemma.

Lemma 5.2. Every $\mathscr{I}$-closed fundamental class $\mathscr{F}$ on $\mathbb{E}$ is stable under pullback by imaginary morphisms.

Proof. Immediately. □

We give a definition of exponential.

Definition 5.3. Let $\mathbb{E}$ be an arbitrary category which has finite product. For two objects $x, y \in \mathbb{E}$, the pair (y^x, ev) where $x \times y^x \xrightarrow{\mathrm{ev}} y$ in $\mathbb{E}$ is said to be an exponential if the following condition holds:

- for a given morphism of the form $x \times z \xrightarrow{f} y$, there exists unique morphism $z \xrightarrow{\hat{f}} y^x$ which makes the following diagram commute.

$$\begin{tikzcd} x \times y^x \arrow[r, "\mathrm{ev}"] & y \\ x \times z \arrow[u, "\mathrm{id}_x \times \hat{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

  Such unique morphism $\hat{f}$ is called transpose of f.

In that case, y^x is called an exponential object of x and y, and ev is called its evaluation.

We can also introduce the notion of weal exponential which is similar with exponential, but is defined without uniqueness condition for transpose. We say that $\mathbb{E}$ has (weak) exponential if there is an (weak) exponential for every two objects.

Example 5.4. `Set` has exponential. For each two sets $x, y \in$ `Set`, their exponential object can be given by usual function space y^x. As evaluation, we define:

$$\begin{array}{rcl} \mathrm{ev} : x \times y^x & \to & y \\ (a, f) & \mapsto & f(a) \end{array}$$

It is easy to see that (y^x, ev) is certainly an exponential.

Example 5.5. Cp has weak exponential. This is a consequence of the existence of a universal computable function, a function with utm-property and smn-property. See [9] for the detail.

In what follows, assume that $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a strictly imaginary extensional pre-effectiveness which has exponential. Let $f = (f, \alpha) : x \xrightarrow{\mathfrak{Im}} y$. We write $\lceil f \rceil_{x\to y}$ in a $(\mathscr{S}, \mathscr{T})$-factorization shown as the right one below.

$$\begin{tikzcd} x \times y^x \arrow[r, "\mathrm{ev}"] & y \\ x \times \alpha \arrow[u, "\mathrm{id}\times\hat{f}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \alpha \arrow[rr, "\hat{f}"] \arrow[dr, "!"'] & & y^x \\ & \cdot \arrow[ur, "\lceil f \rceil_{x\to y}"'] & \end{tikzcd}$$

This $\lceil f \rceil_{x\to y}$ is called code of f. Conversely, suppose that we have an imaginary element $(\alpha' \xrightarrow{e} y^x) \in \mathscr{T}$. We define a morphism $\lfloor e \rfloor_{x\to y}$ in the following commutative diagram.

$$\begin{tikzcd} x \times y^x \arrow[r, "\mathrm{ev}"] & y \\ x \times \alpha' \arrow[u, "\mathrm{id}\times e"] \arrow[ur, "\lfloor e \rfloor_{x\to y}"'] & \end{tikzcd}$$

The imaginary morphism $(\lfloor e \rfloor_{x\to y}, \alpha')$ will be abbreviated, merely, as $\lfloor e \rfloor_{x\to y}$. This $\lfloor e \rfloor_{x\to y}$ is called decode of e.

For each two imaginary morphisms $f = (f, \alpha), g = (g, \beta) : x \xrightarrow{\mathfrak{Im}} y$, we write $f \sim g$ if and only if for every imaginary element a of x, one has $f(a) \cong g(a)$. We establish a point.

Lemma 5.6. Let $f = (f, \alpha, \pi_1)$ be an imaginary morphism and α' be an imaginary with existence of $\alpha' \xrightarrow{!} \alpha$. Suppose that the following diagram commutes.

$$\begin{tikzcd} x \times \alpha \arrow[r, "\pi_1"] & x \\ x \times \alpha' \arrow[u, "\mathrm{id}\times !"'] \arrow[ur, "\pi_1'"'] & \end{tikzcd}$$

If we define $f' = f \circ (\mathrm{id}\times !)$ and abbreviate as $f' = (f', \alpha', \pi_1')$, then $f \sim f'$.

> **Proof.** By Lemma 3.31, we have a concatenation of two pullback diagrams as shown below.
>
> $$\begin{tikzcd} x \times \alpha' \arrow[r, "\pi_2'"] \arrow[d, "\mathrm{id}\times !"'] & \alpha' \arrow[r, "\mathrm{id}"] \arrow[d, "!"] & \alpha' \arrow[d, "!_{\alpha'}"] \\ x \times \alpha \arrow[r, "\pi_2"'] & \alpha \arrow[r, "!_\alpha"'] & 1 \end{tikzcd}$$
>
> Since $!_{\alpha'}$ belongs to $\mathscr{S}$, so is $! \times \mathrm{id}$. Hence, for every imaginary element a of x, we obtain:
>
> $$\begin{aligned} & (! \times \mathrm{id})^{-1}\big[\pi_1^{-1}\big[a\big]\big] \cong (\pi_1')^{-1}\big[a\big] \\ \Longrightarrow \quad & \pi_1^{-1}\big[a\big] \cong (! \times \mathrm{id})\big[(\pi_1')^{-1}\big[a\big]\big] \end{aligned}$$

And thus:

$$\begin{array}{rcllcl} f(a) & = & f\big[\pi_1^{-1}\big[a\big]\big] & \cong & f\big[(!\times \mathrm{id})\big[(\pi_1')^{-1}\big[a\big]\big]\big] & \\ & & & \cong & f'\big[(\pi_1')^{-1}\big[a\big]\big] & = \; f'(a) \end{array}$$

This shows that $f \sim f'$. □

Proposition 5.7. Let $f = (f, \alpha, \pi_1)$ and $g = (g, \beta, \pi_1')$ be two imaginary morphisms both from x to y. Then $f \sim g$ if and only if $\lceil f \rceil_{x\to y} \cong \lceil g \rceil_{x\to y}$.

Proof. Assume that we have $\lceil f \rceil_{x\to y} \cong \lceil g \rceil_{x\to y}$. Let α' and β' be the domains of $\lceil f \rceil_{x\to y}$ and $\lceil g \rceil_{x\to y}$, respectively. Firstly, $\lfloor \lceil f \rceil_{x\to y} \rfloor_{x\to y} \sim \lfloor \lceil g \rceil_{x\to y} \rfloor_{x\to y}$ is trivial. Next by the definition of code, both of $\alpha \xrightarrow{!} \alpha'$ and $\beta \xrightarrow{!} \beta'$ is certainly exists. By Lemma 5.6, $f \sim \lfloor \lceil f \rceil_{x\to y} \rfloor_{x\to y} \sim \lfloor \lceil g \rceil_{x\to y} \rfloor_{x\to y} \sim g$ follows. This is "if" of the desired assertion. Assume now that $f \sim g$. We define $f' = f \circ (\mathrm{id}\times !)$ where $! : \alpha \times \beta \to \alpha$ and abbreviate as $f' = (f', \alpha\times\beta, \pi_1 \circ (\mathrm{id}\times !))$. By Lemma 5.6, $f \sim f'$ follows. Similarly, if $g' = g \circ (\mathrm{id}\times !)$ where $! : \alpha \times \beta \to \beta$ and $g' = (g', \alpha \times \beta, \pi_1' \circ (\mathrm{id}\times !))$, then $g \sim g'$.

$$\begin{tikzcd} x \times \alpha \arrow[r, "f"] & y \arrow[d, equal] \\ x \times (\alpha \times \beta) \arrow[u, "\mathrm{id}\times !"] \arrow[d, "\mathrm{id}\times !"'] \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & y \arrow[d, equal] \\ x \times \beta \arrow[r, "g"'] & y \end{tikzcd}$$

Suppose that $f' \neq g'$. Since $\mathbb{E}$ is strictly imaginary extensional, one can find a morphism k with being an imaginary of its domain such that $f'k \neq g'k$. Let $k = ts$ be a $(\mathscr{S}, \mathscr{T})$-factorization and let us define $a = (\pi_1 \circ (\mathrm{id}\times !))\big[t\big]$. Then:

$$\begin{array}{rcl} f'(a) & = & f'\big[(\pi_1 \circ (\mathrm{id}\times !))^{-1}\big[a\big]\big] \\ & \cong & f'\big[(\pi_1 \circ (\mathrm{id}\times !))^{-1}\big[(\pi_1 \circ (\mathrm{id}\times !))\big[t\big]\big]\big] \;\cong\; f'\big[t\big] \end{array}$$

Note that the following diagram commutes.

$$\begin{tikzcd} x \times (\alpha \times \beta) \arrow[r, "\mathrm{id}\times !"] \arrow[d, "\mathrm{id}\times !"'] & x \times \alpha \arrow[d, "\pi_1"] \\ x \times \beta \arrow[r, "\pi_1'"'] & x \end{tikzcd}$$

Hence we also have:

$$\begin{array}{rcl} g'(a) & = & g'\big[(\pi_1' \circ (\mathrm{id}\times !))^{-1}\big[a\big]\big] \\ & = & g'\big[(\pi_1 \circ (\mathrm{id}\times !))^{-1}\big[a\big]\big] \\ & \cong & g'\big[(\pi_1 \circ (\mathrm{id}\times !))^{-1}\big[(\pi_1 \circ (\mathrm{id}\times !))\big[t\big]\big]\big] \;\cong\; g'\big[t\big] \end{array}$$

This implies $f(a) \cong f'(a) \not\cong g'(a) \cong g(a)$ since $f'\big[t\big] \not\cong g'\big[t\big]$ holds by Lemma 3.41. Contradiction. Therefore $f' = g'$. Now by Lemma 5.6, $\lceil f \rceil_{x \to y} \cong \lceil f' \rceil_{x \to y} \cong \lceil g' \rceil_{x \to y} \cong \lceil g \rceil_{x \to y}$ holds. This is "only if" of the desired assertion. $\square$

Let $f = (f, \alpha) : x \xrightarrow{\mathfrak{Im}} y$ and let $g = (g, \beta) : y \xrightarrow{\mathfrak{Im}} z$. Let us abbreviate as $g \circ f = g \circ (f \times \mathrm{id}_\beta) \circ \mathrm{can}$.

$$x \times (\alpha \times \beta) \xrightarrow{\mathrm{can}} (x \times \alpha) \times \beta \xrightarrow{f \times \mathrm{id}_\beta} y \times \beta \xrightarrow{g} y$$

We define a new imaginary morphism by $g \circ f = (g \circ f, \alpha \times \beta, \pi_1'')$ where π_1'' is implicitly equipped first projection of $x \times (\alpha \times \beta)$.

For every three objects $x, y, z \in \mathbb{E}$, we define a morphism $(- \cdot -) : z^y \times y^x \to z^x$ in the following commutative diagram:

$$\begin{tikzcd}
x \times z^x \arrow[rr, "\mathrm{ev}"] & & z \\
x \times (z^y \times y^x) \arrow[u, "\mathrm{id} \times (- \cdot -)"] \arrow[r, "\mathrm{can}"'] & (x \times y^x) \times z^y \arrow[r, "\mathrm{ev} \times \mathrm{id}"] & y \times z^y \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

where can is the canonical isomorphism. If a and b is two imaginary elements of z^y and y^x, respectively, then we abbreviate as $a \cdot b = (- \cdot -)(a \times b)$. Note, here, that $a \times b$ is an imaginary element of $z^y \times y^x$.

Lemma 5.8. Suppose that $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ is a strictly imaginary extensional pre-effectiveness which has exponential. Let $f = (f, \alpha) : x \xrightarrow{\mathfrak{Im}} y$ and let $g = (g, \beta) : y \xrightarrow{\mathfrak{Im}} z$. One has the following identification.

$$\lceil g \circ f \rceil_{x \to z} \quad \cong \quad \lceil g \rceil_{y \to z} \cdot \lceil f \rceil_{x \to y}$$

Proof. By the definition of $(- \cdot -)$ and $g \circ f$, the following diagram commutes.

$$\begin{tikzcd}
x \times z^x \arrow[r, equal] & x \times z^x \arrow[rrr, "\mathrm{ev}"] & & & z \arrow[d, equal] \\
& x \times (z^y \times y^x) \arrow[u, "\mathrm{id} \times (- \cdot -)"] \arrow[r, "\mathrm{can}"] & (x \times y^x) \times z^y \arrow[r, "\mathrm{ev} \times \mathrm{id}"] & y \times z^y \arrow[r, "\mathrm{ev}"] & z \arrow[d, equal] \\
x \times (\alpha \times \beta) \arrow[uu, "\mathrm{id} \times (g \circ f)^\wedge"] \arrow[r, "\mathrm{id} \times \mathrm{can}"'] \arrow[rrrr, bend right, "g \circ f"'] & x \times (\beta \times \alpha) \arrow[u, "\mathrm{id} \times (\hat{g} \times \hat{f})"] \arrow[r, "\mathrm{can}"'] & (x \times \alpha) \times \beta \arrow[u, "(\mathrm{id} \times \hat{f}) \times \hat{g}"] \arrow[r, "f \times \mathrm{id}"'] & y \times \beta \arrow[u, "\mathrm{id} \times \hat{g}"] \arrow[r, "g"'] & z
\end{tikzcd}$$

Thus universality of exponential implies $(g \circ f)^\wedge = (- \cdot -) \circ (\hat{g} \times \hat{f}) \circ \mathrm{can}$. Now the desired identification:

$$\lceil g \circ f \rceil_{x \to z} \cong \lceil g \rceil_{y \to z} \cdot \lceil f \rceil_{x \to y}$$

follows from essential uniqueness of $(\mathscr{S}, \mathscr{T})$-factorization. $\square$

Corollary 1. Let $f, f' : x \xrightarrow{\mathfrak{Im}} y$ and let $g, g' : y \xrightarrow{\mathfrak{Im}} z$. One has $g \circ f \sim g' \circ f'$ whenever $f \sim f'$ and $g \sim g'$.

Proof. Suppose that $f \sim f'$ and $g \sim g'$. By Proposition 5.7, $\lceil f \rceil_{x \to y} \cong \lceil f' \rceil_{x \to y}$ and $\lceil g \rceil_{y \to z} \cong \lceil g' \rceil_{y \to z}$ holds. Hence by Lemma 5.8 and again by Proposition 5.7, we obtain:

$$\begin{aligned} \lceil f \rceil_{x \to y} \cong \lceil f' \rceil_{x \to y},\ \lceil g \rceil_{y \to z} \cong \lceil g' \rceil_{y \to z} \quad &\Longrightarrow \quad \lceil g \rceil_{y \to z} \cdot \lceil f \rceil_{x \to y} \cong \lceil g' \rceil_{y \to z} \cdot \lceil f' \rceil_{x \to y} \\ &\Longleftrightarrow \quad \lceil g \circ f \rceil_{x \to z} \cong \lceil g' \circ f' \rceil_{x \to z} \\ &\Longleftrightarrow \quad g \circ f \sim g' \circ f' \end{aligned}$$

This is the desired assertion. $\square$

Let $x, y \in \mathbb{E}$. We say that x is imaginary isomorphic to y if there is an imaginary β and an imaginary morphism $h = (h, \alpha) : x \xrightarrow{\mathfrak{Im}} y$ such that $h \times \mathrm{id}_\beta$ is an isomorphism. Such h is called an imaginary isomorphism in that case.

Lemma 5.9. Let $\mathscr{F}'$ be a $\mathscr{I}$-closed fundamental class on $\mathbb{E}$. For every two objects $x, y \in \mathbb{E}$, if $h = (h, \alpha)$ is an imaginary isomorphism from x to y, then $h\big[-\big] : \mathscr{F}'(x) \to \mathscr{F}'(y)$ gives a bijective correspondence.

Proof. One can find an imaginary β which makes $h \times \mathrm{id}_\beta$ to be an isomorphism. Since α is $\mathscr{F}'$-compact by (ii) of Proposition 3.33, $\pi_1\big[-\big]$ gives a bijective correspondence from $x \times \alpha$ to x where $x \times \alpha \xrightarrow{\pi_1} x$. Similar with $\pi_1' : x \times (\alpha \times \beta) \to x$ and with $\pi_1'' : y \times \beta \to y$. And, of course, as $h \times \mathrm{id}_\beta$ is being an isomorphism, $(h \times \mathrm{id}_\beta)\big[-\big]$ is a bijective correspondence from $\mathscr{F}'((x \times \alpha) \times \beta)$ to $\mathscr{F}'(y \times \beta)$. Hence all correspondences appear in the following commutative diagram are bijective.

$$\begin{tikzcd} \mathscr{F}'(x \times \alpha) \arrow[r, "h[-]"] & \mathscr{F}'(y) \\ \mathscr{F}'((x \times \alpha) \times \beta) \arrow[u, "\pi_1'[-]"] \arrow[r, "(h \times \mathrm{id}_\beta)[-]"'] & \mathscr{F}'(y \times \beta) \arrow[u, "\pi_1''[-]"'] \end{tikzcd}$$

Therefore $h\big[-\big] = h\big[\pi_1^{-1}\big[-\big]\big] : \mathscr{F}'(x) \to \mathscr{F}'(y)$ is again an bijective correspondence. $\square$

Exponentials in A Category of Represented Spaces

Lemma 5.10. Let $\mathbb{E}$ be a topological category over `Set` which has exponential. For every two regular epimorphisms f and g, their product $f \times g$ is again a regular epimorphism.

Proof. Let f be a regular epimorphism and let $x \in \mathbb{E}$. First we show that $\mathrm{id}_x \times f$ is regular epimorphic. Suppose that f is a coequalizer of g_1, g_2 and that $h(\mathrm{id}_x \times g_1) = h(\mathrm{id}_x \times g_2)$ in the following situation.

$$\begin{tikzcd} x \times \cdot \arrow[r, shift left, "\mathrm{id}_x \times g_1"] \arrow[r, shift right, "\mathrm{id}_x \times g_2"'] & x \times \cdot \arrow[r, "h"] & y \end{tikzcd}$$

The following diagram commutes where $i = 1, 2$.

$$\begin{tikzcd} x \times y^x \arrow[r, "\mathrm{ev}"] & y \\ x \times \cdot \arrow[u, "\mathrm{id}_x \times \hat{h}"] \arrow[ur, "h"'] & \\ x \times \cdot \arrow[u, "\mathrm{id}_x \times g_i"] \arrow[uur, bend right, "hg_1 = hg_2"'] & \end{tikzcd}$$

Hence $\hat{h}g_1 = \hat{h}g_2$. There exists unique morphism j which makes the following diagram commute.

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "\hat{h}"'] & \cdot \arrow[d, "j"] \\ & y^x \end{tikzcd}$$

We obtain $h = \mathrm{ev} \circ (\mathrm{id}_x \times \hat{h}) = \mathrm{ev} \circ (\mathrm{id}_x \times j) \circ (\mathrm{id}_x \times f)$. So, finally, the following diagram commutes.

$$\begin{tikzcd} x \times \cdot \arrow[r, "\mathrm{id}_x \times f"] \arrow[dr, "h"'] & x \times \cdot \arrow[d, "\mathrm{ev} \circ (\mathrm{id}_x \times j)"] \\ & y \end{tikzcd}$$

Since $f \in \mathrm{RegEpi}_{\mathbb{E}} \subseteq \mathrm{Ext}^{\mathrm{Epi}_{\mathtt{Set}}}$ and since $\mathrm{Ext}^{\mathrm{Epi}_{\mathtt{Set}}}$ is stable under pullback, $\mathrm{id}_x \times f \in \mathrm{Ext}^{\mathrm{Epi}_{\mathtt{Set}}} \subseteq \mathrm{Epi}_{\mathbb{E}}$. This implies the expected uniqueness of $\mathrm{ev} \circ (\mathrm{id}_x \times j)$ immediately. Therefore $\mathrm{id}_x \times f$ a coequalizer of $\mathrm{id}_x \times g_1$ and $\mathrm{id}_x \times g_2$, and thus is regular epimorphic. Now let f and g be two regular epimorphisms. Then both $\mathrm{id} \times f$ and $g \times \mathrm{id}$ are regular epimorphic. Note that regular epicity coincides with quotientness in $\mathbb{E}$ (cf. (ii) of Lemma 4.14). So $f \times g = (\mathrm{id} \times f) \circ (g \times \mathrm{id})$ is regular epimorphic again. $\square$

Proposition 5.11. Let $\mathbb{D}$ be a regular effectiveness over `Set` which has weak exponential and let $\mathbb{E}$ be a topological category over `Set` which has exponential. Suppose that $F \in \mathtt{Reg}_{\mathtt{Set}}(\mathbb{D}, \mathbb{E})$ and $\mathscr{R}F = (\mathscr{R}F \xrightarrow{U} \mathbb{E})$. Every two objects $\delta = (Fx \xrightarrow{\delta} y), \delta' = (Fx' \xrightarrow{\delta'} y') \in \mathscr{R}F$ have an exponential whenever δ is regular epimorphic.

Proof. Let $((x')^x, \mathrm{ev}^*)$ be a weak exponential of x and x' in $\mathbb{D}$. Suppose that δ is a coequalizer of a pair of parallel morphisms $z \overset{f_1}{\underset{f_2}{\rightrightarrows}} Fx$. Let us define four morphisms $g_1, \hat{g}_1$ and $g_2, \hat{g}_2$ in the following commutative diagram, respectively, where $i = 1, 2$.

$$\begin{tikzcd}
z \times (y')^z \arrow[rrr, "\mathrm{ev}"] & & & y' \\
z \times F(x')^x \arrow[u, "\mathrm{id}\times\hat{g}_i"] \arrow[urrr, "g_i"] \arrow[r, "f_i\times\mathrm{id}"'] & Fx \times F(x')^x \arrow[r, "\mathrm{can}"'] & F(x \times (x')^x) \arrow[r, "F\mathrm{ev}^*"'] & Fx' \arrow[u, "\delta'"']
\end{tikzcd}$$

Let $w \xrightarrow{u} F(x')^x$ be an equalizer of $\hat{g}_1$ and $\hat{g}_2$. Define $h = F\mathrm{ev}^* \circ \mathrm{can} \circ (\mathrm{id} \times u)$. The following calculation is valid.

$$\begin{aligned}
\delta' h(f_1 \times \mathrm{id}_w) &= \delta' \circ F\mathrm{ev}^* \circ \mathrm{can} \circ (\mathrm{id} \times u) \circ (f_1 \times \mathrm{id}_w) \\
&= \delta' \circ F\mathrm{ev}^* \circ \mathrm{can} \circ (f_1 \times \mathrm{id}_w) \circ (\mathrm{id}_z \times u) \\
&= g_1 \circ (\mathrm{id}_z \times u) \\
&= \mathrm{ev} \circ (\mathrm{id}_z \times \hat{g}_1) \circ (\mathrm{id}_z \times u) \\
&= \mathrm{ev} \circ (\mathrm{id}_z \times \hat{g}_1 \circ u)
\end{aligned}$$

Similarly one may see that $\delta' h(f_2 \times \mathrm{id}_w) = \mathrm{ev} \circ (\mathrm{id}_z \times \hat{g}_2 \circ u)$ and thus the following identification holds.

$$\delta' h(f_1 \times \mathrm{id}_w) \quad = \quad \delta' h(f_2 \times \mathrm{id}_w)$$

By Lemma 5.10, $\delta \times \mathrm{id}_w$ is a coequalizer of $f_1 \times \mathrm{id}_w$ and $f_2 \times \mathrm{id}_w$. Hence one can find unique morphisms j and $\hat{j}$ which makes the following diagram commute.

$$\begin{tikzcd}
y \times (y')^y \arrow[r, "\mathrm{ev}"] & y' \\
y \times w \arrow[u, "\mathrm{id}\times\hat{j}"] \arrow[r, "j"] & y' \arrow[u, equal] \\
Fx \times w \arrow[u, "\delta\times\mathrm{id}_w"] \arrow[r, "h"'] & Fx' \arrow[u, "\delta'"']
\end{tikzcd}$$

Note that we can find a morphism $(w_* \xrightarrow{v} (x')^x) \in \mathrm{Emb}_{\mathbb{D}}^{\mathrm{Mono}_{\mathrm{Set}}}$ such that $u \cong Fv$. Suppose that can is the canonical isomorphism which makes the following diagram commute.

$$\begin{tikzcd}
w_* \arrow[rr, "\mathrm{can}"] \arrow[dr, "Fv"'] & & w \arrow[dl, "u"] \\
& F(x')^x &
\end{tikzcd}$$

Now let $\hat{j} \circ \mathrm{can} = t\delta_*$ be an $(\mathrm{Ext}_{\mathbb{E}}^{\mathrm{Epi}_{\mathrm{Set}}}, \mathrm{Emb}_{\mathbb{E}}^{\mathrm{Mono}_{\mathrm{Set}}})$-factorization of $\hat{j} \circ \mathrm{can}$ and let us denote by y_* the codomain of δ_*. So $\delta_* = (w_*, y_*, \delta_*)$ is an object of $\mathscr{R}F$. We also define $\mathrm{ev}_* = \mathrm{ev} \circ (\mathrm{id}_y \times t)$.

We show that $(\delta_*, \mathrm{ev}_*)$ is an exponential of δ and δ' in $\mathscr{R}F$. Let $\delta \times \tilde{\delta} \xrightarrow{f} \delta'$ be a morphism in $\mathscr{R}F$ and suppose that $\tilde{\delta}$ is being of the form $\tilde{\delta} = (\tilde{x}, \tilde{y}, \tilde{\delta})$. One can find a morphism k in $\mathbb{D}$ which makes the following diagram commutes where we abbreviate as $\delta \times \tilde{\delta} = \delta \times \tilde{\delta} \circ \mathrm{can}'$.

$$\begin{tikzcd}
 & y \times \tilde{y} \arrow[r, "f"] & y' \\
Fx \times F\tilde{x} \arrow[ur, "\delta\times\tilde{\delta}"] \arrow[r, "\mathrm{can}'"'] & F(x \times \tilde{x}) \arrow[u, "\delta\times\tilde{\delta}"] \arrow[r, "Fk"'] & Fx' \arrow[u, "\delta'"']
\end{tikzcd}$$

Let the following two diagrams be commutative.

$$\begin{tikzcd}
y \times (y')^y \arrow[r, "\mathrm{ev}"] & y' \\
y \times \tilde{y} \arrow[u, "\mathrm{id}_y \times \hat{f}"] \arrow[ur, "f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \times (x')^x \arrow[r, "\mathrm{ev}^*"] & x' \\
x \times \tilde{x} \arrow[u, "\mathrm{id}_x \times \hat{k}"] \arrow[ur, "k"']
\end{tikzcd}$$

Note that the following diagram also commutes.

$$\begin{tikzcd}
Fx \times F(x')^x \arrow[r, "\mathrm{can}"] & F(x \times (x')^x) \\
Fx \times F\tilde{x} \arrow[u, "F\mathrm{id}_x \times F\hat{k}"] \arrow[r, "\mathrm{can}'"'] & F(x \times \tilde{x}) \arrow[u, "F(\mathrm{id}_x \times \hat{k})"']
\end{tikzcd}$$

Then we obtain:

$$\begin{aligned}
g_1(\mathrm{id}_z \times F\hat{k}) &= \delta' \circ F\mathrm{ev}^* \circ \mathrm{can} \circ (f_1 \times \mathrm{id}) \circ (\mathrm{id}_z \times F\hat{k}) \\
&= \delta' \circ F\mathrm{ev}^* \circ \mathrm{can} \circ (F\mathrm{id}_x \times F\hat{k}) \circ (f_1 \times \mathrm{id}) \\
&= \delta' \circ F\mathrm{ev}^* \circ F(\mathrm{id}_x \times \hat{k}) \circ (f_1 \times \mathrm{id}) \\
&= \delta' \circ Fk \circ (f_1 \times \mathrm{id}) \\
&= f \circ (\delta \times \tilde{\delta}) \circ (f_1 \times \mathrm{id}) \\
&= f \circ (\delta f_1 \times \tilde{\delta})
\end{aligned}$$

Similarly one can obtain $g_2(\mathrm{id}_z \times F\hat{k}) = f \circ (\delta f_2 \times \tilde{\delta})$. Since $\delta f_1 = \delta f_2$, the following diagram commutes where $i = 1, 2$.

$$\begin{tikzcd}
z \times (y')^z \arrow[r, "\mathrm{ev}"] & y' \\
z \times F(x')^x \arrow[u, "\mathrm{id}_z \times \hat{g}_i"] \arrow[ur, "g_i"'] & \\
z \times F\tilde{x} \arrow[u, "\mathrm{id}_z \times F\hat{k}"] \arrow[uur, bend right, "g_1(\mathrm{id}_z \times F\hat{k}) = g_2(\mathrm{id}_z \times F\hat{k})"'] &
\end{tikzcd}$$

This implies $g_1 F\hat{k} = g_2 F\hat{k}$. One can find unique morphism l which makes the following diagram commute by universality of pullback.

$$\begin{tikzcd}
w_* \arrow[r, "Fv"] & F(x')^x \\
F\tilde{x} \arrow[u, "l"] \arrow[ur, "\hat{k}"'] &
\end{tikzcd}$$

Note that $\delta \times \mathrm{id}_{F\tilde{x}} \in \mathrm{RegEpi}_{\mathbb{E}} \subseteq \mathrm{Epi}_{\mathbb{E}}$ by Lemma 5.10. Using this fact and universality of exponential, one can see that the following diagram commutes where $h' = F(\mathrm{ev}^* \circ (\mathrm{id}_x \times v) \circ \mathrm{can}'')$ and $\mathrm{can}'' : Fx \times Fw_* \to F(x \times w_*)$.

$$\begin{tikzcd}
y \times (y')^y \arrow[rr, "\mathrm{ev}"] & & y' \arrow[dd, equal] \\
& y \times \tilde{y} \arrow[ul, "\mathrm{id}_y \times \hat{f}"'] \arrow[ur, "f"] & \\
y \times Fw_* \arrow[uu, "\mathrm{id}_y \times (\hat{j} \circ \mathrm{can})"] \arrow[rr, "j"] & & y' \\
& y \times F\tilde{x} \arrow[uu, "\mathrm{id}_y \times \tilde{\delta}"] \arrow[ul, "\mathrm{id}_y \times l"'] \arrow[ur, "j \circ (\mathrm{id}_y \times l)"'] & \\
Fx \times Fw_* \arrow[uu, "\delta \times \mathrm{id}_{Fw_*}"] \arrow[rr, "h'"] & & Fx' \arrow[uu, "\delta'"'] \\
& Fx \times F\tilde{x} \arrow[uu, "\delta \times \mathrm{id}_{F\tilde{x}}"] \arrow[ul, "\mathrm{id}_{Fx} \times l"] \arrow[ur, "Fk \circ \mathrm{can}'"'] &
\end{tikzcd}$$

By orthogonality $\tilde{\delta} \perp t$, there exists unique morphism $\hat{f}_*$ which makes the following diagram commute.

$$\begin{tikzcd}
F\tilde{x} \arrow[r, "\delta_*"] \arrow[d, "\tilde{\delta}"'] & y_* \arrow[d, "t"] \\
\tilde{y} \arrow[r, "\hat{f}"'] \arrow[ur, "\hat{f}_*"] & (y')^y
\end{tikzcd}$$

Now we obtain the following commutative diagram.

$$\begin{tikzcd}
y \times y_* \arrow[r, "t"] \arrow[rr, bend left, "\mathrm{ev}_*"] & y \times (y')^y \arrow[r, "\mathrm{ev}"] & y' \\
& y \times \tilde{y} \arrow[ul, "\mathrm{id}_y \times \hat{f}_*"'] \arrow[u, "\mathrm{id}_y \times \hat{f}"] \arrow[ur, "f"'] & \\
Fx \times Fx_* \arrow[uu, "\delta \times \delta_*"] \arrow[rr, "h'"] & & Fx' \arrow[uu, "\delta'"'] \\
& Fx \times F\tilde{x} \arrow[uu, "\delta \times \tilde{\delta}"] \arrow[ul, "\mathrm{id}_{Fx} \times l"] \arrow[ur, "Fk \circ \mathrm{can}'"'] &
\end{tikzcd}$$

Note that one can find a morphism l' in $\mathbb{D}$ which makes the following diagram commute by the definition of $\mathrm{Emb}_{\mathbb{D}}^{\mathrm{Mono}_{\mathbf{Set}}}$.

$$\begin{tikzcd}
Fx \times Fx_* \arrow[r, "\mathrm{can}''"] & F(x \times x_*) \\
Fx \times F\tilde{x} \arrow[u, "\mathrm{id}_{Fx} \times l"] \arrow[r, "\mathrm{can}'"'] & F(x \times \tilde{x}) \arrow[u, "F(\mathrm{id}_x \times l')"']
\end{tikzcd}$$

This shows that the following diagram commutes in $\mathscr{R}F$.

$$\begin{tikzcd}
\delta \times \delta_* \arrow[r, "\mathrm{ev}_*"] & \delta' \\
\delta \times \tilde{\delta} \arrow[u, "\mathrm{id}_\delta \times \hat{f}_*"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The expected uniqueness of $\hat{f}_*$ is trivial, and thus $(\delta_*, \mathrm{ev}_*)$ is an exponential of δ and δ' in $\mathscr{R}F$. $\square$

5.1.2 Classifier

We give a definition of classifier.

Definition 5.12. Let $\mathbb{E}$ be a pre-effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. A morphism $(1 \xrightarrow{\top} \Omega) \in \mathscr{F}$ is said to be a classifier of $\mathscr{F}$ if the following condition holds:

- for every $t \in \mathscr{F}$, there exists unique morphism $\mathrm{ch}(t)$ which makes the following square pullback.

$$\begin{tikzcd}
\cdot \arrow[r, "!"] \arrow[d, "t"'] & 1 \arrow[d, "\top"] \\
\cdot \arrow[r, "\chi"'] & \Omega
\end{tikzcd}$$

  Such unique $\mathrm{ch}(t)$ will be called character of t.

Let $\mathbb{E}$ be a pre-effectiveness. Suppose that a fundamental class $\mathscr{F}$ on $\mathbb{E}$ has a classifier $1 \xrightarrow{\top} \Omega$. For each $t \in \mathscr{I}\mathscr{F}$, particularly if $t \in \mathscr{I}_\alpha\mathscr{F}$, of course $t \times \alpha$ belongs to $\mathscr{F}$, and we define $\mathrm{ch}_\alpha(t) = \mathrm{ch}(t \circ \alpha)$. We also abbreviate as $\mathrm{ch}(\alpha \times t) = (\mathrm{ch}(\alpha \times t), \alpha, \pi_1)$ and call it an imaginary character of t.

Definition 5.13. Let $\mathbb{E}$ be a pre-effectiveness and let $\mathscr{F}$ be a fundamental class on $\mathbb{E}$. A classifier $1 \xrightarrow{\top} \Omega$ of $\mathscr{F}$ is said to be strong if the following condition holds:

- for every $t \in \mathscr{I}\mathscr{F}$ has essentially unique imaginary character i.e. if both $\mathrm{ch}_\alpha(t)$ and $\mathrm{ch}_\beta(t)$ are imaginary characters of t, then $\mathrm{ch}_\alpha(t) \sim \mathrm{ch}_\beta(t)$.

It is easy to see that if $\mathbb{E}$ is strictly imaginary extensional, a classifier $1 \xrightarrow{\top} \Omega$ of $\mathscr{F}$ is strong whenever Ω has at most two imaginary elements.

Definition 5.14. Let $\mathbb{E}$ be a strictly imaginary extensional pre-effectiveness which has exponential. Suppose that a fundamental class $\mathscr{F}$ on $\mathbb{E}$ has a strong classifier $1 \xrightarrow{\top} \Omega$. In such a situation, we say that $\mathbb{E}$ is pseudo higher orderly $(\mathscr{F}; \Omega, \top)$-structured.

In what follows, let $\mathbb{E}$ be an arbitrarily fixed pseudo higher orderly $(\mathscr{F}; \Omega, \top)$-structured pre-effectiveness. For a given object $x \in \mathbb{E}$, as an abbreviation, we write Hx instead of Ω^x. And for each $(\cdot \xrightarrow{t} x) \in \mathscr{I}\mathscr{F}$, we write $\lceil t \rceil_x$ instead of $\lceil \mathrm{ch}_\alpha(t) \rceil_{x \to \Omega}$ where $\mathrm{ch}_\alpha(t)$ is an imaginary character of t. By the definition of strong classifier and by Proposition 5.7, $\lceil t \rceil_x$ is essentially unique. So it doesn't depend on a choice of $\mathrm{ch}_\alpha(t)$. Conversely, if we given an imaginary element e of Hx, we define $\lfloor e \rfloor_{Hx} = (\lfloor e \rfloor_{x \to \Omega})^{-1}[\top]$ and call it decode of e.

Example 5.15. $\mathtt{Rep}$ is pseudo higher orderly $(\Pi^0_{1,\mathtt{Rep}}; \Omega, \top)$-structured where Ω and $\top$ are from Example 4.19 (see also Proposition 5.11 and Example 5.5).

Proposition 5.16. Let $\mathbb{E}$ be a strictly imaginary extensional pre-effectiveness which has exponential. For every $(\cdot \xrightarrow{t} x), (\cdot \xrightarrow{t'} x) \in \mathscr{I}\mathscr{F}$, one has $t \cong t'$ if and only if $\lceil t \rceil_x \cong \lceil t' \rceil_x$.

> **Proof.** "if" is trivial. And "only if" follows from the definition of strong classifier and Proposition 5.7. □

Let $f = (f, \alpha, \pi_1)$ be an imaginary morphism from x to y and let $(\cdot \xrightarrow{t} y) \in \mathscr{I}\mathscr{F}$. Define $\mathrm{inv}f(\lceil t \rceil_y) = \lceil t \rceil_y \cdot \lceil f \rceil_{x \to y}$. We also define a morphism $\mathrm{inv}f(-) : Hy \to Hx$ by $\mathrm{inv}f(-) = (\lceil f \rceil_{x \to y} \times (- \cdot -)^*) \circ \mathrm{can}'$ where $\alpha' = \mathrm{dom}\lceil f \rceil_{x \to y}$, $\mathrm{can}' : Hy \times \alpha' \cong \alpha' \times Hy$ and $(- \cdot -)^*$ is defined in the following commutative diagram.

$$
\begin{tikzcd}
y^x \times Hx^{y^x} \arrow[rr, "\mathrm{ev}"] & & Hx \arrow[d, equal] \\
y^x \times Hy \arrow[u, "\mathrm{id}\times(-\cdot-)^*"] \arrow[r, "\mathrm{can}"'] & Hy \times y^x \arrow[r, "(-\cdot-)"] & Hx
\end{tikzcd}
$$

The imaginary morphism $(\mathrm{inv}f(-), \alpha', \pi'_1)$ will be abbreviated, merely, as $\mathrm{inv}f(-)$.

Lemma 5.17. Let $f = (f, \alpha, \pi_1)$ be an imaginary morphism from x to y and let $(\cdot \xrightarrow{t} y) \in \mathscr{I}\mathscr{F}$. One can see the following identification.

$$\lceil f^{-1}[t] \rceil_x \cong \mathrm{inv}f(\lceil t \rceil_y)$$

Proof. Let $\mathrm{ch}_\beta(t)$ be an imaginary character of t. The following diagram commutes, particularly both middle and right squares are pullback, by the definitions of $f^{-1}\big[t\big]$ and $\mathrm{ch}_\beta(t) \circ f$.

$$\begin{tikzcd}
\cdot \arrow[d, "f^{-1}\left[t\right]"'] & \cdot \arrow[l] \arrow[r] \arrow[d, "((f\times\beta)\circ\mathrm{can})^{-1}\left[t\times\beta\right]"] & \cdot \arrow[r, "!"] \arrow[d, "t\times\beta"] & 1 \arrow[d, "\top"] \\
x & x \times (\alpha \times \beta) \arrow[l, "\pi_1"] \arrow[r, "(f\times\beta)\circ\mathrm{can}"'] \arrow[rr, bend right, "\mathrm{ch}_\beta(t)\circ f"'] & y \times \beta \arrow[r, "\mathrm{ch}_\beta(t)"'] & \Omega
\end{tikzcd}$$

By the uniqueness which we required in the definition of classifier, one has $\mathrm{ch}_{\alpha\times\beta}(f^{-1}\big[t\big]) = \mathrm{ch}_\beta(t) \circ f$. Now the following holds.

$$\begin{aligned}
\lceil f^{-1}\big[t\big]\rceil_x &\cong \lceil \mathrm{ch}_{\alpha\times\beta}(f^{-1}\big[t\big])\rceil_{x\to\Omega} \\
&\cong \lceil \mathrm{ch}_\beta(t) \circ f\rceil_{x\to\Omega} \\
&\cong \lceil \mathrm{ch}_\beta(t)\rceil_{y\to\Omega} \cdot \lceil f\rceil_{x\to y} \;=\; \mathrm{inv}f(\lceil t\rceil_y)
\end{aligned}$$

This is the desired identification. □

5.2 The Other Direction

5.2.1 Opposite Class

In what follows, let $\mathbb{E}$ be an arbitrarily fixed pseudo higher orderly $(\mathscr{F};\Omega,\top)$-structured effectiveness. For given $(\cdot \xrightarrow{t} x) \in \mathscr{L}\mathscr{F}$ and $(\cdot \xrightarrow{r} Hx) \in \mathscr{T}$, we say that r is a representation of t (in Hx) if the following condition holds:

- for every imaginary element e of Hx, one has $e \in r$ if and only if $t \le \lfloor e\rfloor_{Hx}$.

It is easy to see that if $\mathbb{E}$ is a regular effectiveness over `Set`, then representation of each morphism in $\mathscr{L}\mathscr{F}$ is at most essentially unique i.e. if both r and r' are representations of $t \in \mathscr{L}\mathscr{F}$, then $r \cong r'$.

Lemma 5.18. For each object x, id_x has essentially unique representation. Particularly, $\lceil \mathrm{id}_x\rceil_x$ is a representation of it.

Proof. Let $(\cdot \xrightarrow{t} x) \in \mathscr{I}\mathscr{F}$, first, with $\mathrm{id}_x \le t$. Then obviously $\mathrm{id}_x \cong t$ and thus by Proposition 5.16, we obtain $\lceil \mathrm{id}_x\rceil_x \cong \lceil t\rceil_x$. Next suppose that $\lceil t\rceil_x \le \lceil \mathrm{id}_x\rceil_x$. Since the domains of $\lceil t\rceil_x$ and $\lceil \mathrm{id}_x\rceil_x$ are imaginaries, the unique morphism j which makes the following diagram commute belongs to both of $\mathscr{S}$ and $\mathscr{T}$.

$$\begin{tikzcd}
\cdot \arrow[rr, "j"] \arrow[dr, "\lceil t\rceil_x"'] & & \cdot \arrow[dl, "\lceil \mathrm{id}_x\rceil_x"] \\
& Hx &
\end{tikzcd}$$

Hence $\lceil t \rceil_x \cong \lceil \mathrm{id}_x \rceil_x$ and again by Proposition 5.16, $t \cong \mathrm{id}_x$ follows. So $\lceil \mathrm{id}_x \rceil_x$ is a representation of id_x.
Assume that we are given a representation r of id_x. Obviously $\lceil \mathrm{id}_x \rceil_x \leq r$. Let the following diagram be commutative by a (necessarily unique) morphism j.

$$\begin{tikzcd} \alpha \arrow[rr, "j"] \arrow[dr, "\lceil \mathrm{id}_x \rceil_x"'] & & \beta \arrow[dl, "r"] \\ & Hx & \end{tikzcd}$$

If the domain β of r is an imaginary, j belongs to both $\mathscr{S}$ and $\mathscr{T}$. Then of course j is isomorphic and thus $\lceil \mathrm{id}_x \rceil_x \cong r$. Now suppose that β is not an imaginary. Even in this case, since α is an imaginary and since the following diagram commutes, β remains to be non-empty.

$$\begin{tikzcd} \beta \arrow[r, "!_\beta"] & 1 \\ \alpha \arrow[u, "j"] \arrow[ur, "!_\alpha"'] & \end{tikzcd}$$

Hence by Proposition 3.42, β has at least two essentially distinct imaginary elements i.e. one can find its two imaginary elements p, q and $p \ncong q$. Since α, an imaginary, has essentially unique imaginary element again by Proposition 3.42, either p or q, we temporarily suppose it's p, satisfies $rp \nleq \lceil \mathrm{id}_x \rceil_x$. This rp is an imaginary element of Hx, and hence $\mathrm{id}_x \leq \lfloor rp \rfloor_{Hx}$ because $rp \leq r$ and r is a representation of id_x. This contradicts to our result which has been already obtained. So r must to be isomorphic to $\lfloor \mathrm{id}_x \rfloor_{Hx}$. $\square$

Suppose that a subclass $\mathfrak{T} \subseteq \mathscr{LF}$ is closed under composition with isomorphisms i.e. for every $t \in \mathfrak{T}$ and $h \in \mathrm{Iso}$, one has $th, ht \in \mathfrak{T}$. We define a new subclass ${}^*\mathfrak{T}$ of $\mathscr{LF}$ as follows: for every morphism in $\mathscr{LF}$, it belongs to ${}^*\mathfrak{T}$ if and only if it has a representation which belongs to $\mathfrak{T}$. This ${}^*\mathfrak{T}$ is called opposite class of $\mathfrak{T}$.

Let $f = (f, \alpha) : x \xrightarrow{\mathfrak{Im}} y$ and let $(\cdot \xrightarrow{t} x) \in {}^*\mathscr{IF}$. We write $f\langle t \rangle$ instead of $\mathrm{cl}_{\mathscr{LF}}(f[t])$. This $f\langle t \rangle$ will be called closured image of t by f.

Lemma 5.19. ${}^*\mathscr{IF}$ is closed under closured image by an imaginary morphism $f : x \xrightarrow{\mathfrak{Im}} y$ if $\mathscr{IF}(y) \subseteq \mathscr{LF}(y)$

Proof. Let $f = (f, \alpha, \pi_1)$ be an imaginary morphism from x to y. Assume $\mathscr{IF}(y) \subseteq \mathscr{LF}(y)$. Suppose that r is a representation of $(\cdot \xrightarrow{t} x) \in {}^*\mathscr{IF}$ which belongs to $\mathscr{IF}$.

For every $(\cdot \xrightarrow{t'} y) \in \mathscr{I}\mathscr{F}$, we obtain the following observation by Lemma 5.1.

$$\begin{aligned} f\langle t\rangle \leq t' \iff f\big[t\big] \leq t' &\iff t \leq f^{-1}\big[t'\big] \\ &\iff \lceil f^{-1}\big[t'\big]\rceil_x \leq r \\ &\iff \mathrm{inv}f(\lceil t'\rceil_y) \leq r \\ &\iff \mathrm{inv}f\big[\lceil t'\rceil_y\big] \leq r \iff \lceil t'\rceil_y \leq (\mathrm{inv}f)^{-1}\big[r\big] \end{aligned}$$

This shows that $(\mathrm{inv}f)^{-1}\big[r\big]$ is a representation of $f\langle t\rangle$. Also $(\mathrm{inv}f)^{-1}\big[r\big]$ belongs to $\mathscr{I}\mathscr{F}$ since $\mathscr{I}\mathscr{F}$ is stable under pullback by imaginary morphisms (cf. Lemma 5.2). Hence $f\langle t\rangle$ again belongs to ${}^*\mathscr{I}\mathscr{F}$. □

Let $(\cdot \xrightarrow{t} x) \in {}^*\mathscr{I}\mathscr{F}$ and let r be a representation of t which belongs to $\mathscr{I}\mathscr{F}$. We call $\lceil r\rceil_{Hx}$ an opposite code of t. So an opposite code of t is an imaginary element of HHx.

5.2.2 Final Observation

Let $\mathbb{E} = (\mathbb{E}, \mathscr{S}, \mathscr{T})$ and $\mathbb{E}^* = (\mathbb{E}^*, \mathscr{S}^*, \mathscr{T}^*)$ be two strictly imaginary extensional effectivenesses and let $\mathscr{F}$ and $\mathscr{F}^*$ be two fundamental classes, respectively, on $\mathbb{E}$ and $\mathbb{E}^*$.

Definition 5.20. Each $(\mathscr{F}, \mathscr{F}^*)$-simultaneous functor $G : \mathbb{E} \to \mathbb{E}^*$ is said to be strict if the following two conditions hold:

- G reflects and preserves imaginaries
  i.e. each $\alpha \in \mathbb{E}$ is an imaginary if and only if $G\alpha$ is an imaginary;
- for every imaginary $\alpha^* \in \mathbb{E}$, there is an imaginary $\alpha \in \mathbb{E}$ such that $G\alpha \xrightarrow{!} \alpha^*$ exists;
- for each $x \in \mathbb{E}$, G gives a bijective monotonically increasing correspondence from $\mathscr{T}(x)$ to $\mathscr{T}^*(Gx)$ when we define $G : [t] \mapsto [Gt]$.

Lemma 5.21. Let $G : \mathbb{E} \to \mathbb{E}^*$ be a strict $(\mathscr{F}, \mathscr{F}^*)$-simultaneous functor. The following four statements hold:

(i) G is $(\mathscr{I}_{\mathbb{E}}\mathscr{F}, \mathscr{I}_{\mathbb{E}^*}\mathscr{F}^*)$-simultaneous;

(ii) for another given fundamental class $\mathscr{F}'$ on $\mathbb{E}^*$, all imaginaries of $\mathbb{E}^*$ are $\mathscr{F}'$-compact if $G\alpha$ is $\mathscr{F}'$-compact for each imaginary α of $\mathbb{E}$.

The following statement also hold if $\mathbb{E}^*$ is well-powered:

(iii) G is $(\mathscr{L}_{\mathbb{E}}\mathscr{F}, \mathscr{L}_{\mathbb{E}^*}\mathscr{F}^*)$-simultaneous.

Proof. (i): Let $(\cdot \xrightarrow{t} x) \in \mathscr{T}$. Suppose that $t \in \mathscr{I}_{\mathbb{E}}\mathscr{F}$, and particularly that $t \in \mathscr{I}_{\alpha}\mathscr{F}$ where α is an imaginary. Note that $G\alpha$ is again an imaginary by the definition of strict simultaneous functor. Then $\alpha \times t \in \mathscr{F}$ and hence $G(\alpha \times t) \in \mathscr{F}^*$ since G is $(\mathscr{F}, \mathscr{F}^*)$-simultaneous. Hence $G\alpha \times Gt \in \mathscr{F}^*$ holds and thus $Gt \in \mathscr{I}_{G\alpha}\mathscr{F}^* \leq \mathscr{I}_{\mathbb{E}^*}\mathscr{F}^*$. Next suppose that $Gt \in \mathscr{I}_{\mathbb{E}^*}\mathscr{F}^*$, and particularly that $Gt \in \mathscr{I}_{\alpha}\mathscr{F}^*$ where β is an imaginary. By the definition of strict simultaneous functor, there is an imaginary $\alpha \in \mathbb{E}$ such that $G\alpha \xrightarrow{!} \beta$ exists. So $Gt \in \mathscr{I}_{G\alpha}\mathscr{F}^*$ follows. Then one may obtain $G\alpha \times Gt \in \mathscr{F}^*$ and, at the same time, $G(\alpha \times t) \in \mathscr{F}^*$. This implies $\alpha \times t \in \mathscr{F}$ and thus we have $t \in \mathscr{I}_{\alpha}\mathscr{F}$.
(ii): It is sufficient to see that for two imaginaries $\alpha, \beta \in \mathbb{E}^*$, one obtain $\mathscr{F}'$-compactness of β from that of α whenever $\alpha \xrightarrow{!} \beta$ exists. Suppose that α is $\mathscr{F}'$-compact and that ! really exists. Let $x \in \mathbb{E}^*$. The following diagram commutes.

$$\begin{tikzcd}
\alpha \times x \arrow[rr, "\pi_2"] \arrow[dr, "! \times \mathrm{id}"'] & & x \\
& \beta \times x \arrow[ur, "\pi_2'"'] &
\end{tikzcd}$$

Note that $! \times \mathrm{id}$ belongs to $\mathscr{S}$. For each $(\cdot \xrightarrow{t} \beta \times x) \in \mathscr{F}'$, one has:

$$\begin{aligned}
\pi_2'\big[t\big] &\cong \pi_2'\big[(! \times \mathrm{id})\big[(! \times \mathrm{id})^{-1}\big[t\big]\big]\big] \\
&\cong \pi_2\big[(! \times \mathrm{id})^{-1}\big[t\big]\big]
\end{aligned}$$

Since $(! \times \mathrm{id})^{-1}\big[t\big] \in \mathscr{F}'$ and since α is $\mathscr{F}'$-compact, $\pi_2'\big[t\big] \in \mathscr{F}'$ follows. This shows that β is $\mathscr{F}'$-compact.
(iii): $\mathbb{E}^*$ is well-powered. It is sufficient to see that G is $(\mathscr{L}^{\eta}_{\mathbb{E}}\mathscr{F}, \mathscr{L}^{\eta}_{\mathbb{E}^*}\mathscr{F}^*)$-simultaneous for every ordinal η. The case of $\eta = 0$ follows from the assumption that G is $(\mathscr{F}, \mathscr{F}^*)$-simultaneous. Suppose that $\eta = \eta' + 1$ and that G is $(\mathscr{L}^{\eta'}\mathscr{F}, \mathscr{L}^{\eta'}\mathscr{F}^*)$-simultaneous. Since G gives a bijective monotonically increasing correspondence from $\mathscr{T}(x)$ to $\mathscr{T}^*(Gx)$, one can see that for every $t \in \mathscr{T}$, t belongs to $\overline{\mathscr{L}^{\eta'}\mathscr{F}}$ if and only if Gt belongs to $\overline{\mathscr{L}^{\eta'}\mathscr{F}^*}$. Then, by a mathematical induction, we see that G is $(\mathscr{L}^{\eta}\mathscr{F}, \mathscr{L}^{\eta}\mathscr{F}^*)$-simultaneous. The case of η is a limit ordinal is left, but this case is trivial. □

Corollary 1. Let $G : \mathbb{E} \to \mathbb{E}^*$ be a strict $(\mathscr{F}, \mathscr{F}^*)$-simultaneous functor. For each $x \in \mathbb{E}$, one has the following.

$$\begin{aligned}
G : \mathscr{I}_{\mathbb{E}}\mathscr{F}(x) &\cong \mathscr{I}_{\mathbb{E}^*}\mathscr{F}^*(x) \\
G : \mathscr{L}_{\mathbb{E}}\mathscr{F}(x) &\cong \mathscr{L}_{\mathbb{E}^*}\mathscr{F}^*(x)
\end{aligned}$$

Proof. Immediately. □

Suppose that we are given a functor $G : \mathbb{E} \to \mathbb{E}^*$. We say that $\mathbb{E}$ is pseudo higher orderly $(\mathscr{F}, \mathscr{F}^*)$-structured by G if the following two conditions hold:

- $\mathbb{E}^*$ is pseudo higher orderly $\mathscr{F}^*$-structured;
- G is a strict $(\mathscr{F}, \mathscr{F}^*)$-simultaneous functor.

If $\mathbb{E}$ is pseudo higher orderly $(\mathscr{F}, \mathscr{F}^*)$-structured by G, we define a subclass ${}^*\mathscr{I}\mathscr{F}$ of $\mathscr{L}\mathscr{F}$ as follows: for every $t \in \mathscr{L}\mathscr{F}$, t belongs to ${}^*\mathscr{I}\mathscr{F}$ if and only if Gt belongs to ${}^*\mathscr{I}\mathscr{F}^*$. Each $x \in \mathbb{E}$ is said to be $\mathscr{I}\mathscr{F}$-full if its identity belongs to ${}^*\mathscr{I}\mathscr{F}$ i.e. $\mathrm{id}_x \in {}^*\mathscr{I}\mathscr{F}$.

Example 5.22. Let us think $\mathtt{Rep}_{\mathrm{op}}$ is pseudo higher orderly $(\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}, \Pi^0_{1,\mathtt{Rep}}; \Omega, \top)$-structured by $U : \mathtt{Rep}_{\mathrm{op}} \to \mathtt{Rep}$. We show the following statement:

$(*)$ all objects of $\mathtt{Rep}_{\mathrm{op}}$ is $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$-full.

Proof for $(*)$: Let $\delta = (u, x, \delta) \in \mathtt{Rep}_{\mathrm{op}}$. Since τ_u is second countable, one can find a sequence $\{p_i\}_{i\in\omega}$ on u such that the set $\{p_i : i \in \omega\}$ is dense in u. We define an imaginary $\beta = (\{p\}, \{*\}, !)$ of $\mathtt{Rep}$ where $p \in 2^\omega$ is defined by:

$$p(\langle i, j\rangle) \quad = \quad p_i(j)$$

Here we denote by $\langle -, - \rangle$ the Cantor pairing function. Namely, $\langle i, j\rangle = j + (i + j)(i + j + 1)/2$.
Now let $(H\delta, \mathrm{ev})$ be an exponential of $U\delta$ and Ω. Suppose that $H\delta$ is being of the form $H\delta = (Hu, Hx, H\delta)$. It follows that the function $\chi : Hx \times \{*\} \to 2$ defined as follows is a morphism from $H\delta \times \beta$ to Ω.

$$\chi(e, *) \quad = \quad \begin{cases} 0 & {}^\forall i \in \omega,\ \mathrm{ev}(e, \delta(p_i)) = 0 \\ 1 & \text{otherwise} \end{cases}$$

It is easy to see that $\chi = (\chi, \beta)$ is an imaginary character of $\mathrm{id}_{U\delta}$. Hence δ is $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$-full.

Example 5.23. Similar with the case of Example 5.22, if we think Cp is pseudo higher orderly $(\mathscr{B}_{0,\mathrm{Cp}}, \Pi^0_{1,\mathtt{Rep}}; \Omega, \top)$-structured by $U : \mathrm{Cp} \to \mathtt{Rep}$, it turns out that all objects of Cp is $\mathscr{I}_{0,\mathrm{Cp}}$-full.

Theorem 5.24. Assume that $\mathbb{E}$ is pseudo higher orderly $(\mathscr{F}, \mathscr{F}^*)$-structured by $G : \mathbb{E} \to \mathbb{E}^*$ and that $\mathbb{E}^*$ is well-powered. One has the identification $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ if the following three conditions hold:

(i) all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact;

(ii) all objects of $\mathbb{E}$ are $\mathscr{I}\mathscr{F}$-full;

(iii) ${}^*\mathscr{I}\mathscr{F}$ is included by $\mathscr{I}\mathscr{F}$.

Proof. Assume (i), (ii) and (iii). First, we show $\mathscr{LF} \subseteq {}^*\mathscr{IF}$. Note that by Corollary 1 of Lemma 5.21, the followings hold for every $y \in \mathbb{E}$.

$$\mathscr{IF}(y) \subseteq \mathscr{LF}(y) \iff \mathscr{IF}^*(Gy) \subseteq \mathscr{LF}^*(Gy)$$

Let $(x \xrightarrow{t} y) \in \mathscr{LF}$. Since $\mathrm{id}_x \in {}^*\mathscr{IF}$ by (ii), $G\mathrm{id}_x = \mathrm{id}_{Gx} \in {}^*\mathscr{IF}^*$. We obtain the following from Lemma 5.19 and from (i).

$$\begin{aligned} \mathrm{id}_{Gx} \in {}^*\mathscr{IF}^* &\implies Gt \cong Gt\big[\mathrm{id}_{Gx}\big] \cong Gt\langle \mathrm{id}_{Gx}\rangle \in {}^*\mathscr{IF}^* \\ &\implies t \in {}^*\mathscr{IF} \end{aligned}$$

Hence $\mathscr{LF} \subseteq {}^*\mathscr{IF}$. So we have $\mathscr{IF} \leq \mathscr{LF} = {}^*\mathscr{IF}$. What we need is now ${}^*\mathscr{IF} \leq \mathscr{IF}$, but this exactly is the assertion of (iii). $\square$

Example 5.25. Let us think $\mathtt{Rep}_{\mathrm{op}}$ is pseudo higher orderly $(\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}, \Pi^0_{1,\mathtt{Rep}}; \Omega, \top)$-structured by $U : \mathtt{Rep}_{\mathrm{op}} \to \mathtt{Rep}$. By Theorem 5.24, we obtain the equality $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}} = \mathscr{LI}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. So "oracle co-r.e. closedness is coinside with topological closedness" on each object of $\mathtt{Rep}_{\mathrm{op}}$, a represented topological space with an open representation. To see that, only the condition (iii) have been left to be checked.
Proof for (iii): Let $\delta = (u, x, \delta) \in \mathtt{Rep}_{\mathrm{op}}$ and let $(\cdot \xrightarrow{t} \delta) \in {}^*\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}} = \mathscr{L}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. We define an imaginary as $\gamma = (\{p\}, \{*\}, !)$ where p is being of the form $p = \iota(w_0)\iota(w_1)\iota(w_2)\cdots$ and where the following equivalence holds: for every $w \in 2^*$, $\delta\big[[w] \cap u\big] \cap \mathrm{range}(t) = \emptyset$ if and only if $w = w_i$ for some $i \in \omega$. It follows that $t \in \mathscr{I}_\gamma\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$.
Actually there is no difference between the above proof and a direct proof $\mathscr{L}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}} \leq \mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$. However, condition (iii) provides us to measure the strength of non-effectivity of the concerned inequality in a sense. Recall p from Example 4.26. It can be seen that there is a function which translate an opposite code of a morphism $(\cdot \xrightarrow{t} x) \in {}^*\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$ to a code of it as a morphism in $\mathscr{I}\Pi^0_{1,\mathtt{Rep}_{\mathrm{op}}}$, and which is limit computable from p as an oracle.

Example 5.26. Similar with the case of Example 5.25, if we think Cp is pseudo higher orderly $(\mathscr{B}_{0,\mathrm{Cp}}, \Pi^0_{1,\mathtt{Rep}}; \Omega, \top)$-structured by $U : \mathrm{Cp} \to \mathtt{Rep}$, the equality $\mathscr{I}_{0,\mathrm{Cp}} = \mathscr{L}_{0,\mathrm{Cp}}$ can be obtained by Theorem 5.24. So "oracle co-r.e. closedness is coinside with topological closedness" on each object of Cp, a subset of Cantor space.

6 Concluding Remarks

In this thesis, we reformulated a fundamental result from computable analysis, the equivalence of oracle co-r.e. closedness and topological closedness, as Theorem 5.24, in a pure categorical way. And we showed that the equivalence is valid on every represented topological space with an open representation, by an application of Theorem 5.24.

Our setting did not require a particular choice of an effectivity concept nor of a special kind of space. Therefore our approach and our results does not depend on a particular effectivity concept.

Further Problems We refer to two further problems, labeled as I and II, in what follows. Recall that in the setting of Theorem 5.24, our category $\mathbb{E}$ is supposed to be suitably related to another category $\mathbb{E}^*$ by a functor $G : \mathbb{E} \to \mathbb{E}^*$.

I: It concerns the optimality of the three conditions (i)-(iii) from Theorem 5.24. The question is "Can we find a pair $(\mathbb{E}^*, G)$ which is universal one?". The term universality is used here in the following sense: if another pair $(\mathbb{E}^{**}, G')$ are given such that our category $\mathbb{E}$ is suitably related to $\mathbb{E}^{**}$ by G', then there exists unique functor $H : \mathbb{E}^* \to \mathbb{E}^{**}$ which suitably relate $\mathbb{E}^*$ to $\mathbb{E}^{**}$ and which makes the following diagram commute.

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "G"] \arrow[dr, "G'"'] & \mathbb{E}^* \arrow[d, "H"] \\
 & \mathbb{E}^{**}
\end{tikzcd}$$

If such universal pair $(\mathbb{E}^*, G)$ exists, it turns out that (i)-(iii) from Theorem 5.24 is keeped with the weakest logical strength by $(\mathbb{E}^*, G)$, i.e., (i)-(iii) with respect to $(\mathbb{E}^*, G)$ is weaker than (i)-(iii) with respect to any other pair $(\mathbb{E}^{**}, G')$.

Furthermore, since such universal pair is unique, we can reconstruct it only from $\mathbb{E}$ if it exists. Hence, in that case, all assumptions of Theorem 5.24, originally for $\mathbb{E}$, $\mathbb{E}^*$ and G, can be collected up as only for $\mathbb{E}$.

II: It concerns the possibility of a further analysis for the condition (iii) of Theorem 5.24. The question is "Can we categorically describe extensions of effectivity concepts?".

As we have already explained in Example 5.25, in the case of the category $\texttt{Rep}_{\mathrm{op}}$ of represented topological spaces whose representation is an open map, proofs of satisfaction of condition (iii), essentially, requires constructing a limit computable function. Thus a categorical description of extensions of effectivity concepts, such as computability to limit computability, might be needed to a further analysis for the condition (iii).

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