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A Categorical Description of Relativization

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The aim of this thesis is to give a foundation for computable analysis which does not depend on a particular effectivity concept.

The main purpose in computable analysis is investigations of computational structures appear in analysis, geometry, topology, or any other fields of mathematics. Although many researchers developed foundations for computable analysis, most of those are based on particular effectivity concepts, such as computability, polynomial time computability or limit computability, and are also based on choices of special kind of spaces, such as computable topological space [4], effective uniform neighborhood system [2] or effective equilogical space [1].

Our goal is to reformulate fundamental results from computable analysis without a particular choice of an effectivity concept or of a special kind of space. To do that, we give a description of "relativization to oracles" on a pure category theoretical setting, based on the approach from [3]. Using the description, at the end of this thesis, a corresponding result to the equivalence between oracle co-r.e. closedness and topological closedness will be shown categorically.

Summary of Main Works Let us explain, firstly, our settings. In the following, as a typical but a simple example, the category Cp, whose objects are subsets of Cantor space and whose morphisms are computable total functions, will be used on our explanation.

We work on a (large and well-powered) category $\mathbb{E}$ equipped with a proper factorization system $(\mathscr{S}, \mathscr{T})$, a pair of two classes of morphisms. The class $\mathscr{S}$ is supposed to be stable under pullback and our category $\mathbb{E}$ is supposed to have $\mathscr{T}$-intersection. One can think of $\mathbb{E}$ as a broad generalization of the category of topological spaces. A subclass of $\mathscr{T}$ is called a fundamental class on $\mathbb{E}$ when it contains all isomorphisms, is closed under composition and is stable under pullback. A fundamental class can be thought of as defining a topology-like structure on our category $\mathbb{E}$. This notion is basically from [3]. On Cp, if $\mathscr{S}$ and $\mathscr{T}$ are suitably defined for it, one can define a fundamental class $\mathscr{B}_{0,\mathrm{Cp}}$ which identify the notion of co-r.e. closedness.

Our primal work is a categorical abstraction of the notion of oracle. We call an object with a certain property an imaginary. In the case of Cp, the set of all imaginaries coincides with the set of all oracles. As the next work, we define two closure operators $\mathscr{I}$ and $\mathscr{L}$ for fundamental classes. On the one hand, the action of $\mathscr{I}$ is an abstraction of "relativization to oracles". In the case of Cp, it turns out that $\mathscr{I}\mathscr{B}_{0,\mathrm{Cp}}$ identifies the notion of oracle co-r.e. closedness. On the other hand, the action of $\mathscr{L}$ is an abstraction of "generation of topology". In the case of Cp, it turns out that $\mathscr{L}\mathscr{B}_{0,\mathrm{Cp}}$ identifies the notion of topological closedness.

Two theorems will be shown in this thesis as our main works. Both of them are on a comparison of $\mathscr{I}\mathscr{F}$ and $\mathscr{I}\mathscr{F}$ where $\mathscr{F}$ is a given fundamental class on $\mathbb{E}$.

The first main theorem is stated as follows. For a given fundamental class $\mathscr{F}$ on $\mathbb{E}$, the inclusion $\mathscr{I}\mathscr{F} \subseteq \mathscr{L}\mathscr{F}$ holds if and only if all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact. Therefore this is a complete characterization of the concerned inclusion. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted respectively as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, the concerned inclusion corresponds to the fact that oracle co-r.e. closedness implies topological closedness.

The second main theorem is concerned with a slightly complicated situation. We have to prepare another category $\mathbb{E}^*$ with a certain structure and its equipped factorization system $(\mathscr{S}^*, \mathscr{T}^*)$. Suppose that we are given two fundamental classes $\mathscr{F}$ and $\mathscr{F}^*$ respectively on $\mathbb{E}$ and $\mathbb{E}^*$. Assume also $\mathbb{E}$ is suitably related to $\mathbb{E}^*$ with respect to $\mathscr{F}, \mathscr{F}^*$ and a functor $G : \mathbb{E} \to \mathbb{E}^*$. In this situation, we define another class of morphisms ${}^*\mathscr{I}\mathscr{F}$. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted respectively as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, the class ${}^*\mathscr{I}\mathscr{F}$ identifies what is called r.e. closedness.

The second main theorem is stated as follows. The equality $\mathscr{I}\mathscr{F} = \mathscr{L}\mathscr{F}$ holds if the following three conditions are fulfilled: (i) all imaginaries of $\mathbb{E}$ are $\mathscr{L}\mathscr{F}$-compact; (ii) all objects of $\mathbb{E}$ are $\mathscr{I}\mathscr{F}$-full; (iii) ${}^*\mathscr{I}\mathscr{F}$ is included in $\mathscr{I}\mathscr{F}$. If $\mathbb{E}$ and $\mathscr{F}$ are interpreted as Cp and $\mathscr{B}_{0,\mathrm{Cp}}$, respectively, the concerned equality, of course, corresponds to the fact that oracle co-r.e. closedness coincides with topological closedness. Actually the three conditions (i)-(iii) are certainly fulfilled in Cp.

The category Cp is, as we have already mentioned, a typical and a simple example. However, it is quite narrow in a sense. As a broader category, the category $\mathtt{Rep}_{\mathrm{op}}$, whose objects are represented topological spaces with an open representation and whose morphisms are relatively computable functions, will be constructed. All effective topological spaces can be regarded as objects of this category $\mathtt{Rep}_{\mathrm{op}}$ with respect to standard representation, and similarly, all effective metric spaces can be regarded as objects of this category $\mathtt{Rep}_{\mathrm{op}}$ with respect to Cauchy representation.

At the end of this thesis, $\mathtt{Rep}_{\mathrm{op}}$ will also be applied to our second main theorem, and as a result, it turns out that oracle co-r.e. closedness coincides with topological closedness on each object of $\mathtt{Rep}_{\mathrm{op}}$, a represented topological space with open representation.

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