A Study on Computing the Nucleolus and the Shapley Value of Facility Location Games

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We study facility location games, which are cooperative games that arise from the facility location problem. In general, cooperative games that arise from a combinatorial optimization problem are called combinatorial optimization games and it has been studied since about 1970. Facility location games have many variants, but we concentrate on uncapacitated facility location games, which are cooperative games that arise from an uncapacitated facility location problem. In these games, customers in the uncapacitated facility location problem minimize their cost by cooperation. There are two types of costs, one arises when customers open (or construct) facilities and the other one arises when customers receive services from previously opened facilities, and both types of costs are predefined. The minimized cost is distributed among the customers by applying various solution concepts from cooperative game theory.

Cooperative game theory, proposed by von Neumann and Morgenstern, is a variant of game theory which analyzes cooperative behavior among agents. Cooperative game theory has many applications, for example, operations research, economics, social science, communication network, and so on.

Here is an example of an uncapacitated facility location game. Suppose a new train station is to be jointly constructed by some railway companies.

They need to pay not only the station's construction cost, but also the cost of laying tracks to their own stations. There are many possible locations, but some locations could be disadvantageous for some companies because of their distance, or similar considerations. In this case, a way to decide on the location at which they construct the new station is by minimizing the total cost. This can be formulated as an uncapacitated facility location problem. After this decision, how do they distribute the cost among the companies? One good way is to distribute the cost on the basis of cost distribution concepts coming from cooperative game theory.

There are many types of cost distribution (or benefit distribution) concepts in cooperative game theory, especially the nucleolus and the Shapley value are known as good solutions. The nucleolus and the Shapley value are cost distribution concepts on basis of "excess" and "contributions" respectively, and they have several nice properties, for example, always exists and unique. However, computation of such a solution can be hard in the sense of computational complexity theory. Indeed, computation of the nucleolus and the Shapley value of uncapacitated facility location games is NP-hard, so there is no hope to compute such a solution in reasonable time if the input size increases (unless P = NP). We should concentrate on polynomial solvable cases of uncapacitated facility location games, and our main goal is to reveal instances of uncapacitated facility location games for which the nucleolus and the Shapley value can be computed in reasonable time.

In the literature of facility location games, Goemans and Skutella showed the following propositions.

- The core of several variants of facility location game is non-empty if and only if there is no integrality gap for the LP relaxation of the facility location problem.
- For general instances of an uncapacitated facility location problem, it is NP-complete to decide whether the core is non-empty or not.
- If the core of an uncapacitated facility location game is non-empty, an element of the core can be computed in polynomial time and it can be checked in polynomial time whether a given cost distribution belongs to the core.

For uncapacitated facility location games, they also showed some cases in which the core is nom-empty.

In this thesis, we study uncapacitated facility location games with two cost values in particular and our results in this thesis are as follows.

- We show that the uncapacitated facility location problem is NP-hard even when the cost is restricted to two values.
- When there are at most two facilities and the cost is restricted to two values, we characterize convex games. Convex games are a special type of cooperative game for which the nucleolus can be computed in polynomial time.
- When there are at most two facilities and the cost is restricted to two values, we show that the Shapley value can be computed in polynomial time.
- For general uncapacitated facility location games, we characterize the essential coalition, which has a deep relationship with the nucleolus.
- For general uncapacitated facility location games, we exhibit a case in which the nucleolus and the Shapley value coincide.

We describe some open problems of facility location games which arise from our results in conclusion.

- A characterization of convex games for general instances of an uncapacitated facility location games.
- When there are at most two facilities and the cost is restricted to two values, can the nucleolus be computed in polynomial time in the number of customers?
- For instances of the uncapacitated facility location game, is it NP-complete to decide whether the sufficient condition in which the nucleolus and the Shapley value coincide be satisfied?