Relating Bishop's function spaces to neighbourhood spaces

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Abstract

We extend Bishop's concept of function spaces to the concept of pre-function spaces. We show that there is an adjunction between the category of neighbourhood spaces and the category of Φ-closed pre-function spaces. We also show that there is an adjunction between the category of uniform spaces and the category of Ψ-closed pre-function spaces.

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1 Introduction

In 1967, Bishop [3] proposed two approaches to topology in his constructive mathematics: one approach is based on the idea of a neighbourhood space, and the other is based on the idea of a function space. However, in his book, he did not investigate them in detail.

It turns out that neighbourhood spaces are both formal topologies, as introduced by Sambin [19, 20, 21], and constructive topological spaces (see Aczel [1]). In addition, connections between neighbourhood spaces and other constructive topological notions – in particular the Bridges-Vîţă one of an apartness space [7, 9] – have been explored [14, 13]. On the other hand, the approach to constructive topology based on the idea of a function space has lain relatively dormant for over forty years.

Recently, Bridges [5] has dealt with various aspects of function spaces which revive Bishop's approach to topology based on function spaces. Following Bishop [3, Definition 8, Chapter 3], we define a *function space* X to be a pair $(\underline{X}, \mathcal{F}_X)$ of a set $\underline{X}$ and a set $\mathcal{F}_X$ of functions from $\underline{X}$ to $\mathbf{R}$ satisfying the following conditions.

F1. $\mathcal{F}_X$ contains the constant functions.

F2. Sums and products of elements of $\mathcal{F}_X$ are in $\mathcal{F}_X$.

F3. The composition $\varphi \circ f$ of an element f of $\mathcal{F}_X$ and a continuous function $\varphi : \mathbf{R} \to \mathbf{R}$ is in $\mathcal{F}_X$, where $\varphi : \mathbf{R} \to \mathbf{R}$ is continuous if it is uniformly continuous on every compact interval.

F4. Uniform limits of elements of $\mathcal{F}_X$ are in $\mathcal{F}_X$; that is, if for each $\epsilon > 0$ there exists g in $\mathcal{F}_X$, with $|g(x) - f(x)| \leq \epsilon$ for all x in $\underline{X}$, then $f \in \mathcal{F}_X$.

Bishop called $\mathcal{F}_X$ the *topology* on $\underline{X}$.

In this paper, we first introduce the notion of a pre-function space just as a pair of a set S and a set of real-valued functions on S, and the notion of a function space morphism according to [5]. Then we focus on the condition F3 above, and introduce the notion of a C-complete pre-function space for a set C of functions from $\mathbf{R}$ to $\mathbf{R}$; in the definition of a function space, C is taken to be the set of continuous functions in the above sense. We show that the category of C-complete pre-function spaces with function space morphisms is complete and cocomplete.

We propose a closure condition Φ_S on a set of real-valued functions on a set S, and introduce the notion of a Φ-closed pre-function space. It emerges that each Φ-closed pre-function space is a function space in Bishop's sense. Then we construct an adjunction between the category of neighbourhood spaces with continuous functions in usual sense and the category of Φ-closed pre-function spaces with function space morphisms, which relates Bishop's two approaches to topology, and show that the category of Φ-closed pre-function spaces is complete and cocomplete. We also construct an adjoint equivalence between the category of neighbourhood spaces with a compatible family of pseudometrics and the category of Φ-closed pre-function spaces.

Finally, we introduce another closure condition Ψ_S on a set of real-valued functions on a set S and the corresponding notion of a Ψ-closed pre-function space, and construct an adjunction between the category of uniform spaces with uniformly continuous functions and the category of Ψ-closed pre-function spaces with function space morphisms.

Although the results are presented in informal Bishop-style constructive mathematics [3, 4, 6, 22, 8], it is possible to formalize them in Aczel's constructive Zermelo-Fraenkel set theory (**CZF**) [2] together with Relativized Dependent Choice (RDC).

There are other constructive treatments of topology: see, for example, Grayson [11, 12].

2 Complete pre-function spaces

A *pre-function space* X is a pair $(\underline{X}, \mathcal{F}_X)$ consisting of a set $\underline{X}$ and a set $\mathcal{F}_X$ of functions from $\underline{X}$ to $\mathbf{R}$, called a *function space structure* on $\underline{X}$. According to [5], a *function space morphism* from a pre-function space X into a pre-function space Y is a mapping $f : \underline{X} \to \underline{Y}$ such that

$$\forall g \in \mathcal{F}_Y (g \circ f \in \mathcal{F}_X).$$

We write $f : X \to Y$ to denote that f is a function space morphism from X into Y, and $\mathrm{Hom}(X, Y)$ for the set of function space morphisms from X into Y.

For any set S, there are the pre-function spaces $(S, \mathbf{R}^S)$, where $\mathbf{R}^S$ is the set of functions from S into $\mathbf{R}$, and $(S, \emptyset)$, called the *discrete* function space of S and the *trivial* pre-function space of S, respectively. For each pre-function space Y, any mapping $f : S \to \underline{Y}$ is a function space morphism from the discrete function space of S into Y, and any mapping $f : \underline{Y} \to S$ is a function space morphism from Y into the trivial pre-function space of S.

Let C be a set of functions from $\mathbf{R}$ to $\mathbf{R}$ containing the identity map $\mathrm{id}_{\mathbf{R}}$ and closed under composition. A pre-function space X is *C-complete* if

$$\forall f \in \mathcal{F}_X \forall \varphi \in C (\varphi \circ f \in \mathcal{F}_X).$$

The discrete function spaces and the trivial pre-function spaces are C-complete for any C, and any pre-function space is $\{\mathrm{id}_{\mathbf{R}}\}$-complete. If a pre-function space X is C-complete, then X is C'-complete for any $C' \subseteq C$. Since C is closed under composition, the pre-function space $\mathbf{R}_C = (\mathbf{R}, C)$ is C-complete.

Lemma 2.1. *Let X be a pre-function space. Then*

1. $\mathrm{Hom}(X, \mathbf{R}_C) \subseteq \mathcal{F}_X$,

2. *X is C-complete if and only if $\mathcal{F}_X \subseteq \mathrm{Hom}(X, \mathbf{R}_C)$,*

3. *X is C-complete if and only if $\mathcal{F}_X = \mathrm{Hom}(X, \mathbf{R}_C)$.*

Proof. Straightforward. For (1), note that $\mathrm{id}_{\mathbf{R}} \in C$. □

Especially, $C = \mathrm{Hom}(\mathbf{R}_C, \mathbf{R}_C)$.

Proposition 2.2. *Let X be a pre-function space. Then the pre-function space $\tilde{X} = (\underline{X}, \mathrm{Hom}(X, \mathbf{R}_C))$, called the C-completion of X, is C-complete. Furthermore, $\mathrm{id}_{\underline{X}} : X \to \tilde{X}$, and if Y is a C-complete pre-function space and $f : X \to Y$, then $f : \tilde{X} \to Y$.*

Proof. Let $\varphi \in C$ and $f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Then for each $\psi \in C$, since $\psi \circ (\varphi \circ f) = (\psi \circ \varphi) \circ f$ and $\psi \circ \varphi \in C$, we have $\psi \circ (\varphi \circ f) \in \mathcal{F}_X$, and therefore $\varphi \circ f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Hence $\tilde{X}$ is C-complete.

Since $\mathrm{Hom}(X, \mathbf{R}_C) \subseteq \mathcal{F}_X$, by Lemma 2.1 (1), we have $\mathrm{id}_{\underline{X}} : X \to \tilde{X}$. Let Y be a C-complete pre-function space, and let $f : X \to Y$. Then for each $g \in \mathcal{F}_Y$ and $\varphi \in C$, since $\varphi \circ g \in \mathcal{F}_Y$, we have $\varphi \circ (g \circ f) = (\varphi \circ g) \circ f \in \mathcal{F}_X$, and therefore $g \circ f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Hence $f : \tilde{X} \to Y$. □

Let $\mathbf{Fun}_C$ denote the category of function spaces whose objects are C-complete pre-function spaces and whose morphisms are function space morphisms. For basic notions and results in category theory, we refer the reader to [10, 16, 17, 18].

Note that, in $\mathbf{Fun}_C$, the initial object is the discrete function space of $\emptyset$, the terminal objects are the trivial pre-function spaces of singletons.

Let $\mathbf{I}$ and $\mathbf{C}$ be categories. A *cone* of a functor $H : \mathbf{I} \to \mathbf{C}$ is an object L in $\mathbf{C}$, together with a family of morphisms $\phi_I : L \to H(I)$ for each object I in $\mathbf{I}$, such that $H(i) \circ \phi_I = \phi_J$ for each morphism $i : I \to J$ in $\mathbf{I}$. A cone $\langle L, \phi_I \rangle$ of a functor $H : \mathbf{I} \to \mathbf{C}$ is a *limit* of H if for each cone $\langle X, \psi_I \rangle$ of H there exists a unique morphism $u : X \to L$ such that $\phi_I \circ u = \psi_I$ for each object I in $\mathbf{I}$. We say that $\mathbf{C}$ is *complete* if every functor $H : \mathbf{I} \to \mathbf{C}$ from a small category $\mathbf{I}$ has a limit. A *cocone* of a functor $H : \mathbf{I} \to \mathbf{C}$ is an object L in $\mathbf{C}$, together with a family of morphisms $\phi_I : H(I) \to L$ for each object I in $\mathbf{I}$, such that $\phi_J \circ H(i) = \phi_I$ for each morphism $i : I \to J$ in $\mathbf{I}$. A cocone $\langle L, \phi_I \rangle$ of a functor $H : \mathbf{I} \to \mathbf{C}$ is a *colimit* of H if for each cocone $\langle X, \psi_I \rangle$ of H there exists a unique morphism $u : L \to X$ such that $u \circ \phi_I = \psi_I$ for each object I in $\mathbf{I}$. We say that $\mathbf{C}$ is *cocomplete* if every functor $H : \mathbf{I} \to \mathbf{C}$ from a small category $\mathbf{I}$ has a colimit.

We will show that the category $\mathbf{Fun}_C$ is complete and cocomplete.

Proposition 2.3. *Let S be a set, let $\{X_i\}_{i\in I}$ be a family of C-complete pre-function spaces, and for each $i \in I$ let $f_i : S \to \underline{X_i}$. Then there exists a function space structure $\mathcal{F}$ on S such that the pre-function space $(S, \mathcal{F})$ is C-complete, and if h is a mapping from the underlying set $\underline{Y}$ of a pre-function space Y into S, then $h : Y \to (S, \mathcal{F})$ if and only if $f_i \circ h : Y \to X_i$ for each $i \in I$.*

Proof. Let

$$\mathcal{F} = \{f \circ f_i \mid i \in I, f \in \mathcal{F}_{X_i}\}.$$

Then for each $\varphi \in C$, $i \in I$ and $f \in \mathcal{F}_{X_i}$, since $\varphi \circ (f \circ f_i) = (\varphi \circ f) \circ f_i$ and $\varphi \circ f \in \mathcal{F}_{X_i}$, we have $\varphi \circ (f \circ f_i) \in \mathcal{F}$, and hence the pre-function space $(S, \mathcal{F})$ is C-complete.

It is straightforward to see that $f_i : (S, \mathcal{F}) \to X_i$ for each $i \in I$, and hence if $h : Y \to (S, \mathcal{F})$, then $f_i \circ h : Y \to X_i$ for each $i \in I$. Let Y be a pre-function space and let $h : \underline{Y} \to S$ be such that $f_i \circ h : Y \to X_i$ for each $i \in I$. Then for each $i \in I$ and $f \in \mathcal{F}_{X_i}$, we have $(f \circ f_i) \circ h = f \circ (f_i \circ h) \in \mathcal{F}_Y$, and hence $h : Y \to (S, \mathcal{F})$. $\square$

Proposition 2.4. *Let S be a set, let $\{X_i\}_{i\in I}$ be a family of C-complete pre-function spaces, and for each $i \in I$ let $f_i : \underline{X_i} \to S$. Then there exists a function space structure $\mathcal{F}$ on S such that the pre-function space $(S, \mathcal{F})$ is C-complete, and if h is a mapping from S into the underlying set $\underline{Y}$ of a pre-function space Y, then $h : (S, \mathcal{F}) \to Y$ if and only if $h \circ f_i : X_i \to Y$ for each $i \in I$.*

Proof. Let

$$\mathcal{F} = \{f \in \mathbf{R}^S \mid \forall i \in I(f \circ f_i \in \mathcal{F}_{X_i})\}.$$

Then for each $f \in \mathcal{F}$ and $\varphi \in C$, since $(\varphi \circ f) \circ f_i = \varphi \circ (f \circ f_i) \in \mathcal{F}_{X_i}$, we have $\varphi \circ f \in \mathcal{F}$, and hence the pre-function space $(S, \mathcal{F})$ is C-complete.

It is straightforward to see that $f_i : X_i \to (S, \mathcal{F})$ for each $i \in I$, and hence if $h : (S, \mathcal{F}) \to Y$, then $h \circ f_i : X_i \to Y$ for each $i \in I$. Let Y be a pre-function space and let $h : S \to \underline{Y}$ be such that $h \circ f_i : X_i \to Y$ for each $i \in I$. Then for each $g \in \mathcal{F}_Y$, since $(g \circ h) \circ f_i = g \circ (h \circ f_i) \in \mathcal{F}_{X_i}$ for each $i \in I$, we have $g \circ h \in \mathcal{F}$, and hence $h : (S, \mathcal{F}) \to Y$. $\square$

Theorem 2.5. *The category* $\mathbf{Fun}_C$ *is complete and cocomplete.*

Proof. Let $H : \mathbf{I} \to \mathbf{Fun}_C$ be a functor from a small category $\mathbf{I}$. Then, since the category $\mathbf{Set}$ of sets and mappings is complete, the functor $KH : \mathbf{I} \to \mathbf{Set}$ has a limit $\langle S, \phi_I \rangle$ in $\mathbf{Set}$, where K is the forgetful functor from $\mathbf{Fun}_C$ into $\mathbf{Set}$, taking each C-complete pre-function space X to its underlying set $\underline{X}$ and each function space morphism to itself. By Proposition 2.3, since $\phi_I : S \to \underline{H(I)}$ for each object I in $\mathbf{I}$, there exists a function space structure $\mathcal{F}$ on S such that $L = (S, \mathcal{F})$ is an object in $\mathbf{Fun}_C$, and if h is a mapping from the underlying set $\underline{Y}$ of an object Y in $\mathbf{Fun}_C$ into S, then $h : Y \to L$ if and only if $\phi_I \circ h : Y \to H(I)$ for each object I in $\mathbf{I}$.

Since $\mathrm{id}_S : L \to L$, we have $\phi_I = \phi_I \circ \mathrm{id}_S : L \to H(I)$ for each object I in $\mathbf{I}$, and hence $\langle L, \phi_I \rangle$ is a cone of H in $\mathbf{Fun}_C$. Let $\langle Y, \psi_I \rangle$ be a cone of H in $\mathbf{Fun}_C$. Then $\langle \underline{Y}, \psi_I \rangle$ is a cone of KH in $\mathbf{Set}$, and hence there exists a unique mapping $h : \underline{Y} \to S$ such that $\phi_I \circ h = \psi_I$ for each object I in $\mathbf{I}$. Therefore, since $\phi_I \circ h = \psi_I : Y \to H(I)$ for each object I in $\mathbf{I}$, we have $h : Y \to L$. Thus $\langle L, \phi_I \rangle$ is a limit of H in $\mathbf{Fun}_C$.

Similarly, using Proposition 2.4 instead of Proposition 2.3, we see that $\mathbf{Fun}_C$ is cocomplete. $\square$

3 Neighbourhood and function spaces

A *neighbourhood space* A is a pair $(\underline{A}, \tau_A)$ consisting of a set $\underline{A}$ and an inhabited set τ_A of subsets of $\underline{A}$, called an *open base* on $\underline{A}$, such that

NS1. $\forall x \in \underline{A} \exists U \in \tau_A (x \in U)$,

NS2. $\forall x \in \underline{A} \forall U, V \in \tau_A [x \in U \cap V \implies \exists W \in \tau_A (x \in W \subseteq U \cap V)]$;

see [3, Chapter 3]. A *continuous mapping* f from a neighbourhood space A into a neighbourhood space B is a mapping $f : \underline{A} \to \underline{B}$ such that

$$\forall x \in \underline{A} \forall V \in \tau_B [f(x) \in V \implies \exists U \in \tau_A (x \in U \subseteq f^{-1}(V))].$$

We write $f : A \to B$ to denote that f is a continuous mapping from A into B, and $\mathcal{C}(A, B)$ for the set of continuous mappings from A into B.

For any set S, there are the neighbourhood spaces (S, σ_S), where σ_S is the set of singletons of S, and $(S, \{S\})$, called the *discrete* neighbourhood space of S and the *trivial* neighbourhood space of S, respectively. For each neighbourhood space Y, any mapping $f : S \to \underline{Y}$ is a continuous mapping from the discrete neighbourhood space of S into Y, and any mapping $f : \underline{Y} \to S$ is a continuous mapping from Y into the trivial neighbourhood space of S.

Let **Nbh** denote the category of neighbourhood spaces whose objects are neighbourhood spaces and whose morphisms are continuous mappings. Note that, in **Nbh**, the initial object is the discrete neighbourhood space of $\emptyset$, the terminal objects are the trivial neighbourhood spaces of singletons.

As we will see in Theorem 3.3, the category **Nbh** is complete and cocomplete.

Proposition 3.1. *Let S be a set, let $\{A_i\}_{i\in I}$ be a family of neighbourhood spaces, and for each $i \in I$ let $f_i : S \to \underline{A_i}$. Then there exists an open base τ on S such that if h is a mapping from the underlying set $\underline{B}$ of a neighbourhood space B into S, then $h : B \to (S, \tau)$ if and only if $f_i \circ h : B \to A_i$ for each $i \in I$.*

Proof. Let

$$\tau = \{\textstyle\bigcap_{k=1}^{n} f_{i_k}^{-1}(U_k) \mid i_k \in I, U_k \in \tau_{A_{i_k}}, 1 \le k \le n, 0 \le n\}.$$

Then τ is an open base on S. It is straightforward to see that $f_i : (S, \tau) \to X_i$ for each $i \in I$, and hence if $h : Y \to (S, \tau)$, then $f_i \circ h : Y \to X_i$ for each $i \in I$. Let Y be a neighbourhood space and let $h : \underline{Y} \to S$ be such that $f_i \circ h : Y \to X_i$ for each $i \in I$. If $h(x) \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(U_k)$, then, since $x \in \bigcap_{k=1}^{n}(f_{i_k} \circ h)^{-1}(U_k)$, there exists $V \in \tau_Y$ such that $x \in V \subseteq \bigcap_{k=1}^{n}(f_{i_k} \circ h)^{-1}(U_k) = h^{-1}(\bigcap_{k=1}^{n} f_{i_k}^{-1}(U_k))$. Hence $h : B \to (S, \tau)$. □

Ishihara and Palmgren [15, Theorem 4.3] proved the following proposition in **CZF** with the *Relativized Dependent Choice* (RDC).

Proposition 3.2. *Let S be a set, let $\{A_i\}_{i\in I}$ be a family of neighbourhood spaces, and for each $i \in I$ let $f_i : \underline{A_i} \to S$. Then there exists an open base τ on S such that if h is a mapping from S into the underlying set $\underline{B}$ of a neighbourhood space B, then $h : (S, \tau) \to B$ if and only if $h \circ f_i : A_i \to B$ for each $i \in I$.*

Theorem 3.3. *The category* **Nbh** *is complete and cocomplete.*

Proof. Similar to the proof of Theorem 2.5, using Proposition 3.1 and Proposition 3.2. □

In the following, we shall write, simply, $\mathbf{R}$ for the neighbourhood space $\mathbf{R}$ with the standard open base consisting of open intervals, and $\mathcal{C}(A)$ for $\mathcal{C}(A, \mathbf{R})$.

Let S be a set, and define a relation Φ_S between a function space structure $\mathcal{F}$ on S and $g \in \mathbf{R}^S$ as follows: $\Phi_S(\mathcal{F}, g)$ if and only if for each $x \in S$ and $\epsilon > 0$ there exist $f_1, \ldots, f_n \in \mathcal{F}$ $(n \geq 0)$ and $\delta > 0$ such that

$$\forall y \in S \left(\sum_{i=1}^{n} |f_i(x) - f_i(y)| < \delta \implies |g(x) - g(y)| < \epsilon \right).$$

Then a pre-function space X is Φ-*closed* if

$$\forall g \in \mathbf{R}^{\underline{X}} (\Phi_{\underline{X}}(\mathcal{F}_X, g) \implies g \in \mathcal{F}_X).$$

Note that if X is a Φ-closed pre-function space, then $\mathcal{F}_X$ contains the constant functions, and the pointwise sum of finitely many functions in $\mathcal{F}_X$ belongs to $\mathcal{F}_X$. Furthermore, it is straightforward to show that if X is a Φ-closed pre-function space, then X is $\mathcal{C}(\mathbf{R})$-complete, and uniform limits of functions of $\mathcal{F}_X$ are in $\mathcal{F}_X$. It was shown in [5, Lemma 1] that if X is a $\mathcal{C}(\mathbf{R})$-complete pre-function space such that $\mathcal{F}_X$ is closed under finite pointwise sum, then $\mathcal{F}_X$ is closed under finite pointwise product. Therefore Φ-closed pre-function spaces are function spaces in Bishop's sense.

For each set S, the discrete function space on S is Φ-closed, and the pre-function space $(S, \mathcal{K}_S)$, called the *constant* function space on S, where $\mathcal{K}_S$ is the set of constant functions on S, is Φ-closed. For each Φ-closed pre-function space Y, any mapping $f : \underline{Y} \to S$ is a function space morphism from Y into the constant function space of S.

In the following, we shall call a Φ-closed pre-function space just a *function space*.

Let **Fun** denote the category of pre-function spaces with function spaces as objects and function space morphisms as morphisms. Note that, in **Fun**, the initial object is the discrete function space of $\emptyset$, the terminal objects are the constant function spaces of singletons.

An *adjunction* $\langle F, G, \eta, \varepsilon \rangle$ between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, and natural transformations $\eta : 1_{\mathbf{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{D}}$ such that $\varepsilon_F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta_G = 1_G$. The functor F is the *left-adjoint*, and the functor G is the *right-adjoint*. The natural transformation η is the *unit*, and the natural transformation ε is the *counit*.

The adjunction $\langle F, G, \eta, \varepsilon\rangle$ is called an *adjoint equivalence* if both the unit η and the counit ϵ are natural isomorphisms.

We now aim to prove the following result.

Theorem 3.4. *There exists an adjunction between* **Nbh** *and* **Fun** *whose counit is a natural isomorphism.*

Corollary 3.5. *The category* **Fun** *is complete and cocomplete.*

Proof. Since the category **Nbh** is complete and cocomplete, by Theorem 3.3, it follows from [16, Exercise 7, VI.3] or the dual of Theorem 4.2 in [14]. $\square$

To prove Theorem 3.4, we need a series of lemmas.

Lemma 3.6. *For each neighbourhood space A, the pre-function space $(\underline{A}, \mathcal{C}(A))$ is Φ-closed.*

Proof. Let $g \in \mathbf{R}^{\underline{A}}$, and suppose that $\Phi(\mathcal{C}(A), g)$. Then for each $x \in \underline{A}$ and $\epsilon > 0$ there exist $f_1, \ldots, f_n \in \mathcal{C}(A)$ and $\delta > 0$ such that for each $y \in \underline{A}$, if $\sum_{i=1}^n |f_i(x) - f_i(y)| < \delta$, then $|g(x) - g(y)| < \epsilon$. Hence there exists $U \in \tau_A$ such that $x \in U$ and if $y \in U$, then $\sum_{i=1}^n |f_i(x) - f_i(y)| < \delta$, and hence $|g(x) - g(y)| < \epsilon$. Thus $g \in \mathcal{C}(A)$. $\square$

Lemma 3.7. *Let A and B be neighbourhood spaces. If $f : A \to B$, then $f : (\underline{A}, \mathcal{C}(A)) \to (\underline{B}, \mathcal{C}(B))$.*

Proof. Straightforward. $\square$

For a function space structure $\mathcal{F}$ on a set S, let $\tau_{\mathcal{F}}$ be the set of subsets of S of the form

$$U_{f_1,\ldots,f_n}(x, \epsilon) = \{y \in S \mid \textstyle\sum_{k=1}^n |f_i(x) - f_i(y)| < \epsilon\},$$

where $f_1, \ldots, f_n \in \mathcal{F}$ $(n \geq 0)$, $x \in S$ and $\epsilon > 0$.

Lemma 3.8. *For each pre-function space X, the pair $(\underline{X}, \tau_{\mathcal{F}_X})$ is a neighbourhood space.*

Proof. Straightforward. For (NS1), note that $U(x, \epsilon) = \{y \in \underline{X} \mid 0 < \epsilon\} = \underline{X}$. $\square$

Lemma 3.9. *Let X and Y be pre-function space. If $f : X \to Y$, then $f : (\underline{X}, \tau_{\mathcal{F}_X}) \to (\underline{Y}, \tau_{\mathcal{F}_Y})$.*

Proof. Suppose that $f : X \to Y$, and let $f(x) \in U_{g_1,\ldots,g_n}(y,\epsilon) \in \tau_{\mathcal{F}_Y}$. Then, since $g_i \circ f \in \mathcal{F}_X$ for each $i = 1,\ldots,n$, we have $x \in U_{g_1\circ f,\ldots,g_n\circ f}(x,\delta) \in \tau_{\mathcal{F}_X}$ with

$$\delta = \epsilon - \sum_{i=1}^{n} |g_i(y) - g_i(f(x))|,$$

and if $z \in U_{g_1\circ f,\ldots,g_n\circ f}(x,\delta)$, then $f(z) \in U_{g_1,\ldots,g_n}(y,\epsilon)$. Hence $f : (\underline{X}, \tau_{\mathcal{F}_X}) \to (\underline{Y}, \tau_{\mathcal{F}_Y})$. □

Lemma 3.10. *If A is a neighbourhood space, then $\mathrm{id}_{\underline{A}} : A \to (\underline{A}, \tau_{\mathcal{C}(A)})$.*

Proof. Let $x \in U_{f_1,\ldots,f_n}(y,\epsilon) \in \tau_{\mathcal{C}(A)}$. Then, since $f_1,\ldots,f_n \in \mathcal{C}(A)$, there exists $U \in \tau_A$ such that $x \in U$ and if $z \in U$, then

$$\sum_{i=1}^{n} |f_i(x) - f_i(z)| < \epsilon - \sum_{i=1}^{n} |f_i(y) - f_i(x)|,$$

and hence $z \in U_{f_1,\ldots,f_n}(y,\epsilon)$. Thus $\mathrm{id}_{\underline{A}} : A \to (\underline{A}, \tau_{\mathcal{C}(A)})$. □

Lemma 3.11. *Let X be a pre-function space. Then $\mathcal{F}_X \subseteq \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X})$. Moreover, if X is Φ-closed, then $\mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X}) \subseteq \mathcal{F}_X$.*

Proof. Note that $g \in \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X})$ if and only if $\Phi_{\underline{X}}(\mathcal{F}_X, g)$. Then, trivially, $\mathcal{F}_X \subseteq \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X})$, and if X is Φ-closed, then $\mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X}) \subseteq \mathcal{F}_X$. □

Proof of Theorem 3.4. Define a functor F from **Nbh** to **Fun** by $F(A) = (\underline{A}, \mathcal{C}(A))$ and $F(f) = f$, and define a functor G from **Fun** to **Nbh** by $G(X) = (\underline{X}, \tau_{\mathcal{F}_X})$ and $G(f) = f$. Then F and G are faithful functors, by Lemma 3.9 and Lemma 3.7.

Furthermore, we see that if we let η_A and ϵ_X denote the identity maps on the sets $\underline{A}$ and $\underline{X}$, respectively, then $\eta_A : A \to (\underline{A}, \tau_{\mathcal{C}(A)})$ in **Nbh**, by Lemma 3.10, and

$$\epsilon_X : (\underline{X}, \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X})) \to X \text{ and } \epsilon_X^{-1} : X \to (\underline{X}, \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X}))$$

in **Fun**, by Lemma 3.11. Hence $\eta : 1_{\mathbf{Nbh}} \to GF$ is a natural transformation and $\epsilon : FG \to 1_{\mathbf{Fun}}$ is a natural isomorphism satisfying $\epsilon_F \circ F\eta = 1_F$ and $G\epsilon \circ \eta_G = 1_G$. Therefore $\langle F, G, \eta, \epsilon\rangle$ forms an adjunction between **Nbh** and **Fun**. □

Let S be a set. Then a family $\{d_i\}_{i\in I}$ of pseudometrics on S is *compatible* with an open base τ on S if

1. for each $x \in S$ and $i \in I$, the mapping $y \mapsto d_i(x, y)$ is in $\mathcal{C}(S, \tau)$,
2. for each $x \in S$ and $U \in \tau$ with $x \in U$ there exist $i_1, \ldots, i_n \in I$ $(n \geq 0)$ and $\delta > 0$ such that for each $y \in S$, if $\sum_{k=1}^{n} d_{i_k}(x, y) < \delta$, then $y \in U$.

For a function space structure $\mathcal{F}$ on a set S, let $\{d_f\}_{f \in \mathcal{F}}$ be a family of pseudometrics of S defined by $d_f(x, y) = |f(x) - f(y)|$. Then the family $\{d_f\}_{f \in \mathcal{F}}$ is compatible with $\tau_{\mathcal{F}}$.

A neighbourhood space A *has a compatible family of pseudometrics* if there exists a family $\{d_i\}_{i \in I}$ of pseudometrics on $\underline{A}$ compatible with τ_A.

Lemma 3.12. *If A is a neighbourhood space having a compatible family of pseudometrics, then* $\mathrm{id}_{\underline{A}} : (\underline{A}, \tau_{\mathcal{C}(A)}) \to A$.

Proof. Let $\{d_i\}_{i \in I}$ be a family of pseudometrics on $\underline{A}$ compatible with τ_A. Let $x \in \underline{A}$ and let $U \in \tau_A$ with $x \in U$. Then there exist $i_1, \ldots, i_n \in I$ and $\delta > 0$ such that if $y \in \underline{A}$ and $\sum_{k=1}^{n} d_{i_k}(x, y) < \delta$, then $y \in U$. Since the mapping $y \mapsto d_{i_k}(x, y)$ is in $\mathcal{C}(A)$ for each $k = 1, \ldots, n$, setting $f_k(y) = d_{i_k}(x, y)$, we have $f_k \in \mathcal{C}(A)$ for each $k = 1, \ldots, n$, and if $y \in \underline{A}$ and

$$\sum_{k=1}^{n} d_{i_k}(x, y) = \sum_{k=1}^{n} |f_k(x) - f_k(y)| < \delta,$$

then $y \in U$. Hence $\mathrm{id}_{\underline{A}} : (\underline{A}, \tau_{\mathcal{C}(A)}) \to A$. □

Let $\mathbf{Nbh}_{\mathrm{pms}}$ denote the category of neighbourhood spaces whose objects are neighbourhood spaces having a compatible family of pseudometrics, and whose morphisms are continuous mappings.

Theorem 3.13. *There exists an adjoint equivalence between* $\mathbf{Nbh}_{\mathrm{pms}}$ *and* $\mathbf{Fun}$.

Proof. Let F' be a functor restricting the functor F constructed in the proof of Theorem 3.4 to the category $\mathbf{Nbh}_{\mathrm{pms}}$, and note that the functor G constructed in the proof is a functor from $\mathbf{Fun}$ into $\mathbf{Nbh}_{\mathrm{pms}}$. We see that if we let $\eta_A = \mathrm{id}_{\underline{A}}$ and $\epsilon_X = \mathrm{id}_{\underline{X}}$, then $\eta_A : A \to (\underline{A}, \tau_{\mathcal{C}(A)})$ and $\eta_A^{-1} : (\underline{A}, \tau_{\mathcal{C}(A)}) \to A$ in $\mathbf{Nbh}_{\mathrm{pms}}$, by Lemma 3.12, and

$$\epsilon_X : (\underline{X}, \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X})) \to X \text{ and } \epsilon_X^{-1} : X \to (\underline{X}, \mathcal{C}(\underline{X}, \tau_{\mathcal{F}_X}))$$

in $\mathbf{Fun}$. Hence $\eta : 1_{\mathbf{Nbh}} \to GF'$ and $\epsilon : F'G \to 1_{\mathbf{Fun}}$ are natural isomorphisms satisfying $\epsilon_{F'} \circ F'\eta = 1_{F'}$ and $G\epsilon \circ \eta_G = 1_G$. Therefore $\langle F', G, \eta, \epsilon \rangle$ forms an adjoint equivalence between $\mathbf{Nbh}_{\mathrm{pms}}$ and $\mathbf{Fun}$. □

Corollary 3.14. *The category* $\mathbf{Nbh}_{\mathrm{pms}}$ *is complete and cocomplete.*

4 Uniform and function spaces

In this paper, we define a notion of a uniform space using a base of uniformity as in [13] which is different from the one in [9] and related papers. A *uniform space* A is pair $(\underline{A}, \mathcal{U}_A)$ consisting of a set $\underline{A}$ and an inhabited set $\mathcal{U}_A$ of subsets of $\underline{A} \times \underline{A}$, called a *uniformity* on $\underline{A}$, such that

Ub1. $\forall U, V \in \mathcal{U}_A \exists W \in \mathcal{U}_A (W \subseteq U \cap V)$,

Ub2. $\forall U \in \mathcal{U}_A (\Delta \subseteq U)$,

Ub3. $\forall U \in \mathcal{U}_A \exists V \in \mathcal{U}_A (V \subseteq U^{-1})$,

Ub4. $\forall U \in \mathcal{U}_A \exists V \in \mathcal{U}_A (V \circ V \subseteq U)$.

Here $\Delta = \{(x, x) \mid x \in \underline{A}\}$, and $U^{-1} = \{(x, y) \mid (y, x) \in U\}$ and

$$U \circ V = \{(x, z) \mid \exists y((x, y) \in V \wedge (y, z) \in U)\}$$

for each $U, V \subseteq \underline{A} \times \underline{A}$. A *uniformly continuous mapping* f from a uniform spaces A into a uniform space B is a mapping $f : \underline{A} \to \underline{B}$ such that

$$\forall V \in \mathcal{U}_B \exists U \in \mathcal{U}_A \forall x, y \in \underline{A}[(x, y) \in U \implies (f(x), f(y)) \in V].$$

We write $f : A \to B$ to denote that f is a uniformly continuous mapping from A into B, and $\mathcal{C}_{\mathrm{u}}(A, B)$ for the set of uniformly continuous mappings from A into B.

For any set S, there are the uniform spaces $(S, \{\Delta\})$ and $(S, \{S \times S\})$, called the *discrete* uniform space of S and the *trivial* uniform space of S, respectively. For each uniform space Y, any mapping $f : S \to \underline{Y}$ is a uniformly continuous mapping from the discrete uniform space of S into Y, and any mapping $f : \underline{Y} \to S$ is a uniformly continuous mapping from Y into the trivial uniform space of S.

Let **Uni** denote the category of uniform spaces whose objects are uniform spaces and whose morphisms are uniformly continuous mappings. Note that, in **Uni**, the initial object is the discrete uniform space of $\emptyset$, and the terminal objects are the trivial uniform spaces of singletons.

Proposition 4.1. *Let S be a set, let $\{A_i\}_{i \in I}$ be a family of uniform spaces, and for each $i \in I$ let $f_i : S \to \underline{A_i}$. Then there exists a uniformity $\mathcal{U}$ on S such that if h is a mapping from the underlying set $\underline{B}$ of a uniform space B into S, then $h : B \to (S, \mathcal{U})$ if and only if $f_i \circ h : B \to A_i$ for each $i \in I$.*

Proof. Let

$$\mathcal{U} = \{\bigcap_{k+1}^{n}(f_{i_k} \times f_{i_k})^{-1}(U_k) \mid i_k \in I, U_k \in \mathcal{U}_{A_{i_k}}, 1 \le k \le n, 0 \le n\},$$

where $f_i \times f_i : S \times S \to \underline{A_i} \times \underline{A_i}$ is a mapping with $(f_i \times f_i)(x, y) = (f_i(x), f_i(y))$. Then $\mathcal{U}$ is a uniformity on S.

It is straightforward to see that $f_i : (S, \mathcal{U}) \to X_i$ for each $i \in I$, and hence if $h : Y \to (S, \mathcal{U})$, then $f_i \circ h : Y \to X_i$ for each $i \in I$. Let Y be a uniform space and let $h : \underline{Y} \to S$ be such that $f_i \circ h : Y \to X_i$ for each $i \in I$. Then for each $i_1, \ldots, i_n \in I$ and $U_1 \in \mathcal{U}_{A_{i_1}}, \ldots, U_n \in \mathcal{U}_{A_{i_n}}$, there exists $V \in \mathcal{U}_Y$ such that if $(x, y) \in V$, then $(f_{i_k}(h(x)), f_{i_k}(h(y))) \in U_k$ for each $k = 1, \ldots, n$, and hence

$$(h(x), h(y)) \in \bigcap_{k=1}^{n}(f_{i_k} \times f_{i_k})^{-1}(U_k).$$

Therefore $h : B \to (S, \mathcal{U})$. □

Theorem 4.2. *The category* **Uni** *is complete.*

Proof. Similar to the proof of Theorem 2.5, using Proposition 4.1. □

In the following, we shall write, simply, $\mathbf{R}$ for the uniform space $\mathbf{R}$ with the standard uniformity, and $\mathcal{C}_{\mathrm{u}}(A)$ for $\mathcal{C}_{\mathrm{u}}(A, \mathbf{R})$.

Let S be a set, and define a relation Ψ_S between a function space structure $\mathcal{F}$ on S and $g \in \mathbf{R}^S$ as follows: $\Psi_S(\mathcal{F}, g)$ if and only if for each $\epsilon > 0$ there exist $f_1, \ldots, f_n \in \mathcal{F}$ $(n \ge 0)$ and $\delta > 0$ such that

$$\forall x, y \in S \left(\sum_{i=1}^{n} |f_i(x) - f_i(y)| < \delta \implies |g(x) - g(y)| < \epsilon \right).$$

Then a pre-function space X is Ψ-*closed* if

$$\forall g \in \mathbf{R}^{\underline{X}}(\Psi_{\underline{X}}(\mathcal{F}_X, g) \implies g \in \mathcal{F}_X).$$

Note that if a pre-function space is Φ-closed, then it is Ψ-closed. If X is a Ψ-closed pre-function space, then $\mathcal{F}_X$ contains the constant functions, and the pointwise sum of finitely many functions in $\mathcal{F}_X$ belongs to $\mathcal{F}_X$. Furthermore, it is straightforward to show that if X is a Φ-closed pre-function space, then X is $\mathcal{C}_{\mathrm{u}}(\mathbf{R})$-complete, and uniform limits of functions of $\mathcal{F}_X$ are in $\mathcal{F}_X$.

For each set S, the discrete function space and the constant function space are is Ψ-closed. For each Ψ-closed pre-function space Y, any mapping $f : \underline{Y} \to S$ is a function space morphism from Y into the constant function space of S.

In the following, we shall call a Ψ-closed pre-function space a *uniform function space.*

Let $\mathbf{Fun}_{\mathrm{u}}$ denote the category of pre-function spaces with uniform function spaces as objects and function space morphisms as morphisms. Note that, in $\mathbf{Fun}_{\mathrm{u}}$, the initial object is the discrete function space of $\emptyset$, the terminal objects are the constant function spaces of singletons.

We now aim to prove the following result.

Theorem 4.3. *There exists an adjunction between* **Uni** *and* $\mathbf{Fun}_{\mathrm{u}}$ *whose counit is a natural isomorphism.*

Corollary 4.4. *The category* $\mathbf{Fun}_{\mathrm{u}}$ *is complete.*

To prove Theorem 4.3, we need a series of lemmas.

Lemma 4.5. *For each uniform space A, the pre-function space $(\underline{A}, \mathcal{C}_{\mathrm{u}}(A))$ is Ψ-closed.*

Proof. Let $g \in \mathbf{R}^{\underline{A}}$, and suppose that $\Psi(\mathcal{C}_{\mathrm{u}}(A), g)$. Then for each $\epsilon > 0$ there exist $f_1, \ldots, f_n \in \mathcal{C}_{\mathrm{u}}(A)$ and $\delta > 0$ such that for each $x, y \in \underline{A}$, if $\sum_{i=1}^n |f_i(x) - f_i(y)| < \delta$, then $|g(x) - g(y)| < \epsilon$. Hence there exists $U \in \mathcal{U}_A$ such that if $(x, y) \in U$, then $\sum_{i=1}^n |f_i(x) - f_i(y)| < \delta$, and hence $|g(x) - g(y)| < \epsilon$. Thus $g \in \mathcal{C}_{\mathrm{u}}(A)$. $\square$

Lemma 4.6. *Let A and B be uniform spaces. If $f : A \to B$, then $f : (\underline{A}, \mathcal{C}_{\mathrm{u}}(A)) \to (\underline{B}, \mathcal{C}_{\mathrm{u}}(B))$.*

Proof. Straightforward. $\square$

For a function space structure $\mathcal{F}$ on a set S, let $\mathcal{U}_{\mathcal{F}}$ be the set of subsets of $S \times S$ of the form

$$U_{f_1,\ldots,f_n}(\epsilon) = \{(x, y) \in S \times S \mid \textstyle\sum_{k=1}^n |f_i(x) - f_i(y)| < \epsilon\},$$

where $f_1, \ldots, f_n \in \mathcal{F}$ ($n \geq 0$) and $\epsilon > 0$.

Lemma 4.7. *For each pre-function space X, the pair $(\underline{X}, \mathcal{U}_{\mathcal{F}_X})$ is a uniform space.*

Proof. Straightforward. For (Ub4), note that $U_{f_1,\ldots,f_n}(\epsilon/2) \circ U_{f_1,\ldots,f_n}(\epsilon/2) \subseteq U_{f_1,\ldots,f_n}(\epsilon)$. $\square$

Lemma 4.8. *Let X and Y be pre-function space. If $f : X \to Y$, then $f : (\underline{X}, \mathcal{U}_{\mathcal{F}_X}) \to (\underline{Y}, \mathcal{U}_{\mathcal{F}_Y})$.*

Proof. Suppose that $f : X \to Y$ and $U_{g_1,\ldots,g_n}(\epsilon) \in \mathcal{U}_{\mathcal{F}_Y}$. Then, since $g_i \circ f \in \mathcal{F}_X$ for each $i = 1, \ldots, n$, we have $U_{g_1 \circ f,\ldots,g_n \circ f}(\epsilon) \in \mathcal{U}_{\mathcal{F}_X}$, and if $(x, y) \in U_{g_1 \circ f,\ldots,g_n \circ f}(\epsilon)$, then $(f(x), f(y)) \in U_{g_1,\ldots,g_n}(\epsilon)$. Hence $f : (\underline{X}, \mathcal{U}_{\mathcal{F}_X}) \to (\underline{Y}, \mathcal{U}_{\mathcal{F}_Y})$. $\square$

Lemma 4.9. *If A is a uniform space, then $\mathrm{id}_{\underline{A}} : A \to (\underline{A}, \mathcal{U}_{\mathcal{C}_{\mathrm{u}}(A)})$.*

Proof. Let $U_{f_1,\ldots,f_n}(\epsilon) \in \mathcal{U}_{\mathcal{C}_{\mathrm{u}}(A)}$. Then, since $f_1, \ldots, f_n \in \mathcal{C}_{\mathrm{u}}(A)$, there exists $U \in \mathcal{U}_A$ such that if $(x, y) \in U$, then $\sum_{i=1}^{n} |f_i(x) - f_i(y)| < \epsilon$, and hence $(x, y) \in U_{f_1,\ldots,f_n}(\epsilon)$. Thus $\mathrm{id}_{\underline{A}} : A \to (\underline{A}, \mathcal{U}_{\mathcal{C}_{\mathrm{u}}(A)})$. $\square$

Lemma 4.10. *Let X be a pre-function space. Then $\mathcal{F}_X \subseteq \mathcal{C}_{\mathrm{u}}(\underline{X}, \mathcal{U}_{\mathcal{F}_X})$. Moreover, if X is Ψ-closed, then $\mathcal{C}_{\mathrm{u}}(\underline{X}, \mathcal{U}_{\mathcal{F}_X}) \subseteq \mathcal{F}_X$.*

Proof. Note that $g \in \mathcal{C}_{\mathrm{u}}(\underline{X}, \mathcal{U}_{\mathcal{F}_X})$ if and only if $\Psi_{\underline{X}}(\mathcal{F}_X, g)$. Then, trivially, $\mathcal{F}_X \subseteq \mathcal{C}_{\mathrm{u}}(\underline{X}, \mathcal{U}_{\mathcal{F}_X})$, and if X is Ψ-closed, then $\mathcal{C}_{\mathrm{u}}(\underline{X}, \mathcal{U}_{\mathcal{F}_X}) \subseteq \mathcal{F}_X$. $\square$

We end with the

Proof of Theorem 4.3. Define a functor F from **Uni** to **Fun**$_{\mathrm{u}}$ by $F(A) = (\underline{A}, \mathcal{C}_{\mathrm{u}}(A))$ and $F(f) = f$, and define a functor G from **Fun** to **Uni** by $G(X) = (\underline{X}, \mathcal{U}_{\mathcal{F}_X})$ and $G(f) = f$. Then F and G are faithful functors, by Lemma 4.8 and Lemma 4.6.

Furthermore, we see that if we let η_A and ϵ_X denote the indentity maps on the sets $\underline{A}$ and $\underline{X}$, respectively, then $\eta_A : A \to (\underline{A}, \tau_{\mathcal{C}_{\mathrm{u}}(A)})$ in **Uni**, by Lemma 4.9, and

$$\epsilon_X : (\underline{X}, \mathcal{C}_{\mathrm{u}}(\underline{X}, \tau_{\mathcal{F}_X})) \to X \text{ and } \epsilon_X^{-1} : X \to (\underline{X}, \mathcal{C}_{\mathrm{u}}(\underline{X}, \tau_{\mathcal{F}_X}))$$

in **Fun**$_{\mathrm{u}}$, by Lemma 4.10. Hence $\eta : 1_{\mathbf{Uni}} \to GF$ is a natural transformation and $\epsilon : FG \to 1_{\mathbf{Fun}_{\mathrm{u}}}$ is a natural isomorphism satisfying $\epsilon_F \circ F\eta = 1_F$ and $G\epsilon \circ \eta_G = 1_G$. Therefore $\langle F, G, \eta, \epsilon \rangle$ forms an adjunction between **Uni** and **Fun**$_{\mathrm{u}}$. $\square$

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