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COMMON DEVELOPMENTS OF THREE INCONGRUENT ORTHOGONAL BOXES*

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We investigate common developments that can fold into several incongruent orthogonal boxes. It was shown that there are infinitely many orthogonal polygons that fold into two incongruent orthogonal boxes in 2008. In 2011, it was shown that there exists an orthogonal polygon that folds into three boxes of size $1 \times 1 \times 5$, $1 \times 2 \times 3$, and $0 \times 1 \times 11$. However it remained open whether there exists an orthogonal polygon that folds into three boxes of positive volume. We give an affirmative answer to this open problem. We show how to construct an infinite number of orthogonal polygons that fold into three incongruent orthogonal boxes.

Keywords: Common development; convex polyhedron.

1. Introduction

Since Lubiw and O'Rourke posed the problem,[5] polygons that can fold into a (convex) polyhedron have been investigated. In the book on geometric folding algorithms by Demaine and O'Rourke,[4] many results about such polygons are given. One of the many interesting problems in this area asks whether there exists a polygon that folds into several incongruent orthogonal boxes. This folding problem is very natural in a discrete geometry world; you are given a polygon that consists of unit squares, and the problem asks are there two or more ways to fold it into simple convex polyhedra. (In this paper, we assume that all the creases are also parallel to the polygon sides.) Biedl et al. first gave two polygons that fold into two incongruent orthogonal boxes[3] (see also Figure 25.53 in the book by Demaine and O'Rourke[4]). Later, Mitani and Uehara constructed infinite families of orthogonal polygons that fold into two incongruent orthogonal boxes.[6] Recently, Abel et al. showed an orthogonal polygon that folds into three boxes of size $1 \times 1 \times 5$, $1 \times 2 \times 3$, and $0 \times 1 \times 11$.[1]

*A preliminary version was presented at CCCG 2012.

However, the last "box" has volume zero, a so-called "doubly covered rectangle."[2] Therefore, it remained open to show whether there is a polygon that can fold into three or more boxes of positive volume.

We give an affirmative answer to this open problem; there exists an orthogonal polygon that can fold into three incongruent orthogonal boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$ (Fig. 1)[a].

The construction idea can be generalized. Therefore, we conclude that there exist infinitely many orthogonal polygons that can fold into three incongruent orthogonal boxes.

2. Construction of the common development

Demaine and O'Rourke[4] give a formal definition of the *development* of a polyhedron. Briefly, the development is the unfolding obtained by slicing the surface of the polyhedron, and it forms a single connected simple polygon without self-overlap. The *common development* of two (or more) polyhedra is a single development that can fold into both polyhedra. In this paper, we only consider orthogonal polygons that consist of unit squares as developments, and boxes of size $x \times y \times z$ for some positive integers x, y, z as polyhedra. For a development, every crease line is a line segment through the boundaries of unit squares.

Intuitively, the basic construction idea is simple. We first choose a common development of two different boxes of size $a \times b \times c$ and $a' \times b' \times c'$. We select one of these two boxes; let it have size $a \times b \times c$. We cut the two rectangles of size $a \times b$ (one at the *top*, and another at the *bottom* of the box) into two pieces of size $a \times b/2$ each. Then we squash the box and make these two rectangles of size $a \times b$ into two rectangles of size $(a + b/2) \times b/2 = 2a \times b/2$ (Fig. 2). However, this simple idea immediately comes to a dead end; this operation can be done properly if and only if $a = b/2$, and hence we only change the rectangle of size 1×2 into the other rectangle of size 2×1, which are congruent.

The main trick to avoid this problem is to move pieces of the rectangles of size $a \times b$ of the box to the side rectangles of size $b \times c$ and $a \times c$. That is, after the squash operation above, the surface areas of the resultant top and bottom rectangles decrease, and the side rectangles grow a little. A specific example is given in Fig. 3; in this example, the rectangle of size 8×7 is split into two congruent pieces by a mid zig-zag line; each piece in turn is divided into one central piece (labeled A, B in Fig. 3). The result is a rectangle of size 13×2. (In Fig. 3(a), the bold lines are cut lines, and dotted lines are folding lines to obtain (b). The lines l_1, l_2, l_3, and l_4 are corresponding, and the gray triangles indicate how two squares are arranged by the operation.) Among the 56 squares, $56 - 26 = 30$ squares are moved to the four sides. We note that the perimeter of these two rectangles is not changed since

[a]This figure is also available at `http://www.jaist.ac.jp/~uehara/etc/puzzle/nets/3box.pdf` for ease to cut and fold.

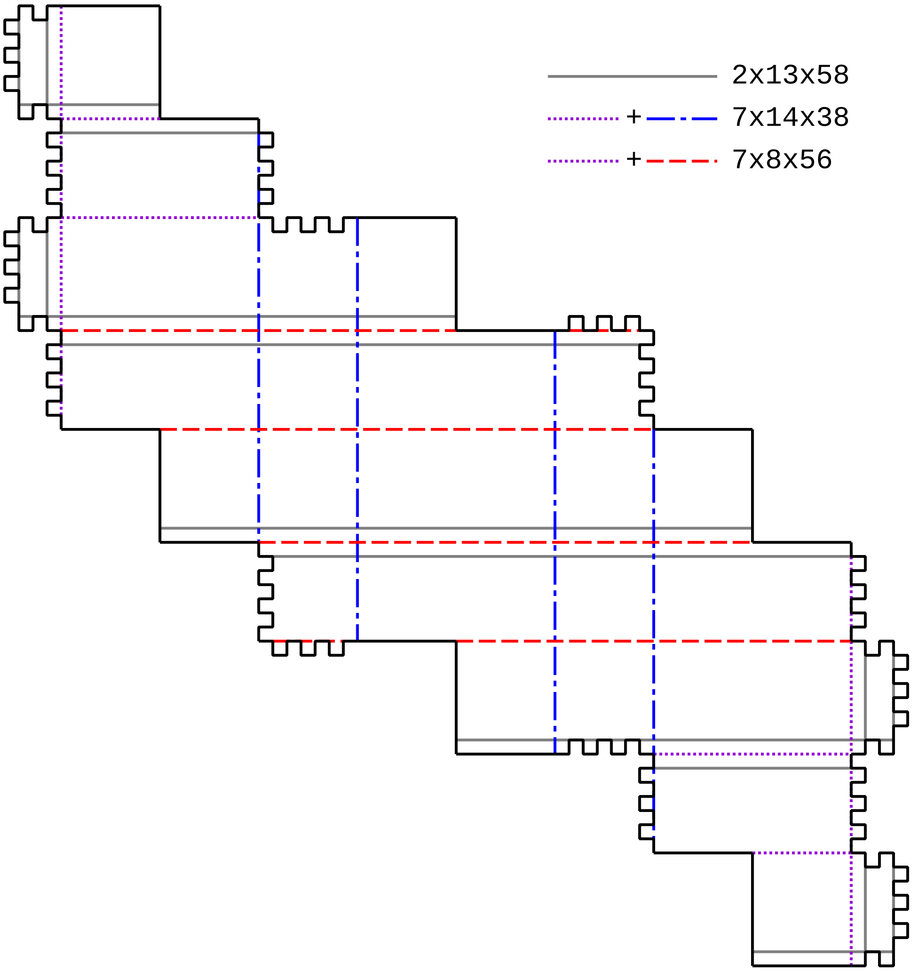

Fig. 1. A common development of three different boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$.

$7 + 8 + 7 + 8 = 2 + 13 + 2 + 13 = 30$.

To apply this idea, we choose a common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$ in Fig. 4. This is a modification of the common development[6] of two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$. To apply the idea, we cut each of the top and bottom rectangles of size $a \times b$ into two congruent rectangles of size $a/2 \times b$. For any integers a and b, the orthogonal polygon in Fig. 4 is a common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$ (the two ways of folding are drawn in bold lines in Fig. 5).

The development in Fig. 4 has useful properties for applying the idea in Fig. 2: (1) we can adjust the size of the top and bottom rectangles to an arbitrary size, and (2) the two ways of folding share several folding lines. In particular, in Fig. 5,

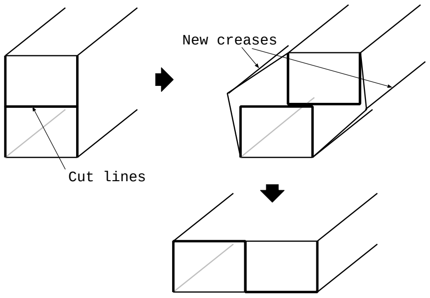

Fig. 2. Basic idea: squash the box.

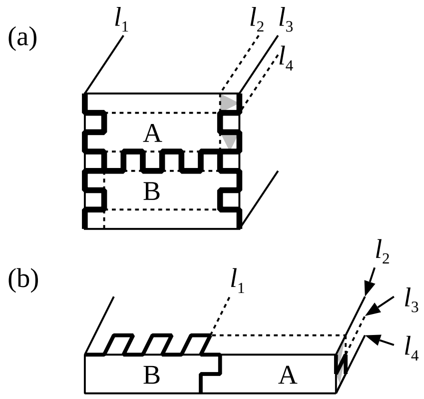

Fig. 3. Squash the box: cut and fold.

each of the two connected gray areas is folded in the same way in both ways of folding. Thus we attach the gadget from Fig. 2 in this neighborhood letting $a = 7$ and $b = 8$. That is, we replace the rectangles of size $a/2 \times b$ by the rectangles A and B surrounded by the zig-zag lines in Fig. 3.

The only problem when applying the gadget is that the zig-zag lines propagate themselves according to the ways of folding. That is, the zig-zag lines are glued to the different edges in some folding. For example, a zig-zag line at the black triangle in Fig. 5(a) is attached to the edge at the black triangle in the manner shown in Fig. 5(b). Thus, these edges must consist of the same zig-zag pattern. On the other hand, this edge is attached to the edge at the black square in the manner shown in Fig. 5(a), which is attached to the black square in Fig. 5(b). Thus, they also must have the same zig-zag pattern. Then the last edge is again attached to the edge with the black circle in Fig. 5(a), and this is attached to the two edges with the smaller black circles in Fig. 5(b). Then the loop of the propagation is closed, and we obtain the set of the edges that have to be represented by the zig-zag pattern.

Checking all the propagations, we finally obtain a common development of three different boxes of size $7 \times 8 \times 56$, $7 \times 14 \times 38$, and $2 \times 13 \times 58$ in Fig. 1.

3. Generalization

In Section 2, we set $a = 7$ and $b = 8$, and changed the rectangle of size 7×8 into 2×13. It is straightforward to generalize this method. For example, setting $a = 11$ and $b = 10$, we can change the rectangle of size 11×10 into 4×17 (see Fig. 6). In general, for each integer $k = 0, 1, 2, \ldots$, setting $a = 4k + 7$ and $b = 2(k + 4)$, we can change the rectangle of size $a \times b$ to $2(k + 1) \times (4k + 13)$ in the same way as in Fig. 3. The difference here from Fig. 1 is in the number of turns of the zig-zags. Therefore, we have the following theorem immediately:

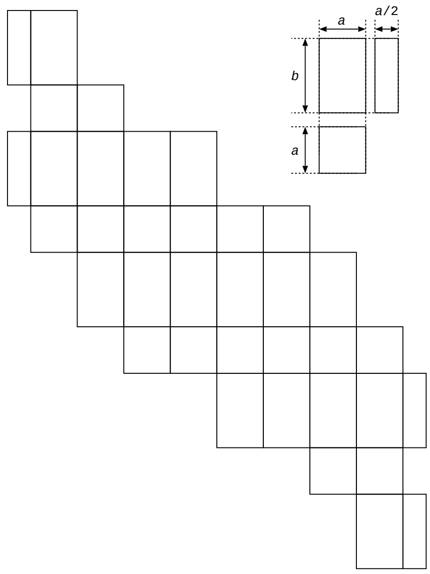

Fig. 4. The base common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$.

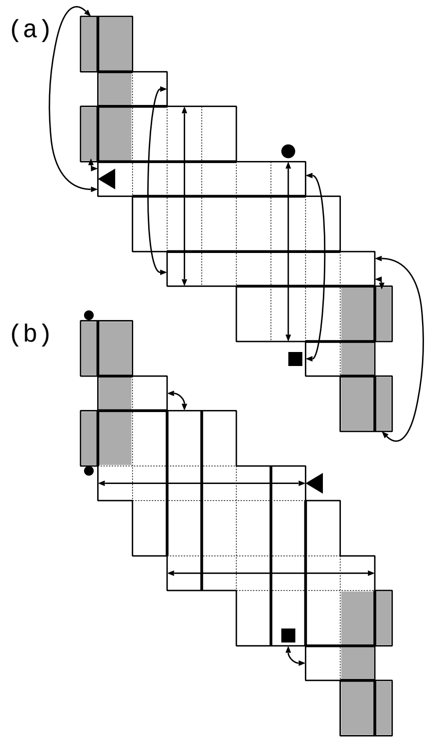

Fig. 5. Some properties of the common development of two boxes of size $a \times b \times 8a$ and $a \times 2a \times (2a + 3b)$.

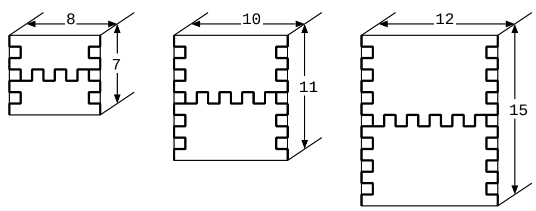

Fig. 6. Generalization of the zig-zag cut.

Theorem 1. *For each integer* $k = 0, 1, 2, \ldots$, *there is a common development that can fold into three different boxes of size* $(4k+7) \times 2(k+4) \times 8(4k+7)$, $(4k+7) \times 2(4k+7) \times 2(7k+19)$, *and* $2(k+1) \times (4k+13) \times 2(16k+29)$.

That is, there exist an infinite number of orthogonal polygons that can fold into three incongruent orthogonal boxes.

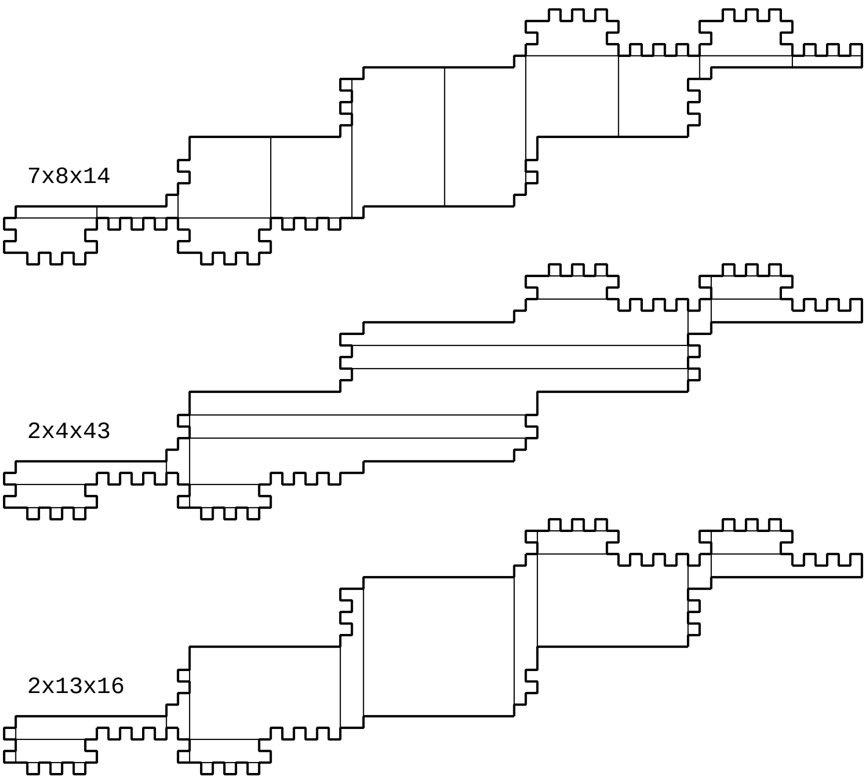

Fig. 7. Another polygon that can fold into three boxes of size $7 \times 8 \times 14$, $2 \times 4 \times 43$, and $2 \times 13 \times 16$.

4. Concluding remarks

It is an open question if a polygon exists that can fold into four or more orthogonal boxes such that all of them have positive volume.

When two boxes of size $a \times b \times c$ and $a' \times b' \times c'$ share a common development, they satisfy a simple necessary condition $ab + bc + ca = a'b' + b'c' + c'a'$ since they have the same surface area. According to our experiments, this necessary condition seems also sufficient for two boxes: for each pair of 3-tuples of integers satisfying the condition, there exist many common developments of two boxes of these sizes.[6] In this sense, the smallest possible surface area that can fold into three different boxes is 46; the area can produce three boxes of size $(1, 1, 11)$, $(1, 2, 7)$, and $(1, 3, 5)$. On the other hand, our construction produces a polygon of large surface area. The polygon in Fig. 1 has area 1792. Applying the same idea to the different common development, we also construct another smaller development of area 532 (Fig. 7). Finding much smaller polygons would be a future work. In particular, is there a common development of area 46 that can fold into three boxes of size $(1, 1, 11)$, $(1, 2, 7)$, and $(1, 3, 5)$?

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