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Description	

# Reverse mathematics and Peano categoricity

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## Abstract

We investigate the reverse-mathematical status of several theorems to the effect that the natural number system is second-order categorical. One of our results is as follows. Define a *system* to be a triple  $A, i, f$  such that  $A$  is a set and  $i \in A$  and  $f : A \rightarrow A$ . A subset  $X \subseteq A$  is said to be *inductive* if  $i \in X$  and  $\forall a (a \in X \Rightarrow f(a) \in X)$ . The system  $A, i, f$  is said to be *inductive* if the only inductive subset of  $A$  is  $A$  itself. Define a *Peano system* to be an inductive system such that  $f$  is one-to-one and  $i \notin \text{range of } f$ . The standard example of a Peano system is  $\mathbb{N}, 0, S$  where  $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$  is the set of natural numbers and  $S : \mathbb{N} \rightarrow \mathbb{N}$  is given by  $S(n) = n + 1$  for all  $n \in \mathbb{N}$ . Consider the statement that all Peano systems are isomorphic to  $\mathbb{N}, 0, S$ . We prove that this statement is logically equivalent to  $\text{WKL}_0$  over  $\text{RCA}_0^*$ . From this and similar equivalences we draw some foundational/philosophical consequences.

Keywords: Reverse mathematics, second-order arithmetic, Peano system, foundations of mathematics, proof theory, second-order logic.

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## 1 Introduction

*Reverse mathematics* is a well known [15, 17] research direction in the foundations of mathematics. The goal of reverse mathematics is to pinpoint the weakest set-existence axioms which are needed in order to prove specific theorems of core mathematics. Such investigations are most fruitfully carried out in the context of subsystems of second-order arithmetic [15]. In that context it frequently happens that the weakest axioms needed to prove a particular theorem are logically equivalent to the theorem, over a weak base theory. For example, the well known theorem that every uncountable closed set in Euclidean space contains a perfect subset is logically equivalent to  $\text{ATR}_0$  over the weak base theory  $\text{RCA}_0$  [15, Theorem V.5.5].

A key theorem in rigorous core mathematics is the categoricity of the natural number system. Stated more precisely and in 20th-century language, the *Peano Categoricity Theorem* [11, Theorem 2.7.1] asserts that any two Peano systems are isomorphic. The Peano Categoricity Theorem was originally proved by Dedekind in 1888 [4, Satz 132], [5, Theorem 132] as a highlight of his rigorous, set-theoretical development [3, 4, 5] of the natural number system  $\mathbb{N}$  and the real number system  $\mathbb{R}$ .

In this paper we investigate the reverse-mathematical and proof-theoretical status of the Peano Categoricity Theorem and related theorems. One of our results is as follows.

The Peano Categoricity Theorem is equivalent to  $\text{WKL}_0$   
over the standard weak base theory  $\text{RCA}_0$ . (1)

Here  $\text{RCA}_0$  and  $\text{WKL}_0$  are familiar [15, 17] subsystems of second-order arithmetic. Namely,  $\text{RCA}_0$  consists of  $\Delta_1^0$  comprehension plus  $\Sigma_1^0$  induction, and  $\text{WKL}_0$  consists of  $\text{RCA}_0$  plus Weak König's Lemma.

Our result (1) offers further confirmation of a point made by Väänänen<sup>1</sup> in a recent talk based on his recent paper [19]. Väänänen observed that various second-order categoricity theorems can be proved without resorting to the full strength of second-order logic. Clearly (1) bears this out, because  $WKL_0$  is a relatively weak<sup>2</sup> subsystem of second-order arithmetic, much weaker than  $ACA_0$  and in fact  $\Pi_2^0$ -equivalent to Primitive Recursive Arithmetic [15, §IX.3]. Since by (1) the Peano Categoricity Theorem is provable in  $WKL_0$ , it follows that the Peano Categoricity Theorem is *finitistically reducible* in the sense of Simpson’s partial realization [14, 16] (see also [1]) of Hilbert’s Program [7].

As a refinement of (1) we obtain the following stronger result.

The Peano Categoricity Theorem is equivalent to  $WKL_0$   
not only over  $RCA_0$  but over the much weaker base theory  $RCA_0^*$ . (2)

Recall from [15, §X.4] and [18] that  $RCA_0^*$  is  $RCA_0$  with  $\Sigma_1^0$  induction weakened to *natural number exponentiation*, i.e., the assertion that  $m^n$  exists for all  $m, n \in \mathbb{N}$ . It is known that  $RCA_0^*$  is  $\Pi_2^0$ -equivalent to Elementary Function Arithmetic [18], hence much weaker than  $RCA_0$  and  $WKL_0$  which are  $\Pi_2^0$ -equivalent to Primitive Recursive Arithmetic [15, §IX.3].

Our stronger result (2) provides some foundational or philosophical insight concerning Dedekind’s construction of the natural number system [4, 5]. Recall that Dedekind’s key technical lemma, the “Satz der Definition durch Induction,” is a straightforward embodiment<sup>3</sup> of the idea of primitive recursion. But at the same time, according to (2), the Peano Categoricity Theorem itself *requires* primitive recursion. Thus (2) constitutes further evidence that primitive recursion is indeed the heart of the matter.

The plan of this paper is as follows. In §2 we prove (1). In §3 we prove (2). In §4 we investigate the reverse-mathematical status of certain variants of the Peano Categoricity Theorem, replacing the Peano system  $\mathbb{N}, 0, S$  by the ordered system  $\mathbb{N}, 0, <$  or the ordered Peano system  $\mathbb{N}, 0, <, S$ . In §5 we summarize our results and state some open questions.

## 2 The role of Weak König’s Lemma

Recall from [15] that  $RCA_0$  is the subsystem of second-order arithmetic consisting of  $\Delta_1^0$  comprehension and  $\Sigma_1^0$  induction. Within  $RCA_0$  one may freely use primitive recursion and minimization to define functions  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  where  $\mathbb{N}$  is the set of natural numbers [15, §II.3]. Recall also [15] that  $WKL_0$  consists of  $RCA_0$  plus Weak König’s Lemma, i.e., the statement that every infinite tree  $T \subseteq \{0, 1\}^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$  has an infinite path.

The purpose of this section is to show that the Peano Categoricity Theorem is equivalent to Weak König’s Lemma, the equivalence being provable in  $RCA_0$ .

<sup>1</sup>We thank Jouko Väänänen for raising the question which is answered by (1).

<sup>2</sup>By the *strength* of a theory  $T$  we mean the set of  $\Pi_1^0$  sentences which are provable in  $T$ .

<sup>3</sup>Dedekind’s “Satz der Definition durch Induction” [4, Satz 126] [5, Theorem 126] may be restated in 20th-century language [11, Theorem 2.2.1] as follows. For any system  $A, i, f$  there is a unique function  $\Phi : \mathbb{N} \rightarrow A$  such that  $\Phi(0) = i$  and  $\Phi(n + 1) = f(\Phi(n))$  for all  $n \in \mathbb{N}$ .

**Definition 2.1.** Within  $\text{RCA}_0$  we make the following definitions. A *system* is a triple  $A, i, f$  such that  $i \in A \subseteq \mathbb{N}$  and  $f : A \rightarrow A$ . A *Peano system* is a system  $A, i, f$  such that  $i \notin \text{rng}(f)$  and  $f$  is one-to-one and

$$(\forall X \subseteq A) ((i \in X \wedge \forall a (a \in X \Rightarrow f(a) \in X)) \Rightarrow X = A).$$

The standard example of a Peano system is  $\mathbb{N}, 0, S$  with  $S : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $S(n) = n + 1$ . A Peano system  $A, i, f$  is said to be *isomorphic to  $\mathbb{N}$*  if there exists a bijection  $\Phi : A \rightarrow \mathbb{N}$  such that  $\Phi(i) = 0$  and  $\Phi(f(a)) = \Phi(a) + 1$  for all  $a \in A$ . A Peano system  $A, i, f$  is said to be *almost isomorphic to  $\mathbb{N}$*  if for each  $a \in A$  there exists  $n \in \mathbb{N}$  such that  $f^n(i) = a$ .

**Lemma 2.2.** The following is provable in  $\text{RCA}_0$ . If a Peano system is almost isomorphic to  $\mathbb{N}$ , it is isomorphic to  $\mathbb{N}$ .

*Proof.* We reason in  $\text{RCA}_0$ . Let  $A, i, f$  be a system. As in [15, §II.3] use  $\Sigma_1^0$  induction to prove that for all  $n \in \mathbb{N}$ ,  $f^n(i)$  exists and  $f^n(i) \in A$ . Use  $\Delta_1^0$  comprehension to prove the existence of the function  $n \mapsto f^n(i) : \mathbb{N} \rightarrow A$ . Assume now that  $A, i, f$  is a Peano system which is almost isomorphic to  $\mathbb{N}$ . Use  $\Delta_1^0$  comprehension to prove the existence of the function  $\Phi : A \rightarrow \mathbb{N}$  given by  $\Phi(a) = \min\{n \mid f^n(i) = a\}$  for all  $a \in A$ . Clearly  $\Phi(i) = 0$  and  $\Phi(f(a)) = \Phi(a) + 1$  for all  $a \in A$ , hence  $\Phi$  is one-to-one. Moreover, because  $A, i, f$  is almost isomorphic to  $\mathbb{N}$ ,  $\Phi$  is onto  $\mathbb{N}$ . Thus  $\Phi$  maps  $A, i, f$  isomorphically onto  $\mathbb{N}$ .  $\square$

**Theorem 2.3.** The following are equivalent over  $\text{RCA}_0$ .

1.  $\text{WKL}_0$ .
2. Every Peano system is isomorphic to  $\mathbb{N}$ .

*Proof.* We first prove  $1 \Rightarrow 2$ . Reasoning in  $\text{WKL}_0$ , let  $A, i, f$  be a Peano system. Recall that  $A \subseteq \mathbb{N}$ . By Lemma 2.2 it suffices to show that for each  $a \in A$  there exists  $n \in \mathbb{N}$  such that  $f^n(i) = a$ . Assume for a contradiction that  $c \in A$  and  $f^n(i) \neq c$  for all  $n \in \mathbb{N}$ . Let  $T$  be the set of all  $t \in \{0, 1\}^{<\mathbb{N}}$  such that

$$(i < \text{lh}(t) \wedge c < \text{lh}(t)) \Rightarrow t(i) \neq t(c)$$

and

$$(\forall a, b < \text{lh}(t)) (f(a) = b \Rightarrow t(a) = t(b)).$$

Clearly  $T$  is a tree. We claim that  $T$  is infinite. To see this, let  $n \in \mathbb{N}$  be given. Define  $t : \{0, \dots, n-1\} \rightarrow \{0, 1\}$  by letting  $t(a) = 1$  if there exists a finite sequence  $a_0, \dots, a_k$  of elements of  $\{0, \dots, n-1\}$  such that  $i = a_0$  and  $f(a_0) = a_1$  and  $f(a_1) = a_2$  and  $\dots$  and  $f(a_{k-1}) = a_k = a$ . If  $a \in \{0, \dots, n-1\}$  and no such finite sequence exists, let  $t(a) = 0$ . Clearly  $t \in T$  and  $\text{lh}(t) = n$  so our claim is proved. By Weak König's Lemma let  $h$  be an infinite path in  $T$ . Letting  $X = \{a \in A \mid h(a) = h(i)\}$  we see that  $i \in X$  and  $f(X) \subseteq X$  and  $c \notin X$  contradicting our assumption that  $A, i, f$  is a Peano system.

Next we prove  $(\neg 1) \Rightarrow (\neg 2)$ . Reasoning in  $\text{RCA}_0$ , assume  $\neg 1$  and let  $T \subseteq \{0, 1\}^{<\mathbb{N}}$  be an infinite tree with no infinite path. Then

$$T' = T \cup \{\underbrace{\langle 1, \dots, 1 \rangle}_n \mid n \in \mathbb{N}\}$$

is a tree with exactly one infinite path. Consider the lexicographic ordering of  $T'$ . The empty sequence  $\langle \rangle$  is the first element of  $T'$ , and  $T$  is an initial segment of  $T'$ , and each  $t \in T'$  has an immediate successor in  $T'$ , and the immediate successor of  $t$  is of the form  $t \hat{\ } \langle 0 \rangle$  or  $t \hat{\ } \langle 1 \rangle$  or  $\langle t(0), \dots, t(m-1), 1 \rangle$  where  $m < \text{lh}(t)$  and  $t(m) = 0$ . Use  $\Delta_1^0$  comprehension to prove the existence of the function  $f : T' \rightarrow T'$  defined by  $f(t) =$  the immediate successor of  $t$ .

We claim that  $T', \langle \rangle, f$  is a Peano system. Otherwise, let  $X \subseteq T'$  be such that  $\langle \rangle \in X$  and  $f(X) \subseteq X$  and  $X \neq T'$ . Then clearly  $T' \not\subseteq X$ , so fix  $t_1 \in T' \setminus X$  and let  $Y = \{t \in T' \mid t \text{ precedes } t_1 \text{ in the lexicographic ordering of } T'\}$ . Use bounded primitive recursion to prove the existence of  $g : \mathbb{N} \rightarrow \{0, 1\}$  defined by

$$g(n) = \begin{cases} 1 & \text{if } \langle g(0), \dots, g(n-1), 1 \rangle \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\langle g(0), \dots, g(n-1) \rangle \in Y$  implies  $\langle g(0), \dots, g(n-1), g(n) \rangle \in Y$ , because otherwise the immediate successor of  $\langle g(0), \dots, g(n-1) \rangle$  would be of the form  $\langle g(0), \dots, g(m-1), 1 \rangle = f(\langle g(0), \dots, g(n-1) \rangle) \in Y$  where  $m < n$  and  $g(m) = 0$ , contradicting the definition of  $g(m)$ . Since  $\langle \rangle \in Y$ , it follows by  $\Delta_1^0$  induction that  $\langle g(0), \dots, g(n) \rangle \in Y$  for all  $n \in \mathbb{N}$ . In particular  $g$  is an infinite path in  $T'$ . This contradiction proves our claim.

Since  $T$  is an infinite initial segment of  $T'$ , the Peano system  $T', \langle \rangle, f$  cannot be isomorphic to  $\mathbb{N}$ . We have now proved  $\neg 2$ , Q.E.D.  $\square$

### 3 The role of $\Sigma_1^0$ induction

Recall from [15, §X.4] and [18] that  $\text{RCA}_0$  consists of  $\text{RCA}_0^*$  plus  $\Sigma_1^0$  induction. In particular  $\text{RCA}_0^*$  is weaker<sup>4</sup> than  $\text{RCA}_0$  and does not support the full use of primitive recursion. However,  $\text{RCA}_0^*$  does support the use of *bounded* primitive recursion, as well as minimization [18]. Recall also [18] that  $\text{WKL}_0^*$  consists of  $\text{RCA}_0^*$  plus Weak König's Lemma.

The purpose of this section is to refine the results of the previous section, using the base theory  $\text{RCA}_0^*$  instead of  $\text{RCA}_0$ . We prove within  $\text{RCA}_0^*$  that Weak König's Lemma and  $\Sigma_1^0$  induction are equivalent to certain statements about Peano systems. As a consequence we show that  $\text{RCA}_0^*$  can replace  $\text{RCA}_0$  in the statement of Theorem 2.3.

**Definition 3.1.** Within  $\text{RCA}_0^*$  we repeat Definition 2.1. Note however that  $\text{RCA}_0^*$  is not strong enough to prove that  $f^n(i)$  exists for all  $n \in \mathbb{N}$  and all systems  $A, i, f$ . Consequently, the notion of a Peano system being almost isomorphic to  $\mathbb{N}$  must be understood somewhat differently.

<sup>4</sup>See footnote 2.

**Lemma 3.2.** The following are equivalent over  $\text{RCA}_0^*$ .

1.  $\text{RCA}_0$ .
2. Every Peano system which is almost isomorphic to  $\mathbb{N}$  is isomorphic to  $\mathbb{N}$ .
3. For each infinite set  $C \subseteq \mathbb{N}$  there exists a one-to-one function  $f : \mathbb{N} \rightarrow C$ .
4. Each infinite subset of  $\mathbb{N}$  includes arbitrarily large finite sets.

*Proof.* We reason within  $\text{RCA}_0^*$ . The implication  $1 \Rightarrow 2$  has already been proved as Lemma 2.2. To prove  $2 \Rightarrow 3$ , let  $C$  be an infinite subset of  $\mathbb{N}$  and apply 2 to the Peano system  $C, c_0, \nu_C$  where  $c_0$  is the least element of  $C$  and  $\nu_C : C \rightarrow C$  is given by  $\nu_C(c) =$  the least  $c' \in C$  such that  $c' > c$ .

The implication  $3 \Rightarrow 4$  is easily proved by means of  $\Sigma_1^0$  bounding [18]. To prove  $4 \Rightarrow 1$  we must prove that 4 implies  $\Sigma_1^0$  induction. Let  $\varphi(n)$  be a  $\Sigma_1^0$  formula such that  $\varphi(0)$  and  $\forall n (\varphi(n) \Rightarrow \varphi(n+1))$  hold. Write  $\varphi(n) \equiv \exists k \theta(k, n)$  where  $\theta(k, n)$  is a  $\Sigma_0^0$  formula. Use  $\Delta_1^0$  comprehension to prove the existence of the set  $C$  consisting of all (codes for) finite sequences  $s = \langle k_0, k_1, \dots, k_n \rangle$  such that  $(\forall m \leq n) (\theta(k_m, m) \wedge \neg(\exists k < k_m) \theta(k, m))$  holds. Our assumptions  $\varphi(0)$  and  $\forall n (\varphi(n) \Rightarrow \varphi(n+1))$  imply that  $C$  has a least element but no greatest element, hence  $C$  is infinite. Now, given  $n \in \mathbb{N}$ , apply 4 to get a finite set  $F \subset C$  of cardinality  $> n$ . Because  $\text{lh} : C \rightarrow \mathbb{N}$  is one-to-one, there exists  $s \in F$  such that  $\text{lh}(s) > n$ . Since  $\text{lh}(s) > n$  and  $s \in C$  it follows that  $\theta(k_n, n)$  holds, hence  $\varphi(n)$  holds. This proves  $\forall n \varphi(n)$ , Q.E.D.  $\square$

**Lemma 3.3.** The following are equivalent over  $\text{RCA}_0^*$ .

1.  $\text{WKL}_0^*$ .
2. Every Peano system is almost isomorphic to  $\mathbb{N}$ .

*Proof.* Our proof of Theorem 2.3 establishes this result.  $\square$

**Theorem 3.4.** The following are equivalent over  $\text{RCA}_0^*$ .

1.  $\text{WKL}_0$ .
2. Every Peano system is isomorphic to  $\mathbb{N}$ .

*Proof.* Combine Lemmas 3.2 and 3.3.  $\square$

## 4 Other categoricity theorems

The Peano Categoricity Theorem may be viewed as a second-order characterization of the natural number system  $\mathbb{N}$  up to isomorphism using the language consisting of the constant  $0 \in \mathbb{N}$  and the successor function  $S : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $S(n) = n + 1$ . It is also possible to study second-order characterizations of  $\mathbb{N}$  in terms of other languages. In this section we consider two languages

which include the order relation  $<$  on  $\mathbb{N}$ . The two languages which we consider are  $0, S, <$  and  $0, <$ . We prove that various categoricity theorems for  $\mathbb{N}$  are equivalent over  $\text{RCA}_0^*$  to various subsystems of second-order arithmetic. The subsystems which we consider are  $\text{RCA}_0$ ,  $\text{WKL}_0$ ,  $\text{WKL}_0^*$ ,  $\text{ACA}_0$ ,  $\text{ADS}_0$ , and  $\text{PFO}_0^*$ . Here  $\text{ACA}_0$  is the well known [15] system consisting of  $\text{RCA}_0$  plus arithmetical comprehension,  $\text{ADS}_0$  is the known [8, 2] system consisting of  $\text{RCA}_0$  plus the ascending/descending sequence principle, and  $\text{PFO}_0^*$  is a new system which we introduce.

**Definition 4.1.** Within  $\text{RCA}_0^*$  we make the following definitions.

1. A *successor system* is a triple  $A, i, f$  such that  $i \in A$  and  $f : A \rightarrow A$  is one-to-one and  $i \notin \text{rng}(f)$ . A successor system is said to be *inductive* if  $(\forall X \subseteq A)((i \in X \wedge \forall a (a \in X \Rightarrow f(a) \in X)) \Rightarrow X = A)$ . Note that an inductive successor system is the same thing as a Peano system.
2. An *ordered system* is a triple  $A, i, <$  such that  $<$  is a linear ordering of  $A$  and  $i$  is the first element of  $A$  with respect to  $<$  and for each  $a \in A$  there exists  $a' \in A$  such that  $a < a'$  and there is no  $b$  such that  $a < b < a'$ . Thus  $a'$  is the *immediate successor* of  $a$  with respect to  $<$ . Note that the *successor function*  $a \mapsto a' : A \rightarrow A$  is not assumed to exist.
3. Let  $A, i, <$  be an ordered system. We say that  $A, i, <$  is *inductive* if

$$(\forall X \subseteq A)((i \in X \wedge \forall a (a \in X \Rightarrow a' \in X)) \Rightarrow X = A).$$

A set  $X \subseteq A$  is said to be  *$<$ -bounded* if  $(\exists c \in A)(\forall a \in X)(a < c)$ . We say that  $A, i, <$  is *strongly inductive* (see Proposition 4.2 below) if each nonempty  $<$ -bounded subset of  $A$  has a first element and a last element with respect to  $<$ . We say that  $A, i, <$  is *isomorphic to  $\mathbb{N}$*  if there exists  $\Phi : A \rightarrow \mathbb{N}$  which is one-to-one and onto  $\mathbb{N}$  such that  $\Phi(i) = 0$  and  $\Phi(a') = \Phi(a) + 1$  for all  $a \in A$ . We say that  $A, i, <$  is *almost isomorphic to  $\mathbb{N}$*  if for each  $c \in A$  the initial segment  $\{a \in A \mid a < c\}$  is finite.

4. An *ordered successor system* is a quadruple  $A, i, <, f$  such that  $A, i, <$  is an ordered system and  $A, i, f$  is a successor system and  $f(a) = a'$  for each  $a \in A$ .
5. Let  $A, i, <, f$  be an ordered successor system. We say that  $A, i, <, f$  is *inductive* if  $A, i, <$  is inductive, or equivalently, if  $A, i, f$  is inductive. We say that  $A, i, <, f$  is *strongly inductive* if  $A, i, <$  is strongly inductive. We say that  $A, i, <, f$  is *isomorphic to  $\mathbb{N}$*  if  $A, i, f$  is isomorphic to  $\mathbb{N}$ , or equivalently, if  $A, i, <$  is isomorphic to  $\mathbb{N}$ . We say that  $A, i, <, f$  is *almost isomorphic to  $\mathbb{N}$*  if  $A, i, f$  is almost isomorphic to  $\mathbb{N}$ , or equivalently, if  $A, i, <$  is almost isomorphic to  $\mathbb{N}$ .

**Proposition 4.2.** It is provable in  $\text{RCA}_0^*$  that every strongly inductive ordered system is inductive.



*Proof.* Let  $A, i, <$  be a strongly inductive ordered system and suppose that  $X \subseteq A$  and  $i \in X$  and  $\forall a (a \in X \Rightarrow a' \in X)$ . It suffices to prove that  $X = A$ . If not, let  $c \in A$  be such that  $c \notin X$ . Then  $Y = \{a \in X \mid a < c\}$  is  $<$ -bounded and nonempty, so let  $a_1$  be the last element of  $Y$  with respect to  $<$ . Then  $a_1 \in X$  and  $a'_1 \notin X$ , a contradiction.  $\square$

**Theorem 4.3.** The following are pairwise equivalent over  $\text{RCA}_0^*$ .

1.  $\text{RCA}_0$ .
2. Every strongly inductive ordered successor system is isomorphic to  $\mathbb{N}$ .
3. Every inductive successor system which is almost isomorphic to  $\mathbb{N}$  is isomorphic to  $\mathbb{N}$ .
4. Every strongly inductive ordered successor system which is almost isomorphic to  $\mathbb{N}$  is isomorphic to  $\mathbb{N}$ .

*Proof.* The equivalence  $1 \Leftrightarrow 3$  has already been proved as Lemma 3.2. The implications  $2 \Rightarrow 4$  and  $3 \Rightarrow 4$  are trivial, and the proof of Lemma 3.2 establishes  $4 \Rightarrow 1$ . It remains to prove  $1 \Rightarrow 2$ . Reasoning in  $\text{RCA}_0$ , let  $A, i, <, f$  be a strongly inductive ordered successor system. By  $\Delta_1^0$  comprehension we have  $g : A \rightarrow \mathbb{N}$  and  $h : \mathbb{N} \rightarrow A$  defined by  $g(a) = \min\{k \mid a < f^k(a)\}$  and  $h(0) = i$  and  $h(n+1) = f^{g(h(n))}(h(n))$ . Since  $h(n) < h(n+1)$  for all  $n$ , we can use  $\Delta_1^0$  comprehension to prove that  $X = \text{rng}(h)$  exists. Since  $h(n) < h(n+1)$  for all  $n$ , we see that  $X$  is cofinal in  $A, <$ . For any  $a = h(n) \in X$  we have  $f^m(i) = a$  where  $m = \sum_{k < n} g(h(k))$ . It follows that  $A, i, f$  is almost isomorphic to  $\mathbb{N}$ . Hence by Lemma 2.2  $A, i, f$  is isomorphic to  $\mathbb{N}$ . This completes the proof.  $\square$

**Theorem 4.4.** The following are pairwise equivalent over  $\text{RCA}_0^*$ .

1.  $\text{WKL}_0^*$ .
2. Every inductive successor system is almost isomorphic to  $\mathbb{N}$ .
3. Every inductive ordered successor system is almost isomorphic to  $\mathbb{N}$ .

*Proof.* Our proof of Theorem 2.3 establishes this result.  $\square$

**Theorem 4.5.** The following are pairwise equivalent over  $\text{RCA}_0^*$ .

1.  $\text{WKL}_0$ .
2. Every inductive successor system is isomorphic to  $\mathbb{N}$ .
3. Every inductive ordered successor system is isomorphic to  $\mathbb{N}$ .

*Proof.* The equivalence  $1 \Leftrightarrow 2$  is Theorem 3.4. The implication  $2 \Rightarrow 3$  is trivial. To prove  $3 \Rightarrow 1$ , note that  $3 \Rightarrow \text{RCA}_0$  by Theorem 4.3, and  $3 \Rightarrow$  Weak König's Lemma by the proof of Theorem 2.3.  $\square$

**Theorem 4.6.** The following are pairwise equivalent over  $\text{RCA}_0$ .

1.  $\text{ACA}_0$ .
2. Every inductive ordered system is isomorphic to  $\mathbb{N}$ .
3. Every strongly inductive ordered system which is almost isomorphic to  $\mathbb{N}$  is isomorphic to  $\mathbb{N}$ .
4. For every strongly inductive ordered system  $A, i, <$  which is almost isomorphic to  $\mathbb{N}$ , there exists  $f : A \rightarrow A$  such that  $A, i, <, f$  is an ordered successor system.

*Proof.* To prove  $1 \Rightarrow 2$  we reason in  $\text{ACA}_0$ . Given an inductive ordered system  $A, i, <$ , use arithmetical comprehension to prove the existence of  $\Psi : \mathbb{N} \rightarrow A$  such that  $\Psi(0) = i$  and  $\Psi(n+1) = \Psi(n)'$  for all  $n \in \mathbb{N}$ . By arithmetical comprehension the set  $\text{rng}(\Psi)$  exists, and then the inductive property implies that  $\text{rng}(\Psi) = A$ . Thus  $\Psi$  is an isomorphism of  $\mathbb{N}$  onto  $A, i, <$  and we have 2.

The implications  $2 \Rightarrow 3$  and  $3 \Rightarrow 4$  are trivial.

To prove  $4 \Rightarrow 1$  we reason in  $\text{RCA}_0$  and assume 4. Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be a one-to-one function. Define a linear ordering  $<_g$  of  $\mathbb{N}$  by letting  $m <_g n$  if and only if  $g(m) < g(n)$ . Using bounded  $\Sigma_1^0$  comprehension [15, Theorem II.3.9], we can easily check that  $\mathbb{N}$  has a first element  $i$  with respect to  $<_g$  and that  $\mathbb{N}, i, <_g$  is a strongly inductive ordered system which is almost isomorphic to  $\mathbb{N}$ . By 4 let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be such that  $\mathbb{N}, i, <_g, f$  is an ordered successor system, i.e., for all  $j \in \mathbb{N}$ ,  $f(j)$  = the immediate successor of  $j$  with respect to  $<_g$ . We then have  $\forall j (\exists n \leq j) (f^n(i) = j)$ , hence

$$\forall m ((\exists j (g(j) = m)) \Leftrightarrow (\exists n \leq m) (g(f^n(i)) = m)),$$

so the range of  $g$  exists by  $\Delta_1^0$  comprehension using  $f$  and  $g$  as parameters. We have now shown that  $\text{rng}(g)$  exists for all one-to-one functions  $g : \mathbb{N} \rightarrow \mathbb{N}$ . By [15, Lemma III.1.3] this implies  $\text{ACA}_0$ , Q.E.D.  $\square$

**Theorem 4.7.** The following are pairwise equivalent over  $\text{RCA}_0^*$ .

1.  $\text{ACA}_0$ .
2. Every inductive ordered system is isomorphic to  $\mathbb{N}$ .
3. Every strongly inductive ordered system which is almost isomorphic to  $\mathbb{N}$  is isomorphic to  $\mathbb{N}$ .

*Proof.* This follows from Theorems 4.3 and 4.6.  $\square$

Our next goal is to determine the reverse-mathematical and proof-theoretical status of the statement that every inductive ordered system is almost isomorphic to  $\mathbb{N}$ . To this end we present Definition 4.8 and Lemmas 4.9 and 4.10 leading to Theorem 4.11.

**Definition 4.8.** Within  $\text{RCA}_0^*$  we make the following definitions (see also [8, Definition 2.3]). Let  $L$  be an infinite linear ordering in which each non-first element has an immediate predecessor and each non-last element has an immediate successor. We say that  $L$  is  $\omega$ -like if each element of  $L$  has finitely many predecessors. We say that  $L$  is  $\omega^*$ -like if each element of  $L$  has finitely many successors. We say that  $L$  is  $\omega + \omega^*$ -like if  $L$  is not  $\omega$ -like or  $\omega^*$ -like and each element of  $L$  has finitely many predecessors or finitely many successors. If  $L$  is  $\omega + \omega^*$ -like, the  $\omega$ -part of  $L$  is the set consisting of all elements of  $L$  with finitely many predecessors, provided this set exists. The  $\omega^*$ -part of  $L$  is defined similarly. Note that the  $\omega$ -part of  $L$  exists if and only if the  $\omega^*$ -part of  $L$  exists. Let OOP be the statement that the  $\omega$ -part of every  $\omega + \omega^*$ -like linear ordering exists. We write  $\text{OOP}_0 = \text{RCA}_0 + \text{OOP}$  and  $\text{OOP}_0^* = \text{RCA}_0^* + \text{OOP}$ .

**Lemma 4.9.**  $\text{OOP}_0$  and  $\text{OOP}_0^*$  are equivalent over  $\text{RCA}_0^*$ .

*Proof.* It suffices to show that  $\Sigma_1^0$  induction is provable in  $\text{OOP}_0^*$ . Reasoning in  $\text{RCA}_0^*$ , suppose  $\Sigma_1^0$  induction fails. By Lemma 3.2 let  $C$  be an infinite subset of  $\mathbb{N}$  which does not have arbitrarily large finite subsets. Let  $c_0$  be the least element of  $C$ . Since  $C$  is unbounded, we may safely assume that  $C$  has no subset of cardinality  $c_0$ . For each  $n$  let  $f(n)$  be cardinality of  $\{c \in C \mid c < n\}$ . Note that  $f(n) < c_0$  for all  $n$ . Let  $L = \mathbb{N}$  and define  $<_L$  on  $L$  by letting  $m <_L n$  if and only if  $m < n \leq c_0$  or  $c_0 < m < n$  or  $m < f(n)$ . In other words, for each pair  $c, c'$  of successive elements of  $C$  we are inserting the interval  $\{n \mid c < n \leq c'\}$  between  $f(c)$  and  $f(c') = f(c) + 1$ . Thus  $L$  is an  $\omega + \omega^*$ -like linear ordering. By OOP let  $B$  be the  $\omega^*$ -part of  $L$ . Then  $B = \{n \leq c_0 \mid C \text{ has no subset of cardinality } n\}$  and  $B$  has a least element, which is clearly impossible, Q.E.D.  $\square$

**Lemma 4.10.**  $\text{OOP}_0$  is equivalent to  $\text{ACA}_0$  over  $\text{RCA}_0$ .

*Proof.* Clearly arithmetical comprehension implies OOP, so it remains to show that  $\text{OOP}_0$  proves arithmetical comprehension. By [15, Lemma III.1.3] it suffices to show that  $\text{OOP}_0$  proves  $\Sigma_1^0$  comprehension. Reasoning in  $\text{OOP}_0$  and letting  $\varphi(m)$  be a  $\Sigma_1^0$  formula, we shall prove the existence of the set  $\{m \mid \varphi(m)\}$ . Write  $\varphi(m) \equiv \exists k \theta(k, m)$  where  $\theta(k, m)$  is  $\Sigma_0^0$ . Let  $D_s = \{m < s \mid (\exists k < s) \theta(k, m)\}$ . Thus  $D_s$ ,  $s = 0, 1, 2, \dots$  is a nondecreasing sequence of finite sets such that  $\forall m (\varphi(m) \Leftrightarrow \exists s (m \in D_s))$ .

Using bounded primitive recursion [18] we shall define an increasing sequence of finite linear orderings  $L_s, <_s$ ,  $s = 0, 1, 2, \dots$ . Actually we shall have  $L_{2s} = \{0, 1, \dots, s-1\}$  so there will be a linear ordering  $L, <_L$  where  $L = \bigcup_s L_s = \mathbb{N}$  and  $<_L = \bigcup_s <_s$ . We shall also define a sequence of partitions  $L_s = A_s \cup B_s$  such that  $(\forall u \in A_s) (\forall v \in B_s) (u <_s v)$ . We shall also have  $(\forall u \in A_s) (\forall v \in A_s) (u <_s v \Leftrightarrow u < v)$  and  $\forall s (B_s \subseteq B_{s+1})$ . Except for trivial cases,  $L$  will be  $\omega + \omega^*$ -like with  $\omega$ -part  $A = \lim_s A_s$  and  $\omega^*$ -part  $B = \lim_s B_s = \bigcup_s B_s$ . In addition we shall define a function  $f : L \rightarrow \mathbb{N}$  with this property:

If  $u_{1,s} < \dots < u_{k,s}$  are the elements of  $A_{2s}$  in increasing order, then  $f(u_{1,s}) < \dots < f(u_{k,s})$  are the first  $|A_{2s}|$  elements of  $\mathbb{N} \setminus D_s$  in increasing order. (3)

Taking the limit as  $s$  goes to infinity, we shall have:

$$\begin{aligned} &\text{If } u_1 < u_2 < \dots \text{ are the elements of } A \text{ in increasing order,} \\ &\text{then } f(u_1) < f(u_2) < \dots \text{ are the elements of } \mathbb{N} \setminus \bigcup_s D_s \text{ in} \quad (4) \\ &\text{increasing order.} \end{aligned}$$

In particular our  $\Sigma_1^0$  formula  $\varphi(m) \equiv \exists s (m \in D_s)$  will be equivalent to the  $\Pi_1^0$  formula  $\forall u (u \in A \Rightarrow f(u) \neq m)$ , and the existence of  $\{m \mid \varphi(m)\}$  will then follow by  $\Delta_1^0$  comprehension.

The inductive construction of  $L$  is as follows.

Stage 0. Let  $L_0 = <_0 = A_0 = B_0 =$  the empty set.

Stage  $2s + 1$ . Let  $L_{2s+1} = L_{2s} \cup \{s\}$  and insert  $s$  between  $A_{2s}$  and  $B_{2s}$ , i.e.,  $<_{2s+1} = <_{2s} \cup \{\langle u, s \rangle \mid u \in A_{2s}\} \cup \{\langle s, v \rangle \mid v \in B_{2s}\}$ . Let  $A_{2s+1} = A_{2s} \cup \{s\}$  and let  $B_{2s+1} = B_{2s}$ . Let  $f(s)$  be the  $(|A_{2s}| + 1)$ -st element of  $\mathbb{N} \setminus D_s$  in increasing order. Our inductive hypothesis (3) implies that  $u_{1,s} < \dots < u_{k,s} < s$  are the elements of  $A_{2s+1}$  in increasing order and  $f(u_{1,s}) < \dots < f(u_{k,s}) < f(s)$  are the first  $|A_{2s+1}|$  elements of  $\mathbb{N} \setminus D_s$  in increasing order.

Stage  $2s + 2$ . Let  $L_{2s+2} = L_{2s+1}$  and  $<_{2s+2} = <_{2s+1}$ . Case 1: If there exists  $u \in A_{2s+1}$  such that  $f(u) \in D_{s+1}$ , let  $a$  be the least such  $u$  and let  $A_{2s+2} = \{u \in A_{2s+1} \mid u < a\}$  and  $B_{2s+2} = L_{2s+2} \setminus A_{2s+2}$ . Case 2: If no such  $u$  exists, let  $A_{2s+2} = A_{2s+1}$  and  $B_{2s+2} = B_{2s+1}$ . In either case it is clear that (3) continues to hold with  $s$  replaced by  $s + 1$ .

We may safely assume that Case 1 holds at infinitely many stages, because otherwise there would be a stage  $t$  such that  $\forall m (\varphi(m) \Leftrightarrow \neg(\exists s > t)(\exists u \in A_s)(f(u) = m))$ , hence  $\{m \mid \varphi(m)\}$  would exist by  $\Delta_1^0$  comprehension. Since Case 1 holds infinitely often, we have  $B_s \subsetneq B_{s+1}$  for infinitely many  $s$ .

We may safely assume that  $\neg\varphi(n)$  holds for infinitely many  $n$ . Let  $n$  be such that  $\neg\varphi(n)$  holds. By bounded  $\Sigma_1^0$  comprehension [15, Theorem II.3.9], the set  $\{m < n \mid \varphi(m)\}$  exists and is finite, so by  $\Sigma_1^0$  bounding [18] let  $s$  be such that  $(\forall m < n)(\varphi(m) \Leftrightarrow m \in D_s)$ . Then  $f(u) = n$  for some  $u \in A_{2s+2n}$ , and then  $u \in A_t$  for all  $t \geq 2s + 2n$ . Since  $\neg\varphi(n)$  holds for infinitely many  $n$ , it follows that  $\lim_s |A_s| = \infty$ .

It is now clear that  $L$  is  $\omega + \omega^*$ -like with  $\omega$ -part  $A = \lim_s A_s$  and  $\omega^*$ -part  $B = \bigcup_s B_s$ . It is also clear that (4) holds, so  $\{m \mid \varphi(m)\}$  exists by  $\Delta_1^0$  comprehension. This completes the proof.  $\square$

**Theorem 4.11.** The following are pairwise equivalent over  $\text{RCA}_0^*$ .

1.  $\text{ACA}_0$ .
2.  $\text{OOP}_0^*$
3. Every inductive ordered system is almost isomorphic to  $\mathbb{N}$ .

*Proof.* The equivalence  $1 \Leftrightarrow 2$  follows from Lemmas 4.9 and 4.10. The implication  $1 \Rightarrow 3$  is clear from Theorem 4.6. It remains to prove  $3 \Rightarrow 2$ . Let  $L$  be  $\omega + \omega^*$ -like such that the  $\omega$ -part of  $L$  does not exist. Let  $i$  be the first element of  $L$ , let  $A$  be the disjoint union of  $L$  and  $\mathbb{N}$ , and extend the given linear ordering

of  $L$  and the standard ordering  $<$  of  $\mathbb{N}$  to a linear ordering  $\prec$  of  $A$  with  $u \prec n$  for all  $u \in L$  and all  $n \in \mathbb{N}$ . Then  $A, i, \prec$  is an ordered system. Let  $X \subseteq A$  be such that  $i \in X$  and  $\forall a (a \in X \Rightarrow a' \in X)$ . If  $L \not\subseteq X$ , fix  $c \in L \setminus X$  and let  $Y = \{a \in X \mid a \prec c\} = \{u \in X \mid u <_L c\}$ . Then  $Y$  is the  $\omega$ -part of  $L$ , a contradiction. Thus  $L \subseteq X$ , and from this it follows that  $X = A$ . Thus  $A, i, \prec$  is an inductive ordered system. It follows by 3 that  $L$  is finite. This contradiction completes the proof.  $\square$

**Remark 4.12.** We thank Richard Shore [13] for showing us a proof of Lemma 4.10. We have modified that proof to obtain our proof above. Our construction yields the following recursion-theoretical results:

1. There exists a recursive linear ordering  $L$  of type  $\omega + \omega^*$  such that the halting problem is Turing reducible to the  $\omega$ -part of  $L$ .
2. Given a recursively enumerable Turing degree  $\mathbf{b}$ , there exists a recursive linear ordering  $L$  of type  $\omega + \omega^*$  such that the  $\omega$ -part of  $L$  is retraceable and the  $\omega^*$ -part of  $L$  is recursively enumerable of degree  $\mathbf{b}$ .

Subsequently Jockusch [10] noted that these results are easily deduced from his 1968 paper [9]. Namely, 1 is implicit in [9, Theorem 5.2], and 2 follows from [9, Theorem 3.2, Corollary 3.3] plus the following characterization [10]:

3. A recursively enumerable set is the  $\omega$ -part of a recursive linear ordering of type  $\omega + \omega^*$  if and only if it is recursive or simple and semirecursive.

We now end this section by commenting on the reverse-mathematical and proof-theoretical status of the statement that every strongly inductive ordered system is almost isomorphic to  $\mathbb{N}$ .

**Definition 4.13.** Within  $\text{RCA}_0^*$  we define a linear ordering  $L$  to be *pseudofinite* if every nonempty subset of  $L$  has a first element and a last element. Let PFO be the statement that every countable pseudofinite linear ordering is finite. We write  $\text{PFO}_0 = \text{RCA}_0 + \text{PFO}$  and  $\text{PFO}_0^* = \text{RCA}_0^* + \text{PFO}$ .

**Theorem 4.14.** The following are equivalent over  $\text{RCA}_0^*$ .

1.  $\text{PFO}_0^*$ .
2. Every strongly inductive ordered system is almost isomorphic to  $\mathbb{N}$ .

*Proof.* We reason in  $\text{RCA}_0^*$ . To prove  $1 \Rightarrow 2$ , assume PFO and let  $A, i, \prec$  be a strongly inductive ordered system. For each  $c \in A$  the initial segment  $\{a \in A \mid a \prec c\}$  is pseudofinite, hence finite, so  $A, i, \prec$  is almost isomorphic to  $\mathbb{N}$ . Thus  $\text{PFO} \Rightarrow 2$ , i.e.,  $1 \Rightarrow 2$ . To prove  $2 \Rightarrow 1$ , assume that  $L$  is a pseudofinite linear ordering. Let  $i$  be the first element of  $L$ . Let  $A$  be the disjoint union of  $L$  and  $\mathbb{N}$ . Extend the given linear ordering of  $L$  and the standard ordering  $<$  of  $\mathbb{N}$  to a linear ordering  $\prec$  of  $A$  with  $a \prec n$  for all  $a \in L$  and all  $n \in \mathbb{N}$ . Then  $A, i, \prec$  is a strongly inductive ordered system. It follows by 2 that  $L$  is finite. Thus  $2 \Rightarrow \text{PFO}$ , i.e.,  $2 \Rightarrow 1$ , Q.E.D.  $\square$

**Remark 4.15.** Yokoyama [20] has shown that  $WKL_0^* + \{RT(k, l) \mid k, l \geq 2\}$  is  $\Pi_2^0$ -equivalent to  $RCA_0^*$ . Here RT stands for Ramsey’s Theorem. Since PFO is provable in  $RCA_0^* + RT(2, 2)$ , Yokoyama’s result implies that  $PFO_0^*$  is much weaker<sup>5</sup> than  $RCA_0$ . We conjecture that  $PFO_0^*$  is  $\Pi_1^1$ -equivalent to  $RCA_0^*$ .

**Definition 4.16.** Within  $RCA_0^*$  let ADS be the ascending/descending sequence principle of Hirschfeldt/Shore [8]: every infinite linear ordering has an infinite ascending sequence or an infinite descending sequence. We write  $ADS_0 = RCA_0 + ADS$  and  $ADS_0^* = RCA_0^* + ADS$ .

**Theorem 4.17.** The following are equivalent over  $RCA_0$ .

1.  $ADS_0$ .
2. Every strongly inductive ordered system is almost isomorphic to  $\mathbb{N}$ .

*Proof.* The proof of [8, Proposition 2.4] shows that our system  $PFO_0$  (Definition 4.13) is equivalent to  $ADS_0$ . Using this observation, Theorem 4.17 follows immediately from Theorem 4.14.  $\square$

**Remark 4.18.** It is easy to see that  $ADS_0^*$  proves  $\Sigma_1^0$  induction and is therefore equivalent to  $ADS_0$ . Chong/Slaman/Yang [2] have shown that  $ADS_0$  is  $\Pi_1^1$ -equivalent to  $RCA_0 + \Sigma_2^0$  bounding, hence  $\Pi_3^0$ -equivalent to  $RCA_0$ , so the strength of  $ADS_0$  is known.

## 5 Summary and open questions

$RCA_0$	isomorphic	almost isomorphic implies isomorphic	almost isomorphic
i.s.s.	$WKL_0$ , 3.4	$RCA_0$ , 3.2	$WKL_0$ , 3.3
i.o.s.	$ACA_0$ , 4.7	$ACA_0$ , 4.7	$ACA_0$ , 4.11
s.i.o.s.	$ACA_0$ , 4.7	$ACA_0$ , 4.7	$ADS_0$ , 4.17, 4.18
i.o.s.s.	$WKL_0$ , 4.5	$RCA_0$ , 4.3	$WKL_0$ , 4.4
s.i.o.s.s.	$RCA_0$ , 4.3	$RCA_0$ , 4.3	$RCA_0$ , 4.3

Table 1: Summary of equivalences over  $RCA_0$ .

Tables 1 and 2 are a summary of our results. Abbreviations are used. For example, s.i.o.s.s. is an abbreviation for “strongly inductive ordered successor system.” Recall also that an i.s.s. or inductive successor system is the same thing as a Peano system. Each entry in Table 1 or 2 stands for one of our results concerning the reverse-mathematical status of a categoricity theorem for  $\mathbb{N}$ . As an example, the entry  $PFO_0^*$ , 4.14, 4.15, 5.1 in Table 2 means that  $RCA_0^*$

<sup>5</sup>See footnote 2.

$RCA_0^*$	isomorphic	almost isomorphic implies isomorphic	almost isomorphic
i.s.s.	$WKL_0$ , 3.4	$RCA_0$ , 3.2	$WKL_0^*$ , 3.3
i.o.s.	$ACA_0$ , 4.7	$ACA_0$ , 4.7	$ACA_0$ , 4.11
s.i.o.s.	$ACA_0$ , 4.7	$ACA_0$ , 4.7	$PFO_0^*$ , 4.14, 4.15, 5.1
i.o.s.s.	$WKL_0$ , 4.5	$RCA_0$ , 4.3	$WKL_0^*$ , 4.4
s.i.o.s.s.	$RCA_0$ , 4.3	$RCA_0$ , 4.3	?????, 5.2

Table 2: Summary of equivalences over  $RCA_0^*$ .

proves  $PFO_0^* \Leftrightarrow$  every strongly inductive ordered system is almost isomorphic to  $\mathbb{N}$ , with references to Theorem 4.14 and Remark 4.15 and Question 5.1.

We now state some open questions which are relevant to Table 2.

**Question 5.1.** Is  $PFO_0^*$   $\Pi_1^1$ -equivalent to  $RCA_0^*$ ? See Remark 4.15.

**Question 5.2.** What is the reverse-mathematical status of the statement that every strongly inductive ordered successor system is almost isomorphic to  $\mathbb{N}$ ? We do not know whether this statement is provable in  $RCA_0^*$ . By Table 2 it is provable in each of the systems  $RCA_0$  and  $PFO_0^*$  and  $WKL_0^*$ .

**Question 5.3.** Does there exist a second-order characterization of  $\mathbb{N}$  which is provable in  $RCA_0^*$ ? More precisely, does  $RCA_0^*$  prove the existence of a second-order sentence or set of sentences  $T$  such that  $\mathbb{N}, 0, S$  is a second-order model of  $T$  and all second-order models of  $T$  are isomorphic to  $\mathbb{N}, 0, S$ ? One may also consider the same question with  $RCA_0^*$  replaced by systems which are  $\Pi_2^0$ -equivalent to  $RCA_0^*$ .

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