Reverse mathematics and Peano categoricity

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First draft: August 18, 2011
This draft: March 28, 2012

Abstract

We investigate the reverse-mathematical status of several theorems to the effect that the natural number system is second-order categorical. One of our results is as follows. Define a *system* to be a triple A, i, f such that A is a set and $i \in A$ and $f : A \to A$. A subset $X \subseteq A$ is said to be *inductive* if $i \in X$ and $\forall a\,(a \in X \Rightarrow f(a) \in X)$. The system A, i, f is said to be *inductive* if the only inductive subset of A is A itself. Define a *Peano system* to be an inductive system such that f is one-to-one and $i \notin$ the range of f. The standard example of a Peano system is $\mathbb{N}, 0, S$ where $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ = the set of natural numbers and $S : \mathbb{N} \to \mathbb{N}$ is given by $S(n) = n+1$ for all $n \in \mathbb{N}$. Consider the statement that all Peano systems are isomorphic to $\mathbb{N}, 0, S$. We prove that this statement is logically equivalent to WKL_0 over RCA_0^*. From this and similar equivalences we draw some foundational/philosophical consequences.

Keywords: Reverse mathematics, second-order arithmetic, Peano system, foundations of mathematics, proof theory, second-order logic.

2010 Mathematics Subject Classification: Primary 03B30; Secondary 03D80, 03F35, 00A30, 03F25.

*Yokoyama's research was supported by a Japan Society for the Promotion of Science postdoctoral fellowship for young scientists, and by a grant from the John Templeton Foundation.

Contents

1 Introduction

Reverse mathematics is a well known [15, 17] research direction in the foundations of mathematics. The goal of reverse mathematics is to pinpoint the weakest set-existence axioms which are needed in order to prove specific theorems of core mathematics. Such investigations are most fruitfully carried out in the context of subsystems of second-order arithmetic [15]. In that context it frequently happens that the weakest axioms needed to prove a particular theorem are logically equivalent to the theorem, over a weak base theory. For example, the well known theorem that every uncountable closed set in Euclidean space contains a perfect subset is logically equivalent to ATR_0 over the weak base theory RCA_0 [15, Theorem V.5.5].

A key theorem in rigorous core mathematics is the categoricity of the natural number system. Stated more precisely and in 20th-century language, the *Peano Categoricity Theorem* [11, Theorem 2.7.1] asserts that any two Peano systems are isomorphic. The Peano Categoricity Theorem was originally proved by Dedekind in 1888 [4, Satz 132], [5, Theorem 132] as a highlight of his rigorous, set-theoretical development [3, 4, 5] of the natural number system $\mathbb{N}$ and the real number system $\mathbb{R}$.

In this paper we investigate the reverse-mathematical and proof-theoretical status of the Peano Categoricity Theorem and related theorems. One of our results is as follows.

The Peano Categoricity Theorem is equivalent to WKL_0 over the standard weak base theory RCA_0. (1)

Here RCA_0 and WKL_0 are familiar [15, 17] subsystems of second-order arithmetic. Namely, RCA_0 consists of Δ^0_1 comprehension plus Σ^0_1 induction, and WKL_0 consists of RCA_0 plus Weak König's Lemma.

Our result (1) offers further confirmation of a point made by Väänänen[1] in a recent talk based on his recent paper [19]. Väänanen observed that various second-order categoricity theorems can be proved without resorting to the full strength of second-order logic. Clearly (1) bears this out, because WKL_0 is a relatively weak[2] subsystem of second-order arithmetic, much weaker than ACA_0 and in fact Π^0_2-equivalent to Primitive Recursive Arithmetic [15, §IX.3]. Since by (1) the Peano Categoricity Theorem is provable in WKL_0, it follows that the Peano Categoricity Theorem is *finitistically reducible* in the sense of Simpson's partial realization [14, 16] (see also [1]) of Hilbert's Program [7].

As a refinement of (1) we obtain the following stronger result.

The Peano Categoricity Theorem is equivalent to WKL_0 not only over RCA_0 but over the much weaker base theory RCA^*_0. (2)

Recall from [15, §X.4] and [18] that RCA^*_0 is RCA_0 with Σ^0_1 induction weakened to *natural number exponentiation*, i.e., the assertion that m^n exists for all $m, n \in \mathbb{N}$. It is known that RCA^*_0 is Π^0_2-equivalent to Elementary Function Arithmetic [18], hence much weaker than RCA_0 and WKL_0 which are Π^0_2-equivalent to Primitive Recursive Arithmetic [15, §IX.3].

Our stronger result (2) provides some foundational or philosophical insight concerning Dedekind's construction of the natural number system [4, 5]. Recall that Dedekind's key technical lemma, the "Satz der Definition durch Induction," is a straightforward embodiment[3] of the idea of primitive recursion. But at the same time, according to (2), the Peano Categoricity Theorem itself *requires* primitive recursion. Thus (2) constitutes further evidence that primitive recursion is indeed the heart of the matter.

The plan of this paper is as follows. In §2 we prove (1). In §3 we prove (2). In §4 we investigate the reverse-mathematical status of certain variants of the Peano Categoricity Theorem, replacing the Peano system $\mathbb{N}, 0, S$ by the ordered system $\mathbb{N}, 0, <$ or the ordered Peano system $\mathbb{N}, 0, <, S$. In §5 we summarize our results and state some open questions.

2 The role of Weak König's Lemma

Recall from [15] that RCA_0 is the subsystem of second-order arithmetic consisting of Δ^0_1 comprehension and Σ^0_1 induction. Within RCA_0 one may freely use primitive recursion and minimization to define functions $f : \mathbb{N}^k \to \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers [15, §II.3]. Recall also [15] that WKL_0 consists of RCA_0 plus Weak König's Lemma, i.e., the statement that every infinite tree $T \subseteq \{0,1\}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ has an infinite path.

The purpose of this section is to show that the Peano Categoricity Theorem is equivalent to Weak König's Lemma, the equivalence being provable in RCA_0.

[1] We thank Jouko Väänänen for raising the question which is answered by (1).

[2] By the *strength* of a theory T we mean the set of Π^0_1 sentences which are provable in T.

[3] Dedekind's "Satz der Definition durch Induction" [4, Satz 126] [5, Theorem 126] may be restated in 20th-century language [11, Theorem 2.2.1] as follows. For any system A, i, f there is a unique function $\Phi : \mathbb{N} \to A$ such that $\Phi(0) = i$ and $\Phi(n+1) = f(\Phi(n))$ for all $n \in \mathbb{N}$.

Definition 2.1. Within RCA_0 we make the following definitions. A *system* is a triple A, i, f such that $i \in A \subseteq \mathbb{N}$ and $f : A \to A$. A *Peano system* is a system A, i, f such that $i \notin \operatorname{rng}(f)$ and f is one-to-one and

$$(\forall X \subseteq A)\,((i \in X \wedge \forall a\,(a \in X \Rightarrow f(a) \in X)) \Rightarrow X = A).$$

The standard example of a Peano system is $\mathbb{N}, 0, S$ with $S : \mathbb{N} \to \mathbb{N}$ defined by $S(n) = n + 1$. A Peano system A, i, f is said to be *isomorphic to* $\mathbb{N}$ if there exists a bijection $\Phi : A \to \mathbb{N}$ such that $\Phi(i) = 0$ and $\Phi(f(a)) = \Phi(a) + 1$ for all $a \in A$. A Peano system A, i, f is said to be *almost isomorphic to* $\mathbb{N}$ if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $f^n(i) = a$.

Lemma 2.2. The following is provable in RCA_0. If a Peano system is almost isomorphic to $\mathbb{N}$, it is isomorphic to $\mathbb{N}$.

Proof. We reason in RCA_0. Let A, i, f be a system. As in [15, §II.3] use Σ^0_1 induction to prove that for all $n \in \mathbb{N}$, $f^n(i)$ exists and $f^n(i) \in A$. Use Δ^0_1 comprehension to prove the existence of the function $n \mapsto f^n(i) : \mathbb{N} \to A$. Assume now that A, i, f is a Peano system which is almost isomorphic to $\mathbb{N}$. Use Δ^0_1 comprehension to prove the existence of the function $\Phi : A \to \mathbb{N}$ given by $\Phi(a) = \min\{n \mid f^n(i) = a\}$ for all $a \in A$. Clearly $\Phi(i) = 0$ and $\Phi(f(a)) = \Phi(a)+1$ for all $a \in A$, hence Φ is one-to-one. Moreover, because A, i, f is almost isomorphic to $\mathbb{N}$, Φ is onto $\mathbb{N}$. Thus Φ maps A, i, f isomorphically onto $\mathbb{N}$. $\square$

Theorem 2.3. The following are equivalent over RCA_0.

1. WKL_0.

2. Every Peano system is isomorphic to $\mathbb{N}$.

Proof. We first prove $1 \Rightarrow 2$. Reasoning in WKL_0, let A, i, f be a Peano system. Recall that $A \subseteq \mathbb{N}$. By Lemma 2.2 it suffices to show that for each $a \in A$ there exists $n \in \mathbb{N}$ such that $f^n(i) = a$. Assume for a contradiction that $c \in A$ and $f^n(i) \neq c$ for all $n \in \mathbb{N}$. Let T be the set of all $t \in \{0,1\}^{<\mathbb{N}}$ such that

$$(i < \operatorname{lh}(t) \wedge c < \operatorname{lh}(t)) \Rightarrow t(i) \neq t(c)$$

and

$$(\forall a, b < \operatorname{lh}(t))\,(f(a) = b \Rightarrow t(a) = t(b)).$$

Clearly T is a tree. We claim that T is infinite. To see this, let $n \in \mathbb{N}$ be given. Define $t : \{0, \ldots, n-1\} \to \{0,1\}$ by letting $t(a) = 1$ if there exists a finite sequence $a_0, \ldots, a_k$ of elements of $\{0, \ldots, n-1\}$ such that $i = a_0$ and $f(a_0) = a_1$ and $f(a_1) = a_2$ and $\ldots$ and $f(a_{k-1}) = a_k = a$. If $a \in \{0, \ldots, n-1\}$ and no such finite sequence exists, let $t(a) = 0$. Clearly $t \in T$ and $\operatorname{lh}(t) = n$ so our claim is proved. By Weak König's Lemma let h be an infinite path in T. Letting $X = \{a \in A \mid h(a) = h(i)\}$ we see that $i \in X$ and $f(X) \subseteq X$ and $c \notin X$ contradicting our assumption that A, i, f is a Peano system.

Next we prove $(\neg 1) \Rightarrow (\neg 2)$. Reasoning in RCA_0, assume $\neg 1$ and let $T \subseteq \{0,1\}^{<\mathbb{N}}$ be an infinite tree with no infinite path. Then

$$T' = T \cup \{\underbrace{\langle 1,\ldots,1\rangle}_{n} \mid n \in \mathbb{N}\}$$

is a tree with exactly one infinite path. Consider the lexicographic ordering of T'. The empty sequence $\langle\rangle$ is the first element of T', and T is an initial segment of T', and each $t \in T'$ has an immediate successor in T', and the immediate successor of t is of the form $t^\frown\langle 0\rangle$ or $t^\frown\langle 1\rangle$ or $\langle t(0),\ldots,t(m-1),1\rangle$ where $m < \mathrm{lh}(t)$ and $t(m) = 0$. Use Δ^0_1 comprehension to prove the existence of the function $f : T' \to T'$ defined by $f(t) =$ the immediate successor of t.

We claim that $T', \langle\rangle, f$ is a Peano system. Otherwise, let $X \subseteq T'$ be such that $\langle\rangle \in X$ and $f(X) \subseteq X$ and $X \neq T'$. Then clearly $T \nsubseteq X$, so fix $t_1 \in T \setminus X$ and let $Y = \{t \in T \mid t$ precedes t_1 in the lexicographic ordering of $T\}$. Use bounded primitive recursion to prove the existence of $g : \mathbb{N} \to \{0,1\}$ defined by

$$g(n) = \begin{cases} 1 & \text{if } \langle g(0),\ldots,g(n-1),1\rangle \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle g(0),\ldots,g(n-1)\rangle \in Y$ implies $\langle g(0),\ldots,g(n-1),g(n)\rangle \in Y$, because otherwise the immediate successor of $\langle g(0),\ldots,g(n-1)\rangle$ would be of the form $\langle g(0),\ldots,g(m-1),1\rangle = f(\langle g(0),\ldots,g(n-1)\rangle) \in Y$ where $m < n$ and $g(m) = 0$, contradicting the definition of $g(m)$. Since $\langle\rangle \in Y$, it follows by Δ^0_1 induction that $\langle g(0),\ldots,g(n)\rangle \in Y$ for all $n \in \mathbb{N}$. In particular g is an infinite path in T. This contradiction proves our claim.

Since T is an infinite initial segment of T', the Peano system $T', \langle\rangle, f$ cannot be isomorphic to $\mathbb{N}$. We have now proved $\neg 2$, Q.E.D. $\square$

3 The role of Σ^0_1 induction

Recall from [15, §X.4] and [18] that RCA_0 consists of RCA^*_0 plus Σ^0_1 induction. In particular RCA^*_0 is weaker[4] than RCA_0 and does not support the full use of primitive recursion. However, RCA^*_0 does support the use of *bounded* primitive recursion, as well as minimization [18]. Recall also [18] that WKL^*_0 consists of RCA^*_0 plus Weak König's Lemma.

The purpose of this section is to refine the results of the previous section, using the base theory RCA^*_0 instead of RCA_0. We prove within RCA^*_0 that Weak König's Lemma and Σ^0_1 induction are equivalent to certain statements about Peano systems. As a consequence we show that RCA^*_0 can replace RCA_0 in the statement of Theorem 2.3.

Definition 3.1. Within RCA^*_0 we repeat Definition 2.1. Note however that RCA^*_0 is not strong enough to prove that $f^n(i)$ exists for all $n \in \mathbb{N}$ and all systems A, i, f. Consequently, the notion of a Peano system being almost isomorphic to $\mathbb{N}$ must be understood somewhat differently.

[4] See footnote 2.

Lemma 3.2. The following are equivalent over RCA_0^*.

1. RCA_0.
2. Every Peano system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.
3. For each infinite set $C \subseteq \mathbb{N}$ there exists a one-to-one function $f : \mathbb{N} \to C$.
4. Each infinite subset of $\mathbb{N}$ includes arbitrarily large finite sets.

Proof. We reason within RCA_0^*. The implication $1 \Rightarrow 2$ has already been proved as Lemma 2.2. To prove $2 \Rightarrow 3$, let C be an infinite subset of $\mathbb{N}$ and apply 2 to the Peano system C, c_0, ν_C where c_0 is the least element of C and $\nu_C : C \to C$ is given by $\nu_C(c) =$ the least $c' \in C$ such that $c' > c$.

The implication $3 \Rightarrow 4$ is easily proved by means of Σ^0_1 bounding [18]. To prove $4 \Rightarrow 1$ we must prove that 4 implies Σ^0_1 induction. Let $\varphi(n)$ be a Σ^0_1 formula such that $\varphi(0)$ and $\forall n\,(\varphi(n) \Rightarrow \varphi(n+1))$ hold. Write $\varphi(n) \equiv \exists k\,\theta(k,n)$ where $\theta(k,n)$ is a Σ^0_0 formula. Use Δ^0_1 comprehension to prove the existence of the set C consisting of all (codes for) finite sequences $s = \langle k_0, k_1, \ldots, k_n\rangle$ such that $(\forall m \le n)\,(\theta(k_m, m) \wedge \neg\,(\exists k < k_m)\,\theta(k,m))$ holds. Our assumptions $\varphi(0)$ and $\forall n\,(\varphi(n) \Rightarrow \varphi(n+1))$ imply that C has a least element but no greatest element, hence C is infinite. Now, given $n \in \mathbb{N}$, apply 4 to get a finite set $F \subset C$ of cardinality $> n$. Because $\mathrm{lh} : C \to \mathbb{N}$ is one-to-one, there exists $s \in F$ such that $\mathrm{lh}(s) > n$. Since $\mathrm{lh}(s) > n$ and $s \in C$ it follows that $\theta(k_n, n)$ holds, hence $\varphi(n)$ holds. This proves $\forall n\,\varphi(n)$, Q.E.D. $\square$

Lemma 3.3. The following are equivalent over RCA_0^*.

1. WKL_0^*.
2. Every Peano system is almost isomorphic to $\mathbb{N}$.

Proof. Our proof of Theorem 2.3 establishes this result. $\square$

Theorem 3.4. The following are equivalent over RCA_0^*.

1. WKL_0.
2. Every Peano system is isomorphic to $\mathbb{N}$.

Proof. Combine Lemmas 3.2 and 3.3. $\square$

4 Other categoricity theorems

The Peano Categoricity Theorem may be viewed as a second-order characterization of the natural number system $\mathbb{N}$ up to isomorphism using the language consisting of the constant $0 \in \mathbb{N}$ and the successor function $S : \mathbb{N} \to \mathbb{N}$ defined by $S(n) = n + 1$. It is also possible to study second-order characterizations of $\mathbb{N}$ in terms of other languages. In this section we consider two languages which include the order relation $<$ on $\mathbb{N}$. The two languages which we consider are $0, S, <$ and $0, <$. We prove that various categoricity theorems for $\mathbb{N}$ are equivalent over RCA_0^* to various subsystems of second-order arithmetic. The subsystems which we consider are RCA_0, WKL_0, WKL_0^*, ACA_0, ADS_0, and PFO_0^*. Here ACA_0 is the well known [15] system consisting of RCA_0 plus arithmetical comprehension, ADS_0 is the known [8, 2] system consisting of RCA_0 plus the ascending/descending sequence principle, and PFO_0^* is a new system which we introduce.

Definition 4.1. Within RCA_0^* we make the following definitions.

1. A *successor system* is a triple A, i, f such that $i \in A$ and $f : A \to A$ is one-to-one and $i \notin \mathrm{rng}(f)$. A successor system is said to be *inductive* if $(\forall X \subseteq A)\,((i \in X \wedge \forall a\,(a \in X \Rightarrow f(a) \in X)) \Rightarrow X = A)$. Note that an inductive successor system is the same thing as a Peano system.

2. An *ordered system* is a triple $A, i, \prec$ such that $\prec$ is a linear ordering of A and i is the first element of A with respect to $\prec$ and for each $a \in A$ there exists $a' \in A$ such that $a \prec a'$ and there is no b such that $a \prec b \prec a'$. Thus a' is the *immediate successor* of a with respect to $\prec$. Note that the *successor function* $a \mapsto a' : A \to A$ is not assumed to exist.

3. Let $A, i, \prec$ be an ordered system. We say that $A, i, \prec$ is *inductive* if

$$(\forall X \subseteq A)\,((i \in X \wedge \forall a\,(a \in X \Rightarrow a' \in X)) \Rightarrow X = A).$$

   A set $X \subseteq A$ is said to be *$\prec$-bounded* if $(\exists c \in A)\,(\forall a \in X)\,(a \prec c)$. We say that $A, i, \prec$ is *strongly inductive* (see Proposition 4.2 below) if each nonempty $\prec$-bounded subset of A has a first element and a last element with respect to $\prec$. We say that $A, i, \prec$ is *isomorphic to* $\mathbb{N}$ if there exists $\Phi : A \to \mathbb{N}$ which is one-to-one and onto $\mathbb{N}$ such that $\Phi(i) = 0$ and $\Phi(a') = \Phi(a) + 1$ for all $a \in A$. We say that $A, i, \prec$ is *almost isomorphic to* $\mathbb{N}$ if for each $c \in A$ the initial segment $\{a \in A \mid a \prec c\}$ is finite.

4. An *ordered successor system* is a quadruple $A, i, \prec, f$ such that $A, i, \prec$ is an ordered system and A, i, f is a successor system and $f(a) = a'$ for each $a \in A$.

5. Let $A, i, \prec, f$ be an ordered successor system. We say that $A, i, \prec, f$ is *inductive* if $A, i, \prec$ is inductive, or equivalently, if A, i, f is inductive. We say that $A, i, \prec, f$ is *strongly inductive* if $A, i, \prec$ is strongly inductive. We say that $A, i, \prec, f$ is *isomorphic to* $\mathbb{N}$ if A, i, f is isomorphic to $\mathbb{N}$, or equivalently, if $A, i, \prec$ is isomorphic to $\mathbb{N}$. We say that $A, i, \prec, f$ is *almost isomorphic to* $\mathbb{N}$ if A, i, f is almost isomorphic to $\mathbb{N}$, or equivalently, if $A, i, \prec$ is almost isomorphic to $\mathbb{N}$.

Proposition 4.2. It is provable in RCA_0^* that every strongly inductive ordered system is inductive.

Proof. Let $A, i, \prec$ be a strongly inductive ordered system and suppose that $X \subseteq A$ and $i \in X$ and $\forall a\,(a \in X \Rightarrow a' \in X)$. It suffices to prove that $X = A$. If not, let $c \in A$ be such that $c \notin X$. Then $Y = \{a \in X \mid a \prec c\}$ is $\prec$-bounded and nonempty, so let a_1 be the last element of Y with respect to $\prec$. Then $a_1 \in X$ and $a_1' \notin X$, a contradiction. □

Theorem 4.3. The following are pairwise equivalent over RCA_0^*.

1. RCA_0.

2. Every strongly inductive ordered successor system is isomorphic to $\mathbb{N}$.

3. Every inductive successor system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.

4. Every strongly inductive ordered successor system which is almost ismorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.

Proof. The equivalence $1 \Leftrightarrow 3$ has already been proved as Lemma 3.2. The implications $2 \Rightarrow 4$ and $3 \Rightarrow 4$ are trivial, and the proof of Lemma 3.2 establishes $4 \Rightarrow 1$. It remains to prove $1 \Rightarrow 2$. Reasoning in RCA_0, let $A, i, \prec, f$ be a strongly inductive ordered successor system. By Δ^0_1 comprehension we have $g : A \to \mathbb{N}$ and $h : \mathbb{N} \to A$ defined by $g(a) = \min\{k \mid a < f^k(a)\}$ and $h(0) = i$ and $h(n+1) = f^{g(h(n))}(h(n))$. Since $h(n) < h(n+1)$ for all n, we can use Δ^0_1 comprehension to prove that $X = \mathrm{rng}(h)$ exists. Since $h(n) \prec h(n+1)$ for all n, we see that X is cofinal in $A, \prec$. For any $a = h(n) \in X$ we have $f^m(i) = a$ where $m = \sum_{k<n} g(h(k))$. It follows that A, i, f is almost isomorphic to $\mathbb{N}$. Hence by Lemma 2.2 A, i, f is isomorphic to $\mathbb{N}$. This completes the proof. □

Theorem 4.4. The following are pairwise equivalent over RCA_0^*.

1. WKL_0^*.

2. Every inductive successor system is almost isomorphic to $\mathbb{N}$.

3. Every inductive ordered successor system is almost isomorphic to $\mathbb{N}$.

Proof. Our proof of Theorem 2.3 establishes this result. □

Theorem 4.5. The following are pairwise equivalent over RCA_0^*.

1. WKL_0.

2. Every inductive successor system is isomorphic to $\mathbb{N}$.

3. Every inductive ordered successor system is isomorphic to $\mathbb{N}$.

Proof. The equivalence $1 \Leftrightarrow 2$ is Theorem 3.4. The implication $2 \Rightarrow 3$ is trivial. To prove $3 \Rightarrow 1$, note that $3 \Rightarrow \mathsf{RCA}_0$ by Theorem 4.3, and $3 \Rightarrow$ Weak König's Lemma by the proof of Theorem 2.3. □

Theorem 4.6. The following are pairwise equivalent over RCA_0.

1. ACA_0.

2. Every inductive ordered system is isomorphic to $\mathbb{N}$.

3. Every strongly inductive ordered system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.

4. For every strongly inductive ordered system $A, i, \prec$ which is almost isomorphic to $\mathbb{N}$, there exists $f : A \to A$ such that $A, i, \prec, f$ is an ordered successor system.

Proof. To prove $1 \Rightarrow 2$ we reason in ACA_0. Given an inductive ordered system $A, i, \prec$, use arithmetical comprehension to prove the existence of $\Psi : \mathbb{N} \to A$ such that $\Psi(0) = i$ and $\Psi(n+1) = \Psi(n)'$ for all $n \in \mathbb{N}$. By arithmetical comprehension the set $\mathrm{rng}(\Psi)$ exists, and then the inductive property implies that $\mathrm{rng}(\Psi) = A$. Thus Ψ is an isomorphism of $\mathbb{N}$ onto $A, i, \prec$ and we have 2.

The implications $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are trivial.

To prove $4 \Rightarrow 1$ we reason in RCA_0 and assume 4. Let $g : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. Define a linear ordering $<_g$ of $\mathbb{N}$ by letting $m <_g n$ if and only if $g(m) < g(n)$. Using bounded Σ^0_1 comprehension [15, Theorem II.3.9], we can easily check that $\mathbb{N}$ has a first element i with respect to $<_g$ and that $\mathbb{N}, i, <_g$ is a strongly inductive ordered system which is almost isomorphic to $\mathbb{N}$. By 4 let $f : \mathbb{N} \to \mathbb{N}$ be such that $\mathbb{N}, i, <_g, f$ is an ordered successor system, i.e., for all $j \in \mathbb{N}$, $f(j) =$ the immediate successor of j with respect to $<_g$. We then have $\forall j\, (\exists n \leq j)\, (f^n(i) = j)$, hence

$$\forall m\, ((\exists j\, (g(j) = m)) \Leftrightarrow (\exists n \leq m)\, (g(f^n(i)) = m))\,,$$

so the range of g exists by Δ^0_1 comprehension using f and g as parameters. We have now shown that $\mathrm{rng}(g)$ exists for all one-to-one functions $g : \mathbb{N} \to \mathbb{N}$. By [15, Lemma III.1.3] this implies ACA_0, Q.E.D. □

Theorem 4.7. The following are pairwise equivalent over RCA_0^*.

1. ACA_0.

2. Every inductive ordered system is isomorphic to $\mathbb{N}$.

3. Every strongly inductive ordered system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.

Proof. This follows from Theorems 4.3 and 4.6. □

Our next goal is to determine the reverse-mathematical and proof-theoretical status of the statement that every inductive ordered system is almost isomorphic to $\mathbb{N}$. To this end we present Definition 4.8 and Lemmas 4.9 and 4.10 leading to Theorem 4.11.

Definition 4.8. Within RCA_0^* we make the following definitions (see also [8, Definition 2.3]). Let L be an infinite linear ordering in which each non-first element has an immediate predecessor and each non-last element has an immediate successor. We say that L is *ω-like* if each element of L has finitely many predecessors. We say that L is *ω^*-like* if each element of L has finitely many successors. We say that L is *$\omega + \omega^*$-like* if L is not ω-like or ω^*-like and each element of L has finitely many predecessors or finitely many successors. If L is $\omega + \omega^*$-like, the *ω-part of* L is the set consisting of all elements of L with finitely many predecessors, provided this set exists. The *ω^*-part of* L is defined similarly. Note that the ω-part of L exists if and only if the ω^*-part of L exists. Let OOP be the statement that the ω-part of every $\omega + \omega^*$-like linear ordering exists. We write $\mathsf{OOP}_0 = \mathsf{RCA}_0 + \mathrm{OOP}$ and $\mathsf{OOP}_0^* = \mathsf{RCA}_0^* + \mathrm{OOP}$.

Lemma 4.9. OOP_0 and OOP_0^* are equivalent over RCA_0^*.

Proof. It suffices to show that Σ^0_1 induction is provable in OOP_0^*. Reasoning in RCA_0^*, suppose Σ^0_1 induction fails. By Lemma 3.2 let C be an infinite subset of $\mathbb{N}$ which does not have arbitrarily large finite subsets. Let c_0 be the least element of C. Since C is unbounded, we may safely assume that C has no subset of cardinality c_0. For each n let $f(n)$ be cardinality of $\{c \in C \mid c < n\}$. Note that $f(n) < c_0$ for all n. Let $L = \mathbb{N}$ and define $<_L$ on L by letting $m <_L n$ if and only if $m < n \leq c_0$ or $c_0 < m < n$ or $m < f(n)$. In other words, for each pair c, c' of successive elements of C we are inserting the interval $\{n \mid c < n \leq c'\}$ between $f(c)$ and $f(c') = f(c) + 1$. Thus L is an $\omega + \omega^*$-like linear ordering. By OOP let B be the ω^*-part of L. Then $B = \{n \leq c_0 \mid C$ has no subset of cardinality $n\}$ and B has a least element, which is clearly impossible, Q.E.D. ∎

Lemma 4.10. OOP_0 is equivalent to ACA_0 over RCA_0.

Proof. Clearly arithmetical comprehension implies OOP, so it remains to show that OOP_0 proves arithmetical comprehension. By [15, Lemma III.1.3] it suffices to show that OOP_0 proves Σ^0_1 comprehension. Reasoning in OOP_0 and letting $\varphi(m)$ be a Σ^0_1 formula, we shall prove the existence of the set $\{m \mid \varphi(m)\}$. Write $\varphi(m) \equiv \exists k\, \theta(k, m)$ where $\theta(k, m)$ is Σ^0_0. Let $D_s = \{m < s \mid (\exists k < s)\, \theta(k, m)\}$. Thus D_s, $s = 0, 1, 2, \ldots$ is a nondecreasing sequence of finite sets such that $\forall m\, (\varphi(m) \Leftrightarrow \exists s\, (m \in D_s))$.

Using bounded primitive recursion [18] we shall define an increasing sequence of finite linear orderings $L_s, <_s$, $s = 0, 1, 2, \ldots$. Actually we shall have $L_{2s} = \{0, 1, \ldots, s-1\}$ so there will be a linear ordering $L, <_L$ where $L = \bigcup_s L_s = \mathbb{N}$ and $<_L = \bigcup_s <_s$. We shall also define a sequence of partitions $L_s = A_s \cup B_s$ such that $(\forall u \in A_s)\, (\forall v \in B_s)\, (u <_s v)$. We shall also have $(\forall u \in A_s)\, (\forall v \in A_s)\, (u <_s v \Leftrightarrow u < v)$ and $\forall s\, (B_s \subseteq B_{s+1})$. Except for trivial cases, L will be $\omega + \omega^*$-like with ω-part $A = \lim_s A_s$ and ω^*-part $B = \lim_s B_s = \bigcup_s B_s$. In addition we shall define a function $f : L \to \mathbb{N}$ with this property:

If $u_{1,s} < \cdots < u_{k,s}$ are the elements of A_{2s} in increasing order, then $f(u_{1,s}) < \cdots < f(u_{k,s})$ are the first $|A_{2s}|$ elements of $\mathbb{N} \setminus D_s$ in increasing order. (3)

Taking the limit as s goes to infinity, we shall have:

$$\text{If } u_1 < u_2 < \cdots \text{ are the elements of } A \text{ in increasing order, then } f(u_1) < f(u_2) < \cdots \text{ are the elements of } \mathbb{N} \setminus \bigcup_s D_s \text{ in increasing order.} \tag{4}$$

In particular our Σ^0_1 formula $\varphi(m) \equiv \exists s\,(m \in D_s)$ will be equivalent to the Π^0_1 formula $\forall u\,(u \in A \Rightarrow f(u) \neq m)$, and the existence of $\{m \mid \varphi(m)\}$ will then follow by Δ^0_1 comprehension.

The inductive construction of L is as follows.

Stage 0. Let $L_0 = <_0 = A_0 = B_0 =$ the empty set.

Stage $2s+1$. Let $L_{2s+1} = L_{2s} \cup \{s\}$ and insert s between A_{2s} and B_{2s}, i.e., $<_{2s+1} = <_{2s} \cup \{\langle u, s\rangle \mid u \in A_{2s}\} \cup \{\langle s, v\rangle \mid v \in B_{2s}\}$. Let $A_{2s+1} = A_{2s} \cup \{s\}$ and let $B_{2s+1} = B_{2s}$. Let $f(s) =$ the $(|A_{2s}|+1)$-st element of $\mathbb{N} \setminus D_s$ in increasing order. Our inductive hypothesis (3) implies that $u_{1,s} < \cdots < u_{k,s} < s$ are the elements of A_{2s+1} in increasing order and $f(u_{1,s}) < \cdots < f(u_{k,s}) < f(s)$ are the first $|A_{2s+1}|$ elements of $\mathbb{N} \setminus D_s$ in increasing order.

Stage $2s+2$. Let $L_{2s+2} = L_{2s+1}$ and $<_{2s+2} = <_{2s+1}$. Case 1: If there exists $u \in A_{2s+1}$ such that $f(u) \in D_{s+1}$, let a be the least such u and let $A_{2s+2} = \{u \in A_{2s+1} \mid u < a\}$ and $B_{2s+2} = L_{2s+2} \setminus A_{2s+2}$. Case 2: If no such u exists, let $A_{2s+2} = A_{2s+1}$ and $B_{2s+2} = B_{2s+1}$. In either case it is clear that (3) continues to hold with s replaced by $s+1$.

We may safely assume that Case 1 holds at infinitely many stages, because otherwise there would be a stage t such that $\forall m\,(\varphi(m) \Leftrightarrow \neg(\exists s > t)\,(\exists u \in A_s)\,(f(u) = m))$, hence $\{m \mid \varphi(m)\}$ would exist by Δ^0_1 comprehension. Since Case 1 holds infinitely often, we have $B_s \subsetneqq B_{s+1}$ for infinitely many s.

We may safely assume that $\neg\varphi(n)$ holds for infinitely many n. Let n be such that $\neg\varphi(n)$ holds. By bounded Σ^0_1 comprehension [15, Theorem II.3.9], the set $\{m < n \mid \varphi(m)\}$ exists and is finite, so by Σ^0_1 bounding [18] let s be such that $(\forall m < n)\,(\varphi(m) \Leftrightarrow m \in D_s)$. Then $f(u) = n$ for some $u \in A_{2s+2n}$, and then $u \in A_t$ for all $t \geq 2s+2n$. Since $\neg\varphi(n)$ holds for infinitely many n, it follows that $\lim_s |A_s| = \infty$.

It is now clear that L is $\omega + \omega^*$-like with ω-part $A = \lim_s A_s$ and ω^*-part $B = \bigcup_s B_s$. It is also clear that (4) holds, so $\{m \mid \varphi(m)\}$ exists by Δ^0_1 comprehension. This completes the proof. $\square$

Theorem 4.11. The following are pairwise equivalent over RCA^*_0.

1. ACA_0.
2. OOP^*_0
3. Every inductive ordered system is almost isomorphic to $\mathbb{N}$.

Proof. The equivalence $1 \Leftrightarrow 2$ follows from Lemmas 4.9 and 4.10. The implication $1 \Rightarrow 3$ is clear from Theorem 4.6. It remains to prove $3 \Rightarrow 2$. Let L be $\omega+\omega^*$-like such that the ω-part of L does not exist. Let i be the first element of L, let A be the disjoint union of L and $\mathbb{N}$, and extend the given linear ordering of L and the standard ordering $<$ of $\mathbb{N}$ to a linear ordering $\prec$ of A with $u \prec n$ for all $u \in L$ and all $n \in \mathbb{N}$. Then $A, i, \prec$ is an ordered system. Let $X \subseteq A$ be such that $i \in X$ and $\forall a\,(a \in X \Rightarrow a' \in X)$. If $L \nsubseteq X$, fix $c \in L \setminus X$ and let $Y = \{a \in X \mid a \prec c\} = \{u \in X \mid u <_L c\}$. Then Y is the ω-part of L, a contradiction. Thus $L \subseteq X$, and from this it follows that $X = A$. Thus $A, i, \prec$ is an inductive ordered system. It follows by 3 that L is finite. This contradiction completes the proof. □

Remark 4.12. We thank Richard Shore [13] for showing us a proof of Lemma 4.10. We have modified that proof to obtain our proof above. Our construction yields the following recursion-theoretical results:

1. There exists a recursive linear ordering L of type $\omega + \omega^*$ such that the halting problem is Turing reducible to the ω-part of L.

2. Given a recursively enumerable Turing degree $\mathbf{b}$, there exists a recursive linear ordering L of type $\omega + \omega^*$ such that the ω-part of L is retraceable and the ω^*-part of L is recursively enumerable of degree $\mathbf{b}$.

Subsequently Jockusch [10] noted that these results are easily deduced from his 1968 paper [9]. Namely, 1 is implicit in [9, Theorem 5.2], and 2 follows from [9, Theorem 3.2, Corollary 3.3] plus the following characterization [10]:

3. A recursively enumerable set is the ω-part of a recursive linear ordering of type $\omega + \omega^*$ if and only if it is recursive or simple and semirecursive.

We now end this section by commenting on the reverse-mathematical and proof-theoretical status of the statement that every strongly inductive ordered system is almost isomorphic to $\mathbb{N}$.

Definition 4.13. Within RCA_0^* we define a linear ordering L to be *pseudofinite* if every nonempty subset of L has a first element and a last element. Let PFO be the statement that every countable pseudofinite linear ordering is finite. We write $\mathsf{PFO}_0 = \mathsf{RCA}_0 + \mathrm{PFO}$ and $\mathsf{PFO}_0^* = \mathsf{RCA}_0^* + \mathrm{PFO}$.

Theorem 4.14. The following are equivalent over RCA_0^*.

1. PFO_0^*.

2. Every strongly inductive ordered system is almost isomorphic to $\mathbb{N}$.

Proof. We reason in RCA_0^*. To prove $1 \Rightarrow 2$, assume PFO and let $A, i, \prec$ be a strongly inductive ordered system. For each $c \in A$ the initial segment $\{a \in A \mid a \prec c\}$ is pseudofinite, hence finite, so $A, i, \prec$ is almost isomorphic to $\mathbb{N}$. Thus PFO $\Rightarrow 2$, i.e., $1 \Rightarrow 2$. To prove $2 \Rightarrow 1$, assume that L is a pseudofinite linear ordering. Let i be the first element of L. Let A be the disjoint union of L and $\mathbb{N}$. Extend the given linear ordering of L and the standard ordering $<$ of $\mathbb{N}$ to a linear ordering $\prec$ of A with $a \prec n$ for all $a \in L$ and all $n \in \mathbb{N}$. Then $A, i, \prec$ is a strongly inductive ordered system. It follows by 2 that L is finite. Thus $2 \Rightarrow$ PFO, i.e., $2 \Rightarrow 1$, Q.E.D. □

Remark 4.15. Yokoyama [20] has shown that $\mathsf{WKL}_0^* + \{\mathrm{RT}(k,l) \mid k,l \geq 2\}$ is Π^0_2-equivalent to RCA_0^*. Here RT stands for Ramsey's Theorem. Since PFO is provable in $\mathsf{RCA}_0^* + \mathrm{RT}(2,2)$, Yokoyama's result implies that PFO_0^* is much weaker[5] than RCA_0. We conjecture that PFO_0^* is Π^1_1-equivalent to RCA_0^*.

Definition 4.16. Within RCA_0^* let ADS be the ascending/descending sequence principle of Hirschfeldt/Shore [8]: every infinite linear ordering has an infinite ascending sequence or an infinite descending sequence. We write $\mathsf{ADS}_0 = \mathsf{RCA}_0 + \mathrm{ADS}$ and $\mathsf{ADS}_0^* = \mathsf{RCA}_0^* + \mathrm{ADS}$.

Theorem 4.17. The following are equivalent over RCA_0.

1. ADS_0.

2. Every strongly inductive ordered system is almost isomorphic to $\mathbb{N}$.

Proof. The proof of [8, Proposition 2.4] shows that our system PFO_0 (Definition 4.13) is equivalent to ADS_0. Using this observation, Theorem 4.17 follows immediately from Theorem 4.14. □

Remark 4.18. It is easy to see that ADS_0^* proves Σ^0_1 induction and is therefore equivalent to ADS_0. Chong/Slaman/Yang [2] have shown that ADS_0 is Π^1_1-equivalent to $\mathsf{RCA}_0 + \Sigma^0_2$ bounding, hence Π^0_3-equivalent to RCA_0, so the strength of ADS_0 is known.

5 Summary and open questions

RCA_0	isomorphic	almost isomorphic implies isomorphic	almost isomorphic
i.s.s.	WKL_0, 3.4	RCA_0, 3.2	WKL_0, 3.3
i.o.s.	ACA_0, 4.7	ACA_0, 4.7	ACA_0, 4.11
s.i.o.s.	ACA_0, 4.7	ACA_0, 4.7	ADS_0, 4.17, 4.18
i.o.s.s.	WKL_0, 4.5	RCA_0, 4.3	WKL_0, 4.4
s.i.o.s.s.	RCA_0, 4.3	RCA_0, 4.3	RCA_0, 4.3

Table 1: Summary of equivalences over RCA_0.

Tables 1 and 2 are a summary of our results. Abbreviations are used. For example, s.i.o.s.s. is an abbreviation for "strongly inductive ordered successor system." Recall also that an i.s.s. or inductive successor system is the same thing as a Peano system. Each entry in Table 1 or 2 stands for one of our results concerning the reverse-mathematical status of a categoricity theorem for $\mathbb{N}$. As an example, the entry PFO_0^*, 4.14, 4.15, 5.1 in Table 2 means that RCA_0^*

[5] See footnote 2.

RCA_0^*	isomorphic	almost isomorphic implies isomorphic	almost isomorphic
i.s.s.	WKL_0, 3.4	RCA_0, 3.2	WKL_0^*, 3.3
i.o.s.	ACA_0, 4.7	ACA_0, 4.7	ACA_0, 4.11
s.i.o.s.	ACA_0, 4.7	ACA_0, 4.7	PFO_0^*, 4.14, 4.15, 5.1
i.o.s.s.	WKL_0, 4.5	RCA_0, 4.3	WKL_0^*, 4.4
s.i.o.s.s.	RCA_0, 4.3	RCA_0, 4.3	?????, 5.2

Table 2: Summary of equivalences over RCA_0^*.

proves $\mathsf{PFO}_0^* \Leftrightarrow$ every strongly inductive ordered system is almost isomorphic to $\mathbb{N}$, with references to Theorem 4.14 and Remark 4.15 and Question 5.1.

We now state some open questions which are relevant to Table 2.

Question 5.1. Is PFO_0^* Π_1^1-equivalent to RCA_0^*? See Remark 4.15.

Question 5.2. What is the reverse-mathematical status of the statement that every strongly inductive ordered successor system is almost isomorphic to $\mathbb{N}$? We do not know whether this statement is provable in RCA_0^*. By Table 2 it is provable in each of the systems RCA_0 and PFO_0^* and WKL_0^*.

Question 5.3. Does there exist a second-order characterization of $\mathbb{N}$ which is provable in RCA_0^*? More precisely, does RCA_0^* prove the existence of a second-order sentence or set of sentences T such that $\mathbb{N}, 0, S$ is a second-order model of T and all second-order models of T are isomorphic to $\mathbb{N}, 0, S$? One may also consider the same question with RCA_0^* replaced by systems which are Π_2^0-equivalent to RCA_0^*.

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