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Notes on the first-order part of Ramsey’s theorem for pairs

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Abstract
We give the $\Pi^0_2$-part, the $\Pi^0_3$-part and the $\Pi^0_4$-part of RT$_2^2$ and related combinatorial principles.

1 Introduction
Determinating the first-order part of WKL$_0 + \text{RT}_2^2$ and other important combinatorial principles is one of the crucial topics in the study of Reverse Mathematics (see, e.g., [2, 4]). The usual approach for these questions is using forcing arguments to construct a second-order part for the target combinatorial principle. On the other hand, there is a traditional way to study the strength of combinatorial principles by using indicator functions. (For the details of indicator functions, see [6].) In [1], Bovykin and Weiermann gave the $\Pi^0_2$-part of WKL$_0 + \text{RT}_2^2$ by means of an indicator function defined by a density notion, using the idea of Paris [7] and Paris/Kirby [8]. Using similar arguments, we can show that the $\Pi^0_2$-part of WKL$_0 + \text{RT}_2^2$ is equivalent to Elementary Function Arithmetic (see [9]). In this paper, we give the $\Pi^0_3$-part and the $\Pi^0_4$-part of WKL$_0 + \text{RT}_2^2$ based on [1]. We will also give several density notions to characterize the $\Pi^0_2$-part, the $\Pi^0_3$-part and the $\Pi^0_4$-part of RT$_{<\infty}^2$, SRT$_2^2$, SRT$_{<\infty}^2$ and EM.

2 The $\Pi^0_2$-part of WKL$_0 + \text{RT}_2^2$
This section is essentially due to Bovykin/Weiermann[1].

Definition 2.1 (within $\Sigma_1$). For a finite set $X$, we define the notion of $n$-density as follows.

- A finite set $X$ is said to be $0$-dense if $|X| > \min X$.
- A finite set $X$ is said to be $n+1$-dense if for any (coloring) function $P : |X|^2 \to 2$, there exists a subset $Y \subseteq X$ such that $Y$ is $n$-dense and $Y$ is $P$-homogeneous, i.e., $P$ is constant on $|Y|^2$.

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Note that “\( X \text{ is } m\text{-dense} \)” can be expressed by a \( \Sigma_0 \)-formula.

**Definition 2.2.** \( n\text{PH}_2 \) asserts that for any \( a \) there exists an \( n\)-dense set \( X \) such that \( \min X > a \).

Define \( T_0 := \{ k\text{PH}_2^2 \mid k \in \omega \} \cup \Sigma_1 \).

**Lemma 2.1.**
- \( \text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2 \) for any \( n \in \omega \).
- \( \Sigma_1 \vdash m\text{PH}_2^2 \rightarrow \text{PH}_{m+1}^2 \).

**Proof.** Easy.

**Lemma 2.2** (Bovykin/Weiermann[1]). Let \( M \) be a countable model of \( \Sigma_1 \), and let \( X \subseteq M \) is a (\( M \)-)finite set which is \( k \)-dense for any \( k \in \omega \). Then, there exists a cut \( I \subseteq M \) such that \( \min X \in I < \max X \), \( X \cap I \) is unbounded in \( I \) and \( (I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2 \).

**Proof.** See [1].

**Theorem 2.3** (Bovykin/Weiermann[1]). A \( \Pi^0_2 \) sentence \( \psi \) is provable in \( \text{WKL}_0 + \text{RT}_2^2 \) if and only if it is provable in \( T_0 \).

**Proof.** See [1].

In fact, we can generalize this theorem as follows.

**Theorem 2.4.** A \( \Pi^0_3 \) formula \( \psi \) (\( \psi \) may contain set parameters) is provable in \( \text{WKL}_0 + \text{RT}_2^2 \) if and only if it is provable in \( \Sigma^0_1 \cup \{ k\text{PH}_2^2 \mid k \in \omega \} \). (Here, \( \Sigma^0_1 \) is a system of second-order arithmetic which contains basic axioms and induction axioms for \( \Sigma^0_1 \)-formulas with set parameters.)

**3 The \( \Pi^0_3 \)-part of \( \text{WKL}_0 + \text{RT}_2^2 \)**

**Definition 3.1.** Let \( \theta(a, x, y) \) be a \( \Sigma_0 \)-formula. We say that a finite set \( X = \{ a_i \mid i \leq l \} \) dominates \( \theta(a, \cdot, \cdot) \) if \( \forall i \leq l \ \forall x \leq a_i \ \exists y \leq a_{i+1} \theta(a, x, y) \) holds. We define several variations of \( \text{PH}_2^2 \) as follows:

- \( \theta-n\text{PH}_2^2 \equiv \forall a (\forall x \exists y \theta(a, x, y) \rightarrow \exists X \ (X \text{ is finite, } n\text{-dense, and dominates } \theta(a, \cdot, \cdot) )) \),
- \( n\text{PH}_2^2 \equiv \forall X (\forall x \exists y \geq x \ y \in X \rightarrow \exists Y \ (Y \text{ is finite, } n\text{-dense, and } Y \subseteq X)) \).

Define \( T_1 := \{ \theta-k\text{PH}_2^2 \mid k \in \omega, \theta \in \Sigma_0 \} \cup \Sigma_1 \) and \( T_1 \) := \( \{ k\text{PH}_2^2 \mid k \in \omega \} \cup \text{RCA}_0 \). Note that \( T_1 \) is a \( \Pi^0_3 \)-theory, i.e., \( T_1 \) is a set of \( \Pi^0_3 \)-sentences.

**Lemma 3.1.** Let \( \theta(a, x, y) \) be a \( \Sigma_0 \)-formula, and let \( n \in \omega \). Then, \( \text{WKL}_0 + \text{RT}_2^2 \vdash \theta-n\text{PH}_2^2 \), and \( \text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2 \).

**Proof.** Easy.
Theorem 3.2. A $\Pi^0_3$ sentence $\psi$ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in $T_1$. Thus, $T_1$ is the $\Pi^0_3$-part of $\text{WKL}_0 + \text{RT}_2^2$.

Proof. We show that $T_1 \nvdash \psi$ implies $\text{WKL}_0 + \text{RT}_2^2 \nvdash \psi$ for any $\Pi^0_3$-sentence $\psi$. Assume that $\psi \equiv \forall a \exists x \forall y \theta(a, x, y)$ is not provable from $T_1$. Then, there exists a nonstandard countable model $M \models T_1$ such that $M \models \forall x \exists y \neg \theta(a, x, y)$ for some $a \in M$. By $(\neg \theta)$-$k\text{PH}_2^1$ and overspill, there exists an $m$-dense set $X$ which dominates $-\theta(a, \cdot, \cdot)$ for some $m \in M \setminus \omega$. By Lemma 2.2, there exists an initial segment $I \subseteq M$ such that $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$ and $I \cap X$ is unbounded in $I$. Since $X$ dominates $-\theta$, for any $x \in I$ there exists $y \in I$ such that $I \models \neg \theta(a, x, y)$. Thus, we have $(I, \text{Cod}(I/M)) \models \neg \psi$, which means that $\text{WKL}_0 + \text{RT}_2^2 \nvdash \psi$. \hfill $\square$

Theorem 3.3. A $\Pi^0_3$ formula $\psi$ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in $\widehat{T}_1$.

Proof. Similar to the proof of Theorem 3.2. \hfill $\square$

Note that $\widehat{T}_1$ is equivalent to $\Sigma^0_1 \cup \{\forall A \forall a(\forall x \exists y \theta(A, a, x, y) \rightarrow \exists X (X \text{ is finite, } n\text{-dense, and dominates } \theta(A, a, \cdot, \cdot))) \mid n \in \omega, \theta \in \Sigma^0_0\}$ with respect to $\Pi^1_1$-sentences.

4 The $\Pi^0_4$-part of $\text{WKL}_0 + \text{RT}_2^2$

Definition 4.1 (within $\Sigma_1$). Let $\theta(a, x, y, z)$ be a $\Sigma_0$-formula. Then, we define the notion of weakly domination as follows.

- A 0-dense set $X$ weakly dominates $\theta(a, \cdot, \cdot, \cdot)$.
- An $n + 1$-dense set $X$ weakly dominates $\theta(a, \cdot, \cdot, \cdot)$ if for any coloring $P : [X]^2 \rightarrow 2$, there exists a $P$-homogeneous set $Y \subseteq X$ such that $\forall x < \min X \exists y < \min Y \forall z < \max Y \theta(a, x, y, z)$, $Y$ is $n$-dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$.

Note that “$X$ is $m$-dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$” can be expressed by a $\Sigma_0$ formula.

Definition 4.2. Let $\theta(a, x, y, z)$ be a $\Sigma_0$-formula. Then, the assertion $\theta^* - \text{PH}_2^1$ is the following

$\forall a \forall b (\forall x \exists y \forall z \theta(a, x, y, z) \rightarrow \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(a, \cdot, \cdot, \cdot) \text{ and } \min X > b))$.

Define $T_2 := \{\theta^* - \text{PH}_2^1 \mid n \in \omega, \theta(a, x, y, z) \in \Sigma_0\} \cup \Sigma_1$. Note that $T_2$ is a $\Pi^0_4$-theory.

Lemma 4.1. Let $\theta(a, x, y, z)$ be a $\Sigma_0$-formula, and let $n \in \omega$. Then, $\text{WKL}_0 + \text{RT}_2^2 \vdash \theta^* - \text{PH}_2^1$.

Proof. Easy. \hfill $\square$
Theorem 4.2. A $\Pi^0_4$ sentence $\psi$ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in $T_2$. Thus, $T_2$ is the $\Pi^0_4$-part of $\text{WKL}_0 + \text{RT}_2^2$.

Proof. We show that $T_2 \not\vdash \psi$ implies $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$ for any $\Pi^0_4$-sentence $\psi$. Assume that $\psi \equiv \forall a \exists x \forall y \forall z \theta(a, x, y, z)$ is not provable from $T_2$. Then, there exists a non-standard countable model $M \models T_2$ such that $M \models \forall x \exists y \forall z \neg \theta(a, x, y, z)$ for some $a \in M$. By $(k, -\theta)\text{PH}_2^2$ and overspill, there exists an $(m, \theta(a, \cdot, \cdot, \cdot))$-dense set $X$ such that $\min X > a$ for some $m \in M \setminus \omega$. As the proof of Theorem 1 of [1], we can construct a descending sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ which satisfies the following:

- $I = \sup\{\min X_i \mid i \in \omega\} \subseteq M$,
- $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$,
- $I \cap X$ is unbounded in $I$,
- $\forall x \leq \min X_i \exists y \leq \min X_{i+1} \forall z \leq \max X_{i+1} - \theta(a, x, y, z)$ for any $i \in \omega$.

Since $\min X_i < \min X_{i+1} < I < \max X_{i+1}$ for any $i \in \omega$, we have $I \models \forall x \exists y \forall z \neg \theta(a, x, y, z)$, i.e., $(I, \text{Cod}(I/M)) \models \neg \psi$. This means that $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$. \hfill $\square$

Remark 4.3. Adding set parameters, we can easily show the following: a $\Pi^0_4$ formula $\psi$ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in

$$\text{I}^0_4 \cup \{\forall A \forall a \forall b (\forall x \exists y \forall z \theta(A, a, x, y, z) \rightarrow \exists X(X \text{ is } n\text{-dense, weakly dominates } \theta(A, a, \cdot, \cdot, \cdot) \text{ and } \min X > b) \mid n \in \omega, \theta \in \Sigma^0_4\}.$$ 

5 \ PH\^2\_2 with stronger largeness notion

In this section, we compare $n\text{PH}_2^2$ with $\text{PH}_2^2$ plus “stronger largeness”.

Definition 5.1 (within $\text{I}^\Sigma_1$).
- A finite set $X$ is said to be 0-large if $X \neq \emptyset$.
- A finite set $X$ is said to be $r + 1$-large if there is a partition $X = \bigsqcup_{i \leq \min X} Y_i$ such that $\max Y_i < \min Y_{i+1}$ for any $i < \min X$ and each $Y_i$ is $r$-large.

Remark 5.1. 1. For any $r \in \omega$, $\text{I}^\Sigma_1$ proves that for any $a$, there exists a finite set $X$ such that $\min X > a$ and $X$ is $r$-large.

2. $Q(a, b) := \max\{r \mid [a, b] \text{ is } r\text{-large}\}$ is an indicator function for $\text{WKL}_0$.

3. More generally, if $M$ is a model of $\text{I}^\Sigma_1$ and $X \subseteq M$ is $r$-large for some $r \in M \setminus \omega$, then there exists a cut $I \subseteq M$ such that $(I, \text{Cod}(I/M)) \models \text{WKL}_0$ and $X \cap I$ is unbounded in $I$.

Definition 5.2. 1. $\text{PH}^2_r$ asserts that for any $a$, there exists a finite set $X$ such that $\min X > a$ and for any coloring $P : [X]^2 \to 2$, there exists a $P$-homogeneous set $Y \subseteq X$ which is $r$-large.
2. \( \widetilde{PH}^2_{2,r} \) asserts that for any infinite set \( A \), there exists a finite set \( X \) such that \( X \subseteq A \) and for any coloring \( P : [X]^2 \to 2 \), there exists a \( P \)-homogeneous set \( Y \subseteq X \) which is \( r \)-large.

3. In general, \( n\widetilde{PH}^2_{2,r} \) asserts that for any infinite set \( A \), there exists a finite set \( X \) such that \( X \subseteq A \) and \( X \) is \((n,r)\)-dense, where the notion of \((n,r)\)-density is defined as follows:

- A finite set \( X \) is said to be \((0,r)\)-dense if \( X \) is \( r \)-large.
- A finite set \( X \) is said to be \((n+1,r)\)-dense if for any coloring \( P : [X]^2 \to 2 \), there exists a \( P \)-homogeneous set \( Y \subseteq X \) which is \((n,r)\)-dense.

**Proposition 5.2.** \( \Sigma_1 \vdash n\widetilde{PH}^2_{2} \to \widetilde{PH}^2_{2,n} \).

**Proof.** Easy. \( \square \)

The strength of \( \widetilde{PH}^2_{2,r} \) is related to the strength of \( n\widetilde{PH}^2_{2} \) in the following meaning.

**Proposition 5.3.** Assume that \( WKL_0 \vdash \widetilde{PH}^2_{2,r} \) for all \( r \in \omega \), then we have \( WKL_0 \vdash n\widetilde{PH}^2_{2} \) for all \( n \in \omega \).

**Proof.** Our assumption is \( WKL_0 \vdash 1\widetilde{PH}^2_{2,r} \) for any \( r \in \omega \). We will show by induction on \( n \) that \( WKL_0 \vdash n\widetilde{PH}^2_{2,r} \) for any \( r \in \omega \) and for any \( n \in \omega \). Let \( WKL_0 \vdash n\widetilde{PH}^2_{2,r} \) for any \( r \in \omega \). Assume for the sake of contradiction that \( WKL_0 \not\vdash (n+1)\widetilde{PH}^2_{2,r} \) for some \( r \in \omega \). Then, there exists a model \((M,S) \models WKL_0 \) and \( A \in S \) such that \( M \not\models \omega \), \( A \) is unbounded in \( M \) and any \((M-)\)finite subset of \( A \) is not \((n+1,r)\)-dense. By the assumption, there exists an \((n,s)\)-dense subset of \( A \) for any \( s \in \omega \). Thus, by overspill, for some \( m \in M \setminus \omega \), we can take an \((n,m)\)-dense subset \( X \subseteq A \). We will show that this \( X \) is in fact \((n+1,r)\)-dense, which leads to a contradiction. By the definition of \((n,m)\)-density, for any coloring \( P : [X]^2 \to 2 \), there exists a \( P \)-homogeneous set \( Y_1 \subseteq X \) which is \((n-1,m)\)-dense, and we can repeat this process \( n \)-times then the result set \( Y_n \) is \( m \)-large. By Remark 5.1.3, there exists a cut \( I \subseteq M \) such that \( (I,\text{Cod}(I/M)) \models WKL_0 \) and \( Y_n \cap I \) is unbounded in \( I \). Thus, there exists a finite subset of \( Y_n \cap I \) which is \((1,r)\)-dense. This means that \( Y_n \) is \((1,r)\)-dense, and hence \( X \) is \((n+1,r)\)-dense.

Thus, if \( WKL_0 \vdash \widetilde{PH}^2_{2,r} \), then \( WKL_0 + RT^2_2 \) is a \( \Pi^0_2 \)-conservative extension of \( WKL_0 \). This may give a new approach to study the proof-theoretic strength of \( WKL_0 + RT^2_2 \).

**Question 5.3.** Is \( \Sigma_1 \cup \{n\widetilde{PH}^2_2 \mid n \in \omega \} \) equivalent to \( \Sigma_1 \cup \{\widetilde{PH}^2_{2,r} \mid r \in \omega \} \)?
6 Other combinatorial principles

In this section, we give several density notions for SRT$_2^2$, RT$_{<\infty}^2$, SRT$_{<\infty}^2$, EM and ADS.
(For the definitions of these combinatorial principles, see [2, 5, 1].) Using these notions, we can characterize $\Pi^0_2$, $\Pi^0_3$ or $\Pi^0_4$ part of the target combinatorial principle as in Sections 2, 3 and 4.

We reason within $\Sigma^0_1$.

**Proposition 6.1.** The $\Pi^0_2$-part, $\Pi^0_3$-part and the $\Pi^0_4$-part of $\text{WKL}_0 + \text{SRT}_2^2$ is characterized by the following density notion.

A finite set $X$ is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$-dense if for any $P : [X]^2 \rightarrow 2$,
  - there exists a $P$-homogeneous subset $Y \subseteq X$ which is $m$-dense, or,
  - there exists $Y = \{y_0 < y_1 < \cdots < y_l\} \subseteq X$ such that $P(y_0, y_i) \neq P(y_0, y_{i+1})$ for any $0 < i < l$ and $Y$ is $m$-dense.

For the strength of SRT$_2^2$, see also Chong/Slaman/Yang [3].

**Proposition 6.2.** The $\Pi^0_2$-part, $\Pi^0_3$-part and the $\Pi^0_4$-part of $\text{WKL}_0 + \text{RT}_2^2$ is characterized by the following density notion.

A finite set $X$ is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$-dense if for any coloring $P : [X]^2 \rightarrow k$ such that $k < \min X$, there exists a $P$-homogeneous subset $Y \subseteq X$ which is $m$-dense.

**Proposition 6.3.** The $\Pi^0_2$-part, $\Pi^0_3$-part and the $\Pi^0_4$-part of $\text{WKL}_0 + \text{SRT}_{<\infty}^2$ is characterized by the following density notion.

A finite set $X$ is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$-dense if for any coloring $P : [X]^2 \rightarrow k$ such that $k < \min X$,
  - there exists a $P$-homogeneous subset $Y \subseteq X$ which is $m$-dense, or,
  - there exists $Y = \{y_0 < y_1 < \cdots < y_l\} \subseteq X$ such that $P(y_0, y_i) \neq P(y_0, y_{i+1})$ for any $0 < i < l$ and $Y$ is $m$-dense.

**Proposition 6.4.** The $\Pi^0_2$-part, $\Pi^0_3$-part and the $\Pi^0_4$-part of $\text{WKL}_0 + \text{EM}$ is characterized by the following density notion.

A finite set $X$ is said to be
• 0-dense if $|X| > \min X$, and

• $m+1$-dense if
  
  for any coloring $P : [X]^2 \to 2$, there exists $Y \subseteq X$ such that $P$ is transitive on $Y$ and $Y$ is $m$-dense, and,
  
  there is a partition $X = \bigsqcup_{i \leq \min X} Y_i$ such that $\max Y_i < \min Y_{i+1}$ for any $i < \min X$ and each $Y_i$ is $m$-dense.

Here, a coloring $P$ is said to be transitive if $P(a, b) = P(b, c) \Rightarrow P(a, c) = P(a, c)$.

**Proposition 6.5.** The $\Pi^0_2$-part, $\Pi^0_3$-part and the $\Pi^0_4$-part of $\text{WKL}_0 + \text{ADS}$ is characterized by the following density notion.

A finite set $X$ is said to be

• 0-dense if $|X| > \min X$, and

• $m+1$-dense if for any transitive coloring $P : [X]^2 \to 2$, there exists a $P$-homogeneous subset $Y \subseteq X$ which is $m$-dense.

In fact, Slaman/Chong/Yang[4] showed that $\text{WKL}_0 + \text{ADS}$ is a $\Pi^1_1$-conservative extension of $\text{B}\Sigma^0_2$. Thus, for any $n \in \omega$, $\text{WKL}_0$ actually proves for any $a$, there exists a finite set $X$ such that $\min X > a$ and $X$ is $n$-dense for ADS.

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**References**


