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Author(s)	吉田，聡
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Description	Supervisor:石原 哉，情報科学研究科，修士

CONNECTIVITY OF METRIC SPACES IN CONSTRUCTIVE MATHEMATICS

Yoshida Satoru

School of Information Science,
Japan Advanced Institute of Science and Technology

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Connectivity of metric spaces is the notion that it cannot be the union of its disjoint open (or closed) subsets. For example, differentiable function and integrable one in complex analysis are defined on such a space, hence it is important in such a theory.

In this paper, we will define three connectivities and consider their properties respectively and the relation on them.

Now, we show at first constructive mathematics, and will define a metric space and consider connectivity of metric spaces.

Classical mathematics, which is called mathematics by the majority mathematicians, is formalized by *classical logic* , and correspondingly *constructive mathematics* is done by *intuitionistuc logic* (see [4]). Actually, the character of constructive mathematics can be showed by the following *BHK* (*Brouwer* − *Heyting* − *Kolmogorov*) − *interpretation*, which is that of logical operators by Brouwer, Heyting and Kolmogorov (see [4] and [3]).

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting a proof of A or B.
- A proof of $A \Rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.
- Absurdity $\perp$ (*contbradiction*) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical poof of A into a proof of a contradiction.
- A proof of $\forall x A(x)$ is a construction which transforms any $d \in D$ (D the intended range of the variable x) into a proof of $A(d)$.

- A proof of $\exists x A(x)$ is given by presenting a $d \in D$ and a proof of $A(d)$.

This interpretation is restricted than that of classical mathematics. Actually, for a proof of $A \vee B$, though it is enough to show that $\neg A \wedge \neg B$ details a contradiction in classical mathematics, it is at least necessary in constructive mathematics either to give a proof of A or to give a proof of B. For a proof of $\exists x A(x)$, we can regard it classically as showing that $\forall x \neg A(x)$ is impossible, but constructively we must present explicitly d with $A(d)$. Therefore, it can be thought that constructive mathematics classifies rules, translations and existences in mathematics under computability.

Now, there are some propositions not to be provable in constructive mathematics but to be provable in classical mathematics. For example, *the Principle of Excluded Middle* $A \vee \neg A$ is unprovable since for a open problem A, we cannot give a proof of A or a proof of $\neg A$ under BHK-interpretation. Then, *Axiom of Choice* cannot be a part of constructive mathematics since this axiom implies the Principle of Excluded Middle, where Axiom of Choice is is as follows.

$$\forall S \subset A \times B[\forall x \in A \exists y \in B((x, y) \in S) \Rightarrow \exists f : A \to B \forall x \in A((x, f(x)) \in S).]$$

But, *Axiom of Countable Choice* is acceptable in constructive mathematics, where this axiom is one of replacing A with the set of natural numbers $\mathbf{N}$ in the above Axiom of Choice.

Now, in constructive mathematics, there are the three main schools, which are *Bishop's constructive mathematics*, *Brouwer's intuitionistic mathematics* and *Markov's constructive mathematics*. Bishop's constructive mathematics is the mathematics accepting Axiom of Countable Choice under the BHK-interpretation, and Brouwer's intuitionistic mathematics can be regarded as Bishop's one added Brouwer's characteristic axioms. Markov's constructive mathematics can be also think of Bishop's one added Church's thesis i.e. "all sequences of natural numbers are recursive" and Markov's principle

MP $\forall (\alpha_n) \in \{0, 1\}^{\mathbf{N}}[\neg\neg\exists n(\alpha_n = 1) \Rightarrow \exists n(\alpha_n = 1)]$.

That is, two other theories are extended from Bishop's constructive mathematics. Still, classical one is also the extension from Bishop's one since classical mathematics is regarded as a system that is added the Principle of Excluded Middle to Bishop's constructive mathematics syntactically. Then, Brouwer' intuitionistic mathematics and Markov's constructive mathematics are inconsistent with classical mathematics respectively, but Bishop's constructive mathematics is not so. Therefore, in this paper, the author considers in Bishop's constructive mathematics. From now on, "constructive mathematics" means "Bishop's constructive mathematics" in this paper.

Well, the following propositions are unprovable in constructive mathematics.

LPO(the Least Principle of Omniscience)

$\forall (\alpha_n) \in \{0, 1\}^{\mathbf{N}}[\exists n(\alpha_n = 1) \vee \neg\exists n(\alpha_n = 1)]$.

LLPO(the Lesser Principle of Omniscience)

$\forall (\alpha_n), (\beta_n) \in \{0, 1\}^{\mathbf{N}}[\neg(\exists n(\alpha_n = 1) \wedge \exists n(\beta_n = 1)) \Rightarrow \neg\exists n(\alpha_n = 1) \vee \neg\exists n(\beta_n = 1)]$.

Actually, we cannot give a proof by the soundness for intuitionistic logic since there exist some models in such that they are false semantically (see [4]), and MP is also unprovable constructively in the same way. But, LPO, LLPO and MP hold in classical mathematics.

In constructive mathematics, "A given proposition does not hold" means that we can prove its negation or that it implies LPO, LLPO, MP and so on.

Next, we define real numbers, where we assume that the readers know rational numbers and its theory.

A sequence of rational numbers is *regular* if $|a_m - a_n| \leq m^{-1} + n^{-1}$ for all positive numbers m and n. A *real number* is a regular sequence of rational numbers, and the set of real numbers is denoted by $\mathbf{R}$.

For example, $a < r \vee r < b$ for all real numbers a ,b and r with $a < b$, but $r < s \vee r \geq s$ for all real numbers r and s is equivalent to LPO. However, $\neg(a > b)$ implies $a \leq b$.

Now, we define metric spaces.

A *metric* on a set X is a function $d : X \times X \to \mathbf{R^{0+}}$ such that for all x, y and z in X (1) $d(x, y) = 0$ if and only if $x = y$, (2) $d(x, y) = d(y, x)$ and (3) $d(x, z) \leq d(x, y) + d(y, z)$. Then, (X, d) is called a *metiric space*.

A *Open ball* of radius $r > 0$ for a point x in a metric space (X, d) is a subset $B(x, r) \equiv \{y \in X | d(x, y) < r\}$ of X. Then, the *interior* and *closure* of a subset A of X are subsets $A^i \equiv \{x \in X | \exists r > 0(B(x, r) \subset A)\}$ and $A^- \equiv \{x \in X | \forall r > 0(B(x, r) \cap A \textit{ is inhabited})\}$ of X respectively, and A is *open* if $A = A^i$ and is *closed* if $A = A^-$.

Next, we define continuity and sequentially continuity for maps on a metric space.

A map f from a metrics space X to Y is *continuous* if for each x in X and positive real number ϵ, there exists a positive real number δ such that $B(x, \delta)$ is contained in $f^{-1}(B(f(x), \epsilon))$. Then, a sequence (x_n) of elements of X *converges* to x in X if for each positive real number ϵ, there exists a positive integer N such that $d(x_n, x) < \epsilon$, and a map $f : X \to Y$ is *sequetially continuous* if for each x in X and (x_n) in X, $f(x_n) \to f(x)$ whenever $x_n \to x$.

Then, we show that for any map f from a metric space X to a metric space Y, f is continuous $\Rightarrow$ f is sequentially continuous $\Rightarrow$ $f^{-1}(F)$ is closed in X for all closed subset F of Y $\Rightarrow$ $f(A^-)$ is *contained* in $f(A)^-$ for all subset A of X.

Here, it is known that a continuous map on a given metric space is sequentially continuous but the converse does not hold (see [2]).

Finally, we define a connected metric space.

A metric space X is *connected* if for all inhabited and open subsets V and W of X with $V \cup W = X$, $V \cap W$ is inhabited. A subset A of a given metric space X is connected if the subspace A of X is connected.

Next, a metric space X is $path - connected$ if for all a and b in X, there exists a continuous map $f : [0, 1] \to X$ with $f(0) = a$ and $f(1) = 1$.

Then, we show that if X is connected and f is a continuous map from X to a metric space Y, then $f(X)$ is connected and that a path-connected set is connected.

Next, a metric space X is $C-connected$ if for all inhabited and closed subsets V and W of X with $V \cup W = X$, $V \cap W$ is inhabited, and a metric space Y is *strongly connected* if for all inhabited subsets S and T of Y with $S \cup T = Y$, $S^- \cap T$ or $S \cap Y^-$ is inhabited. Then, we show that if X is C-connected and f is a map from X to a metric space X' such that $f^{-1}(F)$ is closed in X for all closed subset F of X', then $f(X)$ is C-connected, that if Y is strongly connected and g is a map from Y to a metric space Y' such that $g(A^-)$ is contained in $g(A)^-$ for all subset A of Y, then $g(Y)$ is strongly connected and that a path-connected set is C-connected and strongly connected. Moreover, we show that a strongly connected metric space is connected and C-connected.

Well, the *intermediate value theorem* implies LLPO, hence it does not hold in constructive mathematics, where the intermediate value theorem is as follows:
let X be connected, and let $f : X \to \mathbf{R}$ be continuous. Let a and b be elements of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, there exists a c in X with $f(c) = \gamma$.

Now, we show the following weak version of it.

Let X be connected, and $f : X \to \mathbf{R}$ be continuous. Let a and b be elements of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in X with $|f(c) - \gamma| < \epsilon$.

Moreover, the following theorems for C-connected and strongly connected hold respectively.

Let X be C-connected, and let f be a map from X to $\mathbf{R}$ such that $f^{-1}(F)$ is closed in X for all closed subset F of $\mathbf{R}$. Let a and b be elements of X with $f(a) < f(b)$, and let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in X with $|f(c) - \gamma| < \epsilon$.

Let Y be strongly connected, and let f be a map from Y to $\mathbf{R}$ such that $f^{-1}(F)$ is closed in Y for all closed subset F of $\mathbf{R}$. Let x and y be elements of Y with $f(x) < f(y)$, and let ω be a real number with $f(Y) \leq \omega \leq f(y)$. Then, for any $\epsilon > 0$, there exists z in X with $|f(z) - \omega| < \epsilon$.

Then, the above theorems imply the following corollary since $[a, b]$ is connected.

Let a and b be real numbers with $a < b$, and let $f : [a, b] \to \mathbf{R}$ satisfies that $f(A^-)$ is *contained* in $f(A)^-$ for all subset A of $\mathbf{R}$. Let γ be a real number with $f(a) \leq \gamma \leq f(b)$. Then, for any $\epsilon > 0$, there exists c in X with $|f(c) - \gamma| < \epsilon$.

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