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Description	

Non-additive multi-attribute fuzzy target-oriented decision analysis

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Abstract

As an emerging area considering behavioral aspects of decision making, target-oriented decision model lies in the philosophical root of bounded rationality as well as represents the S-shaped value function. This paper deals with multi-attribute decision analysis from target-oriented viewpoint. First, the basic (random) target-oriented decision model is extended to involve three types of target preferences: *benefit target*, *cost target*, and *equal target*. Next, since applying fuzzy set theory in decision analysis allows the decision maker to specify imprecise aspiration levels, fuzzy target-oriented decision analysis is formulated to model three typical types of fuzzy targets: *fuzzy min*, *fuzzy max*, and *fuzzy equal*. Also, different attitudes are used to derive target achievement functions, which can be viewed as a support for "probability as psychological distance". Furthermore, we have proved that multi-attribute target-oriented decision analysis has a similar structure with discrete fuzzy measure and Choquet integral. Hence, we propose using discrete fuzzy measure and Choquet integral to model non-additive multi-attribute target-oriented decision analysis. In particular, the λ-measure is applied to reduce the difficulty of collecting information via a designed bisection search algorithm. Finally, a new product development example is used to illustrate the effectiveness and advantages of our model. The main advantages of our target-oriented decision model are its abilities to model the fuzzy uncertainty of targets as well as capture the non-additive behaviors among targets by means of discrete fuzzy measure and Choquet integral.

Keywords: Multi-attribute decision analysis; Target-oriented decision; Target preference types; Fuzzy targets; λ-measure; Choquet integral.

1. Introduction

Multi-attribute decision making (MADM) is one of the most widely used decision methodologies in the sciences, business, and engineering worlds. A typical problem in MADM is concerned with the task of

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ranking a finite number of decision alternatives, each of which is explicitly described in terms of different characteristics (also, often called attributes, decision criteria, or objectives), which have to be taken into account simultaneously. Among various MADM methods, multi-attribute utility theory (MAUT) [24] is one widely used one[1]. However, substantial empirical evidence and prior research have shown that it is difficult to build mathematically rigorous utility functions based on attributes [8] and the conventional attribute utility function often does not provide a good description of individual preferences [21]. As a substitute for utility theory, Kahneman and Tversky [21] have proposed an S-shaped value function. Heath et al. [17] have suggested that the inflection point in this S-shaped value function can be interpreted as a target. To develop this concept further, target-oriented decision analysis involves interpreting an increasing, bounded function, properly scaled, as a cumulative distribution function (cdf) and relating it to the probability of meeting or exceeding a target value. The cdf of the uncertain target is viewed as the target achievement (target-oriented utility) function.

The use of the cdf as a utility function recurs in the literature. Borch [7] has used it to study the probability of ruin. Berhold [6] has exploited it to propose a family of natural conjugate utility functions inspired by results in Bayesian statistics. Castagnoli and LiCalzi [10] have proved that expected utility can be expressed in terms of "expected probability", with the utility function interpreted as a cdf in the case of a single attribute (see also [9]). Abbas and Matheson [2] have defined "aspiration equivalents" for the alternatives based on an organization's utility function, drawing an analogy with notion of *satisficing* by seeking an alternative that meets or exceeds an aspiration level [32], and showed that these aspiration equivalents can be used as targets. LiCalzi and Sorato [29] have described a parametric family of utility functions based on Pearson system of distributions. Huynh et al. [18, 19] have proposed a fuzzy target-oriented approach to decision making under uncertainty. In many decision making situations, multiple attributes are of interest, thus it is important to extend basic target-oriented model to the multi-attribute case. Bordley and Kirkwood [8] have considered situations in which a target-oriented approach is natural and defined a target-oriented decision maker (DM) for a single attribute as one with a utility that depends only on whether a target for that attribute is achieved or not. They have then extended this definition to targets for multiple attributes, requiring that the DM's utility for a multidimensional outcome depend only on the subset of attributes for which targets are met. Taking a different tack, Tsetlin and Winkler [37] have considered multi-attribute target-oriented decision making and studied the impact on changes of expected utility in parameters of performance and target distributions via statistics techniques. More research of multi-attribute target-oriented decision analysis and its applications, especially to Kansei evaluations, can be referred to [20, 45, 46, 47].

[1] Other methods involving attributes, utility and relative measurement, include the analytical hierarchy process (AHP) and the simple multi-attribute rating technique (SMART) methods, which are simple versions of MAUT [39].

Despite the great advances in target-oriented decision analysis, there are still some challenges. First, in target-oriented decision analysis, a monotonically increasing preference on attributes is usually assumed in advance to simplify the decision problems. However, there exist two other types of attribute preferences: monotonically decreasing preference and non-monotonic preference. A natural question is whether we can use the cdf to model these two types of target preferences. Next, target-oriented decision analysis assumes the target has a probability distribution. As a mathematical counterpart of the probability theory, possibility theory deals with uncertainty by means of fuzzy set [48]. Applying fuzzy set in decision analysis has the advantage that the DM is allowed to specify imprecise aspiration levels [44]. One natural question that arises is how to solve target-oriented decision analysis using fuzzy targets. Although Huynh et al. [18, 19] have already considered the fuzzy targets, their work only focuses on decision with payoff variables, which are restricted to a bounded domain. They then derive the probability of meeting the fuzzy target regarding the monotonically increasing preference. However, as we shall see in Section 3, the derived value function is counterintuitive and cannot model the other two types of targets. Finally, Tsetlin and Winkler [37]'s approach to model the mutually dependent MATO decision analysis is too complex in real applications. On the other hand, even if in an objective sense the targets are mutually independent (probabilistically independent), the attributes (targets) are not necessarily considered to be independent from the DM's subjective viewpoint. In this regard, traditional analytic methods are inadequate and not applicable for modeling such complex situations.

In light of the above observations, this paper tries to propose a non-additive multi-attribute fuzzy target-oriented decision model. In Section 2, we extend the basic (random) target-oriented decision analysis by involving three types of target preferences. In Section 3, we formulate fuzzy target-oriented decision analysis via the concepts of tolerance level and possibility distribution, and discuss three types of commonly used fuzzy targets: fuzzy min, fuzzy max, and fuzzy equal. Our model also provides some relationships with goal programming (GP) and fuzzy goal programming (FGP). Also, different attitudes are used to derive target achievement functions: fuzzy optimistic target, fuzzy neutral target, and fuzzy pessimistic target. Such attitudinal targets can be viewed as a support for "probability as psychological distance". In Section 4, after formulating MATO decision analysis based on [8, 36, 37], we prove that MATO decision analysis with stochastic independence among targets has a similar structure with discrete fuzzy measure and Choquet integral. Hence, we propose using discrete fuzzy measure and Choquet integral to model non-additive MATO decision analysis. In particular, the λ-measure is applied to reduce the difficulty of collecting information via a designed bisection search algorithm. In Section 5 we apply a new product development problem, borrowed from the literature, to illustrate the effectiveness and advantages of our model. Comparisons with existing research are also given. Finally, some concluding remarks are presented in Section 6.

2. Random target-oriented decision analysis

Suppose that a DM has to rank several possible decisions. For notational convenience, designate a decision attribute by X with a continuous domain, and an arbitrary specific level of that attribute by x. In an uncertain environment, each decision d may lead to different outcomes, usually summarized in a random consequence X_d. Assume for simplicity that the set $\mathcal{O}$ of random consequences is finite. Denote by p_d a probability distribution for the random consequence X_d associated[2] with a decision d. Then the expected utility model suggests that the ranking be obtained by

$$\begin{aligned} V(d) &= \mathrm{EU}(X_d) \\ &= \sum_x U(x) p_d(x), \end{aligned}$$

where $U(x)$ is a von Neumann and Morgenstern (NM) utility function over consequences.

The target-oriented decision model, instead, suggests using the following value function [9, 10]:

$$\begin{aligned} V(d) &= \Pr(X_d \succeq T) \\ &= \sum_x \Pr(x \succeq T) p_d(x), \end{aligned} \tag{1}$$

where $\Pr(x \succeq T)$ is the target achievement (target-oriented utility), i.e., the probability of meeting an uncertain target T and the target T is stochastically independent of the consequence X_d.

Interestingly, despite the differences in approach and interpretation, both the utility-based procedure and target-oriented procedure essentially lead to only one basic model for decision making [9, 10]. The idea that the NM-utility function should be interpreted as a probability distribution may appear unusual but, in fact, NM-utilities are probabilities [1, 10]. Note that target-oriented decision analysis is strictly more general than expected utility, in the sense that equivalence holds under stochastic independence of the target.

With the assumption that the DM's preference function on an attribute X is monotonically increasing, x and t are mutually independent, Bordley and Kirkwood [8] suggest the target achievement (target-oriented utility) function be defined as follows:

$$\Pr(x \succeq T) = \int_{-\infty}^{x} p_T(t) dt, \tag{2}$$

where $p_T(t)$ is the probability density function (pdf) of uncertain target T and $\Pr(x \succeq T)$ is in fact the cdf of the uncertain target T, representing the target achievement function.

Most studies on target-oriented decision analysis assume the DM has a monotonically increasing preference on attributes [e.g., 1, 9, 10, 36, 37], and use the cdf as the target achievement function, as shown in Eq. (2). In general, there are three types of target preferences, introduced as follows.

[2] Formally, let p be the DM's subjective probability distribution on the state space $\mathcal{S}$, the probability distribution p_d is induced by the decision $d : \mathcal{S} \to \mathcal{O}$ through the equality $p_d(X_d = x) = p(\{s | d(S) = x\})$.

- When the DM has a monotonically increasing preference on an attribute X, the target values are adjustable and the more the better. Such a type of targets is usually used in MADM for benefit attributes and will be referred to as "*benefit target*".

- When the DM has a monotonically decreasing preference on an attribute X, the target values are adjustable and the less the better. Such a type of targets is usually used in MADM for cost attributes and will be referred to as "*cost target*".

- When the DM has a non-monotonic preference on an attribute X, the target values are fairly fixed and not subject to much change, i.e., too much or too little is not acceptable. Such a type of targets is usually used in MADM for non-monotonic attributes and will be referred to as "*equal target*".

Remark 1. Note that the shape of the pdf of the random target does not represent the monotonicity of the DM's preference function (utility function). For example, Bordley and LiCalzi [9] consider a situation in which the target T is represented by a normal distribution. In their example, the DM has a monotonically increasing preference on an attribute X, the normally distributed target T, therefore, is a benefit target. As a generalization, the normally distributed target T can also be a cost target or an equal target, depending on the DM's preference function on the attribute X.

For notational convenience in the context of MADM, we only consider the target achievement function. The main problem now is how to use the cdf to represent the target achievement functions with respect to these three types of targets in a general representation.

2.1. Random target-oriented decision with different types of target preferences

We first define the preference relation $\succeq$ as $\geq$, $\leq$, and $\cong$ for benefit target, cost target, and equal target, respectively. Recall that target-oriented decision analysis assumes that the pdf $p_T(t)$ has a mode value t_m (location of peak point) and views the mode value t_m as a reference point (the inflection point of the cdf of the uncertain target T) [9]. Furthermore, a DM is said to be target oriented for a single-attribute decision if his utility for an outcome depends only on whether a target is achieved or not (there are only two levels of utilities: 1 or 0). With respect to different types of target preferences, the DM will have different utility functions. Therefore, we define a unified target achievement function $\Pr(x \succeq T)$ as follows:

$$\Pr(x \succeq T) = \xi_1 \int_{-\infty}^{t_m} u_1(x,t)p_T(t)dt + \xi_2 \int_{t_m}^{t_m} u_2(x,t)p_T(t)dt + \xi_3 \int_{t_m}^{+\infty} u_3(x,t)p_T(t)dt. \tag{3}$$

where $u_{(\cdot)}(x,t)$ and $\xi_{(\cdot)}$ are the utility and adjustment parameter over different intervals, respectively, both of which are determined by the types of target preferences.

If T is a benefit target, i.e., the DM has a monotonically increasing preference on an attribute X, the DM has only one utility function such that

$$u_1(x,t) = u_2(x,t) = u_3(x,t) = \begin{cases} 1, & \text{if } x \geq t; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Furthermore, we set the adjustment parameters as $\xi_1 = \xi_2 = \xi_3 = 1$. Substituting $u_{(\cdot)}(x,t)$ and $\xi_{(\cdot)}$ into Eq. (3), we can obtain the target achievement function as follows:

$$\Pr(x \geq T) = \int_{-\infty}^{x} p_T(t)dt, \tag{5}$$

which is equivalent to the traditional one [9, 10], i.e., the target-oriented model views the cdf as the target achievement function.

Similar with the benefit target, for a cost target (the DM has a monotonically decreasing preference on an attribute X) the DM also has only one utility function such that

$$u_1(x,t) = u_2(x,t) = u_3(x,t) = \begin{cases} 1, & \text{if } x \leq t; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Furthermore, we set $\xi_1 = \xi_2 = \xi_3 = 1$. Substituting $u_{(\cdot)}(x,t)$ and $\xi_{(\cdot)}$ into Eq. (3), we can obtain the target achievement function as

$$\begin{aligned} \Pr(x \leq T) &= \int_{x}^{+\infty} p_T(t)dt \\ &= 1 - \int_{-\infty}^{x} p_T(t)dt, \end{aligned} \tag{7}$$

which also uses the cdf to express the target achievement function.

If T is an equal target (the DM has a non-monotonic preference on the attribute X), the reference point t_m will be the aspiration point. In this case, the DM has a monotonically increasing preference on X when $x \in (-\infty, t_m)$ and a monotonically decreasing preference on X when $x \in (t_m, +\infty)$. Accordingly, the DM has three utility functions such that

$$\begin{aligned} &u_1(x,t) = 1, \text{ if } x \geq t \text{ and } x \in (-\infty, t_m); \quad 0, \text{ otherwise.} \\ &u_2(x,t) = 1, \text{ if } x = t_m; \quad 0, \text{ otherwise.} \\ &u_3(x,t) = 1, \text{ if } x \leq t \text{ and } x \in (t_m, +\infty); \quad 0, \text{ otherwise.} \end{aligned} \tag{8}$$

Moreover, due to the boundary property of $\Pr(x \succeq T)$, i.e., $\Pr(x \succeq T) \in [0,1]$, the parameters ξ_1, ξ_2, ξ_3 are defined to adjust $\Pr(x \cong T)$ such that

$$\begin{aligned} &\xi_1 = \frac{1}{\int_{-\infty}^{t_m} p_T(t)dt}, \text{ if } \exists p_T(t) \neq 0; \quad 0, \text{ otherwise.} \\ &\xi_2 = \frac{1}{\int_{t_m}^{t_m} p_T(t)dt}. \\ &\xi_3 = \frac{1}{\int_{t_m}^{+\infty} p_T(t)dt}, \text{ if } \exists p_T(t) \neq 0; \quad 0, \text{ otherwise.} \end{aligned} \tag{9}$$

Substituting Eq. (8) and Eq. (9) into Eq. (3), we can obtain the target achievement function $\Pr(x \cong T)$ of meeting an equal target as

$$\Pr(x \cong T) = \begin{cases} \xi_1 \int_{-\infty}^{x} p_T(t)dt, & \text{if } x \in (-\infty, t_m); \\ \xi_2 \int_{t_m}^{t_m} p_T(t)dt = 1, & \text{if } x = t_m; \\ \xi_3 \int_{t_m}^{x} p_T(t)dt, & \text{if } x \in (t_m, +\infty). \end{cases} \tag{10}$$

with the notation that $\frac{0}{0} = 1$. Roughly, when $x \leq t_m$ the attribute X can be viewed as a pseudo-benefit attribute; when $x > t_m$ the attribute X can be viewed as a pseudo-cost attribute. We know $p_T(t)$ is monotonically non-decreasing for $t \leq t_m$ and monotonically non-increasing for $t > t_m$, respectively. Thus, when $x \leq t_m$ or $x \geq t_m$, Eq. (10) is convex shaped which indicates that the convex functions can be viewed as losses relative to the reference point t_m. Note that the benefit and cost targets have the same adjustment parameter as $\xi_1 = \xi_2 = \xi_3 = \frac{1}{\int_{-\infty}^{+\infty} p_T(t)dt} = 1$.

Remark 2. Obviously, $\Pr(x \cong T)$ is not a traditional probability measure, but a combination of the cdf of the random target T. The main reason why we define Eq. (10) is that there exists a reference point t_m of the target-oriented utility. For the equal target, the utility (probability of meeting the target) of the reference point should be one. Therefore, we defined Eq. (10) by adding some adjustment parameters.

As a generalization of the non-monotonic target preference, the mode value may be an interval range, denoted as $t_m \equiv [t_{ml}, t_{mr}]$. An example of this case is the uncertain target having a trapezoidal probability distribution [12]. Similar with Eq. (10), we can induce the target achievement function with respect to a non-monotonic target having an interval mode as follows:

$$\Pr(x \cong T) = \begin{cases} \xi_1 \int_{-\infty}^{x} p_T(t)dt, & \text{if } x \in (-\infty, t_{ml}); \\ \xi_2 \int_{t_{ml}}^{t_{mr}} p_T(t)dt = 1, & \text{if } x \in [t_{ml}, t_{mr}]; \\ \xi_3 \int_{t_{mr}}^{x} p_T(t)dt, & \text{if } x \in (t_{mr}, +\infty). \end{cases} \tag{11}$$

where

$$\begin{aligned} \xi_1 &= \frac{1}{\int_{-\infty}^{t_{ml}} p_T(t)dt}, \text{ if } \exists p_T(t) \neq 0; \quad 0, \text{ otherwise.} \\ \xi_2 &= \frac{1}{\int_{t_{ml}}^{t_{mr}} p_T(t)dt}. \\ \xi_3 &= \frac{1}{\int_{t_{mr}}^{+\infty} p_T(t)dt}, \text{ if } \exists p_T(t) \neq 0; \quad 0, \text{ otherwise.} \end{aligned} \tag{12}$$

2.2. Discussion: Degree of achievement

Bordley and Kirkwood [8] have generalized the "degree of achievement" of targets to a more general case by the following loss functions

$$u(x,t) = \begin{cases} -a(t-x), & x < t, \\ b - c(x-t), & \text{otherwise,} \end{cases} \tag{13}$$

where $a \geq 0$, $b \geq 0$, and c is a real value. In Eq. (13), if $a > 0$ there is added loss of value for missing the target on the low side by greater amounts, and either added value, no change in value, or added loss for exceeding the target by greater amounts depending on whether $c > 0, c = 0, c < 0$. For example,

- if the DM has a monotonically increasing preference, we can set $a = 0, b = 1, c = 0$;
- if the DM prefers non-monotonic preference (equal target), we can set $a > 0, b = 0, c > 0$.

However, this approach is debatable. As pointed by Bordley and Kirkwood, an expected utility DM is defined to be target oriented for a single attribute decision if the DM's utility for an outcome depends only on whether a target is achieved with respect to X [8, p. 824]. Thus we shall have only two utility levels $u(x,t) = 1$ or $u(x,t) = 0$. The above functions allow more than two utility levels, thus there exists some inconsistency in Bordley and Kirkwood's approach.

Bordley and Kirkwood [8] first consider the generalized "degree of achievement" of targets, Eq. (13), with respect to a crisp target t. They then apply it to the case of uncertain target. However, Bordley and Kirkwood's method of the "degree of achievement" of targets to a more general case by the loss functions is not suitable. For example, consider the uncertain target with a normal distributed pdf. If we assume the DM has an equal target preference on an attribute X, we will always obtain the non-positive (negative or zero) value as

$$\Pr(x \cong T) = \int_{-\infty}^{+\infty} u(x,t) \cdot p_T(t) dt.$$

The main reason is that $u(x,t)$ is a loss function, i.e., $u(x,t) \leq 0$.

Although the target uncertainty has been discussed in both our work and Bordley and Kirkwood's work, the "degree of achievement" of targets is different. Instead of using loss functions to model target achievement function, we have used the cdf to express the target achievement function, in which we have only two utility levels (1 or 0) in the case of crisp targets.

3. Fuzzy target-oriented decision model

As a mathematical counterpart of probability theory, possibility theory [49] deals with uncertainty via fuzzy sets [48]. Formally, the soft constraint imposed on a variable V is a statement "V is A", where A is a fuzzy set and $\nu_A(x)$ is the membership function of A. The fuzzy sets can be considered as inducing a possibility distribution π on the domain of A such that $\nu_A(x) = \pi_A(x)$ for each x. In this paper, we shall use the "possibility distribution" and "membership function" interchangeably. Since the introduction of possibility, the relationship between possibility and probability has received much attention from the research community. Particularly, the issue of associating probability distributions with possibility distributions has been discussed extensively. Yager [40] has proposed a proportional method for instantiating a possibility

variable over a discrete domain by converting its possibility distribution into a probability distribution as $p(x) = \pi(x)/\sum_x \pi(x)$. This transformation method has been extended into a continuous context as

$$p(x) = \frac{\pi(x)}{\int_x \pi(x)dx}. \tag{14}$$

In the following we will use this conversion of a possibility distribution into a probability distribution for decision making with fuzzy targets.

Remark 3. The possibility-probability consistency principle is a heuristic relationship between possibilities and probabilities. The normalization based transformation approach satisfies Zadeh's consistency principle [49]. However, even Yager [40, p. 265] himself pointed out that:

> "We should note using this normalizing approach to possibility distribution-probability distribution conversion the probability measure obtained is not always dominated by the possibility measure. That is, using this approach to generate a probability distribution and assuming an additive probability measure situations can arise in which the probability measure of a subset of outcomes is greater than the possibility measure of the subset. . . . . ."

In this paper, we prefer Yager's transformation method due to the following reasons. From a theoretical point of view, the possibility-probability transformation method proposed by Yager [40] relates to the defuzzification method used in [49]. In this section, we will consider the fuzziness of the target, therefore we believe a defuzzification characteristic in the transformation is natural, whereas the alpha-cuts based transformation method [13, 15] does not display this characteristic [40]. In addition, it is easily and explicitly to analyze the properties of the utility function with respect to a fuzzy target by using Yager's transformation method, which is important for our work. From an applicable point of view, the possibility-probability transformation method proposed by Yager [40], has been widely used in the literature [3, 11, 18, 19, 22, 26, 27, 31, 41, 43]. The main reasons are twofold. First, this method is easy to use. Second, due to the nature of the problem to be solved, such a method has good properties in application [3, 11, 22, 31].

3.1. General formulation of fuzzy target-oriented decision

Assume that a DM specifies a target T for an attribute X. If there is no impreciseness about his judgment, we shall denote it as t_m. However, this is a difficult task for the DM. Applying fuzzy set theory allows the DM to specify imprecise target values. To build possibility distribution of a target, aspiration level t_m and the tolerance level δ should be determined first. The tolerance can be chosen either subjectively by the DM or objectively by a technical process [44]. The left and right tolerance values relative to t_m are denoted as δ^- and δ^+ ($\delta^-, \delta^+ \geq 0$), respectively. Without loss of generalization, we define $t_m \equiv [t_{ml}, t_{mr}], t_{ml} \leq t_{mr}$, with $t_{ml} = t_{mr}$ as a special case. Accordingly, we can set $t_{\min} = t_{ml} - \delta^-, t_{\max} = t_{mr} + \delta^+$. Then the fuzzy

target can be expressed in the canonical form of a fuzzy number [25] as follows:

$$\pi_T(t) = \begin{cases} f_T(t), & t_{\min} \leq t < t_{ml}, \\ 1, & t \in [t_{ml}, t_{mr}], \\ g_T(t), & t_{mr} < t \leq t_{\max}, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

where $\pi_T(t)$ is the possibility distribution of T, $f_T(t)$ and $g_T(t)$ are real-valued monotonically non-decreasing and non-increasing functions, respectively. In addition, if we assume f_T and g_T are linear functions, the fuzzy targets in Eq. (15) can also be represented by a trapezoidal fuzzy number such that

$$T = (t_{\min}, t_{ml}, t_{mr}, t_{\max}). \tag{16}$$

Possibility distributions are known to be somewhat limited in expressiveness when compared with other models such as belief function [38] and possibility theory can deal with uncertainty via fuzzy sets [48]. In addition, trapezoidal fuzzy sets are usually used in most situations [42]. Therefore, trapezoidal fuzzy numbers are used to express different fuzzy targets in form of Eq. (16). Based on the proportional possibility-probability conversion method in Eq. (14), we can derive a pdf $p_T(t)$ of the fuzzy target T.

If the DM has monotonic preferences on an attribute X, substituting $p_T(t)$ into Eq. (5) and Eq. (7), we can induce the target achievement function as follows:

$$\Pr(x \geq T) = \frac{\int_{-\infty}^{x} \pi_T(t)dt}{\int_{-\infty}^{+\infty} \pi_T(t)dt}, \text{ benefit target;} \tag{17}$$

$$\Pr(x \leq T) = \frac{\int_{x}^{+\infty} \pi_T(t)dt}{\int_{-\infty}^{+\infty} \pi_T(t)dt}, \text{ cost target.} \tag{18}$$

Since the canonical form of a fuzzy number is used to express a fuzzy target, the pdf p_T derived from the possibility distribution π_T has a mode range $t_m \equiv [t_{ml}, t_{mr}]$. If the DM has a non-monotonic preference on an attribute X, substituting p_T into Eq. (11), we can induce the following target achievement function

$$\Pr(x \cong T) = \begin{cases} \frac{\int_{t_{\min}}^{x} \pi_T(t)dt}{\int_{t_{\min}}^{t_{ml}} \pi_T(t)dt}, & t_{\min} \leq x < t_{ml}; \\ 1, & t_{ml} \leq x \leq t_{mr}; \\ \frac{\int_{x}^{t_{\max}} \pi_T(t)dt}{\int_{t_{mr}}^{t_{\max}} \pi_T(t)dt}, & t_{mr} < x \leq t_{\max}; \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

with the notation $\frac{0}{0} = 1$.

3.2. Three typical types of fuzzy targets

In decision making involving fuzzy targets, three typical types of fuzzy targets are: "**fuzzy min** t_m", "**fuzzy max** t_m", "**fuzzy equal** t_m" [5, 50]. In this section, we discuss these three types of fuzzy targets in the framework of our target-oriented decision model.

3.2.1. Fuzzy min t_m

If a DM specifies a fuzzy min type target, i.e., fuzzy at least $t_m(t_{ml} = t_{mr})$, there will be a monotonically increasing preference. Also, there is no right tolerance δ^+ relative to t_m such that $\delta^+ = 0$, we can easily obtain $t_{\min} < t_m = t_{ml} = t_{mr} = t_{\max}$.

First, similar with the membership function of the fuzzy target in FGP [44], we can build the possibility distribution for a fuzzy target T_1^{bene} as

$$\pi_{T_1^{\text{bene}}}(t) = \begin{cases} \frac{t-t_{\min}}{t_m-t_{\min}}, & t_{\min} \leq t \leq t_m; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

which can also be expressed as $T_1^{\text{bene}} = (t_{\min}, t_m, t_m, t_m)$. Here, the term "bene" is used to represent a benefit target. Substituting Eq. (20) into Eq. (17), we can induce the following target achievement function

$$\Pr(x \geq T_1^{\text{bene}}) = \begin{cases} 0, & x < t_{\min}, \\ \frac{\int_{t_{\min}}^{x}(t-t_{\min})dt}{\int_{t_{\min}}^{t_m}(t-t_{\min})dt}, & t_{\min} \leq x \leq t_m, \\ 1, & x > t_m. \end{cases} \tag{21}$$

Since $(t - t_{\min})$ increases with t over the interval $[t_{\min}, t_m]$, thus $\Pr(x \geq T_1^{\text{bene}})$ has a convex shaped function over the interval $[t_{\min}, t_m]$, as shown in Fig. 1 (indexed by $\Pr(x \geq T_1^{\text{bene}})$). Based on Eq. (14), T_1^{bene} derives a pdf having a mode value t_m. In target-oriented decision framework, t_m is the reference point, all the attribute values below t_m are viewed as losses. This observation is consistent with the psychological finding that people tend to be risk seeking over losses, in other words, convex over losses [21]. Also, T_1^{bene} implies that the DM assesses higher possibility about his target toward the maximal value t_m, which corresponds to the attitude that the DM believes that 'best thing may happen'. Since the convex shaped function reflects the DM's optimistic attitude, T_1^{bene} will also be called *fuzzy optimistic target* with respect to a benefit attribute.

Conversely, a DM may assess higher possibility about his target toward the reservation point $t_{\min}$, which corresponds to the attitude that the DM believes that the best thing that may happen is $t_{\min}$. Such a fuzzy target is referred to as *fuzzy pessimistic target* and can be expressed by $T_2^{\text{bene}} = (t_{\min}, t_{\min}, t_{\min}, t_m)$. Substituting T_2^{bene} into Eq. (17), we can induce the following target achievement function

$$\Pr(x \geq T_2^{\text{bene}}) = \begin{cases} 0, & x < t_{\min}; \\ \frac{\int_{t_{\min}}^{x}(t_m-t)dt}{\int_{t_{\min}}^{t_m}(t_m-t)dt}, & t_{\min} \leq x \leq t_m; \\ 1, & x > t_m. \end{cases} \tag{22}$$

Since $(t_m - t)$ decreases with t in $[t_{\min}, t_m]$, $\Pr(x \geq T_2^{\text{bene}})$ is a concave shaped function over the interval $[t_{\min}, t_m]$, as shown in Fig. 1 (indexed by $\Pr(x \geq T_2^{\text{bene}})$). In this case, $t_{\min}$ is the reference point, all the attribute values greater than $t_{\min}$ are viewed as gains. This observation is consistent with the psychological finding that people tend to be risk averse over gains, in other words, concave over gains [21].

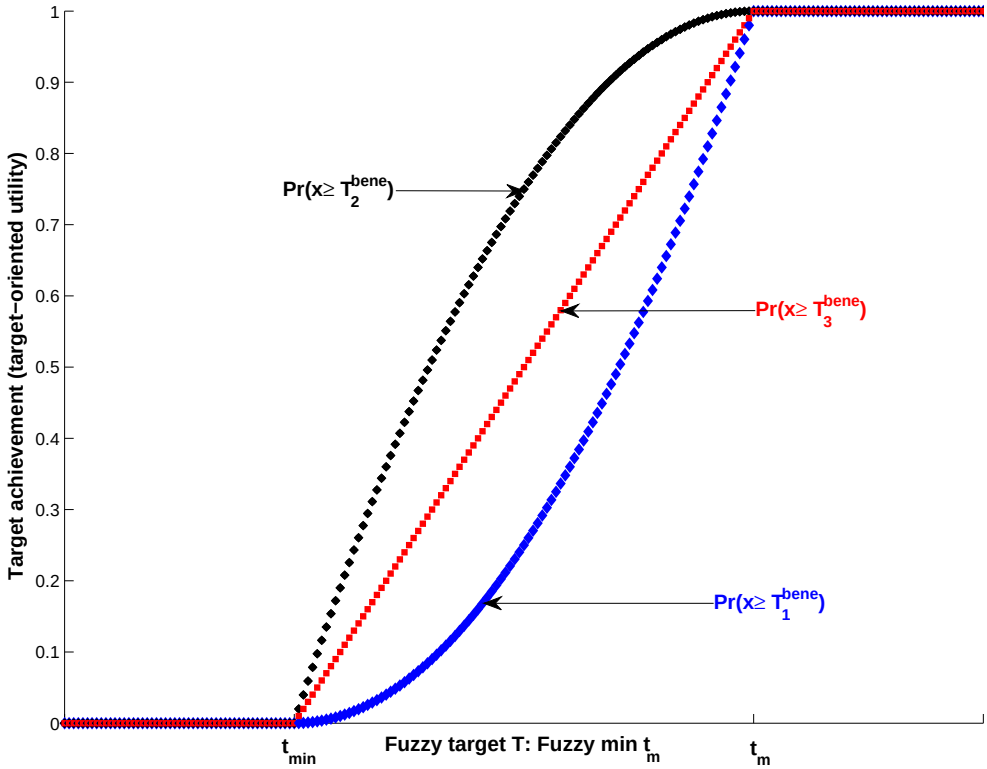


Figure 1: Target achievement function of fuzzy min target

If the DM assesses a uniform possibility distribution about his target over the interval $[t_{\min}, t_m]$, it implies that the DM has a *fuzzy neutral target*, expressed as $T_3^{\text{bene}} = (t_{\min}, t_{\min}, t_m, t_m)$. Substituting T_3^{bene} into Eq. (17), we can simply derive the target achievement function as follows:

$$\Pr(x \geq T_3^{\text{bene}}) = \begin{cases} 0, & x < t_{\min}; \\ \frac{x - t_{\min}}{t_m - t_{\min}}, & t_{\min} \leq x \leq t_m; \\ 1, & x > t_m. \end{cases} \tag{23}$$

It is clearly seen that $\Pr(x \geq T_3^{\text{bene}})$ is equivalent to the utility function of fuzzy min type target in FGP problems [50], as shown in Fig. 1 (indexed by $\Pr(x \geq T_3^{\text{bene}})$).

3.2.2. Fuzzy max t_m

In this case, the DM has a monotonically decreasing preference on an attribute X. There is no left tolerance relative to t_m such that $\delta^- = 0$, we can easily obtain $t_m = t_{ml} = t_{mr} = t_{\min} < t_{\max}$. We also first consider the possibility distribution used in FGP [44], which is expressed as $T_1^{\text{cost}} = (t_m, t_m, t_m, t_{\max})$.

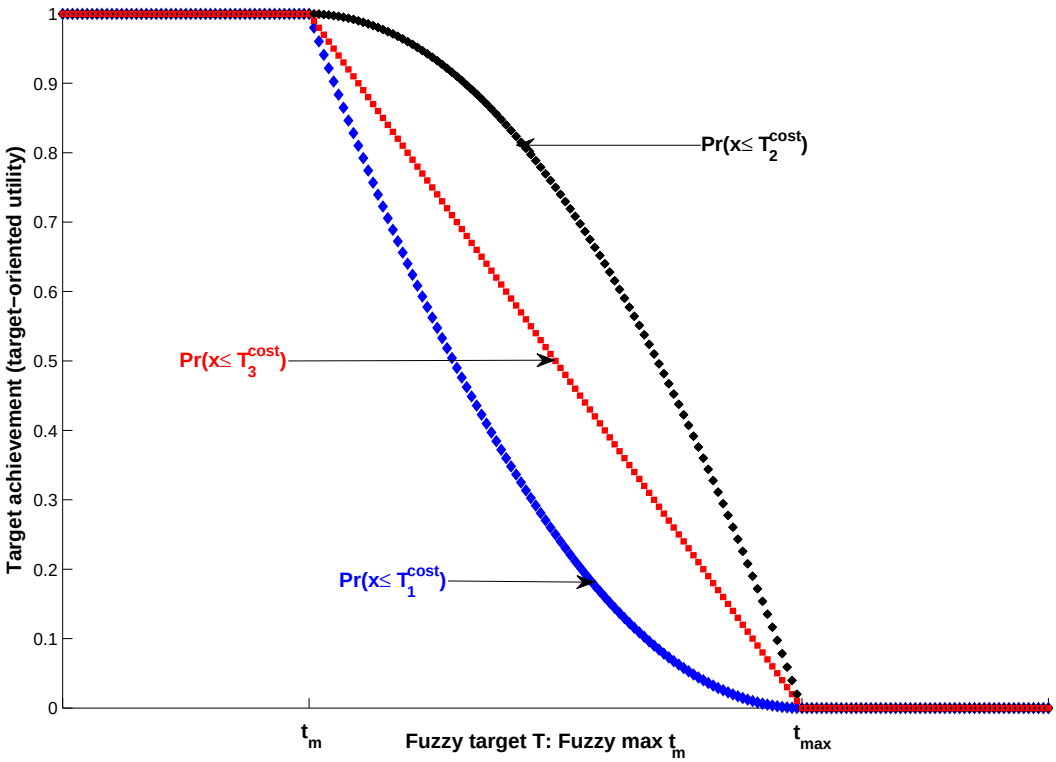


Figure 2: Target achievement function of fuzzy max target

Substituting T_1^{cost} into Eq. (18), $\Pr(x \leq T_1^{\text{cost}})$ is derived by

$$\Pr(x \leq T_1^{\text{cost}}) = \begin{cases} 1, & x < t_m; \\ 1 - \frac{\int_{t_m}^{x} (t_{\max} - t)dt}{\int_{t_m}^{t_{\max}} (t_{\max} - t)dt}, & t_m \leq x \leq t_{\max}; \\ 0, & x > t_{\max}. \end{cases} \tag{24}$$

Since $(t_{\max} - t)$ decreases with t in $[t_m, t_{\max}]$, $1 - \frac{\int_{t_m}^{x} (t_{\max} - t)dt}{\int_{t_m}^{t_{\max}} (t_{\max} - t)dt}$ is a convex shaped function in $[t_m, t_{\max}]$. The psychological semantic behind this observation is that since t_m is the reference point, all the attribute values upper than t_m are viewed as losses (for a monotonically decreasing preferences), i.e., convex over losses, as shown in Fig. 2. The convex shaped value function in Eq. (24) implies that the DM has an optimistic attitude toward a cost attribute, in other words, the DM believes that the best thing may happen is t_m.

Now let us consider a DM's pessimistic and neutral attitudes with respect to the fuzzy max target. Similar with the fuzzy min target, we can define the following possibility distributions on $[t_m, t_{\max}]$:

$$T_2^{\text{cost}} = (t_m, t_{\max}, t_{\max}, t_{\max}), \text{ for pessimistic target;} \tag{25}$$

$$T_3^{\text{cost}} = (t_m, t_m, t_{\max}, t_{\max}), \text{ for neutral target.} \tag{26}$$

Substituting them into Eq. (18), we can derive the following target achievement functions

$$\Pr(x \leq T_2^{\text{cost}}) = \begin{cases} 1, & x < t_m; \\ 1 - \frac{\int_{t_m}^{x}(t-t_m)dt}{\int_{t_m}^{t_{\max}}(t-t_m)dt}, & t_m \leq x \leq t_{\max}; \\ 0, & x > t_{\max}. \end{cases} \tag{27}$$

$$\Pr(x \leq T_3^{\text{cost}}) = \begin{cases} 1, & x < t_m; \\ \frac{t_{\max}-x}{t_{\max}-t_m}, & t_m \leq x \leq t_{\max}; \\ 0, & x > t_{\max}. \end{cases} \tag{28}$$

In Eq. (27), since $(t - t_m)$ increases with t over $[t_m, t_{\max}]$, we can conclude that $\Pr(x \leq T_2^{\text{cost}})$ has a concave shaped value function, which implies that $t_{\max}$ is the most possible target value, thus value less than $t_{\max}$ in $[t_m, t_{\max}]$ will be viewed a gain, i.e., concave over gains, as shown in Fig. 2. $\Pr(x \leq T_3^{\text{cost}})$ in Eq. (28) is equivalent to the utility function of fuzzy max type target in FGP [44], as shown in Fig. 2.

3.2.3. Fuzzy equal t_m

The third case is the "fuzzy equal" type target, which means a non-monotonic preference on an attribute X. In this case, there exist both the left and right tolerances δ^-, δ^+ relative to $t_m \equiv [t_{ml}, t_{mr}]$. We can obtain $t_{\min} = t_{ml} - \delta^-, t_{\max} = t_{mr} + \delta^+$, and $T^{\text{equal}} = (t_{\min}, t_{ml}, t_{mr}, t_{\max})$. Here, the term "equal" is used to represent the fuzzy equal target. Substituting T^{equal} into Eq. (10), we can derive the target achievement function as follows:

$$\Pr(x \cong T^{\text{equal}}) = \begin{cases} \frac{\int_{t_{\min}}^{x}(t-t_{\min})dt}{\int_{t_{\min}}^{t_{ml}}(t-t_{\min})dt}, & t_{\min} \leq x < t_{ml}; \\ 1, & t_{ml} \leq x \leq t_{mr}; \\ 1 - \frac{\int_{t_{mr}}^{x}(t_{\max}-t)dt}{\int_{t_{mr}}^{t_{\max}}(t_{\max}-t)dt}, & t_{mr} < x \leq t_{\max}; \\ 0, & \text{otherwise}. \end{cases} \tag{29}$$

Since $t_m \equiv [t_{ml}, t_{mr}]$ is the reference point, all the attribute values below or upper than t_m will be viewed losses, which indicates a convex shaped function. Especially, when $t_{ml} = t_{mr}$, we have $t_m = t_{ml} = t_{mr}$.

Note that the canonical form of fuzzy numbers is used to represent the fuzzy target in the previous cases, which derives a pdf with a mode of T, thus we can only obtain convex shaped value functions for the fuzzy equal target. As a generalization, if we allow the DM to specify separate possibility distributions for the left and right sides relative to $t_m \equiv [t_{ml}, t_{mr}]$, we can obtain two fuzzy numbers for left and right hands relative to t_m. Since when $x \leq t_m$ the attribute is a pseudo-benefit attribute and when $x \geq t_m$ the attribute is a pseudo-cost attribute, we can derive the target achievement functions for the left and right hands based on the fuzzy min and max targets, respectively. For example, if the DM specifies the fuzzy targets for left and right hands as

$$T_l^{\text{equal}} = (t_{\min}, t_{\min}, t_{\min}, t_{ml}), \text{ left side relative to } t_m;$$

$$T_r^{\text{equal}} = (t_{mr}, t_{mr}, t_{mr}, t_{\max}), \text{ right side relative to } t_m.$$

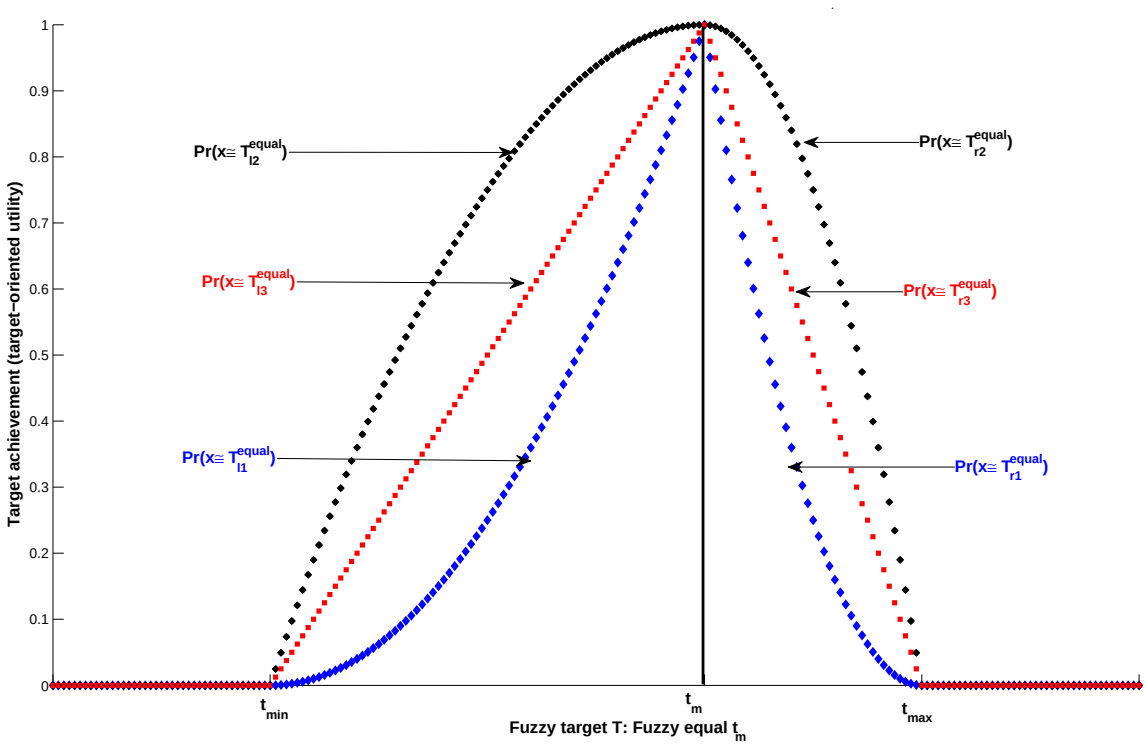


Figure 3: Target achievement function of fuzzy equal target

By using Eqs. (17)-(18) we can derive value functions for left and right hands as follows:

$$\Pr(x \cong T^{\text{equal}}) = \begin{cases} \frac{\int_{t_{\min}}^{x}(t_{ml}-t)dt}{\int_{t_{\min}}^{t_{ml}}(t_{ml}-t)dt}, & t_{\min} \leq x < t_{ml}; \\ 1, & t_{ml} \leq x \leq t_{mr}; \\ 1 - \frac{\int_{t_{mr}}^{x}(t_{\max}-t)dt}{\int_{t_{mr}}^{t_{\max}}(t_{\max}-t)dt}, & t_{mr} < x \leq t_{\max}; \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

We know that T_l^{equal} decreases with t in $[t_{\min}, t_{ml}]$, thus $\Pr(x \cong T_l^{\text{equal}})$ is concave shaped, which indicates that the DM is pessimism oriented in the left side. Similarly, as T_r^{equal} decrease with t in $[t_{mr}, t_{\max}]$, $\Pr(x \cong T_r^{\text{equal}})$ is convex shaped, which incidents that the DM has a optimistic attitude toward the right side with respect to an equal target.

Based on the fuzzy min and fuzzy max targets, for each side, we can obtain three kinds of target achievement functions, as shown in Fig. 3. In Fig. 3, we have assumed that $t_m = t_{ml} = t_{mr}$. In the left side relative to t_m, $\Pr(x \cong T_{l1}^{\text{equal}}), \Pr(x \cong T_{l2}^{\text{equal}}), \Pr(x \cong T_{l3}^{\text{equal}})$ represent the fuzzy optimistic, pessimistic, and neutral attitudes, respectively. Similarly, in the right side relative to t_m, $\Pr(x \cong T_{r1}^{\text{equal}}), \Pr(x \cong T_{r2}^{\text{equal}}), \Pr(x \cong T_{r3}^{\text{equal}})$ represent the fuzzy optimistic, pessimistic, and neutral attitudes, respectively.

3.3. Comparative analysis with related research

In this section, we analyze the main differences of target achievement function between our model and GP, FGP. The GP model has been developed to respond to the DM's desire to satisfy many objective at the same time. Considering only one objective, the achievement function is formulated as

$$\begin{aligned} &\min\ \delta^+ + \delta^- \\ &\text{s.t. } x - t_m = \delta^+ - \delta^-, \end{aligned}$$

where $\delta^-, \delta^+ \geq 0$ are the left and right distance to the crisp target value t_m. The main idea is to use a distance based optimization function $|x - t_m| < \varepsilon$. However, in several application situations the DM is not able to establish exactly the goal value associated with each objective [44]. The FGP has the advantage of allowing for the vague aspirations of a DM, which can be quantified using some natural language or vague phenomena. The FGP can be formulated as follows

$$\begin{array}{lll} \max & \gamma & \\ \text{s.t.} & \gamma \leq 1 - \frac{(x-t_m)}{\delta^+}, & \textbf{or} \quad \text{s.t.} \quad \gamma - \pi_T(x) \leq 0. \\ & \gamma \leq 1 - \frac{(t_m - x)}{\delta^-}. & \end{array}$$

where $\gamma \geq 0$ is an additional continuous variable and δ is the tolerance level specified by the DM or technical process. The main idea behind FGP is using the membership function to represent the DM's utility based on a linear transformation. In fact, FGP is based on the seminal work on fuzzy decision making introduced by Bellman and Zadeh [5]. As pointed out by Beliakov and Warren [4], many researchers, including Zadeh himself, refer to membership functions as 'a kind of utility functions'. We shall call the membership degree based utility as Bellman-Zadeh's paradigm.

Compared with GP, our model relaxes the crisp target to a fuzzy target. Instead of using the distance based approach, we have used the target-oriented utility as the achievement function. We can derive different shaped achievement functions according to a DM's preferences. In this regard, our model provides a support for "probability as a psychological distance" [35].

Compared with Bellman-Zadeh paradigm, our model also utilizes the fuzzy set to capture the impreciseness of the target. The main differences between our model and Bellman-Zadeh's paradigm are twofold.

1. First, the semantics of membership functions are different. Bellman-Zadeh framework views the membership function as 'a kind of utilities', whereas our approach views the membership function as a kind of uncertainty representations, possibility distribution. In fact, according to the context of problems, membership degrees can be interpreted as similarity, preference, or uncertainty [14].
2. Second, the rules governing operations in fuzzy set theory are fairly specific, whereas in our model there are virtually no constraints (other than monotonicity) on how one ought to model the costs of falling short of a target. In our approach, even the same shaped fuzzy number can have more

than one semantic depending on DM's preferences. Whereas, Bellman-Zadeh framework considers only one semantic. For example, the fuzzy min target $T = (t_{\min}, t_m, t_m, t_m)$ for a benefit attribute in Section 3.1, can be viewed as a optimistic fuzzy target. However, the same shaped fuzzy set, e.g., $[t_m, t_{\max}, t_{\max}, t_{\max}]$, is viewed as a fuzzy pessimistic target with respect to a cost attribute. The linear utility function is FGP is equivalent to target-oriented utility derived by the fuzzy neutral target.

In a similar but different framework, Huynh et al. [18, 19] have also considered fuzzy target-oriented decision analysis under uncertainty, in which only payoff variables are considered. They assume a monotonically increasing preference on an attribute, thus their approach cannot model cost and equal targets. Moreover, they assume that a payoff variable is restricted to a bounded domain $[x_{\min}, x_{\max}]$, and then assume that the target is possibly distributed in $[x_{\min}, x_{\max}]$. For example, they defined the fuzzy min target as

$$T^{\text{bene}} = \begin{cases} (t - x_{\min})/(t_m - x_{\min}), & x_{\min} \leq x < t_m; \\ 1, & t_m \leq x \leq x_{\max}. \end{cases}$$

However, such a formulation of the fuzzy min type target is debatable. First, the tolerance level is not necessarily $(t_m - x_{\min})$, it may be provided by the DM or the technical process. Even if we can use $(t_m - x_{\min})$ as the tolerance level, there is no right tolerance level relative to t_m for the fuzzy min type target. Thus we believe that the possibility distribution of fuzzy target T is zero when $x \in (t_m, x_{\max}]$, whereas Huynh et al.'s formulation is one when $t_m \leq x \leq x_{\max}$. In general, Huynh et al have used the target achievement function in FGP to represent the possibility distribution of a target. As t_m is the aspiration value and there is no right tolerance relative to t_m, the $\Pr(x \geq T^{\text{bene}})$ should be 1 if $x \geq t_m$. However, as we see in [19], only if $x \to x_{\max}$, $\Pr(x \geq T^{\text{bene}}) = 1$.

4. Non-additive multi-attribute target-oriented decision analysis based on λ-measure and Choquet integral

In this section, after formulating multi-attribute target-oriented (MATO) decision model based on [36, 37], we prove that MATO decision analysis has a similar structure with discrete fuzzy measure and Choquet integral, especially in the case of mutually stochastic independence among targets. Hence, we propose using discrete fuzzy measure and Choquet integral to model non-additive MATO decision analysis. Moreover, in order to reduce the difficulty of collecting information λ fuzzy measure is applied via a designed bisection search algorithm.

4.1. Formulation of multi-attribute target-oriented function

The consequences in decision making often involve multiple attributes. Suppose a set of N attributes $\mathcal{X} = \{X_1, \ldots, X_n, \ldots, X_N\}$ are of interest and the arbitrary specific levels of a decision d for that attributes set $\mathcal{X}$ are represented by $\mathbf{x} = (x_1, \ldots, x_n, \ldots, x_N)$, denoted as outcome vector. The targets for the attributes

set $\mathcal{X}$ are represented by $\mathbf{T} = (T_1, \ldots, T_n, \ldots, T_N)$. Then the target achievement function for a decision with an outcome vector $\mathbf{x}$ is defined as follows:

$$V(d) = \Pr(\mathbf{x} \succeq \mathbf{T}). \tag{31}$$

If the DM cares only about meeting targets, his utility function should reflect that. Following Bordley and Kirkwood [8], we say a DM is defined to be *target oriented* if his utility for a decision d with an outcome $\mathbf{x} = (x_1, \ldots, x_n, \ldots, x_N)$ depends only on which targets are met by that outcome (i.e., for which $x_n \succeq T_n$). The utility function for a target-oriented DM is completely specified by 2^N constants where these constants are the utilities of achieving specific combinations of the various targets.

Although Bordley and Kirkwood [8] give a general form of MATO function, there is no detailed general representation. For notational convenience, we formulate the general expression of MATO based on Tsetlin and Winkler [36, 37]. Formally, let $\mathbf{I} = (I_1, \ldots, I_n, \ldots, I_N)$ be a vector of indicator variables, where

$$I_n = \begin{cases} 1, & \text{if } x_n \succeq T_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then a target-oriented DM has a function $U_I(\mathbf{I})$ assigning utilities to the 2^N possible values of $\mathbf{I}$. Let $U_I(\mathbf{I}) = \mu_A$, where A is the set of indices $\{n | I_n = 1\}$ corresponding to the attributes in $\mathbf{I}$ for which the targets are met. Without possibility of confusion, the set of indices is used to represent a set of attributes. For example, $U_I(1, 0, \ldots, 0) = \mu_1$, $U_I(0, 1, 1, \ldots, 0) = \mu_{2,3}$ and so on. If $A_1 \subseteq A_2$, then $\mu_{A_1} \leq \mu_{A_2}$; utility can never be reduced by meeting additional targets. We also know that $0 \leq \mu_A \leq 1$ for all A, with $\mu_\varnothing = U_I(0, \ldots, 0, \ldots, 0)$ and $\mu_{1,\ldots,n,\ldots,N} = U_I(1, \ldots, 1, \ldots, 1) = 1$, leaving $2^N - 2$ utilities μ_A to be assessed. Consider a simple example with $N = 2$, we know

$$\begin{aligned} \Pr(\mathbf{x} \succeq \mathbf{T}) &= U_I(\mathbf{I}) \\ &= \mu_\varnothing I_\varnothing + \mu_1 I_1 + \mu_2 I_2 + (1 - \mu_1 - \mu_2) I_1 I_2. \end{aligned}$$

Recall that I_n depends on whether $x_n \succeq T_n$ and $\mu_\varnothing = 0$, thus by integrating out the uncertainty about targets $\mathbf{T}$, we can get

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \mu_1 \mathrm{Pr}_1 + \mu_2 \mathrm{Pr}_2 + (1 - \mu_1 - \mu_2) \mathrm{Pr}_{1,2}, \tag{32}$$

where $\mathrm{Pr}_{1,2}$ is the joint target-oriented utility (joint probability of meeting targets T_1 and T_2), Pr_1 and Pr_2 are the target-oriented utilities of meeting targets T_1 and T_2, respectively. Extending this to N targets, the target-oriented function for a decision d with the outcome $\mathbf{x} = (x_1, \ldots, x_n, \ldots, x_N)$ is as follows

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \sum_{A \subseteq \mathcal{X}} \omega_A \cdot \mathrm{Pr}_{\{n | n \in A\}}, \tag{33}$$

where $\sum_A \omega_A = 1$. The weight ω_A is a linear combination of μ_B terms ($B \subseteq A$), with $\omega_A = \mu_A$ as a special case. In Eq. (32), for example, $\omega_n = \mu_n$ for $n = 1, 2$, but $\omega_{1,2} = \mu_{1,2} - \mu_1 - \mu_2$.

Assessment of 2^N possible μ_A is usually time-consuming and the mutual dependence among targets will lead to complexity and inconvenience in real applications. Thus, Bordley and Kirkwood [8] have applied multi-additive value function to a new product development problem while assuming the mutual independence and additive preference among targets such that

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \sum_{n=1}^{N} \mu_n \cdot \Pr_n. \tag{34}$$

Since positive or negative dependence among targets could occur in the excellent sample, Tsetlin and Winkler [37] have considered the interdependence in multi-attribute target-oriented decision model by means of statistics analysis. They assume targets have some predefined probability distributions (e.g., normal distribution), and then model the interaction among targets using a function of correlations through an example. However, even if, in an objective sense the targets are mutually independent (probabilistically mutually independent), the attributes (targets) are not necessarily considered to be independent from the DM's subjective viewpoint. In this regard, traditional analytic methods are inadequate and not applicable for modeling such complex situations.

4.2. Modeling subjective interdependence among attributes based on fuzzy measure and Choquet integral

The fuzzy measure and Choquet integral (see the appendix part) have been widely applied in MADM problems. One natural question is that whether we can apply them in MATO decision problems. In the sequel, we shall provide an axiomatic approach to interdependent MATO decision model.

Proposition 4.1. *The DM's utility function μ in MATO decision model is a fuzzy measure.*

Proof. The DM's utility function μ in MATO decision function in Eq. (33) satisfies the following axioms of fuzzy measure:

1. boundary, $\mu_\varnothing = 0$ ($\varnothing$ is the empty set) and $\mu_{1,2,\ldots,N} = 1$;
2. and monotonic, if $A_1 \subseteq A_2$, then $\mu_{A_1} \leq \mu_{A_2}$.

Here, A_1 and A_2 are two sets of indices $\{n|I_n = 1\}$ corresponding to the attributes in $\mathbf{I}$ for which the targets are met. Thus, we can model a DM's utility function μ_A over A via the fuzzy measure. □

Proposition 4.2. *The weight information ω_A in Eq. (33) acts as the interaction among targets.*

Proof. Following Eq. (33), we know that $\omega_n = \mu_n (n = 1, 2)$ and $\omega_{1,2} = 1 - \mu_1 - \mu_2$. With three attributes, Eq. (33) becomes

$$\begin{aligned}\Pr(\mathbf{x} \succeq \mathbf{T}) =& \mu_1 \Pr_1 + \mu_2 \Pr_2 + \mu_3 \Pr_3 + \\ & (\mu_{1,2} - \mu_1 - \mu_2)\Pr_{1,2} + (\mu_{1,3} - \mu_1 - \mu_3)\Pr_{1,3} + (\mu_{2,3} - \mu_2 - \mu_3)\Pr_{2,3} + \\ & (1 - \mu_{1,2} - \mu_{1,3} - \mu_{2,3} + \mu_1 + \mu_2 + \mu_3)\Pr_{1,2,3}\end{aligned}$$

which implies that

$$\omega_n = \mu_n (n = 1, 2, 3),$$
$$\omega_{n,l} = \mu_{n,l} - \mu_n - \mu_l (n \neq l, n, l = 1, 2, 3),$$
$$\omega_{1,2,3} = 1 - \mu_{1,2} - \mu_{1,3} - \mu_{2,3} + \mu_1 + \mu_2 + \mu_3.$$

Recursively extending this to N attributes, we can have

$$\omega_A = \sum_{B \subseteq A} (-1)^{|A|-|B|} \cdot \mu_B, A \subseteq \mathcal{X}. \tag{35}$$

Since μ is a fuzzy measure and Eq. (35) is equivalent to Möbius transform of μ, ω_A can be viewed as the interaction index among targets. □

Proposition 4.3. *The MATO decision function in Eq. (33) is linear with respect to the DM's utility function μ.*

Proof. Following Propositions 4.1-4.2, the DM's utility function μ can be expressed in a unique way as

$$\mu_A = \sum_{B \subseteq A} \omega_B, A \subseteq \mathcal{X},$$

which is equivalent to Eq. (.2) in the appendix. The function $\Pr(\mathbf{x} \succeq \mathbf{T})$ is linear with respect to the weight information ω_A [36]. Since conversion formulas between μ and ω are linear, we can obtain another formulation of Eq. (33) as

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \sum_{A \subseteq \mathcal{X}} \mu_A \cdot f_A,$$

where there exist 2^N functions f_A. Therefore, multi-attribute target-oriented function $\Pr(\mathbf{x} \succeq \mathbf{T})$ is linear with respect to the DM's utility function μ. □

As we want to model the mutual dependence among targets from the DM's subjective viewpoint, we assume the set of targets are stochastically mutually independent, but mutually dependent from the DM's subjective viewpoint. Then, the general target-oriented function, Eq. (33), reduces to the following function:

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \sum_{A \subseteq \mathcal{X}} \omega_A \cdot \prod\nolimits_{n \in A} \Pr_n. \tag{36}$$

Proposition 4.4. *Non-additive MATO decision function can be modeled by the Choquet integral while assuming mutually stochastic independence among targets.*

Proof. Propositions 4.1-4.3 are necessary conditions for the Choquet integral, but not sufficient conditions. In fact, the Choquet integral using the Möbius transform in our research context can be expressed by

$$\Pr(\mathbf{x} \succeq \mathbf{T}) = \sum_{A} \omega_A \cdot \inf\nolimits_{n \in A} \Pr_n. \tag{37}$$

In general, the operation inf can be the minimum operation or product operation, see [30]. Since Eq. (36) is a special case of Eq. (33), Eq. (36) satisfies Propositions 4.1-4.3. The function in Eq. (36) is nothing else than the Choquet integral, Eq. (.1), expressed in terms of the Möbius transform. □

Due to the above propositions, for an outcome vector $\mathbf{x} = (x_1, \ldots, x_N)$ with its associated partial target achievements $\mathrm{Pr} = (\mathrm{Pr}_1, \ldots, \mathrm{Pr}_N)$, we are now able to model the interdependence among attributes by means of fuzzy measure and Choquet integral as follows:

$$\mathrm{Pr}(\mathbf{x} \succeq \mathbf{T}) = \sum_{n=1}^{N} [\mathrm{Pr}_{(n)} - \mathrm{Pr}_{(n-1)}] \cdot \mu_{A_{(n)}}, \tag{38}$$

where $(\cdot)$ indicates a permutation of $\mathcal{X}$ such that $\mathrm{Pr}_{(1)} \leq \cdots \leq \mathrm{Pr}_{(n)} \leq \cdots \leq \mathrm{Pr}_{(N)}$, $\mathrm{Pr}_{(0)} = 0$, and $A_{(n)} = \{X_{(n)}, \ldots, X_{(N)}\}$.

4.3. Using λ-measure to induce utility values μ

The use of fuzzy measures requires the values for all subsets in $\mathcal{X}$, which is rather unrealistic to assume that the $2^N - 2$ coefficients can be provided by the DM. Therefore, Sugeno and Terano [34] have incorporated the λ-additive axiom to reduce the difficulty of collecting information. Such a fuzzy measure is referred to as λ-measure, which is a special case of fuzzy measures defined iteratively such that

$$\mu_{A \cup B} = \mu_A + \mu_B + \lambda \mu_A \cdot \mu_B, \tag{39}$$

where $\forall A, B \subseteq \mathcal{X}, A \cap B = \emptyset$. The λ-measure has the following properties.

- If $\lambda < 0$, then $\mu_{A \cup B} < \mu_A + \mu_B$, which represents the substitutive effect between A and B.
- If $\lambda = 0$, then $\mu_{A \cup B} = \mu_A + \mu_B$, which represents the additive effect between A and B.
- If $\lambda > 0$, then $\mu_{A \cup B} > \mu_A + \mu_B$, which represents the multiplicative effect between A and B.

Extending this to N attributes, the lambda fuzzy measure can be formulated as follows [28]:

$$\mu_{\mathcal{X}} = \sum_{n=1}^{N} \mu_n + \lambda \sum_{n=1}^{N-1} \sum_{l=n+1}^{N} \mu_n \cdot \mu_l + \cdots + \lambda^{n-1} \prod_{n=1}^{N} \mu_n + \frac{1}{\lambda} \left[\prod_{n=1}^{N} (1 + \lambda \mu_n) - 1 \right],$$

which can also be denoted as

$$G(\lambda) = \prod_{n=1}^{N} (1 + \lambda \cdot \mu_n) - \lambda - 1,$$

where $-1 \leq \lambda < \infty$ and μ_n is used to denote the fuzzy measure with respect to a singleton attribute set $\{X_n\}$. Since the boundary conditions $\mu_{\mathcal{X}} = 1$, the parameter λ can be uniquely determined by

$$\lambda + 1 = \prod_{n=1}^{N} (1 + \lambda \cdot \mu_n).$$

Particularly, we assume the importance weights for the attributes set are given by $\mathbf{W} = (W_1, \ldots, W_n, \ldots, W_N)$ such that $\sum_{n=1}^{N} W_n = 1$ and $W_n \geq 0$. The DM can also provide a λ-value to represent his subjective viewpoint. Since $\prod_{n=1}^{N} (1 + \lambda \cdot \mu_n)$ is a convex function of λ, $G(\lambda)$ is also a convex function [28]. Thus, given a set of fuzzy measures $\mu_n (n = 1, \ldots, N)$ with respect to singleton attribute sets $\{X_n\}(n = 1, \ldots, N)$, there exists only one λ value.

With the λ-value and original weight vector $\mathbf{W}$, we have proposed a bisection search method to find $\mu(\mathcal{X}) = 1$, the pseudocode is shown in Fig. 4. This method is used to identify the fuzzy measures with respect to singleton attribute sets while satisfying $\mu_{\mathcal{X}} = 1$. Due to the boundary condition of fuzzy measure, we first normalize the importance weight in order to derive the initial fuzzy measures with respect to singleton attribute sets such that

$$\mu_n = \frac{W_n}{\max_{n=1,\cdots,N}\{W_n\}}. \tag{40}$$

Moreover, we define a variable $\kappa \in (0, 1]$ to adjust the derived fuzzy measures μ proportionally. If we can find a κ value satisfying $\mu_{\mathcal{X}} = 1$, then the new fuzzy measure with respect to a singleton attribute set is $\kappa \cdot \mu_n$. At the initial step, κ is initialized to be 0.5, the lower and upper variables are set to be $\texttt{lower} = 0$ and $\texttt{upper} = 1$, respectively. Using the new fuzzy measures $\mu_n \leftarrow \kappa \cdot \mu_n$, we then proceed as follows:

- if $\mu_{\mathcal{X}} = 1$, then κ is the final adjustment parameter;
- if $\mu_{\mathcal{X}} > 1$, we set $\texttt{upper} \leftarrow \kappa$, $\kappa \leftarrow (\texttt{lower} + \kappa)/2$, respectively;
- if $\mu_{\mathcal{X}} < 1$, we set $\texttt{lower} \leftarrow \kappa$ and $\kappa \leftarrow (\kappa + \texttt{upper})/2$, respectively.

The algorithm will proceed iteratively until a parameter κ exists while satisfying $\mu(\mathcal{X}) = 1$. Note that normalizing the original weight vector in Eq. (40) makes $\kappa \in (0, 1]$. The case where $\kappa = 1$ only exists when $\lambda = -1$. By Eq. (40) we know that there is a fuzzy measure $\mu_{n^*} = 1$ with respect to a singleton attribute set $\{X_{n^*}\}$. Therefore we have

$$\mu_{\mathcal{X}} = \mu_{n^*} + \mu_A + \lambda \mu_{n^*} \cdot \mu_A,$$

where $A \cup X_{n^*} = \mathcal{X}$ and $A \cap \{X_{n^*}\} = \emptyset$. Since $\mu_{\mathcal{X}} = 1, \mu > 0, \mu_{n^*} = 1$, it is easily seen that $\lambda = -1$. Also, the main idea of our algorithm is based on the single solution of λ identification. Fig. 4 shows the binary search method with a complexity of $O(\log K)$.

Example 1. Assume a set of four attributes $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$ are of interest, the weigh vector for that attributes set is $\mathbf{W} = (0.2, 0.3, 0.1, 0.4)$, and λ is set to be 1.5, which represents multiplicative effect among the four attributes. First, we normalize the original weight vector as $(0.5, 0.75, 0.25, 1)$. Second, by means of the algorithm in Fig. 4, we found it takes 12 iterations to find a κ satisfying $\mu_{\mathcal{X}} = 1$, as shown in Table 1. Note that, the parameter κ and $\mu_{\mathcal{X}}$ are after rounding operation in our computer programming.

Input: Importance weights $\mathbf{W}$ and λ value.

Output: A set of fuzzy measures $\mu_n(n = 1, \ldots, N)$ with respect to singleton attribute sets $\{X_n\}(n = 1, \ldots, N)$.

1: Normalize weights such that $\mu_n = \frac{W_n}{\max_{n=1,\cdots,N}\{W_n\}}$.
2: Initialize $\texttt{lower} = 0$, $\kappa = 0.5$, $\texttt{upper} = 1$
3: Specify $\mu_n \leftarrow \kappa \cdot \mu_n$
4: **for** $2 \leq n \leq N$ **do**
5: $\mu_{A_n} \leftarrow \mu_{A_{n-1}} + \mu_n + \lambda\mu_{A_{n-1}} \cdot \mu_n (n = 2, \cdots, N)$, where $A_n = \{X_1, X_2, \cdots, X_n\}$
6: **if** $\mu_{A_n} > 1$ **then**
7: $\texttt{upper} \leftarrow \kappa$
8: $\kappa \leftarrow (\texttt{lower} + \kappa)/2$
9: go to 3
10: **else**
11: continue
12: **end if**
13: **end for**
14: **if** $\mu_{A_N} < 1$ **then**
15: $\texttt{lower} \leftarrow \kappa$
16: $\kappa \leftarrow (\kappa + \texttt{upper})/2$
17: go to 3
18: **else if** $\mu_{A_N} = 1$ **then**
19: $\mu_n \leftarrow \kappa \cdot \mu_n$
20: **end if**
21: **return** $\mu_n(n = 1, \cdots, N)$

Figure 4: A bisection search method to find $\mu_{\mathcal{X}} = 1$

In summary, with N attributes and N targets, for the partial target achievements $\mathrm{Pr} = (\mathrm{Pr}_1, \ldots, \mathrm{Pr}_N)$ of an outcome $\mathbf{x} = (x_1, \ldots, x_N)$, we proceed as follows:

- Use λ-fuzzy measure to express the fuzzy measures of each individual attributes group.
  1. Specify a λ value.
  2. Identify the fuzzy measures of individual attributes group with a given λ value according to the algorithm in Fig. 4.
- Use Eq. (38) to obtain the overall target achievements by fuzzy measure and Choquet integral.

Table 1: Iteration list of finding κ while satisfying $\mu_{\mathcal{X}} = 1$

Iteration times	κ	$\mu_{\mathcal{X}}$	Iteration times	κ	$\mu_{\mathcal{X}}$
Loop 1	0.5	2.3098	Loop 7	0.2734	0.9666
Loop 2	0.25	0.8588	Loop 8	0.2773	0.9851
Loop 3	0.375	1.4979	Loop 9	0.2793	0.9944
Loop 4	0.3125	1.1583	Loop 10	0.2803	0.9991
Loop 5	0.2813	1.0038	Loop 11	0.2808	1.001
Loop 6	0.2656	0.9301	Loop 12	0.2805	1.0

5. An illustrative example–New product development

5.1. Problem descriptions

Over the past decades, the integrated circuit industry has gone through a cycle of birth, explosive growth, and currently, has moved into a phase of severe competition. A well-known Silicon Valley company, with an established reputation for producing high quality manufacturing, test, and control equipment, had developed a technical breakthrough that, they felt, would give them a significant cost advantage in manufacturing test equipment for very large scale integrated circuits. The company wanted to assess how prospective customers would evaluate a proposed new tester for very large-scale integrated circuits. To do so, following a review of the technical literature and several meetings with technical and marketing staff in the company, they identified four categories of evaluation criteria (technical, economic, software, and vendor support) with a total of 17 evaluation attributes, as shown in Column 1 of Table 2. Both the within-category weights and the category weights (which are 0.52, 0.14, 0.32, and 0.02) are showed in the third column of the table, given by Keeney and Lilien [23] [3]. The preference monotonicity for each evaluation attribute is shown in Column 3 of the table, and the performance scores the evaluation attributes are shown in Columns $4-6$ of the table for the proposed new tester `OR 9000` and its two competitors: the `J941` and the `Sentry 50`.

5.2. Previous research

Keeney and Lilien [23] assessed the measurable value function for a lead user at a primary customer company for this testing equipment. This lead user first assessed a minimum acceptability level and a maximum desirability level for each evaluation attribute. Keeney and Lilien then confirmed that the user's preferences were describable by a weighted additive measurable value function, and they assessed a single

[3] Instead of using the absolute within-category weights given by Keeney and Lilien [23], we use the relative within-category weights. However, this does not change the evaluation ranking results since it is a proportional transformation. For example, the weight of evaluation attribute X_{11} used by Keeney and Lilien is 15, we use 15/100 to represent the weight information of attribute X_{11}.

Table 2: New Product Development: Data

Evaluation attribute	Weight	Monotonicity	Tester ratings		
			`OR 9000`	`J941`	`Sentry 50`
Technical X_1	0.52				
Pin capacity X_{11}	0.15	Increasing	160	96	256
Vector depth X_{12}	0.20	Increasing	0.128	0.256	0.064
Data rate X_{13}	0.10	Increasing	50	20	50
Timing accuracy X_{14}	0.35	Decreasing	1,000	1,000	600
Pin capacitance X_{15}	0.10	Decreasing	55	50	40
Programmable measurement units X_{16}	0.10	Increasing	8	2	4
Economic X_2	0.14				
Price X_{21}	0.50	Decreasing	1.4	1	2.8
Uptime X_{22}	0.20	Increasing	98	95	95
Delivery time X_{23}	0.30	Decreasing	3	6	6
Software X_3	0.32				
Software translator X_{31}	0.15	Increasing	90	90	90
Networking: Communications X_{32}	0.20	Increasing	1	1	1
Networking: Open X_{33}	0.20	Increasing	1	0	0
Development time X_{34}	0.30	Decreasing	3	4	4
Data analysis software X_{35}	0.15	Increasing	1	1	1
Vendor support X_4	0.02				
Vendor service X_{41}	0.30	Decreasing	2	4.75	6
Vendor performance X_{42}	0.30	Decreasing	4	4	4
Customer applications X_{43}	0.40	Increasing	1	1	1

dimensional value function except for the attributes $X_{32}, X_{33}, X_{35}, X_{43}$, as shown in the third column of Table 3. For example, the mathematically measurable value function for the evaluation attribute X_{16} was finally expressed as $1.309[1 - \exp(0.1203(4 - x))]$. With the weights of attributes in Table 2, the assessed weighted additive measurable value function was then used to evaluate the `OR 9000` against its two competitors (the `J941` and `Sentry 50`), the results were obtained as

$$V(\texttt{Sentry 50}) = 0.154 > V(\texttt{OR 9000}) = 0.133 > V(\texttt{J941}) = -0.180,$$

and served as input to determine that the `OR 9000` was competitive enough to `J941`, but not competitive to `Sentry 50`.

Table 3: New product development: Existing research

Attribute	Keeney and Lilien [23]				Bordley and Kirkwood [8]			
	Value function	Scaled values			Target	Target achievements		
		OR 9000	J941	Sentry 50		OR 9000	J941	Sentry 50
X_1								
X_{11}	$1.929[1-\exp(0.0065(144-x))]$	0.191	-0.706	0.998	256	0	0	1
X_{12}	$-0.9736\mathrm{E}-09*[1-\exp(6.917(x-1))]$	0.0	0.0	0.0	0.256	0	1	0
X_{13}	$-0.3091[1-\exp(0.02406(x-40))]$	0.084	-0.118	0.084	50	1	0	1
X_{14}	$(500-x)/250$	-2.0	-2.0	-0.4	600	0	0	1
X_{15}	$(100-x)/70$	0.643	0.714	0.85	40	0	0	1
X_{16}	$1.309[1-\exp(0.1203(4-x))]$	0.5	-0.356	0.0	4	1	0	1
X_2								
X_{21}	$2.5-x$	1.1	1.5	-0.3	1	0	1	0
X_{22}	$(x-98)/2$	0.0	-1.5	-1.5	95	1	1	1
X_{23}	$(6-x)/2$	1.5	0.0	0.0	6	1	1	1
X_3								
X_{31}	$2.768[1-\exp(0.00498(10-x))$	0.91	0.91	0.91	90	1	1	1
X_{32}	Use the original data	1.0	1.0	1.0	1	1	1	1
X_{33}	Use the original data	1.0	0.0	0.0	0	1	1	1
X_{34}	$(4-x)/2$	0.5	0.0	0.0	4	1	1	1
X_{35}	Use the original data	1.0	1.0	1.0	1	1	1	1
X_4								
X_{41}	$-0.3091[1-\exp(0.4811(4-x))]$	0.5	-0.094	-0.191	4.75	1	1	0
X_{42}	$-0.3091[1-\exp(0.4811(4-x))]$	0.0	0.0	0.0	4	1	1	1
X_{43}	Use the original data	1.0	1.0	1.0	1	1	1	1
		0.133	-0.180	0.154		0.514	0.584	0.820
		Overall Value				Overall Value		

Keeney and Lilien's approach is a bit complex in practice since users have to build mathematically rigorous value functions based on attributes. For this decision, it is natural to think in terms of performance targets because the explicit purpose of the analysis was to determine whether the `OR 9000` was attractive against its two competitors (the `J941` and the `Sentry 50`) or not. Thus, *the performance of these two testers*

(the J941 and the Sentry 50) sets targets against which the OR 9000 is judged. Bordley and Kirkwood [8] used the performance targets to valuate the multi-attribute analysis by specifying a crisp target for each evaluation attribute with the following functions

$$T_n = \max\{x_n(\texttt{J941}), x_n(\texttt{Sentry 50})\}, \text{ for benefit attributes;}$$

$$T_n = \min\{x_n(\texttt{J941}), x_n(\texttt{Sentry 50})\}, \text{ for cost attributes.}$$

where $x_n(\cdot)$ is used to denote the performance value of a tester (decision alternative) on the evaluation attribute X_n. The targets of different evaluation attributes are showed in the sixth column of Table 3. The target achievement function for an attribute with an increasing preference is then defined as

$$\Pr_n(\cdot) = \begin{cases} 1, & \texttt{if } x_n(\cdot) \geq T_n; \\ 0, & \texttt{otherwise}. \end{cases}$$

i.e., the more the better; whereas the target achievement function with respect to an attribute with a decreasing preference has a contrary function, i.e., the less the better. With the performance scores, we can only obtain binary partial target achievements (0 or 1) of those three testers, as shown in Columns 7-9 of Table 3. For the sake of simplicity, Bordley and Kirkwood also used the weighted additive function to obtain the overall values without further considering the interdependence among targets. The results were obtained as

$$V(\texttt{Sentry 50}) = 0.820 > V(\texttt{J941}) = 0.584 > V(\texttt{OR 9000}) = 0.514,$$

it is obvious that OR 9000 was *not* competitive *at all* against its two competitors (the J941 and the Sentry 50). Also, the ranking result generated by Bordley and Kirkwood's approach is inconsistent with the one by Keeney and Lilien's approach. However, it may be natural to use Bordley and Kirkwood's approach in practice, since it is quite natural for people to consider targets of the testers. Moreover, Bordley and Kirkwood's approach is easy of use in practice since users do not need to define/specify the complex value functions based on attributes.

Target-oriented decision model assumes there exists some uncertainty of the target. In our example, there is no random uncertainty about the performance targets, therefore Bordley and Kirkwood have defined a crisp target for each evaluation attribute. However, there exists some fuzzy uncertainty about the performance targets. For example, the performance scores of J941 and Sentry 50 with respect to the attribute X_{11} (Pin capacity) are $x_{11}(\texttt{J941}) = 96$ and $x_{11}(\texttt{Sentry 50}) = 256$, respectively. Since X_{11} is a benefit attribute, Bordley and Kirkwood set 256 as the performance target for attribute X_{11}. Recall that the performance of these two testers sets targets against which the OR 9000 is judged. If 256 is set to be the target for X_{11}, how about 96 or other possible values in $(96, 256)$? This observation leads us to use fuzzy targets in our illustrative example. In addition, the above two approaches have used the weighted additive function to obtain a global value function for each tester. As discussed in Section 4, even if, in an objective

sense the targets are mutually independent (probabilistically mutually independent), the attributes (targets) are not necessarily considered to be independent from the DM's subjective viewpoint. In this sense, it may be natural and convenient to use our model to capture the non-additive behaviors among targets.

5.3. Non-additive multi-attribute fuzzy target-oriented decision analysis

In this section, we shall show how to use our non-additive multi-attribute fuzzy target-oriented decision model to assess how prospective customers would evaluate a proposed new tester for very large-scale integrated circuits.

5.3.1. Inclusion of fuzzy targets into the new product development

We first obtain the minimal and maximal values for each attribute according to performance scores of `J941` and `Sentry 50` as

$$t_n^{\min} = \min\{x_n(\texttt{J941}), x_n(\texttt{Sentry 50})\},$$
$$t_n^{\max} = \max\{x_n(\texttt{J941}), x_n(\texttt{Sentry 50})\}.$$

Such functions are based on the idea that *the performance of these two testers (the `J941` and the `Sentry 50`) sets targets against which the `OR 9000` is judged*. As discussed in Section 3, we can build three types of possibility distributions for benefit and cost attributes. By assuming that the company is optimism oriented, we can induce the following fuzzy targets

$$T_n^{\text{optim}} = \left(t_n^{\min}, t_n^{\max}, t_n^{\max}, t_n^{\max}\right), \text{ for benefit attributes;}$$
$$T_n^{\text{optim}} = \left(t_n^{\min}, t_n^{\min}, t_n^{\min}, t_n^{\max}\right), \text{ for cost attributes.}$$

where $(\cdot,\cdot,\cdot,\cdot)$ is used to represent a trapezoidal fuzzy number. Note that if $t_n^{\min} = t_n^{\max}$, we can only obtain a crisp target for attribute X_n. In this case, our approach is equivalent to Bordley and Kirkwood's approach. The derived fuzzy optimistic targets with respect to the 17 attributes are showed in Column 3 of Table 4. With the performance data of the three testers, we can obtain partial target achievements for benefit and cost attributes via Eqs. (17)-(18), as shown in Columns 4-6 of Table 4, respectively. There are three different partial values between our approach and Bordley and Kirkwood's approach. Taking the attribute X_{11} as an example, the crisp target defined by Bordley and Kirkwood is 256. It is clearly that $x_{11}(\texttt{J941}) < x_{11}(\texttt{OR 9000}) < x_{11}(\texttt{Sentry 50}) = 256$, thus `OR 9000` performs better than `J941`, but worse than `Sentry 50` regarding X_{11}. However, according to Bordley and Kirkwood's approach (Table 3), we know that there is no difference between `OR 9000` and `J941` regarding the attribute X_{11}.

If the company is neutral or pessimism oriented, we can also build its fuzzy targets as follows:

	Benefit attribute	Cost attribute
Fuzzy neutral: T_n^{neut}	$\left(t_n^{\min}, t_n^{\min}, t_n^{\max}, t_n^{\max}\right)$	$\left(t_n^{\min}, t_n^{\min}, t_n^{\max}, t_n^{\max}\right)$
Fuzzy pessimistic: T_n^{pess}	$\left(t_n^{\min}, t_n^{\min}, t_n^{\min}, t_n^{\max}\right)$	$\left(t_n^{\min}, t_n^{\max}, t_n^{\max}, t_n^{\max}\right)$

Table 4: New product development: Fuzzy target-oriented decision analysis

Criteria	Attribute	Fuzzy optimistic target: T_n^{optim}	Target achievements		
			`OR 9000`	`J941`	`Sentry 50`
X_1	X_{11}	$(96, 256, 256)$	**0.16**	0	1
	X_{12}	$(0.064, 0.256, 0.256)$	**0.1111**	1	0
	X_{13}	$(20, 50, 50)$	1	0	1
	X_{14}	$(600, 600, 1000)$	0	0	1
	X_{15}	$(40, 40, 50)$	0	0	1
	X_{16}	$(2, 4, 4)$	1	0	1
X_2	X_{21}	$(1, 1, 2.8)$	**0.6049**	1	0
	X_{22}	95	1	1	1
	X_{23}	6	1	1	1
X_3	X_{31}	90	1	1	1
	X_{32}	1	1	1	1
	X_{33}	0	1	1	1
	X_{34}	4	1	1	1
	X_{35}	1	1	1	1
X_4	X_{41}	$(4.75, 4.75, 6)$	1	1	0
	X_{42}	4	1	1	1
	X_{43}	1	1	1	1

Since the partial target achievements of `J941` and `Sentry 50` are either 1 or 0, we only consider `OR 9000`. Also, the performance scores of the `OR 9000` are either outside the target range or the targets are crisp values except for the three attributes X_{11}, X_{12}, X_{21}, thus for the fuzzy neutral and fuzzy pessimistic targets we only list the target achievements of `OR 9000` with respect to these three attributes as follows:

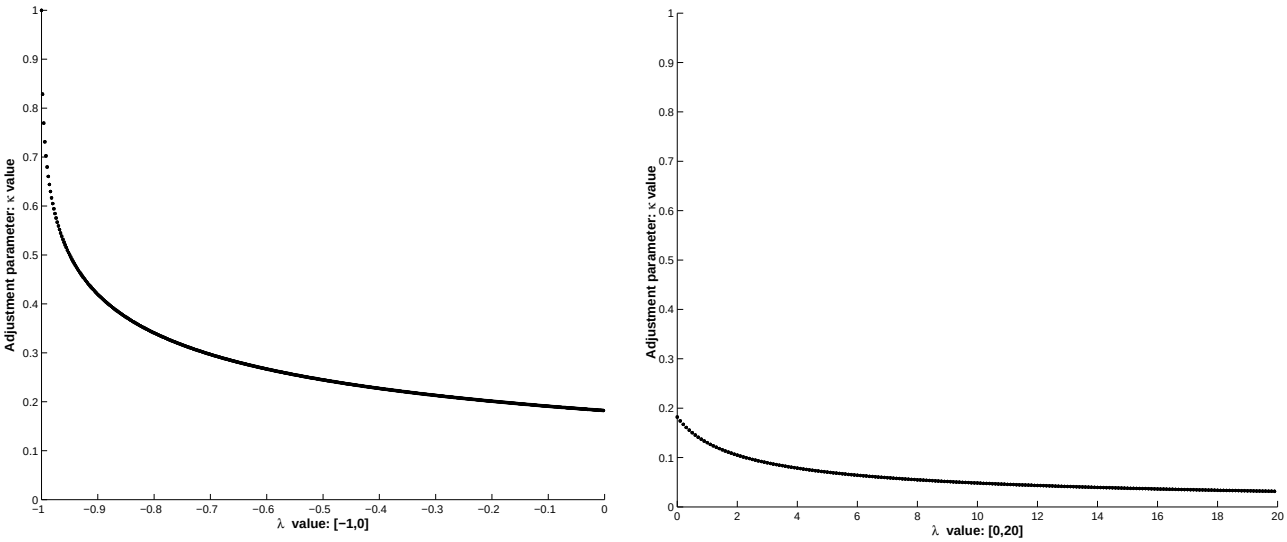
$$\Pr_{11}^{\text{neut}}(\texttt{OR 9000}) = 0.40, \quad \Pr_{12}^{\text{neut}}(\texttt{OR 9000}) = 0.3333, \quad \Pr_{21}^{\text{neut}}(\texttt{OR 9000}) = 0.7778;$$

$$\Pr_{11}^{\text{pess}}(\texttt{OR 9000}) = 0.64, \quad \Pr_{12}^{\text{pess}}(\texttt{OR 9000}) = 0.5556, \quad \Pr_{21}^{\text{pess}}(\texttt{OR 9000}) = 0.9506.$$

It is obvious that different attitudes will lead to different target achievements with respect to X_{11}, X_{12}, X_{21}.

5.3.2. Non-additive aggregation

We now consider the non-additive aggregation by means of fuzzy measure and Choquet integral. The weight vector for the 17 attributes in Column 2 of Table 2 is first normalized into initial fuzzy measures with respect to singleton attribute sets via Eq. (40). Given a λ value, we can find the adjustment parameter



(a) $-1 \leq \lambda \leq 0$

(b) $0 \leq \lambda \leq 20$

Figure 5: Values of the adjustment parameter κ with respect to different λ values

κ satisfying $\mu_{\mathcal{X}} = 1$ and derive the fuzzy measures with respect to singleton attribute sets under λ. The λ-additive axiom in Eq. (39) is then used to induce the fuzzy measures with respect to different attribute sets. The research sets the λ value ranging from -1 to 20. Also we know that

- if $-1 \leq \lambda < 0$, there are substitutive effects among the attributes (targets);
- if $\lambda = 0$, there is no interdependence among the attributes (targets);
- if $0 < \lambda \leq 20$, there is multiplicative effects among the attributes (targets).

Fig. 5 shows the values of the adjustment parameter κ with respect to different lambda values ranging from -1 to 20.

For example, we assume λ is set to be -0.5, which means that the company prefers substitutive effect among the attributes. According to the algorithm in Fig. 4, the adjustment parameter is $\kappa = 0.2447$, and the final fuzzy measures with respect to the singleton attribute sets are shown as follows:

$$X_1: \ \mu_{11} = 0.1049, \mu_{12} = 0.1398, \mu_{13} = 0.0699, \mu_{14} = 0.2447, \mu_{15} = 0.0699, \mu_{16} = 0.0699$$

$$X_2: \ \mu_{21} = 0.0941, \mu_{22} = 0.0376, \mu_{23} = 0.0565$$

$$X_3: \ \mu_{31} = 0.0645, \mu_{32} = 0.086, \mu_{33} = 0.086, \mu_{34} = 0.1291, \mu_{35} = 0.0645$$

$$X_4: \ \mu_{41} = 0.0081, \mu_{42} = 0.0081, \mu_{43} = 0.0108.$$

The fuzzy measure on attributes set $\{X_{11}, X_{12}\}$ is induced as

$$\mu_{\{X_{11},X_{12}\}} = \mu_{11} + \mu_{12} - 0.5\mu_{11}\mu_{12} = 0.2374.$$

Recursively using the λ-additive axiom, we are able to induce the fuzzy measures on different attributes set. If the company is fuzzy optimism oriented, we can obtain the global target achievements via Choquet integral as

$$V(\texttt{Sentry 50}) = 0.8672 > V(\texttt{J941}) = 0.6621 > V(\texttt{OR 9000}) = 0.6553,$$

which is consistent with Bordley and Kirkwood's result that `OR 9000` was *not* competitive *at all* against its two competitors (the `J941` and the `Sentry 50`).

Since individual target achievements of `J941` and `Sentry 50` are either 1 or 0 regardless of the company's attitude, we plot the global target achievements of `OR 9000` with different attitude targets (fuzzy optimistic, fuzzy neutral, and fuzzy pessimistic targets) and the ones of `J941` and `Sentry 50` in one figure. Also, since the λ value is not symmetric, we divide λ values into two domains: $[-1, 0]$ and $[0, 20]$. Fig. 6 shows the non-additive aggregation results with respect to fuzzy optimistic, fuzzy neutral, and fuzzy pessimist targets, with λ values ranging from -1 to 20. Looking at Fig. 6, it is known that the global target achievements of `Sentry 50` and `J941` remain the same regardless of the company's attitudes, under a given *lambda* value. In addition, when the compony is optimism oriented, the global target achievement of `OR 9000` is indexed by `OR 9000` (Optimistic target), in which the ranking of the three testers has three cases:

- when $-1 \leq \lambda < \lambda_1 \approx 0.9$, the ranking is $\texttt{Sentry 50} \succ \texttt{J941} \succ \texttt{OR 9000}$;
- when $\lambda = 0.9$, the ranking is $\texttt{Sentry 50} \succ \texttt{J941} \sim \texttt{OR 9000}$;
- when $0.9 < \lambda \leq 20$, the ranking is $\texttt{Sentry 50} \succ \texttt{OR 9000} \succ \texttt{J941}$.

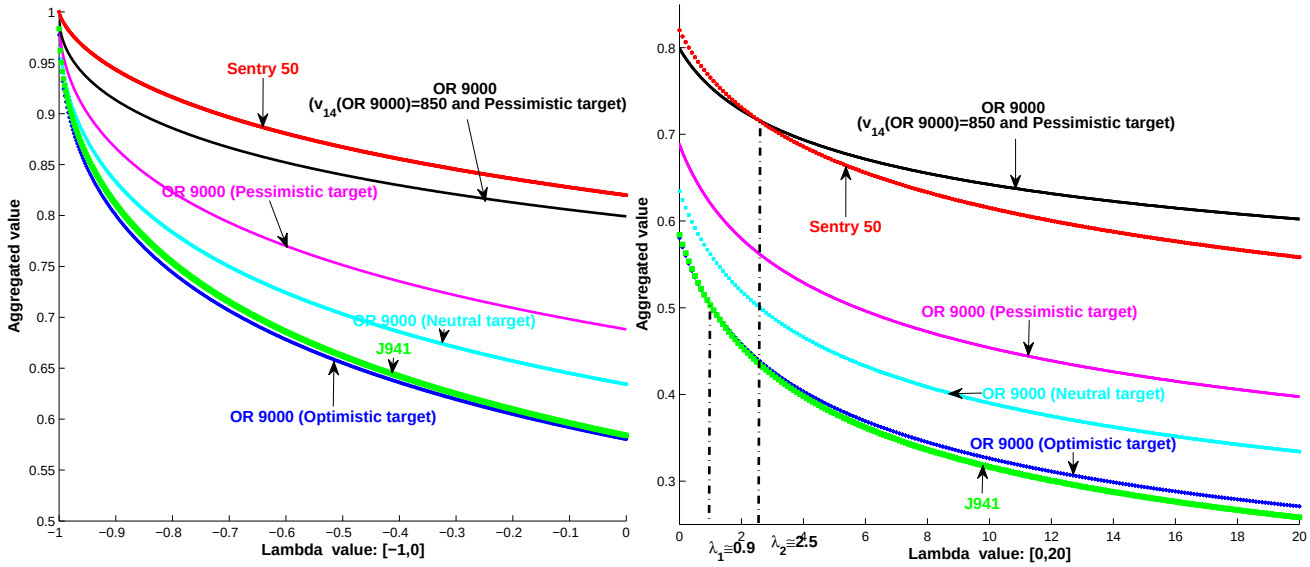
which indicates that `OR 9000` is not competitive enough to the market. In addition, when the company is neutral or pessimism oriented, we always obtain the ranking as $\texttt{Sentry 50} \succ \texttt{OR 9000} \succ \texttt{J941}$, which is consistent with Keeney and Lilien's ranking.

5.3.3. Changing performance score of OR 9000 on attribute X_{14}

The original performance scores on X_{14} of the three testers are $x_{14}(\texttt{OR 9000}) = 1000, x_{14}(\texttt{J941}) = 1000, x_{14}(\texttt{Sentry 50}) = 600$, respectively. We now consider a variation by replacing the performance score 1000 of `OR 9000` on X_{14} with 850. Using Keeney and Lilien's approach, the final ranking is

$$V(\texttt{OR 9000}) = 0.243 > V(\texttt{Sentry 50}) = 0.154 > V(\texttt{J941}) = -0.180,$$

which indicates that `OR 9000` is competitive enough to its two competitors. The main reason is that the varied score of $x_{14}(\texttt{OR 9000})$ has a higher scaled value and X_{14} is the most important attribute among the 17 ones.



(a) $-1 \leq \lambda \leq 0$

(b) $0 \leq \lambda \leq 20$

Figure 6: Aggregation values of three testers with different targets and λ values

However, by using Bordley and Kirkwood's approach, the final ranking is still $V(\texttt{Sentry 50}) = 0.820 > V(\texttt{J941}) = 0.584 > V(\texttt{OR 9000}) = 0.514$, which is quite inconsistent with Keeney and Lilien's approach. The main reason is that the evaluation attribute X_{14} is a cost attribute and $850 > 600$, the target achievement of `OR 9000` on X_{14} is 0 by Bordley and Kirkwood's approach, which is the same as the result by the original performance score.

Using our model, if we assume the company is pessimism oriented, we will obtain target achievement as $\Pr_{14}^{\text{pess}}(\texttt{OR 9000}) = 0.609$. Since $x_{14}(\texttt{Sentry 50}) < 850 < x_{14}(\texttt{J941})$, and `Sentry 50`, `J941` set the performance targets, `OR 9000` should be better than `J941` but worse than `Sentry 50` regarding X_{14}. The non-additive aggregation results with respect to different λ values are shown in Fig. 6, indexed by `OR 9000` ($x_{14}(\texttt{OR 9000}) = 850$ and pessimistic target). It is clearly seen that

- when $-1 \leq \lambda < 2.5$, $\texttt{Sentry 50} \succ \texttt{OR 9000} \succ \texttt{J941}$;
- when $\lambda = 2.5$, the ranking is $\texttt{Sentry 50} \sim \texttt{OR 9000} \succ \texttt{J941}$;
- when $\lambda > 2.5$, the ranking is $\texttt{OR 9000} \succ \texttt{Sentry 50} \succ \texttt{J941}$.

Thus, if the company prefers strong multiplicative effects among targets, `OR 9000` is competitive enough against the market.

5.4. Summary

Keeney and Lilien's multi-attribute value analysis is a bit complex is practice since the users have to build their mathematically rigorous value functions based on attributes, which is a quite difficult task. Such a task may create an obstacle for users in company to use multi-attribute analysis in the new product development problem. Since the explicit purpose of the analysis was to determine whether the `OR 9000` was attractive against its two competitors (the `J941` and the `Sentry 50`) or not, *the performance of these two testers (the `J941` and the `Sentry 50`) sets targets against which the `OR 9000` is judged*. It becomes natural to think of targets in this example. There is no random uncertainty about the performance targets, therefore Bordley and Kirkwood have defined a crisp target for each evaluation attribute. However, there exists some fuzzy uncertainty about the performance targets, which is a critical drawback of Bordley and Kirkwood's approach. As illustrated in Section 5.3, on one hand, our model can reduce the burden of defining mathematically rigorous value functions based on attributes; on the other hand, it can capture the fuzzy uncertainties of targets as well as represent different decision attitudes of the company.

Furthermore, both Keeney and Lilien [23] and Bordley and Kirkwood [8] used the multi-additive value function without further considering the mutual dependence among targets. However, they are not necessarily considered to be independent from the company's subjective viewpoint. The λ-fuzzy measure and Choquet integral are used to model non-additive multi-attribute target-oriented function in our research. As illustrated in Section 5.3, our model provides a way to model the mutually dependence among targets/attributes from company's subjective viewpoint.

6. Concluding remarks and future work

In this paper, we have first extended the basic (random) target-oriented decision model to involve three types of targets. In order to allow the DM to specify imprecise aspiration levels, fuzzy target-oriented decision analysis has been then formulated to model three types of fuzzy targets: fuzzy min, fuzzy max, and fuzzy equal. Also, different attitudes have been used to derive target achievement functions. Our fuzzy target-oriented model also provides some relationships with the GP and FGP models such that the derived target achievement functions can be viewed as a support for "probability as psychological distance" proposed by [35]. The Bellman-Zadeh paradigm [5] based fuzzy decision making utilizes the membership function as "a kind of utilities", in which the rules governing operations are fairly specific. Our approach allows the DM to model the target achievements with virtually no constraints (other than monotonicity), in which the membership functions of fuzzy sets are interpreted as possibility distributions. Finally, we have shown a similar structure between MATO decision model and non-additive Choquet integral. Thus, the fuzzy measure can be used to induce the possible combinations of indices of meeting targets and fuzzy integral

is used to model the non-additive multi-attribute target-oriented model. Particularly, λ fuzzy measure is applied to reduce the difficulty of collecting information via a designed bisection search algorithm.

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Discrete fuzzy measure and Choquet integral

Given a finite attributes set $\mathcal{X}$, the power set $\mathrm{P}(\mathcal{X})$ is a class of all of the subsets of $\mathcal{X}$. The discrete fuzzy measure[4] is defined as follows [33]:

Definition 1. A discrete fuzzy measure on $\mathcal{X}$ is a set function $\mu : \mathrm{P}(\mathcal{X}) \to [0, 1]$ satisfying the following conditions:

- *Axiom 1: boundary conditions*, $\mu_\emptyset = 0$ ($\emptyset$ is the empty set) and $\mu_\mathcal{X} = 1$;
- *Axiom 2: monotonic*, if $A \subseteq B$, then $\mu_A \leq \mu_B$, $\forall A, B \in \mathrm{P}(\mathcal{X})$.

For each subset of the attributes $A \subseteq \mathcal{X}$, μ_A can then be interpreted as the weight or the importance of the coalition A. The monotonicity of μ means that the weight of a subset of the attributes can only increase when one adds new attributes to it. For all $A, B \subseteq \mathcal{X}, A \cap B = \emptyset$, the discrete fuzzy measure is further said to be [30]:

1. *additive* whenever $\mu_{A\cup B} = \mu_A + \mu_B$;
2. *multiplicative* whenever $\mu_{A\cup B} > \mu_A + \mu_B$;
3. *substitutive* whenever $\mu_{A\cup B} < \mu_A + \mu_B$.

When using a fuzzy measure to model the importance of each subset of attributes, a suitable aggregation operator is the discrete Choquet integral [16], which is defined as follows [30]:

[4]Fuzzy measure can be continuous and discrete.

Definition 2. Let μ be a discrete fuzzy measure on $\mathcal{X}$ and h be a positive real-valued function from $\mathcal{X}$ to $[0,1]$, where $\mathcal{X}$ is a finite attributes set. The Choquet integral C_μ of h with respect to μ (in the discrete case), is defined by

$$\begin{aligned} C_\mu(h) &= \sum_{n=1}^{N} h(X_{(n)}) \cdot \left[\mu_{A_{(n)}} - \mu_{A_{(n+1)}}\right] \\ &= \sum_{n=1}^{N} \mu_{A_{(n)}} \cdot \left[h(X_{(n)}) - h(X_{(n-1)})\right], \end{aligned} \tag{.1}$$

where $(\cdot)$ indicates a permutation of $\mathcal{X}$ such that $h(X_{(1)}) \leq \cdots \leq h(X_{(n)}) \leq \cdots \leq h(X_{(N)})$. Also $A_{(n+1)} = \emptyset$, $h(X_{(0)}) = 0$, and $A_{(n)} = \{X_{(n)}, \ldots, X_{(N)}\}$.

Any fuzzy measure μ on $\mathcal{X}$ can uniquely be expressed in terms of its Möbius representation [30] by

$$\mu(A) = \sum_{B \subseteq A} a^\mu(B), \forall A \subseteq \mathcal{X}, \tag{.2}$$

where the set function a^μ is called the *Möbius transform or Möbius representation* of μ and is given by

$$a^\mu(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \mu(B), \forall A \subseteq \mathcal{X}. \tag{.3}$$

$a^\mu(A)$ can be interpreted as the *interaction index* of the attributes in the subset A [30].

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