Title	直観主義様相論理の意味論的研究
Author(s)	橋本，安司
Citation	
Issue Date	1998-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/1171
Rights	
Description	Supervisor:小野 寛晰，情報科学研究科，修士

A Semantical Study of Intuitionistic Modal Logics

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A thesis submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
Graduate Program in Information Science

Written under the direction of
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February 13, 1998

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Chapter 1

Introduction

Modal logics based on classical logic **Cl** have been investigated well. Classical logics are too strong from the computer scientific or constructive mathematical point of view. So we want to weaken logics. But the negation of classical logics is stronger than that of intuitionistic logics. Hence, in classical logics $\Box p \leftrightarrow \neg\Diamond\neg p$ holds, but in intuitionistic logics $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ do not generally hold. This provides more possibilities for defining intuitionistic modal logics. We will consider intuitionistic modal logics to be independent $\Box$ and $\Diamond$.

Let $\mathcal{L}_{\Box\Diamond}$ be the language of propositional modal logic with countably many propositional variables, $p, q, r, \ldots$ and the connectives $\wedge, \vee, \rightarrow, \bot, \Box, \Diamond$. Let $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ be the set of all formulas of $\mathcal{L}_{\Box\Diamond}$. The formula $\neg\alpha$ is defined as $\alpha \rightarrow \bot$ and $\top$ as $\bot \rightarrow \bot$.

How to define an intuitionistic modal analogue of classical normal modal logic **K** ? Much work has been done in the field.

By the study of correspondence to the bi-modal logic with two box operators, Fischer Servi [8][9] constructed a logic **FS** by imposing a weak connection between $\Box$ and $\Diamond$ operators. **FS** is the least set of formulas of $\mathcal{L}_{\Box\Diamond}$ which contains axioms (1)–(6) and is closed under the rules of inference (a)–(c)

(1) **Int**

(2) $\Box(p \rightarrow \Box q) \rightarrow (\Box p \rightarrow \Box q)$

(3) $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$

(4) $\neg\Diamond\bot$

(5) $\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$

(6) $(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$

(a) modus ponens $\frac{\vdash \alpha \rightarrow \beta \quad \vdash \alpha}{\vdash \beta}$ (**MP**)

(b) substitution (**Sub**)

(c) $\frac{\vdash \alpha}{\vdash \Box\alpha}$ (**RN**)

Various extension of **FS** were studied by Bull [5], Ono [11], Fischer Servi [7][8][9], Wolter and Zakharyaschev [21], Wolter[18].

The well-known **MIPC** is introduced by Prior [13]. **MIPC** is obtained by adding to **FS** the axioms

$$\Box p \to p,\ \Box p \to \Box\Box p,\ \Diamond p \to \Box\Diamond p,$$

$$p \to \Diamond p,\ \Diamond\Diamond p \to \Diamond p,\ \Diamond\Box p \to \Box p.$$

From the relation to intuitionistic predicate logics, Bull [5], Ono [11],Ono and Suzuki [12], Suzuki [16] and Bezhanishvili [1] investigate **MIPC**.

Wolter and Zakharyaschev introduced the weakest intuitionistic modal logic $\mathbf{IntK}_{\Box\Diamond}$. $\mathbf{IntK}_{\Box\Diamond}$ is the least set of formulas of $\mathcal{L}_{\Box\Diamond}$ which contains axioms (1)–(3) and is closed under the rules of inference (a)–(c).

(1) the intuitionistic logic **Int**,

(2)
$(2_\Box)$ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ and $(2_\Diamond)$ $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$,

(3)
$(3_\Box)$ $\Box\top$ and $(3_\Diamond)$ $\neg\Diamond\bot$,

(a) modus ponens $\frac{\vdash \alpha\to\beta \quad \vdash\alpha}{\vdash \beta}$ (**MP**),

(b) substitution (**Sub**),

(c)
$\frac{\vdash \alpha\to\beta}{\vdash\Box\alpha\to\Box\beta}$ $(\mathbf{RR}_\Box)$ and $\frac{\vdash \alpha\to\beta}{\vdash\Diamond\alpha\to\Diamond\beta}$ $(\mathbf{RR}_\Diamond)$.

Our goal is that by extending from the weakest logic $\mathbf{IntK}_{\Box\Diamond}$, we investigate what properties each logic has, and determine which logic is the best in some sense.

In relation to $\mathbf{IntK}_{\Box\Diamond}$, we note some remarks. It is easily seen that the converses $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ and $(\Diamond p \vee \Diamond q) \to \Diamond(p \vee q)$ of (2) are derivable in $\mathbf{IntK}_{\Box\Diamond}$.

We can take alternative definitions. For example, Axiom $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ is equivalent to the formula $\Box(p \to q) \to (\Box p \to \Box q)$. The rule of inference $(\mathbf{RR}_\Box)$ is equivalent to the rule of inference $\frac{\vdash \alpha}{\vdash\Diamond\alpha}$ (**RN**) under the formula $\Box(p \to q) \to (\Box p \to \Box q)$.

But operator $\Diamond$ is deferent. Axiom $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ is not equivalent to the formula $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$. The rule of inference $(\mathbf{RR}_\Diamond)$ is not equivalent to the rule of inference $\frac{\vdash \alpha}{\vdash\Diamond\alpha}$ even under the formula $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$.

A set L of formulas of $\mathcal{L}_{\Box\Diamond}$ is said an *intuitionistic modal logic* if L contains $\mathbf{IntK}_{\Box\Diamond}$ and is closed under the rules of inference (a)–(c). We denote by $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ the set of all normal intuitionistic modal logics. Moreover, for a intuitionistic modal logic L, We denote by $\mathrm{NExt}L$ the set of all normal intuitionistic modal logics containing L.

Let L_i be any logics or sets of formulas. We denote by $\bigoplus_{i \in I} L_i$ the smallest logic which contains all of L_i's.

Then, it is easy to see that $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \oplus, \cap)$ forms a complete lattice.

In Chapter 4, we will investigate the property of $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \oplus, \cap)$ further.

Acknowledgements
I would like to thank my advisor Hiroakira Ono. I am so much obliged to Guram Bezhanishvili for advising much matter concerning this thesis.

Chapter 2

Preliminaries

In this chapter we study two type semantics of intuitionistic modal logics, which has been investigated [21][15].

2.1 Algebraic semantics

First we well introduce algebraic semantics for intuitionistic modal logics. By translating the language of logic into that of algebra, we will get algebraic semantics which will be an adequate semantics for these logics.

Definition 2.1 ([21]) *An algebra $\boldsymbol{A} = (A, \Box, \Diamond)$ is called a $\Box\Diamond$-*modal Heyting algebra *if the following conditions are satisfied.*

(1) A *is a* Heyting algebra*,*

(2) $\Box(a \wedge b) = \Box a \wedge \Box b$ *and* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$*,*

(3) $\Box\top = \top$ *and* $\Diamond\bot = \bot$*.*

By $\boldsymbol{m_{\Box\Diamond}HA}$ *, we denote the variety of all* $\Box\Diamond$*-modal Heyting algebras.*

Notice that from (ii) $\Box$ and $\Diamond$ are monotone operators, i.e.

(1) $a \leq b \Rightarrow \Box a \leq \Box b$,

(2) $a \leq b \Rightarrow \Diamond a \leq \Diamond b$.

Definition 2.2 ([21])

(1) A valuation v *on a modal Heyting algebra* $\boldsymbol{A}$ *is a function* : $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond}) \to A$ *which satisfies the following conditions;*

(i) $v(\bot) = \bot$*,*

(ii) $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$*,*

(iii) $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,

(iv) $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$,

(v) $v(\Box\alpha) = \Box v(\alpha)$,

(vi) $v(\Diamond\alpha) = \Diamond v(\alpha)$.

(2) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$, *any* $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$ *and any valuation* v *on* $\boldsymbol{A}$, α *is* true *in* $\boldsymbol{A}$ *under* v *(in symbol,* $(\boldsymbol{A}, v) \models \alpha$*) if* $v(\alpha) = \top$.

(3) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any* $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, α *is* valid *in* $\boldsymbol{A}$ *(in symbol,* $\boldsymbol{A} \models \alpha$ *) if* $v(\alpha) = \top$ *for any valuation* v *on* $\boldsymbol{A}$.

(4) *For any* $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any class* $\mathcal{K} \subset \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, α *is* valid *in* $\mathcal{K}$ *(in symbol,* $\mathcal{K} \models \alpha$*) if* $\boldsymbol{A} \models \alpha$ *for any* $\boldsymbol{A}$ *in* $\mathcal{K}$.

Note that the values of a given valuation v is uniquely determined only by its values of propositional variables.

When A is a Boolean algebra, by taking $\Diamond = \neg\Box\neg$, $\boldsymbol{A}$ is a (classical) modal algebra.

Proposition 2.3

(1) *Let* $\boldsymbol{A}$ *be a* $\Box\Diamond$*-modal Heyting algebra. The set of formulas which are valid in* $\boldsymbol{A}$ *is an intuitionistic modal logic.*

(2) *Let* $\mathcal{K}$ *be a class of* $\Box\Diamond$*-modal Heyting algebras. The set of formulas which are valid in all algebras in* $\mathcal{K}$ *is an intuitionistic modal logic.*

So these logics are called the logic characterized by $\boldsymbol{A}$ and the logic characterized by $\mathcal{K}$ and denoted by $L(\boldsymbol{A})$ and $L(\mathcal{K})$, respectively.

Proof. Since $L(\mathcal{K}) = \bigcap_{A \in \mathcal{K}} L(\boldsymbol{A})$, it's enough to show (1). Let v be any valuation on $\boldsymbol{A}$. $v((\Box p \wedge \Box q) \to \Box(p \wedge q)) = (\Box v(p) \wedge \Box v(q)) \to \Box(v(p) \wedge v(q)) = \top$.
Next, suppose $v(\alpha \to \beta) = \top$. Then since $v(\alpha) \leq v(\beta)$,by monotonicity $\Box v(\alpha) \leq \Box v(\beta)$.Therefore $v(\Box\alpha \to \Box\beta) = \top$. The other axioms and rules can be proved in the same way. ∎

Let $\mathcal{K}$ be a class of $\Box\Diamond$*-modal Heyting algebra*. Then, $\boldsymbol{H}(\mathcal{K})$, $\boldsymbol{S}(\mathcal{K})$ and $\boldsymbol{P}(\mathcal{K})$ denote the class of all homomorphic images of algebras from $\mathcal{K}$, the class of all subalgebras of algebras from $\mathcal{K}$ and the class of all direct products of algebras from $\mathcal{K}$, respectively.

When $\boldsymbol{H}(\mathcal{K}) = \boldsymbol{S}(\mathcal{K}) = \boldsymbol{P}(\mathcal{K}) = \mathcal{K}$ holds, $\mathcal{K}$ is said to be a *variety* . It is well-known that $\mathcal{K}$ is a *variety* iff $\boldsymbol{HSP}(\mathcal{K}) = \mathcal{K}$, because $\boldsymbol{SH}(\mathcal{K}) \subset \boldsymbol{HS}(\mathcal{K}), \boldsymbol{PH}(\mathcal{K}) \subset \boldsymbol{HP}(\mathcal{K}), \boldsymbol{PS}(\mathcal{K}) \subset \boldsymbol{SP}(\mathcal{K})$[17].

We denote by $\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$ the set of all subvarieties of $\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$.

Proposition 2.4 *The set* $(\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}), \vee, \wedge)$ *forms a complete lattice, where* $\mathcal{K}_1 \vee \mathcal{K}_2$ *is* $\boldsymbol{HSP}(\mathcal{K}_1 \cup \mathcal{K}_2)$ *and* $\mathcal{K}_1 \wedge \mathcal{K}_2$ *is* $\mathcal{K}_1 \cap \mathcal{K}_2$.

Proof. It is easy to see that $(\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}\ ), \vee, \wedge)$ is closed with respect to infinite intersections and $\boldsymbol{HSP}(\bigcup_{i\in I}\mathcal{K}_i)$ is the least variety containing all $\mathcal{K}_i$'s. ∎

The following proposition holds, like the case of (classical) modal algebra.

Proposition 2.5 *Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\Box\Diamond$-modal Heyting algebras.*

(1) If $\boldsymbol{B}$ is a homomorphic image of $\boldsymbol{A}$, then $L(\boldsymbol{A}) \subset L(\boldsymbol{B})$.

(2) If $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, then $L(\boldsymbol{A}) \subset L(\boldsymbol{B})$.

(3) If $\boldsymbol{A}$ is a direct product of $\{\boldsymbol{A}_i\}_{i\in I}$, then $L(\boldsymbol{A}) = \bigcap_{i\in I} L(\boldsymbol{A}_i)$.

Proof. (1). Let f be the homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$. There exists a map g of $\boldsymbol{B}$ to $\boldsymbol{A}$ such that $f \circ g = id_{\boldsymbol{B}}$. For each valuation v on $\boldsymbol{B}$, define the valuation $v_{\boldsymbol{A}}$ on $\boldsymbol{A}$ by $g \circ v(p)$ for each propositional variable p. Then we have $f(v_{\boldsymbol{A}}(\alpha)) = v(\alpha)$ for every formula α. So, if $v_{\boldsymbol{A}} = \top$ then $v(\alpha) = \top$. Thus, we have (1).

(2). If $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, any valuation on $\boldsymbol{B}$ is also a valuation on $\boldsymbol{A}$.

(3). Let v be a valuation on $\boldsymbol{A}$. Since the projection π_i is a homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_i$, $\pi_i \circ v$ is a valuation on $\boldsymbol{A}_i$. Moreover, the valuation v can be represented $(\pi_i \circ v)_{i\in I}$. Thus, we have $L(\boldsymbol{A}) \supset \bigcap_{i\in I} L(\boldsymbol{A}_i)$. The Converse direction follows from (1). ∎

Corollary 2.6 *For a given $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, let $\mathcal{K}(L)$ be $\{\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}\ ; \boldsymbol{A} \models L\}$. Then, $\mathcal{K}(L)$ is a variety.*

Our algebraic semantics is *adequate*, since the following theorem holds.

Theorem 2.7 (Completeness) *For any $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $L \vdash \alpha$ iff $\mathcal{K}(L) \models \alpha$.*

It is clear that $L \vdash \alpha$ implies $\mathcal{K}(L) \models \alpha$. To prove the converse, we define the algebra $\boldsymbol{A}_L$ called the Lindenbaum algebra of a logic $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$. First given a logic L, define a congruence relation $\sim_L$ on formulas by taking

$$\alpha \sim_L \beta \ \text{ iff } \ L \vdash (\alpha \to \beta) \wedge (\alpha \leftarrow \beta)$$

Then the Lindenbaum algebra $\boldsymbol{A}_L = (\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})/_{\sim_L}, \wedge, \vee, \to, \bot, \Box, \Diamond)$ is constructed by taking

$$\begin{array}{rcl}
\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})/_{\sim_L} & := & \{|\alpha|_L;\ \alpha \ \text{ a formula }\}, \\
|\alpha|_L \wedge |\beta|_L & := & |\alpha \wedge \beta|_L, \\
|\alpha|_L \vee |\beta|_L & := & |\alpha \vee \beta|_L, \\
|\alpha|_L \to |\beta|_L & := & |\alpha \to \beta|_L, \\
\bot & := & |\bot|_L, \\
\Box|\alpha|_L & := & |\Box\alpha|_L, \\
\Diamond|\alpha|_L & := & |\Diamond\alpha|_L.
\end{array}$$

The fact that $\boldsymbol{A}_L$ is indeed a modal Heyting algebra is easily shown by using the axioms and rules of $\mathbf{IntK}_{\Box\Diamond}$. Also, $\top$ of $\boldsymbol{A}_L$ consists of all provable formulas. Now, define a valuation v_L by

$$v_L(\alpha) = |\alpha|_L, \quad \text{for each formula } \alpha.$$

Then we have

$$v_L(\alpha) = \top \ \text{ iff } \ \alpha \in L.$$

Furthermore, for each valuation v on $\boldsymbol{A}_L$, $v(\alpha)$ is $|\beta|_L$ for some substitution instance β of α. So, in particular if $\alpha \in L$ then $v_L(\alpha) = |\beta|_L = \top$. Thus, $\boldsymbol{A}_L$ validates L. Now suppose that $L \nvdash \alpha$. Then $v_L(\alpha) \neq \top$ and $\boldsymbol{A}_L \in \mathcal{K}(L)$. Thus, $\mathcal{K}(L) \not\models \alpha$ The proof of completeness theorem is completed.

Proposition 2.8

(1) For any $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, $L(\mathcal{K}(L)) = L$.

(2) For any $\mathcal{K} \in \Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$, $\mathcal{K}(L(\mathcal{K})) = \mathcal{K}$.

(3) For any $L_1, L_2 \in \Lambda(\mathbf{IntK}_{\Box\Diamond})$, $L_1 \subset L_2$ *iff* $\mathcal{K}(L_2) \subset \mathcal{K}(L_1)$

Proof. (1). This is the completeness theorem itself.

(2). By Birkhoff's theorem [3], any variety $\mathcal{K}$ is of the form $\mathcal{K}(L)$ for some logic L, so that
$\mathcal{K}(L(\mathcal{K})) = \mathcal{K}(L(\mathcal{K}(L))) = \mathcal{K}(L) = \mathcal{K}$.

(3). By the definition, both $\mathcal{K}(\cdot)$ and $L(\cdot)$ are monotone decreasing. ∎

Since (dually) order isomorphic implies (dually) lattice isomorphic, we have following corollary.

Corollary 2.9 $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ *is dually isomorphic to* $\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA})$.

2.2 Kripke type semantics

In this section we will consider Kripke-type semantics for intuitionistic modal logics.

Definition 2.10 ([21][15])

(1) A structure $\mathcal{F} = (W, R, R_\Box, R_\Diamond)$ *is called an* intuitionistic modal frame *if the following conditions are satisfied.*

(i) $W \neq \emptyset$,

(ii) R *: a partial order on* W,

(iii) $R_\Box, R_\Diamond$*: binary relations on* W,

(iv) $R \circ R_\Box \circ R = R_\Box$, *where* $R_1 \circ R_2$ *is the* relational product *of* R_1, R_2 *defined by* $x(R_1 \circ R_2)y$ *iff there is a* z *such that* $xR_1z \& zR_2y$.

(v) $R^{-1} \circ R_\Diamond \circ R^{-1} = R_\Diamond$,   *where R^{-1} is the reverse of R.*

(2) $\mathcal{CONW}$ *is the set of all cones of* W*, i.e.* $\mathcal{CONW} = \{V \subset W; (x \in V \ \&\ xRy) \Rightarrow y \in V\}$

(3) A valuation v *on* $\mathcal{F}$ *is a function* : $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond}) \rightarrow \mathcal{CONW}$ *which satisfies the following conditions*

(i) $v(\bot) = \emptyset$,

(ii) $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$,

(iii) $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$,

(iv) $v(\alpha \rightarrow \beta) = \{x \in W; \forall y((xRy \ \&\ y \in v(\alpha)) \Rightarrow y \in v(\beta))\}$,

(v) $v(\Box\alpha) = \{x \in W; \forall y(xR_\Box y \Rightarrow y \in v(\alpha))\}$,

(vi) $v(\Diamond\alpha) = \{x \in W; \exists y(xR_\Diamond y \text{ and } y \in v(\alpha))\}$.

(4) A pair $\mathcal{M} = (\mathcal{F}, v)$ *of* $\mathcal{F} \in$ ***IMF*** *and a valuation* v *on* $\mathcal{F}$ *is called a* model*.*

(5) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$*, any model* $\mathcal{M}$ *and any* $x \in W, \alpha$ *is* true *at* x *in* $\mathcal{M}$ *(in simbol,* $(\mathcal{M}, x) \models \alpha$ *or simply* $x \models \alpha$ *if* $\mathcal{M}$ *is understood) if* $x \in v(\alpha)$ *.*

(6) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any model* $\mathcal{M}, \alpha$ *is* true *in* $\mathcal{M}$ *(in simbol,* $\mathcal{M} \models \alpha$*) if* $W = v(\alpha)$*.*

(7) For any $\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ *and any* $\mathcal{F} \in$ ***IMF****,* α *is* valid *in* $\mathcal{F}$ *(in simbol,* $\mathcal{F} \models \alpha$*) if* $W = v(\alpha)$ *for any valuation* v *on* $\mathcal{F}$*.*

Note that the values of a given valuation v is uniquely determined only by its values of propositional variables.

We denote by ***IMF*** the set of all intuitionistic modal frames.

Proposition 2.11 *(1) Let* $\mathcal{F}$ *be an intuitionistic modal frame. The set of formulas which are valid in* $\mathcal{F}$ *is an intuitionistic modal logic.*

(2) Let $\mathcal{C}$ *be a class of intuitionistic modal frames. The set of formulas which are valid in all frames in* $\mathcal{C}$ *is an intuitionistic modal logic.*

So these logics are called the logic characterized by $\mathcal{F}$ and the logic characterized by $\mathcal{C}$ and denoted by $L(\mathcal{F})$ and $L(\mathcal{C})$, respectively.

Proof. Since $L(\mathcal{C}) = \bigcap_{\mathcal{F} \in \mathcal{C}} L(\mathcal{F})$, it's enough to show (1). Let v be any valuation on $\mathcal{F}$. If xRy, $y \models \Box p$ and $y \models \Box q$ then $z \models p$ and $z \models q$ for any z such that $yR_\Box z$. Hence $x \models (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ at any x.
Next, suppose $x \models p \rightarrow q$. If $xRyR_\Box z$ and $y \models \Box p$ then $z \models p$ and hence $z \models \Box q$. So, $y \models \Box q$. Therefore $x \models \Box p \rightarrow \Box q$. The other axioms and rules can be proved in the same

way. ∎

We denote by $\Box^n\alpha$ and $\Diamond^n\alpha$ the formulas $\underbrace{\Box\cdots\Box}_{n}\alpha$ and $\underbrace{\Diamond\cdots\Diamond}_{n}\alpha$, respectively. When $n=0$, we define that both $\Box^0\alpha$ and $\Diamond^0\alpha$ are just α. We denote also by $\Box^{(n)}\alpha$ and $\Diamond^{(n)}\alpha$ the formulas $\Box^0\wedge\cdots\wedge\Box^n\alpha$ and $\Diamond^0\vee\cdots\vee\Diamond^n\alpha$, respectively. In particular, we denote $\Box^{(1)}\alpha$ and $\Diamond^{(1)}\alpha$ by $\Box^+\alpha$ and $\Diamond^+\alpha$, respectively.

On the other hand, for $n>0$ we denote $\underbrace{R_\Box\circ\cdots\circ R_\Box}_{n}$ and $\underbrace{R_\Diamond\circ\cdots\circ R_\Diamond}_{n}$ by $R_\Box^n$ and $R_\Diamond^n$, respectively. We understand $R_\Box^0$ and $R_\Diamond^0$ as R and R^{-1}, respectively. We denote also by $R_\Box^{(n)}$ and $R_\Diamond^{(n)}$ the binary relations $R_\Box^0\cup\cdots\cup R_\Box^n$ and $R_\Diamond^0\cup\cdots\cup R_\Diamond^n$, respectively. In particular,we denote $R_\Box^{(1)}$ and $R_\Diamond^{(1)}$ by $R_\Box^+$ and $R_\Diamond^+$, respectively.

The binary relation $\hat{S}$ denotes the *transitive closure* $\bigcup_{n>0} S^n$ of a given binary relation S.

The frames validating a number of formulas are characterized as follows.

Proposition 2.12 ([15]) *For any intuitionistic modal frame $\mathcal{F}$, $\mathcal{F}$ validates each formula in the following list iff $\mathcal{F}$ satisfies the condition of the list .*

$\Box(p\to q)\to(\Diamond p\to\Diamond q)$	$yR_\Diamond x\Rightarrow\exists z(xRz\ \&\ yR_\Diamond z\ \&\ yR_\Box z)$	(2.1)
$\Diamond(p\to q)\to(\Box p\to\Diamond q)$	$yR_\Diamond x\Rightarrow\exists z(xRz\ \&\ yR_\Diamond z\ \&\ yR_\Box z)$	(2.2)
$(\Diamond p\to\Box q)\to\Box(p\to q)$	$xR_\Box y\Rightarrow\exists z(xRz\ \&\ zR_\Diamond y\ \&\ zR_\Box y)$	(2.3)
$\Box^+(p\to q)\to(\Diamond p\to\Diamond q)$	$yR_\Diamond x\Rightarrow\exists z(xRz\ \&\ yR_\Diamond z\ \&\ yR_\Box^+ z)$	(2.4)
$\Box^{(m)}(p\to q)\to(\Diamond p\to\Diamond q)$	$yR_\Diamond x\Rightarrow\exists z(xRz\ \&\ yR_\Diamond z\ \&\ yR_\Box^{(m)} z)$	(2.5)
$\Box p\to p$	$R_\Box$: *reflexive*	(2.6)
$p\to\Diamond p$	$R_\Diamond$: *reflexive*	(2.7)
$\Box p\to\Box\Box p$	$R_\Box$: *transitive*	(2.8)
$\Diamond\Diamond p\to\Diamond p$	$R_\Diamond$: *transitive*	(2.9)
$\Diamond^k\Box^l p\to\Box^m\Diamond^n p$	$(xR_\Box^m y\ \&\ xR_\Diamond^k z)\Rightarrow\exists u(yR_\Diamond^n u\ \&\ zR_\Box^l u)$	(2.10)
$\Box p\vee\Box\neg\Box p$	$(xR_\Box y\ \&\ xR_\Box z)\Rightarrow yR_\Box z$	(2.11)
$\Box(\Box p\vee q)\to(\Box p\vee\Box q)$	$(xR_\Box y\ \&\ xR_\Box z)\Rightarrow\exists u(xR_\Box u\ \&\ uRz\ \&\ uR_\Box y)$	(2.12)
$\Box(\Box p\to q)\vee\Box(\Box q\to p)$	$(xR_\Box y\ \&\ xR_\Box z)\Rightarrow(yR_\Box z$ *or* $zR_\Box y)$	(2.13)

Proof. We will take up several of them. The rest can be checked similarly.

(2.5). Suppose that $\mathcal{M}=(\mathcal{F},v)$ is a countermodel for it. Then $y\models\Box^{(m)}(p\to q)$ and $y\models\Diamond p$ and $y\not\models\Diamond q$, for some y in $\mathcal{F}$. Since $x\models p$ for some x such that $yR_\Diamond x$, if there exist z and number n such that xRz, $yR_\Diamond z$, $yR_\Box^n z$ and $0\le n\le m$, then we have $z\models p$ and $z\models p\to q$. Hence $z\models q$. This is a contradiction. Conversely, suppose that there are x, y such that $yR_\Diamond x$ and there is no point z for which xRz, $yR_\Diamond z$ and $yR_\Box^{(m)}z$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w; xRw\}, v(q) = \{w; xRw \;\&\; yR_\Box^{(m)}w\}$. Then for any $n : 0 \leq n \leq m$ if $xR_\Box^n w$ and $w \models p$ then $w \models q$. Hence $y \models \Box(p \to q)$. we can also show $y \models \Diamond p$ and $y \not\models \Diamond q$, since $x \models p$ and there is no point z for which $yR_\Diamond z$, xRz and $yR_\Box^{(m)}z$. Thus, $y \not\models \Box(p \to q) \to (\Diamond p \to \Diamond q)$.

(2.10). Suppose that $\mathcal{M} = (\mathcal{F}, v)$ is a countermodel for it. Then $x \models \Diamond^k \Box^l p$ and $x \not\models \Box^m \Diamond^n p$, for some x in $\mathcal{F}$. Hence there are y, z such that $xR_\Box^m y$, $y \not\models \Diamond^n p$ and $xR_\Diamond^k z$, $z \models \Box^l p$. If there is u such that $yR_\Diamond^n u$ and $zR_\Box^l u$ then this is a contradiction. Conversely, suppose that there are x, y, z such that $xR_\Box^m y$, $xR_\Diamond^k z$ and there is no point u for which $yR_\Diamond^n u$ and $zR_\Box^l u$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w; zR_\Box^l w\}$. Then we can show $z \models \Box^l p$ and $y \not\models \Diamond^n p$, whence $x \models \Diamond^k \Box^l p$ and $x \not\models \Box^m \Diamond^n p$. Thus, $x \not\models \Diamond^k \Box^l p \to \Box^m \Diamond^n p$.

(2.12). Suppose that $\mathcal{M} = (\mathcal{F}, v)$ is a countermodel for it. Then $x \models \Box(\Box p \vee q)$ and $x \not\models \Box p \vee \Box q$, for some x in $\mathcal{F}$. Hence there are y, z such that $xR_\Box y, y \not\models p$ and $xR_\Box z, z \not\models q$. If there is u such that $xR_\Box u$, uRz and $uR_\Box y$ then $u \not\models \Box p$ and $u \not\models q$. Hence $u \not\models \Box p \vee q$. This is a contradiction. Conversely, suppose that there are x, y, z such that $xR_\Box y$, $xR_\Box z$ and there is no point u, for which $xR_\Box u$, uRz and $uR_\Box y$. Define a valuation v in $\mathcal{F}$ by taking $v(p) = \{w; \neg wRy\}, v(q) = \{w; \neg wRz\}$. Then we can show $y \not\models p$, $z \not\models q$ and $u \models \Box p \vee q$. Indeed, there is no point u such that $xR_\Box u$, $u \models \Box p$ and $u \models q$. Hence $x \models \Box(\Box p \vee q)$ and $x \not\models \Box p \vee \Box q$. Thus, $x \not\models \Box(\Box p \vee q) \to (\Box p \vee \Box q)$. ∎

Define several logics, by taking
$\mathbf{IntK}^+_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \Box^+(p \to q) \to (\Diamond p \to \Diamond q)$,
$\mathbf{IntK}^*_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \Box(p \to q) \to (\Diamond p \to \Diamond q)$,
$\mathbf{IntK4}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Box p \to \Box\Box p,\ \Diamond\Diamond p \to \Diamond p\}$,
$\mathbf{IntS4}_{\Box\Diamond} = \mathbf{IntK4}_{\Box\Diamond} \oplus \{\Box p \to p,\ p \to \Diamond p\}$,
$\mathbf{IntS4.3}_{\Box\Diamond} = \mathbf{IntS4}_{\Box\Diamond} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$,
$\mathbf{IntK5}_{\Box\Diamond} = \mathbf{IntK}_{\Box\Diamond} \oplus \{\Diamond\Box p \to \Box p,\ \Diamond p \to \Box\Diamond p\}$,
$\mathbf{IntS5}_{\Box\Diamond} = \mathbf{IntK5}_{\Box\Diamond} \oplus \{\Box p \to p,\ p \to \Diamond p\}$.

Then, the frames validating these logics are as follows.

$\mathbf{IntK}^+_{\Box\Diamond}$	$yR_\Diamond x \Rightarrow \exists z(xRz \;\&\; yR_\Diamond z \;\&\; yR^+_\Box z)$
$\mathbf{IntK}^*_{\Box\Diamond}$	$yR_\Diamond x \Rightarrow \exists z(xRz \;\&\; yR_\Diamond z \;\&\; yR_\Box z)$
FS	$yR_\Diamond x \Rightarrow \exists z(xRz \;\&\; yR_\Diamond z \;\&\; yR_\Box z)$
	$xR_\Box y \Rightarrow \exists z(xRz \;\&\; zR_\Diamond y \;\&\; zR_\Box y)$
$\mathbf{IntK4}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: transitive
$\mathbf{IntS4}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: reflexive and transitive
$\mathbf{IntS4.3}_{\Box\Diamond}$	$R_\Box, R_\Diamond$: reflexive and transitive
	$(xR_\Box y \;\&\; xR_\Box z) \Rightarrow (yR_\Box z \textit{ or } zR_\Box y)$
$\mathbf{IntK5}_{\Box\Diamond}$	$(xR_\Box y \;\&\; xR_\Diamond z) \Rightarrow (zR_\Box y \;\&\; yR_\Diamond z)$
$\mathbf{IntS5}_{\Box\Diamond}$	$R_\Diamond = R_\Box^{-1}$ and $R_\Box$:reflexive and transitive
MIPC	$R_\Diamond = R_\Box^{-1}$ and $R_\Box$:reflexive and transitive
	$xR_\Box y \Rightarrow \exists z(xRz \;\&\; yR_\Box z \;\&\; zR_\Box y)$

Truth-preserving operations

In this subsection we will introduce three important operations on intuitionistic modal frames which preserve validity.

Definition 2.13 ([21])

(1) A frame $\mathcal{F}_1$ is called a generated subframe *of a frame $\mathcal{F}_2$ if the following conditions are satisfied.*

(i) $W_1 \subset W_2$,

(ii) R_1 *and* $R_{\Box_1}$ *and* $R_{\Diamond_1}$, *is the restriction of* R_2 *and* $R_{\Box_2}$ *and* $R_{\Diamond_2}$ *to* W_1, *respectively,*

(iii) $x \in W_1$ & $xR_2y \Rightarrow y \in W_1$,

(iv) $x \in W_1$ & $xR_{\Box_2}y \Rightarrow y \in W_1$,

(v) $x \in W_1$ & $xR_{\Diamond_2}y \Rightarrow \exists z \in W_1 \ : \ xR_{\Diamond_1}z$ & yR_2z.

(2) A map $f : W_1 \to W_2$ is said to be a p-morphism *if for all $x \in W_1, y \in W_2$,*

(i) $f(x)R_2y \Leftrightarrow \exists z \in W_1 \ : \ xR_1z$ & $f(z) = y$,

(ii) $f(x)R_{\Box_2}y \Leftrightarrow \exists z \in W_1 \ : \ xR_{\Box_1}z$ & $f(z) = y$,

(iii) $xR_{\Diamond_1}y \Rightarrow f(x)R_{\Diamond_2}f(y)$,

(iv) $f(x)R_{\Diamond_2}y \Rightarrow \exists z \in W_1 \ : \ xR_{\Diamond_1}z$ & $yR_2f(z)$.

(3) A frame $\mathcal{F}_1$ is called reducible *to a frame $\mathcal{F}_2$ if there exists a onto p-morphism (say* reduction*) $f : W_1 \to W_2$.*

(4) The frame $\mathcal{F} = (\sum_{i\in I} W_i, \bigcup_{i\in I} R_i, \bigcup_{i\in I} R_{\Box_i}, \bigcup_{i\in I} R_{\Diamond_i})$ is called the disjoint union *of a disjoint family $\{\mathcal{F}_i : i \in I\}$.*

Note that each $\mathcal{F}_i$ is a generated subframe of the disjoint union of $\{\mathcal{F}_i : i \in I\}$.

Theorem 2.14 (Generation) *Suppose $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, and suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are a model of $\mathcal{F}_1$ and a model of $\mathcal{F}_2$, respectively. If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \ \textit{ iff } \ (\mathcal{M}_2, x) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \ \textit{ iff } \ (\mathcal{M}_2, x) \models \alpha.$$

Proof. We prove by induction on the construction of α. The basis of induction is obvious. Let $\alpha = \Diamond\beta$. If $(\mathcal{M}_1, x) \models \Diamond\beta$ then there is a point $y \in W_1$ such that $xR_{\Diamond_1}y$ and $(\mathcal{M}_1, y) \models \beta$. By the induction hypothesis, $(\mathcal{M}_2, y) \models \beta$, and by (1)(ii) of Definition 2.13, $xR_{\Diamond_2}y$. Therefore $(\mathcal{M}_2, x) \models \Diamond\beta$. Conversely, suppose $(\mathcal{M}_2, x) \models \Diamond\beta$. Then there is a point $y \in W_2$ such that $xR_{\Diamond_2}y$ and $(\mathcal{M}_2, y) \models \beta$. By (1)(v) of Definition 2.13, There is $z \in W_1$ such that $xR_{\Diamond_1}z$ and yR_2z. Since $(\mathcal{M}_2, z) \models \beta$, by the induction hypothesis, $(\mathcal{M}_1, z) \models \beta$, whence $(\mathcal{M}_1, x) \models \Diamond\beta$.

The cases $\alpha = \beta \to \gamma$ and $\alpha = \Box\beta$ are similar, and the cases $\alpha = \beta \wedge \gamma$ and $\alpha = \beta \vee \gamma$ are trivial. ∎

Corollary 2.15 *If $\mathcal{F}_1$ is a generated subframe of $\mathcal{F}_2$, then $L(\mathcal{F}_2) \subset L(\mathcal{F}_1)$.*

Proof. Suppose $\mathcal{F}_1 \not\models \alpha$. Then $((\mathcal{F}_1, v_1), x) \not\models \alpha$ for some v_1 on $\mathcal{F}_1$ and $x \in \mathcal{F}_1$. Define a valuation v_2 on $\mathcal{F}_2$ by taking

$$v_2(p) := v_1(p) \text{ for all propositional variables } p.$$

By Theorem 2.14, $((\mathcal{F}_2, v_2), x) \not\models \alpha$. Therefore, $\mathcal{F}_2 \not\models \alpha$. ∎

Theorem 2.16 (Reduction) *Suppose f is a reduction of $\mathcal{F}_1$ to $\mathcal{F}_2$, and suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are a model of $\mathcal{F}_1$ and a model of $\mathcal{F}_2$, respectively. If for every propositional variable p and every x in $\mathcal{F}_1$*

$$(\mathcal{M}_1, x) \models p \quad \textit{iff} \quad (\mathcal{M}_2, f(x)) \models p$$

then for every formula α and every x in $\mathcal{F}_1$

$$(\mathcal{M}_1, x) \models \alpha \quad \textit{iff} \quad (\mathcal{M}_2, f(x)) \models \alpha.$$

Proof. We prove by induction on the construction of α. The basis of induction is obvious. Let $\alpha = \Diamond\beta$. If $(\mathcal{M}_1, x) \models \Diamond\beta$ then there is a point $y \in W_1$ such that $xR_{\Diamond_1}y$ and $(\mathcal{M}_1, y) \models \beta$. By the induction hypothesis, $(\mathcal{M}_2, f(y)) \models \beta$, and by (2)(iii) of Definition 2.13, $f(x)R_{\Diamond_2}f(y)$. Therefore $(\mathcal{M}_2, x) \models \Diamond\beta$. Conversely, suppose $(\mathcal{M}_2, f(x)) \models \Diamond\beta$. Then there is a point $y \in W_2$ such that $f(x)R_{\Diamond_2}y$ and $(\mathcal{M}_2, y) \models \beta$. By (2)(iv) of Definition 2.13, There is $z \in W_1$ such that $xR_{\Diamond_1}z$ and $yR_2f(z)$. Since $(\mathcal{M}_2, f(z)) \models \beta$, by the induction hypothesis, $(\mathcal{M}_1, z) \models \beta$, whence $(\mathcal{M}_1, f(x)) \models \Diamond\beta$.

The cases $\alpha = \beta \to \gamma$ and $\alpha = \Box\beta$ are similar, and the cases $\alpha = \beta \wedge \gamma$ and $\alpha = \beta \vee \gamma$ are trivial. ∎

Corollary 2.17 *If $\mathcal{F}_1$ is reducible to $\mathcal{F}_2$, then $L(\mathcal{F}_1) \subset L(\mathcal{F}_2)$.*

Proof. Let f be a reduction of $\mathcal{F}_1$ to $\mathcal{F}_2$. Suppose $\mathcal{F}_2 \not\models \alpha$. Then $((\mathcal{F}_2, v_2), f(x)) \not\models \alpha$ for some v_2 on $\mathcal{F}_2$ and $x \in \mathcal{F}_1$. Define a valuation v_1 on $\mathcal{F}_1$ by taking

$$v_1(p) := f^{-1}(v_2(p)) \quad \text{for all propositional variables } p.$$

By Theorem 2.16, $((\mathcal{F}_1, v_1), x) \not\models \alpha$. Therefore, $\mathcal{F}_1 \not\models \alpha$. ∎

Again, as a corollary of Theorem 2.14 we have the following.

Corollary 2.18 *If $\mathcal{F}$ is the disjoint union of a family $\{\mathcal{F}_i : i \in I\}$, then $L(\mathcal{F}) = \bigcap_{i \in I} L(\mathcal{F}_i)$.*

2.3 Correspondence between algebraic semantics and Kripke type semantics

In the following, we will show the relation between modal Heyting algebras and intuitionistic modal frames.

Definition 2.19 ([21])

(1) The map $(\ \cdot\ )^+ : \boldsymbol{IMF} \to \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$ is defined as follows :
For any $\mathcal{F} = (W, R, R_\Box, R_\Diamond), \mathcal{F}^+ = (\ \mathcal{CONW}, \Box, \Diamond)$, where for any $a, b \in \mathcal{CONW}$,

(i) $a \vee b = a \cup b$,

(ii) $a \wedge b = a \cap b$,

(iii) $a \to b = \{x \in W; \forall y \in W(xRy \ \&\ y \in a \Rightarrow y \in b)\}$,

(iv) $\Box a = \{x \in W; \forall y \in W(xR_\Box y \Rightarrow y \in a)\}$,

(v) $\Diamond a = \{x \in W; \exists y \in W : xR_\Diamond y \ \&\ y \in a\}$.

We call $\mathcal{F}^+$ the dual of $\mathcal{F}$.

(2) The map $(\ \cdot\ )_+ : \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA} \to \boldsymbol{IMF}$ is defined as follows :
For any $\boldsymbol{A} = (A, \Box, \Diamond), \boldsymbol{A}_+ = (W, R, R_\Box, R_\Diamond)$, where

(i) W is the set of all prime filters of A ($PF(A)$, in symbol),

(ii) $xRy \overset{def}{\Leftrightarrow} x \subset y$,

(iii) $xR_\Box y \overset{def}{\Leftrightarrow} \forall a \in A(\Box a \in x \Rightarrow a \in y)$,
in other words, $x_\Box \subset y$, where $x_\Box := \{a : \Box a \in x\}$,

(iv) $xR_\Diamond y \overset{def}{\Leftrightarrow} \forall a \in A(a \in y \Rightarrow \Diamond a \in x)$,
in other words, $y \subset x_\Diamond$, where $x_\Diamond := \{a : \Diamond a \in x\}$.

We call $\boldsymbol{A}_+$ the dual of $\boldsymbol{A}$.

Proposition 2.20

(1) For every intuitionistic modal frame $\mathcal{F}$, its dual $\mathcal{F}^+$ is modal Heyting algebra.

(2) For every modal Heyting algebra $\boldsymbol{A}$, its dual $\boldsymbol{A}_+$ is intuitionistic modal frame.

Proof. It is routine to check our proposition. Compare (1) with Proposition 2.11. ∎

In relation to the prime filter, the following theorem using Zorn's Lemma is well-known.

Theorem 2.21 *Let G and J be a filter and an ideal of a distributive lattice A such that $G \cap J = \emptyset$. Then there exists a prime filter F such that $G \subset F$ and $F \cap J = \emptyset$.*

The least filter $[X)$ containing a given non-empty set X in a lattice A is called the *filter generated by* X and represented by

$$[X) = \{y \in A : x_1 \wedge \cdots \wedge x_n \leq y, \text{ for some } x_1, \ldots, x_n \in X\}.$$

The least ideal $(X]$ containing a given non-empty set X in a lattice A is called the *ideal generated by* X and represented by

$$(X] = \{y \in A : y \leq x_1 \vee \cdots \vee x_n, \text{ for some } x_1, \ldots, x_n \in X\}.$$

Proposition 2.22 ([21])

(1) For every intuitionistic modal frame $\mathcal{F}$, $\mathcal{F}$ is embedded into $(\mathcal{F}^+)_+$.

(2) For every modal Heyting algebra $\boldsymbol{A}$, $\boldsymbol{A}$ is embedded into $(\boldsymbol{A}_+)^+$.

Proof. (1). Let $\mathcal{F} = (W, R, R_\Box, R_\Diamond)$ and $(\mathcal{F}^+)_+ = (W', R', R'_\Box, R'_\Diamond)$. Define a map f from W into W' by taking, for all $x \in W$

$$f(x) := \{a \in \mathcal{CONW} : x \in a\} \in W'(= PF(\mathcal{CONW})).$$

Suppose xRy. If $a \in f(x)$, then $y \in a$ since $x \in a$ and $a \in \mathcal{CON}W$. Therefore $a \in f(y)$, whence $f(x)R'f(y)$. Conversely, suppose $f(x) \subset f(y)$. Since $\{z : xRz\} \in f(x)$, $\{z : xRz\} \in f(y)$. Therefore $y \in \{z : xRz\}$, whence xRy.

Suppose $xR_\Box y$. If $\Box a \in f(x)$, then $y \in a$ since $x \in \Box a$. Therefore $a \in f(y)$, whence $f(x)R'_\Box f(y)$. Conversely, suppose $(f(x))_\Box \subset f(y)$. Since $\Box\{z : xR_\Box z\} \in f(x)$, $\{z : xR_\Box z\} \in f(y)$. Therefore $y \in \{z : xR_\Box z\}$, whence $xR_\Box y$.

Suppose $xR_\Diamond y$. If $a \in f(y)$, then $x \in \Diamond a$ since $y \in a$. Therefore $\Diamond a \in f(x)$, whence $f(x)R'_\Diamond f(y)$. Conversely, suppose $f(y) \subset (f(x))_\Diamond$. Since $\{z : yRz\} \in f(y)$, $\Diamond\{z : yRz\} \in f(x)$. Therefore since $x \in \Diamond\{z : yRz\}$, there is z such that $xR_\Diamond z$ and yRz, whence $xR_\Diamond y$.

(2). Let $\boldsymbol{A} = (A, \Box, \Diamond)$ and $\boldsymbol{A} = (A', \Box', \Diamond')$. Define a map h from A into A' by taking, for all $a \in A$

$$h(a) := \{x \in PF(A) : a \in x\} \in A'(= \mathcal{CON}PF(A)).$$

Let $a \not\leq b$. Since there is a prime filter z in A such that $a \in z$ and $b \notin z$ by Theorem 2.21, $h(a) \not\subset h(b)$. therefore h is a injection.

Let's check that h preserves the operations. It is easy to show h preserves $\wedge, \vee, \perp$.

Suppose $x \in h(a \to b)$. If $x \subset y$ and $a \in y$, then $b \in y$ since $a \to b \in x$. Therefore $x \in h(a) \to h(b)$. Conversely, suppose $x \notin h(a \to b)$. If $a \wedge c \leq b$ for some $c \in x$ then $c \leq a \to b$ so it is contradiction. Therefore since $[x \cup \{a\})$ and $(\{b\}]$ are disjoint, there is a prime filter y such that $x \subset y$, $a \in y$ and $b \notin y$ by Theorem 2.21. This means $x \notin h(a) \to h(b)$.

Suppose $x \in h(\Box a)$. If $x_\Box \subset y$, then $a \in y$. Therefore $x \in \Box h(a)$. Conversely, suppose $x \notin h(\Box a)$. Since the filter $x_\Box$ does not contain a, there is a prime filter y such that $x_\Box \subset y$ and $a \notin y$ by Theorem 2.21. This means $x \notin \Box h(a)$.

Suppose $x \in h(\Diamond a)$. Since the ideal $(x_\Diamond)^c$ does not contain a, there is a prime filter y such that $y \subset x_\Diamond$ and $a \in y$ by Theorem 2.21. This means $x \in \Diamond h(a)$. Conversely, suppose $x \in \Diamond h(a)$. There is a prime filter y such that $y \subset x_\Diamond$ and $h(y) \in a$. Then $\Diamond a \in x$, since $a \in y$. Therefore $x \in h(\Diamond a)$. ∎

Proposition 2.23

(1) For every intuitionistic modal frame $\mathcal{F}$, $L(\mathcal{F}^+) = L(\mathcal{F})$.

(2) For every modal Heyting algebra $\boldsymbol{A}$, $L(\boldsymbol{A}_+) \subset L(\boldsymbol{A})$.

Proof. (1). By definition, a valuation on $\mathcal{F}$ is at the same time a valuation on $\mathcal{F}^+$.

(2). Since $\boldsymbol{A}$ is isomorphic to a subalgebra of $(\boldsymbol{A}_+)^+$ by Proposition 2.22 (2), $L((\boldsymbol{A}_+)^+) \subset L(\boldsymbol{A})$ by corollary 2.15. And by (1), $L((\boldsymbol{A}_+)^+) = L(\boldsymbol{A}_+)$. Hence $L(\boldsymbol{A}_+) \subset L(\boldsymbol{A})$. ∎

Chapter 3

Kripke completeness

On algebraic semantics we showed completenes theorem. In this chapter we will consider two method for completenes on Kripke type semantics. One of them is the *method of canonical models*, and the other is *filtration method*.

Definition 3.1 *A logic L is called* Kripke complete *if there is a class $\mathcal{C}$ of frame such that*

$$L \vdash \alpha \quad \textit{iff} \quad \mathcal{C} \models \alpha.$$

3.1 Canonical logics

In this section, we will construct a model refuting formulas outside of L. To get Kripke completeness of L, it is morreover necessary that this frame validate L.

Definition 3.2

*(1) Let L be a intuitionistic modal logic. A set T of formulas is said to be a L-*theory *if*

  (i) $L \subset T$,

  (ii) $\alpha, \alpha \to \beta \in T \Rightarrow \beta \in T$.

(2) A L-theory T is consistent *iff* $\perp \notin T$.

(3) A L-theory T is prime *iff*

  (i) T is consistent,

  (ii) $\alpha \vee \beta \in T \Rightarrow \alpha \in T$ *or* $\beta \in T$.

Definition 3.3

(1) Let L be a intuitionistic modal logic. The canonical frame $\mathcal{F}_L = (W_L, R_L, R_{\Box_L}, R_{\Diamond_L})$ *is defined as follows.*

  (i) W_L is the set of all prime L-theories,

(ii) $T_1 R_L T_2 \overset{def}{\Leftrightarrow} T_1 \subset T_2$,

(iii) $T_1 R_{\Box_L} T_2 \overset{def}{\Leftrightarrow} \forall\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})(\Box a \in T_1 \Rightarrow a \in T_2)$, *in other words*, $(T_1)_\Box \subset T_2$, *where* $(T_1)_\Box := \{\alpha : \Box\alpha \in T_1\}$,

(iv) $T_1 R_{\Diamond_L} T_2 \overset{def}{\Leftrightarrow} \forall\alpha \in \mathcal{F}orm(\mathcal{L}_{\Box\Diamond})(a \in T_2 \Rightarrow \Diamond a \in T_1)$, *in other words*, $T_2 \subset (T_1)_\Diamond$, *where* $(T_1)_\Diamond := \{\alpha : \Diamond\alpha \in T_1\}$.

(2) *The* canonical model $\mathcal{M}_L = (\mathcal{F}_L, v_L)$ *is defined by taking, for every propositional variable* p,

$$v_L(p) := \{x \in W_L : p \in x\}.$$

We can regard a (prime) L-theory as a (prime) filter of the Lindenbaum algebra $\boldsymbol{A}_L$. Compare Definition 3.3 with Definition 2.19(2), and we can also consider the canonical frame $\mathcal{F}_L$ as the dual $(\boldsymbol{A}_L)_+$ of the Lindenbaum algebra $\boldsymbol{A}_L$. Actually a map : $T \mapsto T_L := \{|\alpha|_L : \alpha \in T\}$ is a isomorphism.

Definition 3.4 *Let L be a intuitionistic modal logic. For a given non-empty set X of formulas, we define $[X)$ and $(X]$ by*

$$[X) = \{\beta : L \vdash \alpha_1 \wedge \cdots \wedge \alpha_n \to \beta, \text{ for some } \alpha_1, \ldots, \alpha_n \in X\},$$

$$(X] = \{\beta : L \vdash \beta \to \alpha_1 \vee \cdots \vee \alpha_n, \text{ for some } \alpha_1, \ldots, \alpha_n \in X\}.$$

Therefore similarly to Theorem 2.21, the following theorem holds.

Theorem 3.5 *Let L be a intuitionistic modal logic. Given non-empty sets X and Y of formulas such that $[X) \cap (Y] = \emptyset$, there exists a prime L-theory T such that $X \subset T$ and $T \cap Y = \emptyset$.*

Theorem 3.6 *Let $\mathcal{M}_L = (\mathcal{F}_L, v_L)$ be the canonical model. Then for every formula α,*

$$v_L(\alpha) = \{x \in W_L : \alpha \in x\}.$$

Proof. It is similar to the proof of Proposition 2.22(2), by using v_L instead of h. ∎

Theorem 3.7 *For any $L \in$ NExt**IntK**$_{\Box\Diamond}$, $L \vdash \alpha$ iff $\mathcal{M}_L \models \alpha$.*

Proof. For any $x \in W_L$, $L \subset x$. So $\alpha \in L$ implies $x \models \alpha$. Conversely, suppose $L \nvdash \alpha$. By Theorem 3.5 there is a prime L-theory x such that $x \notin \alpha$. ∎

In order to show logic L is Kripke complete, it is sufficient that the canonical frame validates L. So we call such logic L a *canonical logic.*

Proposition 3.8

(1) If logic L_1 is a extension of logic L_2, then the canonical frame $\mathcal{F}_{L_1}$ is a generated subframe of $\mathcal{F}_{L_2}$.

(2) If L_i is canonical logic for every $i \in I$, then $\bigoplus_{i\in I} L_i$ is canonical logic.

Proof. (1). Clearly W_{L_2} contains W_{L_1}. Let $x \in W_{L_1}$ and $y \in W_{L_2}$. If $x \subset y$ then $L_1 \subset y$, and if $x_\Box \subset y$ then $L_1 \subset y$, too, since $L_1 \subset (L_1)_\Box$. Suppose $y \subset x_\Diamond$. For formula $\alpha \in y$ and formula β, if $\alpha \to \beta \in L_1$ then $\Diamond\beta \in x$ since $\Diamond\alpha \to \Diamond\beta \in L_1 \subset x$. Therefore since $[L_1 \cup y) \subset x_\Diamond$, by Theorem 3.5 there is $z \in W_{L_1}$ such that $z \subset x_\Diamond$ and $y \subset z$.

(2). By (1), $\mathcal{F}_{\oplus L_i}$ is a generated subframe of $\mathcal{F}_{L_i}$ for every $i \in I$. Since $\mathcal{F}_{L_i} \models L_i$, $\mathcal{F}_{\oplus L_i} \models L_i$ by Corollary 2.15. Thus, $\mathcal{F}_{\oplus L_i} \models \bigoplus_{i\in I} L_i$ ∎

Theorem 3.9 ([15][21]) $\mathbf{IntK}_{\Box\Diamond}\oplus\mathbf{\Gamma}$ *is a canonical logic , if* $\mathbf{\Gamma}$ *consists of some formulas in the following list.*

$$\Box(p \to q) \to (\Diamond p \to \Diamond q), \tag{3.1}$$
$$\Diamond(p \to q) \to (\Box p \to \Diamond q), \tag{3.2}$$
$$(\Diamond p \to \Box q) \to \Box(p \to q), \tag{3.3}$$
$$\Box^+(p \to q) \to (\Diamond p \to \Diamond q), \tag{3.4}$$
$$\Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q), \tag{3.5}$$
$$\Box p \to p, \tag{3.6}$$
$$p \to \Diamond p, \tag{3.7}$$
$$\Box p \to \Box\Box p, \tag{3.8}$$
$$\Diamond\Diamond p \to \Diamond p, \tag{3.9}$$
$$\Diamond^k\Box^l p \to \Box^m\Diamond^n p, \tag{3.10}$$
$$\Box p \vee \Box\neg\Box p, \tag{3.11}$$
$$\Box(\Box p \vee q) \to (\Box p \vee \Box q) \tag{3.12}$$
$$\Box(\Box p \to q) \vee \Box(\Box q \to p) \tag{3.13}$$

Therefore, $\mathbf{IntK}_{\Box\Diamond} \oplus \mathbf{\Gamma}$ is Kripke complete.

Proof. We are not going to check all these and take up what is shown in Proposition 2.12. The rest can be checked similarly.

(3.5). Let $\Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q) \in L$, and show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.5), i.e.

$$x \subset y_\Diamond \Rightarrow \exists z \exists n(x \subset z \ \&\ z \subset y_\Diamond \ \&\ y_{\Box^n} \subset z \ \&\ 0 \le n \le m).$$

Suppose $x \subset y_{\Diamond}$. Then we will show $[x \cup y_{\Box^n}) \subset y_{\Diamond}$ for some n. Suppose otherwise. Then there are formulas $\alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m, \gamma_0, \dots, \gamma_m$ such that $\alpha_n \in x$, $\Box^n\beta_n \in y, \Diamond\gamma \notin y$ and $\alpha_n \wedge \beta_n \to \gamma_n \in L$. Since

$$(\beta_0 \vee \cdots \vee \beta_m) \to ((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in L,$$

hence,

$$\Box^n(\beta_0 \vee \cdots \vee \beta_m) \to \Box^n((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in L.$$

Since $\Box^n\beta_n \in y$ implies $\Box^n(\beta_0 \vee \cdots \vee \beta_m) \in y$, hence,

$$\Box^n((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in y.$$

Since also

$$\Box^{(m)}((\alpha_0 \wedge \cdots \wedge \alpha_m) \to (\gamma_0 \vee \cdots \vee \gamma_m)) \in y,$$

hence,

$$\Diamond(\alpha_0 \wedge \cdots \wedge \alpha_m) \to \Diamond(\gamma_0 \vee \cdots \vee \gamma_m) \in y,$$

by using the axiom (3.5). Since $\alpha_0 \wedge \cdots \wedge \alpha_m \in x$ and $x \subset y_{\Diamond}$, hence, $\Diamond(\alpha_0 \wedge \cdots \wedge \alpha_m) \in y$. Therefore

$$\Diamond\gamma_0 \vee \cdots \vee \Diamond\gamma_m \in y.$$

This is a contradiction. Thus we have $[x \cup y_{\Box^n}) \subset y_{\Diamond}$ for some n. Therefore Theorem 3.5 guarantees the existence of L-theory z such that $x \subset z$, $z \subset y_{\Diamond}$ and $y_{\Box^n} \subset z$ for some n.

(3.10). Let $\Diamond^k\Box^l p \to \Box^m\Diamond^n p \in L$, for some $k, l, m, n \geq 0$, and show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.10), i.e.

$$(x_{\Box^m} \subset y \ \&\ z \subset x_{\Diamond^k}) \Rightarrow \exists u(u \subset y_{\Diamond^n} \ \&\ z_{\Box^l} \subset u).$$

Suppose $x_{\Box^m} \subset y$ and $z \subset x_{\Diamond^k}$. Hence, $x_{\Box^m\Diamond^n} \subset y_{\Diamond^n}$ and $z_{\Box^l} \subset x_{\Diamond^k\Box^l}$. On the other hand $x_{\Diamond^k\Box^l} \subset x_{\Box^m\Diamond^n}$ since $\Diamond^k\Box^l\alpha \to \Box^m\Diamond^n\alpha \in x$. Thus $z_{\Box^l} \subset y_{\Diamond^n}$. Therefore Theorem 3.5 guarantees the existence of L-theory u such that $u \subset y_{\Diamond^n}$ and $z_{\Box^l} \subset u$.

(3.12). Let $\Box(\Box p \vee q) \to (\Box p \vee \Box q) \in L$, and show that the canonical frame $\mathcal{F}_L$ satisfies the condition of (2.12), i.e.

$$(x_{\Box} \subset y \ \&\ x_{\Box} \subset z) \Rightarrow \exists u(x_{\Box} \subset u \ \&\ u \subset z \ \&\ u_{\Box} \subset y).$$

Suppose $x_{\Box} \subset y$ and $x_{\Box} \subset z$. Then we will show $x_{\Box} \cap (\{\Box\beta : \beta \notin y\} \cup z^c] = \emptyset$. Suppose otherwise. Then there are formulas $\alpha, \beta_1, \dots, \beta_n, \gamma$ such that $\Box\alpha_n \in x$, $\beta_1, \dots, \beta_n \notin y$ $\gamma \notin z$, and

$$\alpha \to \Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma \in L.$$

Since

$$\Box\alpha \to \Box(\Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma) \in L,$$

hence,

$$\Box(\Box\beta_1 \vee \ldots \vee \Box\beta_n \vee \gamma) \in x.$$

So, by using the axiom (3.12),

$$\Box\beta_1 \vee \Box(\beta_2 \vee \ldots \vee \Box\beta_n \vee \gamma) \in x.$$

By iterating this,

$$\Box\beta_1 \vee \beta_2 \vee \ldots \vee \Box\beta_n \vee \Box\gamma \in x.$$

Therefore, $\Box\beta_i \in x$ for some i or $\Box\gamma \in x$. In both cases, this implies a contradiction. Thus we have $x_\Box \cap (\{\Box\beta : \beta \notin y\} \cup z^c] = \emptyset$. Therefore Theorem 3.5 guarantees the existence of L-theory u such that $x_\Box \subset u$, $u \subset z$ and $u \cap \{\Box\beta : \beta \notin y\} = \emptyset$. The third condition implies $u_\Box \subset y$.

∎

Corollary 3.10 *In particular,* $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntK}^{+}_{\Box\Diamond}$, $\mathbf{IntK}^{*}_{\Box\Diamond}$, **FS**, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$, $\mathbf{IntS4.3}_{\Box\Diamond}$, $\mathbf{IntK5}_{\Box\Diamond}$, $\mathbf{IntS5}_{\Box\Diamond}$ *and* **MIPC** *are Kripke complete.*

3.2 Finite model property

The canonical model of L refutes all the formulas which do not belong to L. It contains continuum many points. But it is better that *finite frame* refutes all the formulas which do not belong to L. The logic L is said *finitely axiomatizable* if $L = \mathbf{IntK}_{\Box\Diamond} \oplus \mathbf{\Gamma}$ for some finite set $\mathbf{\Gamma}$ of formulas.

If L is moreover finitely axiomatizable, then it is *decidable.*

Definition 3.11 *A logic L enjoys the* finite model property *if for every non-theorem φ of L, there exists a finite frame $\mathcal{F}$ such that $\mathcal{F} \models L$ and $\mathcal{F} \not\models \varphi$.*

In the following, we will show some logic enjoys the finite model property by using filtration method.

Definition 3.12 (Filtration)

(1) Let $\mathcal{M}$ be a model and $\mathbf{\Sigma}$ be a set of formulas closed under subformulas, i.e., $\mathbf{Sub}\varphi \subset \mathbf{\Sigma}$ *whenever $\varphi \in \mathbf{\Sigma}$, where* $\mathbf{Sub}\varphi$ *is the set of all subformulas of φ. We define an equivalence relation $\sim_{\mathbf{\Sigma}}$ on W, by taking*

$$x \sim_{\mathbf{\Sigma}} y \overset{def}{\Leftrightarrow} (\mathcal{M}, x) \models \varphi \ \textit{ iff } \ (\mathcal{M}, y) \models \varphi \ , \textit{for every } \varphi \in \mathbf{\Sigma},$$

and we say x, y are $\mathbf{\Sigma}$-equivalent *in $\mathcal{M}$. We denote by $[x]_{\mathbf{\Sigma}}$ the equivalence class generated by x. We write simply $[x]$ if understood.*

(2) A model $\mathcal{M}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, R_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}, v_{\mathbf{\Sigma}})$ is called a filtration of $\mathcal{M}$ through $\mathbf{\Sigma}$ if the following conditions are satisfied.

(i) $W_{\mathbf{\Sigma}} = \{[x] : x \in W\}$,

(ii) $v_{\mathbf{\Sigma}}(p) = \{[x] : x \in v(p)\}$, *for every propositional variable* $p \in \mathbf{\Sigma}$,

(iii) xRy *implies* $[x]R_{\mathbf{\Sigma}}[y]$, *for all* $x, y \in W$,

(iv) $xR_{\Box}y$ *implies* $[x]R_{\Box\mathbf{\Sigma}}[y]$, *for all* $x, y \in W$,

(v) $xR_{\Diamond}y$ *implies* $[x]R_{\Diamond\mathbf{\Sigma}}[y]$, *for all* $x, y \in W$,

(vi) if $[x]R_{\mathbf{\Sigma}}[y]$ *then* $y \models \varphi$ *whenever* $x \models \varphi$, *for* $x, y \in W$ *and* $\varphi \in \mathbf{\Sigma}$,

(vii) if $[x]R_{\Box\mathbf{\Sigma}}[y]$ *then* $y \models \varphi$ *whenever* $x \models \Box\varphi$, *for* $x, y \in W$ *and* $\Box\varphi \in \mathbf{\Sigma}$,

(viii) if $[x]R_{\Diamond\mathbf{\Sigma}}[y]$ *then* $x \models \Diamond\varphi$ *whenever* $y \models \varphi$, *for* $x, y \in W$ *and* $\Diamond\varphi \in \mathbf{\Sigma}$.

Theorem 3.13 (Filtration)

Let $\mathcal{M}_{\mathbf{\Sigma}}$ *be a filtration of a model* $\mathcal{M}$ *through a set* $\mathbf{\Sigma}$ *of formulas. Then for every* x *in* $\mathcal{M}$ *and every formula* $\varphi \in \mathbf{\Sigma}$,

$$(\mathcal{M}, x) \models \varphi \Leftrightarrow (\mathcal{M}_{\mathbf{\Sigma}}, [x]) \models \varphi.$$

Proof. we prove by induction on the construction of φ. The basis of induction follows from (ii). Now let $\varphi = \psi \to \chi \in \mathbf{\Sigma}$. Suppose $x \models \psi \to \chi$, $[x]R_{\mathbf{\Sigma}}[y]$ and $[y] \models \psi$. Then, by (vi), $y \models \psi \to \chi$ and, by the induction hypothesis, $y \models \psi$. Hence, $y \models \chi$, again by the induction hypothesis, $[y] \models \chi$. Thus, $[x] \models \psi \to \chi$. Conversely, suppose $[x] \models \psi \to \chi$, xRy and $y \models \psi$. Then, by (iii), $[x]R_{\mathbf{\Sigma}}[y]$ and, by the induction hypothesis, $[y] \models \psi$. Hence, $[y] \models \chi$, again by the induction hypothesis, $y \models \chi$. Thus, $x \models \psi \to \chi$.

Next let $\varphi = \Box\psi \in \mathbf{\Sigma}$. Suppose $x \models \Box\psi$ and $[x]R_{\Box\mathbf{\Sigma}}[y]$. Then, by (vii), $y \models \psi$ and, by the induction hypothesis, $[y] \models \psi$. Thus, $[x] \models \Box\psi$. Conversely, suppose $[x] \models \Box\psi$ and xRy. Then, by (iv), $[x]R_{\Box\mathbf{\Sigma}}[y]$ and so $[y] \models \psi$. Hence, by the induction hypothesis, $y \models \psi$. Thus, $x \models \Box\psi$.

Let $\varphi = \Diamond\psi \in \mathbf{\Sigma}$. Suppose $x \models \Box\psi$. Then, there is y such that $xR_{\Diamond}y$ and $y \models \psi$. Hence, by (v), $[x]R_{\Diamond\mathbf{\Sigma}}[y]$ and, by the induction hypothesis, $[y] \models \psi$. Thus, $[x] \models \Diamond\psi$. Conversely, suppose $[x] \models \Diamond\psi$. Then, there is $[y]$ such that $[x]R_{\Diamond\mathbf{\Sigma}}[y]$ and $[y] \models \psi$. Hence, by the induction hypothesis, $y \models \psi$ and, by (viii), $x \models \Diamond\psi$. ∎

In general, the conditions (iii)–(viii) do not determine the binary relations uniquely. Actually, they allow us to choose any relations $R_{\mathbf{\Sigma}}, R_{\Box\mathbf{\Sigma}}, R_{\Diamond\mathbf{\Sigma}}$ in the interval $\underline{R}_{\mathbf{\Sigma}} \subset R_{\mathbf{\Sigma}} \subset \overline{R}_{\mathbf{\Sigma}}$, $\underline{R}_{\Box\mathbf{\Sigma}} \subset R_{\Box\mathbf{\Sigma}} \subset \overline{R}_{\Box\mathbf{\Sigma}}$, $\underline{R}_{\Diamond\mathbf{\Sigma}} \subset R_{\Diamond\mathbf{\Sigma}} \subset \overline{R}_{\Diamond\mathbf{\Sigma}}$, where

$$\begin{aligned}
\underline{R}_{\mathbf{\Sigma}} &= \{([x],[y]) : \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \ \&\ y \sim_{\mathbf{\Sigma}} y' \ \&\ x'Ry')\},\\
\underline{R}_{\Box\mathbf{\Sigma}} &= \{([x],[y]) : \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \ \&\ y \sim_{\mathbf{\Sigma}} y' \ \&\ x'R_{\Box}y')\},\\
\underline{R}_{\Diamond\mathbf{\Sigma}} &= \{([x],[y]) : \exists x', y'(x \sim_{\mathbf{\Sigma}} x' \ \&\ y \sim_{\mathbf{\Sigma}} y' \ \&\ x'R_{\Diamond}y')\},\\
\overline{R}_{\mathbf{\Sigma}} &= \{([x],[y]) : \forall\varphi \in \mathbf{\Sigma}(x \models \varphi \Rightarrow y \models \varphi)\},\\
\overline{R}_{\Box\mathbf{\Sigma}} &= \{([x],[y]) : \forall\Box\varphi \in \mathbf{\Sigma}(x \models \Box\varphi \Rightarrow y \models \varphi)\},\\
\overline{R}_{\Diamond\mathbf{\Sigma}} &= \{([x],[y]) : \forall\Diamond\varphi \in \mathbf{\Sigma}(y \models \varphi \Rightarrow x \models \Diamond\varphi)\}.
\end{aligned}$$

Indeed, if $[x]R_{\mathbf{\Sigma}}[y]$, $[x]R_{\Box\mathbf{\Sigma}}[y]$ and $[x]R_{\Diamond\mathbf{\Sigma}}[y]$ hold then, by (vi), (vii) and (viii), $[x]\overline{R}_{\mathbf{\Sigma}}[y]$, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[y]$ and $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[y]$, respectively. And if $[x]\underline{R}_{\mathbf{\Sigma}}[y]$, $[x]\underline{R}_{\Box\mathbf{\Sigma}}[y]$ and $[x]\underline{R}_{\Diamond\mathbf{\Sigma}}[y]$

then $x'Ry'$, $x'R_\Box y'$ and $x'R_\Diamond y'$ for some $x' \in [x], y' \in [y]$, and so, by (iii), (iv) and (v), $[x]R_{\mathbf{\Sigma}}[y]$, $[x]R_{\Box\mathbf{\Sigma}}[y]$ and $[x]R_{\Diamond\mathbf{\Sigma}}[y]$, respectively. The fact that $[x]\underline{R}_{\mathbf{\Sigma}}[y]$, $[x]\underline{R}_{\Box\mathbf{\Sigma}}[y]$ and $[x]\underline{R}_{\Diamond\mathbf{\Sigma}}[y]$ satisfy (vi), (vii) and (viii), respectively, and $[x]\overline{R}_{\mathbf{\Sigma}}[y]$, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[y]$ and $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[y]$, satisfy (iii), (iv) and (v), respectively, follows directly from definition of the valuation.

Definition 3.14 *The filtration on the frame* $\underline{\mathcal{F}}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, \underline{R}_{\mathbf{\Sigma}}, \underline{R}_{\Box\mathbf{\Sigma}}, \underline{R}_{\Diamond\mathbf{\Sigma}})$ *is called the* finest *filtration of* $\mathcal{M}$ *through* $\mathbf{\Sigma}$*,while the filtration on the frame* $\overline{\mathcal{F}}_{\mathbf{\Sigma}} = (W_{\mathbf{\Sigma}}, \overline{R}_{\mathbf{\Sigma}}, \overline{R}_{\Box\mathbf{\Sigma}}, \overline{R}_{\Diamond\mathbf{\Sigma}})$ *is called the* coarsest *filtration of* $\mathcal{M}$ *through* $\mathbf{\Sigma}$.

If $\mathbf{\Sigma}$ is finite then $W_{\mathbf{\Sigma}}$ is finite (at most $2^{|\mathbf{\Sigma}|}$), too. So, to prove the finite model property, it suffices to show the condition of the following definition.

Definition 3.15 *For every non-theorem* φ *of* L *and a model* $\mathcal{M}$ *of* L *such that* $\mathcal{M} \not\models \varphi$, *if there exists a filtration of* $\mathcal{M}$ *through a finite set* $\mathbf{\Sigma}$ *containing* φ *such that* $\mathcal{F}_{\mathbf{\Sigma}} \models L$ *then we say that* L admits filtration.

Since the following logics are sound with respect to the class of frames satisfying a property $\mathcal{P}$ and their canonical frames satisfy $\mathcal{P}$, to prove that they admit filtration it is sufficient to show that a filtration $\mathcal{F}_{\mathbf{\Sigma}}$ of $\mathcal{M}$ satisfying $\mathcal{P}$ through a finite set $\mathbf{\Sigma}$ containing a φ satisfies $\mathcal{P}$.

Theorem 3.16 $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$, $\mathbf{IntS5}_{\Box\Diamond}$ *admit filtration and so enjoy the finite model property.*

Proof. ($\mathbf{IntK}_{\Box\Diamond}$). Let $\mathcal{M}$ be a intuitionistic modal frame and $\mathbf{\Sigma}$ be $\mathbf{Sub}\varphi$.

First, we will check that the coarsest filtration is intuitionistic modal frame. We need to check whether $\overline{R}_{\mathbf{\Sigma}}$ is a partial order, $\overline{R}_{\mathbf{\Sigma}} \circ \overline{R}_{\Box\mathbf{\Sigma}} \circ \overline{R}_{\mathbf{\Sigma}} = \overline{R}_{\Box\mathbf{\Sigma}}$, and $\overline{R}_{\mathbf{\Sigma}}^{-1} \circ \overline{R}_{\Diamond\mathbf{\Sigma}} \circ \overline{R}_{\mathbf{\Sigma}}^{-1} = \overline{R}_{\Diamond\mathbf{\Sigma}}$. The reflexivity of $\overline{R}_{\mathbf{\Sigma}}$ follows from (iii). The anti-symmetry of $\overline{R}_{\mathbf{\Sigma}}$ follows from (vi) and the definition of $\mathbf{\Sigma}$-equivalent. Suppose $[x]\overline{R}_{\mathbf{\Sigma}}[y]\overline{R}_{\mathbf{\Sigma}}[z]$ and $\psi \in \mathbf{\Sigma}$. If $x \models \psi$, then $y \models \psi$. Hence $z \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[z]$. Suppose $[x]\overline{R}_{\mathbf{\Sigma}}[y]\overline{R}_{\Box\mathbf{\Sigma}}[z]\overline{R}_{\mathbf{\Sigma}}[w]$, and $\Box\psi \in \mathbf{\Sigma}$. If $x \models \Box\psi$, then $y \models \Box\psi$. Hence $z \models \psi$. So, $w \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[w]$. Suppose $[x]\overline{R}_{\mathbf{\Sigma}}^{-1}[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[z]\overline{R}_{\mathbf{\Sigma}}^{-1}[w]$, and $\Diamond\psi \in \mathbf{\Sigma}$. If $w \models \psi$, then $z \models \psi$. Hence $y \models \Diamond\psi$. So, $x \models \Diamond\psi$. Thus, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[w]$.

Next, we will consider general filtration. The reflexivity and the anti-symmetry of $R_{\mathbf{\Sigma}}$ is shown in the same way. But, $R_{\mathbf{\Sigma}}$ may be non-transitive. To construct a transitive relation we can take the transitive closure $\hat{R}_{\mathbf{\Sigma}}$. Clearly, $\hat{R}_{\mathbf{\Sigma}}$ satisfies (iii). By the transitivity of $\overline{R}_{\mathbf{\Sigma}}$, $\hat{R}_{\mathbf{\Sigma}}$ satisfies (vi). We also take $R^{\star}_{\Box\mathbf{\Sigma}}$ and $R^{\star}_{\Diamond\mathbf{\Sigma}}$ by

$$R^{\star}_{\Box\mathbf{\Sigma}} = \hat{R}_{\mathbf{\Sigma}} \circ R_{\Box\mathbf{\Sigma}} \circ \hat{R}_{\mathbf{\Sigma}}$$

and

$$R^{\star}_{\Diamond\mathbf{\Sigma}} = \hat{R}_{\mathbf{\Sigma}}^{-1} \circ R_{\Diamond\mathbf{\Sigma}} \circ \hat{R}_{\mathbf{\Sigma}}^{-1}.$$

It is easily shown that these satisfy (iv), (v), (vii) and (viii). The frame $(W_{\mathbf{\Sigma}}, \hat{R}_{\mathbf{\Sigma}}, R^{\star}_{\Box\mathbf{\Sigma}}, R^{\star}_{\Diamond\mathbf{\Sigma}})$ is what we desire.

($\mathbf{IntK4}_{\Box\Diamond}$). Let $\mathcal{M}$ be a $R_\Box, R_\Diamond$ transitive frame and

$$\mathbf{\Sigma} = \mathbf{Sub}\varphi \cup \{\Box\Box\psi : \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi : \Diamond\psi \in \mathbf{Sub}\varphi\}.$$

First, we will check that the coarsest filtration is $\overline{R}_{\Box\mathbf{\Sigma}}, \overline{R}_{\Diamond\mathbf{\Sigma}}$ transitive frame.

Suppose $[x]\overline{R}_{\Box\mathbf{\Sigma}}[y]\overline{R}_{\Box\mathbf{\Sigma}}[z]$ and $\Box\psi \in \mathbf{\Sigma}$. In the case $\Box\psi \in \mathbf{Sub}\varphi$, if $x \models \Box\psi$, then $x \models \Box\Box\psi$ by transitivity. Since $\Box\Box\psi \in \mathbf{\Sigma}$, $y \models \Box\psi$. Hence, $z \models \psi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[z]$. In the case $\Box\psi = \Box\Box\chi \in \{\Box\Box\chi : \Box\chi \in \mathbf{Sub}\varphi\}$, if $x \models \Box\Box\chi$, then $y \models \Box\chi$, since $\Box\Box\chi \in \mathbf{\Sigma}$. By transitivity, $y \models \Box\Box\chi$. Hence, $z \models \Box\chi$. Thus, $[x]\overline{R}_{\Box\mathbf{\Sigma}}[z]$.

Suppose $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[z]$ and $\Diamond\psi \in \mathbf{\Sigma}$. In the case $\Diamond\psi \in \mathbf{Sub}\varphi$, if $z \models \psi$, then $y \models \Diamond\psi$. Since $\Diamond\Diamond\psi \in \mathbf{\Sigma}$, $x \models \Diamond\Diamond\psi$. Hence, $x \models \Diamond\psi$, by transitivity. Thus, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[z]$. In the case $\Diamond\psi = \Diamond\Diamond\chi \in \{\Diamond\Diamond\chi : \Diamond\chi \in \mathbf{Sub}\varphi\}$, if $z \models \Diamond\chi$, then $y \models \Diamond\Diamond\chi$. By transitivity, $y \models \Diamond\chi$. Since $\Diamond\Diamond\chi \in \mathbf{\Sigma}$, $x \models \Diamond\Diamond\chi$. Thus, $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[z]$.

Next, we will consider finest filtration. We take $\underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}}$ and $\underline{R}^{\hat{\star}}_{\Diamond\mathbf{\Sigma}}$ by

$$\underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}} = \hat{\underline{R}}_{\mathbf{\Sigma}} \circ (\widehat{\underline{R}_{\Box\mathbf{\Sigma}} \circ \hat{\underline{R}}_{\mathbf{\Sigma}}})$$

and

$$\underline{R}^{\hat{\star}}_{\Diamond\mathbf{\Sigma}} = \hat{\underline{R}}_{\mathbf{\Sigma}}^{-1} \circ (\widehat{\underline{R}_{\Diamond\mathbf{\Sigma}} \circ \hat{\underline{R}}_{\mathbf{\Sigma}}^{-1}}).$$

It is easily shown that these satisfy (iv) and (v).
Suppose $[x]\underline{R}_{\Box\mathbf{\Sigma}}[y]$, $\Box\psi \in \mathbf{\Sigma}$. Then, there exist x', y' such that $x' \in [x], y' \in [y]$ and $x'R_\Box y'$. If $x \models \Box\psi$, $x' \models \Box\psi$. Since $R_\Box$ is transitive, $y' \models \Box^+\psi$. Hence $y \models \Box^+\psi$. By iterating this argument, for any $\Box\psi \in \mathbf{\Sigma}$, if $[x]\underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}}[y]$ and $x \models \Box\psi$, then $y \models \Box^+\psi$. Hence, $y \models \Box^+\psi$. Thus, $\underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}}$ satisfies (vii).
$\underline{R}^{\hat{\star}}_{\Diamond\mathbf{\Sigma}}$ satisfies (viii) in the same way.
The frame $(W_{\mathbf{\Sigma}}, \hat{\underline{R}}_{\mathbf{\Sigma}}, \underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}}, \underline{R}^{\hat{\star}}_{\Diamond\mathbf{\Sigma}})$ is what we desire.

($\mathbf{IntS4}_{\Box\Diamond}$). $R_\Box$ and $R_\Diamond$ are reflexive, then by (iv) and (v), any filtration $R_{\Box\mathbf{\Sigma}}$ and $R_{\Diamond\mathbf{\Sigma}}$ are reflexive, respectively. Moreover, $\underline{R}^{\hat{\star}}_{\Box\mathbf{\Sigma}} = \hat{\underline{R}}_{\Box\mathbf{\Sigma}}$ and $\underline{R}^{\hat{\star}}_{\Diamond\mathbf{\Sigma}} = \hat{\underline{R}}_{\Diamond\mathbf{\Sigma}}$.

($\mathbf{IntS5}_{\Box\Diamond}$). Let $\mathcal{M}$ be a $R_\Box, R_\Diamond$ reflexive transitive frame such that $R_\Box = R_\Diamond^{-1}$. Put

$$\mathbf{\Sigma} = \mathbf{Sub}\varphi \cup \{\Box\Box\psi, \Diamond\Box\psi : \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi, \Box\Diamond\psi : \Diamond\psi \in \mathbf{Sub}\varphi\}.$$

First, we will check that the coarsest filtration satisfy $\overline{R}_{\Box\mathbf{\Sigma}} = \overline{R}^{-1}_{\Diamond\mathbf{\Sigma}}$. Notice that by properties of $\mathbf{IntS5}_{\Box\Diamond}$, if $\psi = \Box\chi$ or $\Diamond\chi$ for some χ then

$$x \models \psi \text{ iff } x \models \Box\psi \text{ iff } x \models \Diamond\psi.$$

Suppose $[x]\overline{R}_{\Box\mathbf{\Sigma}}[y]$ and $\Diamond\psi \in \mathbf{\Sigma}$. In the case $\Diamond\psi \in \mathbf{Sub}\varphi$, if $x \models \psi$, then by the reflexivity of $R_\Diamond$, $x \models \Diamond\psi$. Hence, $x \models \Box\Diamond\psi$. So, since $\Box\Diamond\psi \in \mathbf{\Sigma}$, $y \models \Diamond\psi$. Thus, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[x]$. In the case $\Diamond\psi \in \{\Box\Box\psi, \Diamond\Box\psi : \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi, \Box\Diamond\psi : \Diamond\psi \in \mathbf{Sub}\varphi\}$, if $x \models \psi$, then $x \models \Box\psi$. Since $\Box\psi \in \mathbf{\Sigma}$, $y \models \psi$. Hence, $y \models \Diamond\psi$. Thus, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[x]$.
Suppose $[x]\overline{R}_{\Diamond\mathbf{\Sigma}}[y]$ and $\Box\psi \in \mathbf{\Sigma}$. In the case $\Box\psi \in \mathbf{Sub}\varphi$, if $y \models \Box\psi$, then $x \models \Diamond\Box\psi$, since $\Diamond\Box\psi \in \mathbf{\Sigma}$. Hence, $x \models \Box\psi$. By the reflexivity of $R_\Box$, $x \models \psi$. Thus, $[y]\overline{R}_{\Box\mathbf{\Sigma}}[x]$. In the case $\Diamond\psi \in \{\Box\Box\psi, \Diamond\Box\psi : \Box\psi \in \mathbf{Sub}\varphi\} \cup \{\Diamond\Diamond\psi, \Box\Diamond\psi : \Diamond\psi \in \mathbf{Sub}\varphi\}$, if $y \models \Box\psi$, then $y \models \psi$. Since $\Diamond\psi \in \mathbf{\Sigma}$, $x \models \Diamond\psi$. Hence, $x \models \psi$. Thus, $[y]\overline{R}_{\Diamond\mathbf{\Sigma}}[x]$.

Next, we will consider finest filtration. Since it is easily seen that $\underline{R}_{\Diamond\mathbf{\Sigma}} = \underline{R}_{\Diamond\mathbf{\Sigma}}^{-1}$, so $\hat{\underline{R}}_{\Box\mathbf{\Sigma}} = \hat{\underline{R}}_{\Diamond\mathbf{\Sigma}}^{-1}$.
Therefore, the frame $(W_{\mathbf{\Sigma}}, \hat{\underline{R}}_{\mathbf{\Sigma}}, \hat{\underline{R}}_{\Box\mathbf{\Sigma}}, \hat{\underline{R}}_{\Diamond\mathbf{\Sigma}})$ is what we desire.

∎

Remark So far, we have treated the logics of $\mathcal{L}_{\Box\Diamond}$. But, if we restrict the modal operators only to $\Box$ operator, Some logics can admit filtration. For example, we fail to show that $\mathbf{IntS4.3}_{\Box\Diamond}$ admits filtration. But $\mathbf{IntS4.3}_{\Box}$ on which the modal operator is restricted to $\Box$ operator admits filtration, because we can take $\Box$-rooted countermodel.

Chapter 4

Algebraic properties

As previously stated, $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \oplus, \cap)$ and its dually isomorphic $(\Lambda(\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}), \vee, \wedge)$ are complete lattices. In this chapter, we will further investigate properties about them for example, distributivity.

4.1 The deduction theorem

Unfortunately, $\mathbf{IntK}_{\Box\Diamond}$ have bad properties. The deduction theorem which implies distributivity does not hold for $\mathbf{IntK}_{\Box\Diamond}$. Recall that a *derivation* of φ from assumption $\mathbf{\Gamma}$ is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that $\varphi_n = \varphi$ and for every i, $1 \le i \le n$, φ_i is either an axiom, an assumption or obtained from some of the preceding formulas in the sequence by one of the inference rules. The deduction theorem for modal logic L is as follows.

The deduction theorem for modal logic L

Suppose $\mathbf{\Gamma}, \psi \vdash \varphi$ and there exists a derivation of φ from the assumptions $\mathbf{\Gamma} \cup \{\psi\}$. Then

$$\mathbf{\Gamma}, \psi \vdash \varphi \ \text{ iff } \ \mathbf{\Gamma} \vdash \Box^{(k)}\psi \to \varphi, \ \text{for some } k \in \boldsymbol{N}$$

The deduction theorem for intuitionistic modal logic L holds under the following condition.

Theorem 4.1 (Bezhanishvili and Hasimoto) *For any $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$, L enjoys the deduction theorem iff*

$$\vdash \Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q), \ \textit{for some } m \in \boldsymbol{N}$$

Proof. Suppose the deduction theorem holds. Since $p \to q \vdash \Diamond p \to \Diamond q$, so $\vdash \Box^{(m)}(p \to q) \to (\Diamond p \to \Diamond q)$, for some $m \in \boldsymbol{N}$. Conversely, we consider a derivation $\varphi_1, \ldots, \varphi_n$ of φ from $\mathbf{\Gamma} \cup \{\psi\}$, and show by induction on i that

$$\mathbf{\Gamma} \vdash \Box^{(m_i)}\psi \to \varphi_i, \ \text{for some } m_i \in \boldsymbol{N}.$$

The cases when φ_i is a substitution instance of an axiom φ_j or belongs to $\mathbf{\Gamma} \cup \{\psi\}$ and obtained from **MP** or $\mathbf{RR}_{\Box}$ are justified in the same way as in classical modal logic. Suppose $\varphi_i = \Diamond\varphi_j \to \Diamond\varphi_k$ is obtained from $\varphi_j \to \varphi_k$ by $\mathbf{RR}_{\Diamond}$. Then, by the induction hypothesis,

$$\mathbf{\Gamma} \vdash \Box^{(m_1)}\psi \to (\varphi_j \to \varphi_k), \text{ for some } m_1 \in \boldsymbol{N}.$$

Hence,

$$\mathbf{\Gamma} \vdash \Box^{(m_1+m)}\psi \to \Box^{(m)}(\varphi_j \to \varphi_k).$$

Thus,

$$\mathbf{\Gamma} \vdash \Box^{(m_1+m)}\psi \to \Diamond(\varphi_j \to \varphi_k).$$

∎

Corollary 4.2 $\mathbf{IntK}^*_{\Box\Diamond}$ *enjoys the deduction theorem.*

4.2 Filters and congruences

We consider this in the algebraic point of view. In this situation, $\Diamond$ is neglected in some sence. Hence, we can use the theory of the algebra with $\Box$.

Definition 4.3 *A filter F in a modal Heyting algebra $\boldsymbol{A}$ is said to be a* $\Box$-filter *if*

$$a \to b \in F \text{ implies } \Box a \to \Box b \in F.$$

If F also satisfies

$$a \to b \in F \text{ implies } \Diamond a \to \Diamond b \in F,$$

then F is said to be a modal filter. *We denote by $\boldsymbol{F}_{\Box}(\boldsymbol{A})$ and $\boldsymbol{F}_M(\boldsymbol{A})$ all $\Box$-filters and all modal filters in $\boldsymbol{A}$, respectively.*

It is easily seen that if a filter F in a modal Heyting algebra $\boldsymbol{A}$ with $\Box(a \to b) \leq \Diamond a \to \Diamond b$ (or $\Box^{(n)}(a \to b) \leq \Diamond a \to \Diamond b$, for some $n \in \boldsymbol{N}$), then,

$$F \text{ is a modal filter} \quad \text{iff} \quad F \text{ is a } \Box\text{-filter} \qquad \text{(c.f. Theorem 4.1).}$$

By $\boldsymbol{m}^*_{\Box\Diamond}\boldsymbol{HA}$, we denote the variety of all modal Heyting algebras with $\Box(a \to b) \leq \Diamond a \to \Diamond b$. For $n \in \boldsymbol{N}$, we denote by $\boldsymbol{m}^{(n)}_{\Box\Diamond}\boldsymbol{HA}$, the variety of all modal Heyting algebras with $\Box^{(n)}(a \to b) \leq \Diamond a \to \Diamond b$.

Proposition 4.4 *For any $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, the set $(\boldsymbol{F}_M(\boldsymbol{A}), \vee, \cap)$ forms a complete lattice, where $F_1 \vee F_2$ is the smallest modal filter containing F_1 and F_2. And the set $(\boldsymbol{F}_{\Box}(\boldsymbol{A}), \vee, \cap)$ forms a complete distributive lattice, where $F_1 \vee F_2$ is the smallest $\Box$-filter containing F_1 and F_2.*

Proof. We will show the distributivity. Since $F_1 \vee F_2$ is $[F_1 \cup F_2)$ in $\Box$-filters, we will show $[F_1 \cup F_2) \cap F_3 \subset [(F_1 \cap F_3) \cup (F_2 \cap F_3))$. Suppose $a \in F_3$ such that $b \wedge c \leq a$ for some $b \in F_1$ and $c \in F_2$. Then, $(b \vee a) \wedge (c \vee a) \leq a$. Since $b \vee a \in F_1 \cap F_3$ and $c \vee a \in F_2 \cap F_3$, $a \in [(F_1 \cap F_3) \cup (F_2 \cap F_3))$. ∎

For any $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, we denote by $\Theta(\boldsymbol{A})$ the set of all congruence relations on $\boldsymbol{A}$.

Proposition 4.5 *For any $\boldsymbol{A} \in \boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$, the set $(\Theta(\boldsymbol{A}), \vee, \cap)$ forms a complete distributive lattice, where $\theta_1 \vee \theta_2$ is the smallest congruence relation containing θ_1 and θ_2.*

Proof. We will show the distributivity. It is easily seen that $\theta_1 \vee \theta_2 = (\widehat{\theta_1 \cup \theta_2})$. So if $a((\theta_1 \vee \theta_2) \cap \theta_3)b$, then there is sequence $a = z_0, z_1, \ldots, z_n = b$ such that $a\theta_3 b$ and $z_0\eta_1 z_1\eta_2 \ldots \eta_n z_n$, for $\eta_i = \theta_1$ or θ_2. Define

$$m(a, b, z_i) := (a \wedge b) \vee (b \wedge z_i) \vee (z_i \wedge a).$$

Then $m(a, b, a) = a$ and $m(a, b, b) = b$. If $z\eta_i z'$ then $m(a, b, z)\eta_i m(a, b, z')$. We claim that $a\theta_3 b$ implies $m(a, b, z)\theta_3 m(a, b, z')$. Suppose $a\theta_3 b$. Then, $a \vee (z \wedge a)\theta_3(a \wedge b) \vee (z \wedge a)$. Since $(z \wedge a)\theta_3(z \wedge b)$ also holds, $a \vee (z \wedge a)\theta_3(a \wedge b) \vee (z \wedge b) \vee (z \wedge a)$. Hence, $a\theta_3 m(a, b, z)$. Therefore, $m(a, b, z)\theta_3 m(a, b, z')$. Thus, there is sequence $a = m(a, b, z_0), m(a, b, z_1), \ldots, m(a, b, z_n) = b$ such that $m(a, b, z_0)(\eta_1 \cap \theta_3)m(a, b, z_1)(\eta_2 \cap \theta_3) \ldots (\eta_n \cap \theta_3)m(a, b, z_n)$. Therefore, $a((\theta_1 \cap \theta_3) \vee (\theta_2 \cap \theta_3))b$ ∎

The correspondence between these complete distributive lattices is as follows.

Theorem 4.6 *Suppose $\boldsymbol{A} \in \boldsymbol{m}^*_{\Box\Diamond}\boldsymbol{HA}$ (or $\boldsymbol{m}^{(n)}_{\Box\Diamond}\boldsymbol{HA}$ for some $n \in \boldsymbol{N}$). Then the map*

$$F \mapsto \theta_F := \{(a, b) : (a \to b) \wedge (b \to a) \in F\}$$

is an isomorphism from $(\boldsymbol{F}_M(\boldsymbol{A}), \vee, \cap)$ onto $(\Theta(\boldsymbol{A}), \vee, \cap)$. Here the inverse map from $(\Theta(\boldsymbol{A}), \vee, \cap)$ onto $(\boldsymbol{F}_M(\boldsymbol{A}), \vee, \cap)$ is given by

$$\theta \mapsto F_\theta := \{a \in A : a\theta\top\}.$$

Proof. By the definition, the monotonicities of both maps are clear. The relation θ_F is an equivalence relation which is compatible with $\wedge, \vee$ and $\to$. We will show that it is compatible with $\Box$ and $\Diamond$. Suppose $a\theta_F b$. Then, $a \to b \in F$ and $b \to a \in F$. Hence, $\Box a \to \Box b \in F$, $\Box b \to \Box a \in F$, $\Diamond a \to \Diamond b \in F$ and $\Diamond b \to \Diamond a \in F$. Thus, $\Box a\ \theta_F \Box b$ and $\Diamond a\ \theta_F \Diamond b$. It is easily seen that F_θ is a filter. We will show that it is a $\Box$-filter, which implies that it is a modal filter. Suppose $a \to b \in F_\theta$. Then, $a \to b\theta\top$. Hence, $\Box(a \to b)\ \theta\Box\top$. Therefore, $\Box a \to \Box b\ \theta\top$. Thus, $\Box a \to \Box b \in F_\theta$. We will show that $F_{\theta_F} = F$. By the definition, $a \in F_{\theta_F}$ iff $a\theta_F\top$ iff $(a \to \top) \wedge (\top \to a) \in F$ iff $a \in F$. We will show that $\theta_{F_\theta} = \theta$. By the definition,

$$\begin{array}{lll} a\theta_{F_\theta} b & \text{iff} & (a \to b) \wedge (b \to a) \in F_\theta \\ & \text{iff} & (a \to b) \wedge (b \to a)\theta\top \\ & \text{iff} & a \to b\ \theta\ \top\ \&\ b \to a\ \theta\ \top \\ & \text{iff} & a\theta a \vee b\ \&\ b\theta a \vee b \\ & \text{iff} & a\theta b \end{array}$$

$\blacksquare$

4.3 Congruence distributivity

In Proposition 4.5 we showed $\Theta(\boldsymbol{A})$ is distributive.

Definition 4.7 *A variety $\mathcal{K}$ is* congruence distributive *if for every $\boldsymbol{A}\in \mathcal{K}$, $\Theta(\boldsymbol{A})$ is distributive.*

This property is useful. Recall that an algebra $\boldsymbol{A}$ is *subdirectly irreducible* if it has a smallest non-trivial congruence. Denote by $\mathcal{K}_{SI}$ the subdirectly irreducible members of a variety $\mathcal{K}$. When a variety is congruence distributive, subdirectly irreducible algebras play an important role because we can apply Jónsson's Lemma [10]. By using Jónsson's Lemma,

$$(\mathcal{K}_1 \vee \mathcal{K}_2)_{SI} = (\mathcal{K}_1)_{SI} \cup (\mathcal{K}_2)_{SI}$$

for any subvarieties $\mathcal{K}_1, \mathcal{K}_2$ of a congruence distributive variety $\mathcal{K}$ [10]. Therefore, for any subvarieties $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$,

$$((\mathcal{K}_1 \vee \mathcal{K}_2) \cap \mathcal{K}_3)_{SI} = ((\mathcal{K}_1 \cap \mathcal{K}_3) \vee (\mathcal{K}_2 \cap \mathcal{K}_3))_{SI}.$$

Since the subdirectly irreducible members of a variety $\mathcal{K}$ determine the variety $\mathcal{K}$, we have the following.

Proposition 4.8 ([10]) *If a variety $\mathcal{K}$ is congruence distributive then the lattice of all subvarieties of $\mathcal{K}$ is distributive.*

Proposition 4.9 $(\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}, \oplus, \cap)$ *is a complete distributive algebraic lattice. Moreover the join infinite distributive law*

$$L \cap \bigoplus_{i\in I} L_i = \bigoplus_{i\in I}(L \cap L_i)$$

holds.

Proof. It is easily verified $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ is algebraic lattice whose compact element is finitely axiomatizable logic. We will show join infinite distributivity. Since $\mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ is algebraic, there are compact elements L'_j such that

$$\bigoplus_{j\in J} L'_j = L \cap \bigoplus_{i\in I} L_i.$$

Hence, $L'_j \subset L$ and $L'_j \subset \bigoplus_{i\in I} L_i$. Since L'_j is compact, there is a finite subset I_0 of I such that $L'_j \subset \bigoplus_{i\in I_0} L_i$. Therefore,

$$\begin{aligned} L'_j &\subset L \cap \bigoplus_{i\in I_0} L_i \\ &= \bigoplus_{i\in I_0}(L \cap L_i) \\ &\subset \bigoplus_{i\in I}(L \cap L_i). \end{aligned}$$

Thus,

$$L \cap \bigoplus_{i\in I} L_i = \bigoplus_{i\in I}(L \cap L_i).$$

∎

But the meet infinite distributive low

$$L \oplus \bigcap_{i\in I} L_i = \bigcap_{i\in I}(L \oplus L_i)$$

does not hold in NExt$\mathbf{IntK}_{\Box\Diamond}$. Indeed, suppose otherwise. Then all logics in NExt**MIPC** which is a complete sublattice of NExt$\mathbf{IntK}_{\Box\Diamond}$ enjoy finite model property [2], which is not the case.

4.4 Congruence extension property

Definition 4.10 *A variety $\mathcal{K}$ has the* congruence extension property *if each congruence in a subalgebra of an algebra can be extended to a congruence of the algebra itself.*

An important property of variety $\mathcal{K}$ with congruence extension property is that $\boldsymbol{HS}(\mathcal{K}_0) = \boldsymbol{SH}(\mathcal{K}_0)$, for each set $\mathcal{K}_0 \subset \mathcal{K}$.

Theorem 4.11 *The variety $\boldsymbol{m}^{*}_{\Box\Diamond}\boldsymbol{HA}$ (and $\boldsymbol{m}^{(n)}_{\Box\Diamond}\boldsymbol{HA}$ for any $n \in \boldsymbol{N}$) has the congruence extension property.*

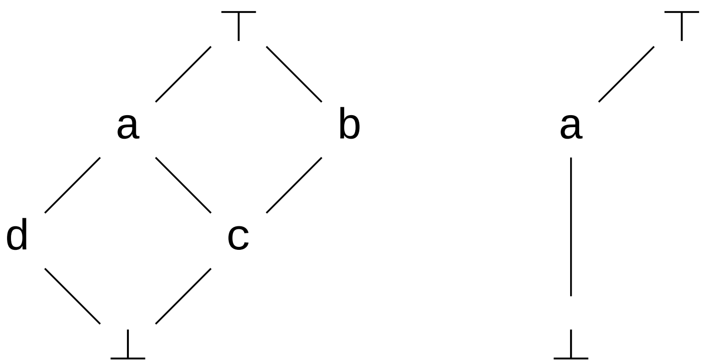
Proof. Since $\Theta(\boldsymbol{A})$ and $\boldsymbol{F}_M(\boldsymbol{A})$ are isomorphic, it is sufficient to show that each modal filter in a subalgebra of an algebra can be extended to a modal filter of the algebra itself. Suppose that $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$ and F is a modal filter in $\boldsymbol{A}$. Define $[F)$ by taking

$$[F) = \{b \in \boldsymbol{B} : a \leq b \text{ for some } a \in \boldsymbol{A}\}.$$

Then, $[F)$ is a $\Box$-filter (hence, modal filter) and $[F) \cap F = F$. ∎

Theorem 4.12 *The variety $\boldsymbol{m}_{\Box\Diamond}\boldsymbol{HA}$ does not have the congruence extension property.*

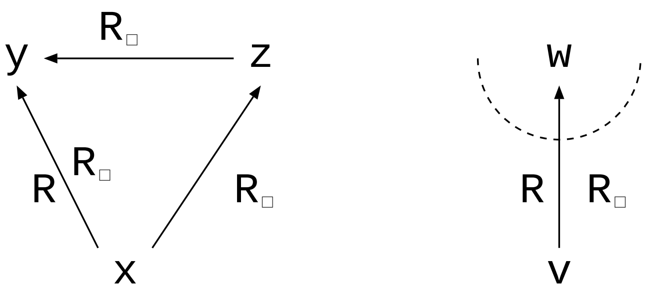
Proof. Denote the algebras below by $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, from left to right.



$\Box\top = \top$	$\Diamond\top = \top$
$\Box a = a$	$\Diamond a = a$
$\Box b = c$	$\Diamond b = \top$
$\Box c = c$	$\Diamond c = c$
$\Box d = \bot$	$\Diamond d = a$
$\Box\bot = \bot$	$\Diamond\bot = \bot$

$\boldsymbol{A}_2$ is a subalgebra of $\boldsymbol{A}_1$. $\boldsymbol{A}_1$ has two congruences, namely the least one $\{(a, a) : a \in \boldsymbol{A}_1\}$ and the greatest one $\boldsymbol{A}_1^2$. But $\boldsymbol{A}_2$ has three congruences. ∎

Using duality, we will see this example again. Denote the algebras below by $\mathcal{F}_1$ and $\mathcal{F}_2$, from left to right.



$x \mapsto v$
$y \mapsto w$
$z \mapsto w$

All the points are reflexive. $R_\Diamond$ is the reverse of $R_\Box$. $\boldsymbol{A}_1$ and $\mathcal{F}_1$ are dual each other and $\boldsymbol{A}_2$ and $\mathcal{F}_2$ are dual each other. $\mathcal{F}_2$ is reducible to $\mathcal{F}_1$ by the map: $x \mapsto v, y \mapsto w$ and $z \mapsto w$. $\{w\}$ is a generated subframe of $\mathcal{F}_2$. But any generated subframe of $\mathcal{F}_1$ is not reducible to $\{w\}$ by the restriction of the map.

This example also shows that the variety corresponding to $\mathbf{IntS5}_{\Box\Diamond}$ does not have the congruence extension property.

Chapter 5

Conclusions and remarks

In this thesis, we verified that like the classical modal case the intuitionistic modal logics adequate algebraic semantics, and algebraic semantics corresponds to Kripke type semantics. We also verified that important logics are canonical logics, so they are Kripke complete.

We have shown that $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$ and $\mathbf{IntS5}_{\Box\Diamond}$ enjoy the finite model property. But, in the classical modal case, it is known that much more logics enjoy the finite model property. For each logic, It is proved that the logic enjoys the finite model property, not only by the filtration method, but by various methods — by algebraic method and by well selecting points from frame. In the intuitionistic modal caes, owing to $\Diamond$ operator, it is more complecated. But, it is interesting future subject to invesigate that much more logics enjoy the finite model property.

Unfortunately, $\mathbf{IntK}_{\Box\Diamond}$ does not satisfy the deduction theorem and does not enjoy congruence extension property. But it seems to $\Diamond$ operator is not essential in $\mathbf{IntK}^*_{\Box\Diamond}$.

Therefore, we have to invesigate $\mathbf{IntK}_{\Box\Diamond}$ in the various points of view in order to see how $\Diamond$ operator behave.

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