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A Semantical Study of Intuitionistic Modal Logics

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1 Introduction

Modal logics based on classical logic **Cl** have been investigated well. Classical logics are too strong from the computer scientific or constructive mathematical point of view. So we want to weaken logics. But the negation of classical logics is stronger than that of intuitionistic logics. Hence, in classical logics $\Box p \leftrightarrow \neg\Diamond\neg p$ holds, but in intuitionistic logics $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ do not generally hold. This provides more possibilities for defining intuitionistic modal logics. We will consider intuitionistic modal logics to be independent $\Box$ and $\Diamond$.

Let $\mathcal{L}_{\Box\Diamond}$ be the language of propositional modal logic with countably many propositional variables, $p, q, r, \ldots$ and the connectives $\wedge, \vee, \rightarrow, \bot, \Box, \Diamond$. Let $\mathcal{F}orm(\mathcal{L}_{\Box\Diamond})$ be the set of all formulas of $\mathcal{L}_{\Box\Diamond}$. The formula $\neg\alpha$ is defined as $\alpha \rightarrow \bot$ and $\top$ as $\bot \rightarrow \bot$.

How to define an intuitionistic modal analogue of classical normal modal logic **K** ? Much work has been done in the field.

By the study of correspondence to the bi-modal logic with two box operators, Fischer Servi [1][2] constructed a logic **FS** by imposing a weak connection between $\Box$ and $\Diamond$ operators. **FS** is the least set of formulas of $\mathcal{L}_{\Box\Diamond}$ which contains axioms (1)–(6) and is closed under the rules of inference (a)–(c)

(1) the intuitionistic logic **Int**,

(2) $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$,

(3) $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$,

(4) $\neg\Diamond\bot$,

(5) $\Diamond(p \to q) \to (\Box p \to \Diamond q)$,

(6) $(\Diamond p \to \Box q) \to \Box(p \to q)$,

(a) modus ponens $\frac{\vdash \alpha \to \beta \quad \vdash \alpha}{\vdash \beta}$ (**MP**),

(b) substitution (**Sub**),

(c) $\frac{\vdash \alpha}{\vdash \Box\alpha}$ (**RN**).

On the other hand, Wolter and Zakharyaschev[3][4] introduced the weakest intuitionistic modal logic $\mathbf{IntK}_{\Box\Diamond}$. $\mathbf{IntK}_{\Box\Diamond}$ is the least set of formulas of $\mathcal{L}_{\Box\Diamond}$ which contains axioms (1)–(3) and is closed under the rules of inference (a)–(c).

(1) the intuitionistic logic **Int**,

(2)

$(2_\Box)$ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ and $(2_\Diamond)$ $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$,

(3)

$(3_\Box)$ $\Box\top$ and $(3_\Diamond)$ $\neg\Diamond\bot$,

(a) modus ponens $\frac{\vdash \alpha \to \beta \quad \vdash \alpha}{\vdash \beta}$ (**MP**),

(b) substitution (**Sub**),

(c)

$\frac{\vdash \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta}$ $(\mathbf{RR}_\Box)$ and $\frac{\vdash \alpha \to \beta}{\vdash \Diamond\alpha \to \Diamond\beta}$ $(\mathbf{RR}_\Diamond)$.

Our goal is that by extending from the weakest logic $\mathbf{IntK}_{\Box\Diamond}$, we investigate what properties each logic has, and determine which logic is the best in some sense.

2 Semantics

A set L of formulas of $\mathcal{L}_{\Box\Diamond}$ is said an *intuitionistic modal logic* if L contains $\mathbf{IntK}_{\Box\Diamond}$ and is closed under the rules of inference (a)–(c).

For intuitionistic modal logics, we consider algebraic semantics and Kripke type semantics. Algebraic semantics connects each formula with a element in modal Heyting algebra. An algebra $\boldsymbol{A} = (A, \Box, \Diamond)$ is called a $\Box\Diamond$-*modal Heyting algebra* if the following conditions are satisfied.

(1) A is a *Heyting algebra*,

(2) $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,

(3) $\Box\top = \top$ and $\Diamond\bot = \bot$.

On the other hand, Kripke type semantics connect each formula with a cone on intuitionistic modal frame. A structure $\mathcal{F} = (W, R, R_\Box, R_\Diamond)$ is called an intuitionistic modal frame if the following conditions are satisfied.

(1) $W \neq \emptyset$,

(2) R : *a partial order on* W,

(3) $R_\Box, R_\Diamond$: binary relations on W,

(4) $R \circ R_\Box \circ R = R_\Box$,

(5) $R^{-1} \circ R_\Diamond \circ R^{-1} = R_\Diamond$, where R^{-1} is the reverse of R.

3 Kripke completeness

Algbraic semantics is adequate for logics, but Kripke type semantics is not adequate. So, we are interested in Kripke completeness.

By correspondence between logics and frame conditions, it was shown that important logics are *canonical logics.*

But with regard to finite model property, $\Diamond$ operator behaves badly. By the filtration method, we show the followings.

Theorem $\mathbf{IntK}_{\Box\Diamond}$, $\mathbf{IntK4}_{\Box\Diamond}$, $\mathbf{IntS4}_{\Box\Diamond}$ and $\mathbf{IntS5}_{\Box\Diamond}$ admit filtration and so enjoy the finite model property.

4 Conclusions and remakes

Although intuitionistic modal logics have some properties like the classical case, $\mathbf{IntK}_{\Box\Diamond}$ have bad properties. The deduction theorem does not hold for $\mathbf{IntK}_{\Box\Diamond}$.

The Variety corresponding to $\mathbf{IntK}_{\Box\Diamond}$ does not have *congruence extension property.* But these may be hints about the properties of $\Diamond$ operator. In the future, we want to show the finite model properties of more logics.

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