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Investigation of Modal Logics with Application to Agent Communication

By Norihiro Arai

A project paper submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
for the degree of
Master of Information Science
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Written under the direction of
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Professor Satoshi Tojo

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Abstract

This paper presents basic knowledge of modal logics as a preliminary stage of studying epistemic logic. Modal logic is a logic based on classical logic. The most important characteristic of this logic is that it could deal with necessity ("it is necessary") and probability ("it is possible"), often represented by the symbol $\Box$ and $\Diamond$, respectively. A modern modal logic is created by Lewis(1932)[1]. In the early 1960s, Kripke introduce a semantics called "Kripke sematics" , which is widely used on modal logic today[2][3][4]. It is well-known that, when we denote "a person A knows φ" and "a person A believe φ" by "$K_A\varphi$" and "$B_A\varphi$" , respectively, there is a lot of common properties between K or B and $\Box$. The logic which deal with knowledge and belief is called epistemic logic, and there are lots of researches which aim to represent agent communication with this logic itself or some variations of this logic, where agent communication is the communication between multi-agent environment: an exchange of messages on Facebook is one of the example of agent communication, which is studied by K. Sano and S. Tojo(2013)[6]. Epistemic logic is based on modal logic, as mentioned above, thus it is important to understand basic knowledge of modal logic before studying agent communication.

In the first part of this thesis, we define the syntax of modal logic K: we denote the propositional variables by the lower-case letters, possibly with subscripts of superscripts; the lower-case Greek letters are reserved as formulas, and capital Greek letters are used for denoting sets of formulas. Syntactically, the only difference is that one unary symbol $\Box$ is added to Cl; a formula of the form $\Diamond\varphi$ is defined as the abbreviation of $\neg(\Box(\neg\varphi))$.

Second, we see Kripke semantics and its basic properties. One of the strong point of this semantics is that the model in this semantics could be drawn in graph, which could be intuitively understood without deep mathematical knowledge. A model of this semantics is a triple $\langle W, R, \mathfrak{V}\rangle$, where W is a non-empty set, R is an arbitrary binary relation on W, and $\mathfrak{V}$ is an arbitrary map from Var$\mathcal{ML}$ to $\mathcal{P}(W)$. A pair $\langle W, R\rangle$ is called a frame and so a model of Kripke semantics could be also regarded as a pair of a frame and a valuation. After that, we see the truth-relation of this semantics, which is natural extension of that of classical logic, and some properties of modal symbols $/Box$ and $\Diamond$. We also see a few well-known operations for models which never change the truth-value of each formula. The operations we check are called generation and reduction: generation is defined by taking a generated submodel; and reduction is defined by taking a map which is called p-morphism. We could consider many classes of frames with their property, and the property of those classes are correspond to specific modal formula. We see some major classes and those corresponding formulas: the class of reflexive, transitive, symmetric and serial frames are correspond to the formula $\Box p \to p$, $\Box p \to \Box\Box p$, $p \to \Box\Diamond p$ and $\Box p \to \Diamond p$, respectively. There is a great tool, which is called Hintikka system, to construct a counter model. Hintikka system is a pair $\mathfrak{h} = \langle T, S\rangle$ where T is a set of a pair $t = (\Gamma, \Delta)$ of formulas, which is called tableau, with some restrictions and S a binary relation on T. With this system, we can also infer the decidability of this semantics.

Third, we introduce a Hilbert-style deduction system, which is called calculus K. The only difference between classical Hilbert-style system is that there is an additional axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and inference rule RN: given a formula φ, we infer $\Box\varphi$. As calculs K is based on classical Hilbert-style system, a formula which is valid in classical logic is also valid in this system. There is a famous theorem in classical logic, which is called deduction theorem, i.e., $\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \to \psi$. With some restrictions to RN, we can formulate deduction theorem for modal logic as it was formulated for classical and intuitionistic logic, however, the formulation of this theorem for modal logic is a bit different. Calculus K is sound and complete with respect to Kripke semantics of course. On proving the completeness, we use Hintikka system and a specific construction of tableaux. There could be many extensions of logic K, and soundness and completeness theorem alos holds for some of them. On provng these theorem, canonical models and filtration are useful; we briefly see these techniques.

Forth and the last, we briefly see another semantics which is called algebraic semantics. There are lots of semantics other than Kripke semantics, and algebraic semantics is one of them. As algebraic semantics is based on algebra and algebra is very abstract, this semantics could be possibly apply to many fields.

Almost all of the contents in this thesis are mainly based on A. Chagrov and M. Zakharyaschev(1997)[7].

Contents

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Chapter 1

Introduction

The inference we often do in our daily life is based on the imperfect knowledge and information about the situation we are facing. The knowledge and information we have shall change every moment, and the truth of them could be overthrown in the future. Moreover, the consequence of our inference directly links to our action so that it is required to infer something, even if the consequence is not satisfying. It is not too much to say that our daily inference is vague, and this is why the importance of a logic available to deal with such a vagueness is growing in the field of agent communication.

Modal logic is a logic which could deal with such a vagueness. In the sense of mathematics, a modal logic is the logic which two connectives $\Box$ and $\Diamond$ are added to classical logic: $\Box$ and $\Diamond$ are often used to represent "it is necessary" and "it is possible", respectively. A modern modal logic is created by Lewis(1932)[1]. In the early 1960s, Kripke introduce a semantics called "Kripke sematics", which is widely used on modal logic[2][3][4]. It is well-known that, when we denote "a person A knows φ" and "a person A believe φ" by "$K_A\varphi$" and "$B_A\varphi$", respectively, there is a lot of common properties between K or B and $\Box$. The logic which deal with knowledge and belief is called epistemic logic. Recently, many researches are done which aim to represent agent communication using epistemic logic itself or some variations of this logic. S. Tojo(2012)[5] uses a variation of epistemic logic to analyze how the court correct his/her knowledge in a trial. K. Sano and S. Tojo(2013)[6] represent agent communication with a communication channel.

This thesis presents basic knowledge of modal logics as a preliminary stage of studying epistemic logic. First we define the syntax. Second we see one of the major semantics on modal logic, i.e. Kripke semantics and its basic properties. After that, we introduce a Hilbert-style calculus, which is called calculus K, and prove the soundness and completeness theorem. We also see another semantics, algebraic semantics to deepen our understanding. In the end, we interrelate a modal logic to epistemic logic, which is often used on researching agent communication. The contents of this thesis are mainly based on A.Chagrov and M. Zakharyaschev(1997)[7].

Chapter 2

Kripke Semantics

2.1 Preliminaries

In this section, we define syntax and some terms of our logic. First of all, we define the syntax.

Definition 2.1.1 (Language $\mathcal{ML}$)**.** *Fix the propositional modal language $\mathcal{ML}$ whose primitive symbols (alphabets) are:*

- *the propositional variables $p_0, p_1, \ldots$;*
- *the propositional constant $\perp$ (falsehood);*
- *the propositional connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\Box$ (necessity);*
- *the punctuation marks (and),*

and the formulas of $\mathcal{ML}$ (or $\mathcal{ML}$-formulas, or simply formulas if $\mathcal{ML}$ is understood) are defined inductively as

- *all the variables in $\mathcal{ML}$ and the constant $\perp$ are atomic $\mathcal{ML}$ formulas;*
- *if φ and ψ are $\mathcal{ML}$-formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\Box\psi)$ are also $\mathcal{ML}$-formulas;*
- *a sequence of primitive symbols in $\mathcal{ML}$ is a formula iff this follows from the two preceding items.*

We will denote propositional variables by the lower-case letters p, q, r, possibly with subscripts or superscripts; the lower-case Greek letters φ, ψ, χ and may be some others are reserved as formulas, and capital Greek letters like Γ, Δ, Σ are used for denoting sets of formulas. There are countably many propositional variables, however, other symbols are countable. The set of all variables in $\mathcal{ML}$ is denoted by $\mathrm{Var}\mathcal{ML}$. Unless otherwise indicated, we will assume $\mathrm{Var}\mathcal{ML}$ to be countable. The set of all formulas in $\mathcal{ML}$ is denoted by $\mathrm{For}\mathcal{ML}$.

Definition 2.1.2 (Subformula). *Let φ and ψ be formulas. We say that a formula χ is a subformula of φ if one of the following is satisfied:*

- $\chi \equiv \varphi$;
- $\varphi \equiv \psi \odot \chi$ *or* $\varphi \equiv \chi \odot \psi$, *where* $\odot \in \{\wedge, \vee, \to\}$;
- $\varphi \equiv \Box\chi$;
- χ *is a subformula of* ψ *and* ψ *is a subformula of* φ.

We denote the set of all subformulas of φ and all variables in φ by $\mathrm{Sub}\varphi$ *and* $\mathrm{Var}\varphi$, *respectively.*

The propositional connectives $\neg$ (negation), $\leftrightarrow$ (equivalence), $\Diamond$ (probability) and the constant $\top$ (truth) can be defined as abbreviations:

$$
\begin{aligned}
(\neg\varphi) &= (\varphi \to \bot),\\
(\varphi \leftrightarrow \psi) &= (\varphi \to \psi) \wedge (\psi \to \varphi),\\
(\Diamond\varphi) &= (\neg(\Box(\neg\varphi))),\\
\top &= (\bot \to \bot).
\end{aligned}
$$

We shall use the following standard conventions on representation of formulas: we assume $\neg$, $\Box$, and $\Diamond$ connect stronger than $\wedge$ and $\vee$, which is stronger than $\to$ and $\leftrightarrow$, and omits those brackets which we can recover without any confusion. We shall write $\varphi_1, \wedge \cdots \wedge \varphi_n$ or $\bigwedge_{1\le i\le n} \varphi_i$ instead of $(\cdots((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \wedge \cdots \wedge \varphi_n)$ and $\varphi_1, \vee \cdots \vee \varphi_n$ or $\bigvee_{1\le i\le n} \varphi_i$ instead of $(\cdots((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \cdots \vee \varphi_n)$; $\bigvee_{i\in\emptyset} \varphi_i$ and $\bigwedge_{i\in\emptyset} \varphi_i$ mean $\bot$ and $\top$, respectively.

2.2 Possible World Semantics

Next, we define the semantics of modal logics. We use possible world semantics, one of the most major semantics in modal logics.

Definition 2.2.1 (Modal Kripke frame). *A modal Kripke frame $\mathfrak{F}$ is a pair $\langle W, R\rangle$ where*

- W *is a non-empty set;*
- R *is an arbitrary binary relation on* W.

Elements of W are called worlds or points. Let $x, y \in W$. If xRy, we say that y is accessible from x, x sees y, y is a successor of x, x is a predecessor of y, $y \in x\uparrow$ or $x \in y\downarrow$.

Definition 2.2.2 (Valuation). *Let $\mathfrak{F} = \langle W, R\rangle$. A valuation $\mathfrak{V}$ on $\mathfrak{F}$ is a map such that* $\mathfrak{V} : \mathrm{Var}\mathcal{ML} \to \mathcal{P}(W)$.

Definition 2.2.3 (Kripke model). *A Kripke model of $\mathcal{ML}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{F} = \langle W, R\rangle$ is a modal frame and $\mathfrak{V}$ a valuation on $\mathfrak{F}$.*

Definition 2.2.4 (Truth-relation)**.** *Let $\mathfrak{F} = \langle W, R\rangle$ be a modal frame, x be a point in W, and $\mathfrak{V}$ be a valuation on $\mathfrak{F}$. By induction on the construction of a formula φ, we define a truth-relation $(\mathfrak{M}, x) \models \varphi$ by taking*

$$
\begin{array}{lll}
(\mathfrak{M}, x) \models p & \textit{iff} & x \in \mathfrak{V}(p), \\
(\mathfrak{M}, x) \models \psi \wedge \chi & \textit{iff} & (\mathfrak{M}, x) \models \psi \textit{ and } (\mathfrak{M}, x) \models \chi, \\
(\mathfrak{M}, x) \models \psi \vee \chi & \textit{iff} & (\mathfrak{M}, x) \models \psi \textit{ or } (\mathfrak{M}, x) \models \chi, \\
(\mathfrak{M}, x) \models \psi \to \chi & \textit{iff} & (\mathfrak{M}, x) \models \psi \textit{ implies } (\mathfrak{M}, x) \models \chi, \\
(\mathfrak{M}, x) \not\models \bot, & & \\
(\mathfrak{M}, x) \models \Box\psi & \textit{iff} & (\mathfrak{M}, y) \models \psi \textit{ for all } y \in W \textit{ such that } xRy,
\end{array}
$$

and so

$$
\begin{array}{lll}
(\mathfrak{M}, x) \models \neg\psi & \textit{iff} & (\mathfrak{M}, x) \not\models \psi, \\
(\mathfrak{M}, x) \models \Diamond\psi & \textit{iff} & (\mathfrak{M}, y) \models \psi \textit{ for some } y \in W \textit{ such that } xRy.
\end{array}
$$

Now let us see some basic properties of Kripke semantics.

Definition 2.2.5 (Accessibility)**.** *Let $\mathfrak{F} = \langle W, R\rangle$ be a modal frame and $x, y \in W$. Say that y is accessible from x by $n \geq 0$ steps and denote $xR^n y$ or $y \in x\uparrow^n$ or $x \in y\downarrow_n$ if there exists (not necessarily distinct) points $z_1, \ldots, z_{n-1} \in W$ such that $xRz_1R \cdots Rz_{n-1}Ry$.*

Note that xR^0y, $y \in x\uparrow^0$ and $x \in y\downarrow_0$ are understood as $x = y$.

Proposition 2.2.6. *For every $n \geq 0$,*

(a) $(\mathfrak{M}, x) \models \Box^n\psi$ *iff* $\forall y \in x\uparrow^n [(\mathfrak{M}, y) \models \psi]$,

(b) $(\mathfrak{M}, x) \models \Diamond^n\psi$ *iff* $\exists y \in x\uparrow^n [(\mathfrak{M}, y) \models \psi]$.

Proof. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$. We prove these two by induction on n.

(a) $(\Rightarrow)$

$n = 0$
: Suppose that $(\mathfrak{M}, x) \models \Box^0\psi$. Fix any $y \in W$ such that $y \in x\uparrow^0$. Our goal is to show $(\mathfrak{M}, y) \models \psi$, and this is obvious since $y \in x\uparrow^0$ means $y = x$.

$n = k + 1$ $(k \geq 0)$
: Suppose that $(\mathfrak{M}, x) \models \Box^{k+1}\psi$. Fix any $y \in W$ such that $y \in x\uparrow^{k+1}$. Our goal is to show $(\mathfrak{M}, y) \models \psi$. By the assumption, we have $(\mathfrak{M}, x) \models \Box^k(\Box\psi)$ and so, by the induction hypothesis, $\forall z \in x\uparrow^k [(\mathfrak{M}, z) \models \Box\psi]$ holds. Since $y \in x\uparrow^{k+1}$, there is a point $z \in W$ such that $z \in x\uparrow^k$ and zRy. Therefore we have

$$
\begin{aligned}
&(\mathfrak{M}, z) \models \Box\psi \\
\Leftrightarrow &\forall y \in W[zRy \Rightarrow (\mathfrak{M}, y) \models \psi],
\end{aligned}
$$

and since zRy, $(\mathfrak{M}, y) \models \psi$ holds.

($\Leftarrow$)

$n = 0$

Suppose that $\forall y \in x\uparrow^0 [(\mathfrak{M}, y) \models \psi]$, i.e., $(\mathfrak{M}, y) \models \psi$. It is obvious that $(\mathfrak{M},\, x) \models \Box^0\psi$.

$n = k + 1$ $(k \geq 0)$

Suppose that $\forall y \in x\uparrow^{k+1} [(\mathfrak{M}, y) \models \psi]$. Our goal is to show

$$\begin{aligned} &(\mathfrak{M},\, x) \models \Box^{k+1}\psi \\ \Leftrightarrow &\forall z \in \mathfrak{F}[xRz \Rightarrow (\mathfrak{M}, z) \models \Box^k\psi]. \end{aligned}$$

Fix any $z \in \mathfrak{F}$ such that xRz. By the assumption, we have $\forall y \in z\uparrow^k [(\mathfrak{M}, y) \models \psi]$ and so, by the induction hypothesis, $(\mathfrak{M}, z) \models \Box^k\psi$ holds.

(b) ($\Rightarrow$)

$n = 0$

Suppose that $(\mathfrak{M},\, x) \models \Diamond^0\psi$. Our goal is to show there is a point $y \in x\uparrow^0$ such that $(\mathfrak{M}, y) \models \psi$, and this is obvious since $y \in x\uparrow^0$ means $y = x$.

$n = k + 1$ $(k \geq 0)$

Suppose that $(\mathfrak{M},\, x) \models \Diamond^{k+1}\psi$. Our goal is to show that there is a point $y \in x\uparrow^{k+1}$ such that $(\mathfrak{M}, y) \models \psi$. By the assumption,

$$\begin{aligned} &(\mathfrak{M},\, x) \models \Diamond(\Diamond^k\psi) \\ \Leftrightarrow &\exists z \in \mathfrak{F}[xRz \text{ and } (\mathfrak{M}, z) \models \Diamond^k\psi] \end{aligned}$$

holds and so, by the induction hypothesis, we have $\exists y \in z\uparrow^k [(\mathfrak{M}, z) \models \psi]$. Since xRz and $z \in y\uparrow^k$, it is obvious that $y \in x\uparrow^{k+1}$.

($\Leftarrow$)

$n = 0$

Suppose that $\exists y \in x\uparrow^0 [(\mathfrak{M}, y) \models \psi]$, i.e., $(\mathfrak{M}, x) \models \psi$. It is obvious that $(\mathfrak{M},\, x) \models \Diamond^0\psi$.

$n = k + 1$ $(k \geq 0)$

Suppose that $\exists y \in x\uparrow^{k+1} [(\mathfrak{M}, y) \models \psi]$. Then there is a point $z \in \mathfrak{F}$ such that $xRzR^ky$ and so, by the induction hypothesis, $(\mathfrak{M}, z) \models \Diamond^k\psi$ holds. Hence we have $(\mathfrak{M}, x) \models \Diamond^{k+1}\psi$.

$\Box$

If xR^ny does not hold for any point y in a frame $\mathfrak{F}$, i.e., $x\uparrow^n = \emptyset$, then $(\mathfrak{F},\, \mathfrak{V},\, x) \models \Box^n\phi$ and $(\mathfrak{F},\, \mathfrak{V},\, x) \not\models \Diamond^n\phi$ for every formula ϕ and valuation $\mathfrak{V}$. In particular, a point x is called dead end and satisfies $(\mathfrak{F},\, \mathfrak{V},\, x) \models \Box^n\phi$ and $(\mathfrak{F},\, \mathfrak{V},\, x) \not\models \Diamond^n\phi$ for all n.

Proposition 2.2.7. *Suppose that $\mathfrak{M}$ is a model on a transitive frame. Then, for every point x in $\mathfrak{M}$ and every formula φ,*

(i) $\forall y \in x\uparrow [(\mathfrak{M}, x) \models \Box\varphi \Rightarrow (\mathfrak{M}, y) \models \Box\varphi]$;

(ii) $\forall y \in x\downarrow [(\mathfrak{M}, x) \models \Diamond\varphi \Rightarrow (\mathfrak{M}, y) \models \Diamond\varphi]$.

Proof.

(i) Fix any point y such that xRy. Suppose that $(\mathfrak{M}, x) \models \Box\varphi$ and, for contradiction, $(\mathfrak{M}, y) \not\models \Box\varphi$. Then there is a point $z \in \mathfrak{M}$ such that yRz and $(\mathfrak{M}, z) \not\models \varphi$. However, since xRy and yRz, and $\mathfrak{M}$ is a model on transitive frame, we have xRz and so $(\mathfrak{M}, x) \not\models \Box\varphi$, contrary to the assumption.

(ii) Fix any y such that yRx and suppose that $(\mathfrak{M}, x) \models \Diamond\varphi$. Then there is a point z such that xRz and $(\mathfrak{M}, z) \models \varphi$. Since yRx and xRz, and $\mathfrak{M}$ is a model on a transitive frame, we have yRz and hence $(\mathfrak{M}, y) \models \Diamond\varphi$.

□

Definition 2.2.8 (Cluster)**.** *Let $\mathfrak{F} = \langle W, R\rangle$ be a transitive frame. Define on W an equivalence relation $\sim$ by taking, for every $x, y \in W$,*

$$(x \sim y) \quad \textit{iff} \quad (x = y \textit{ or } (xRy \textit{ and } yRx))].$$

The equivalence classes with respect to $\sim$ are called clusters. The cluster containing a point x will be denoted by $C(x)$.

We distinguish three types of clusters: a degenerate cluster consisting of a single irreflexive point; a simple cluster consisting of a single reflexive point; and a proper cluster containing at least two points.

Proposition 2.2.9. *Suppose that x is a point in a model $\mathfrak{M}$ built on a transitive frame and φ an arbitrary formula. Then,*

(i) $\forall y \in C(x)[(\mathfrak{M}, x) \models \Box\varphi \quad \textit{iff} \quad (\mathfrak{M}, y) \models \Box\varphi]$;

(ii) $\forall y \in C(x)[(\mathfrak{M}, x) \models \Diamond\varphi \quad \textit{iff} \quad (\mathfrak{M}, y) \models \Diamond\varphi]$.

Proof. Fix any $y \in C(x)$. There are two cases, the case when both xRy and yRx holds, and the case when $x = y$. We only consider the former case since the latter is trivial.

(i) ($\Rightarrow$)
Suppose that $(\mathfrak{M}, x) \models \Box\varphi$. Our goal is to show $(\mathfrak{M}, y) \models \Box\varphi$ and it is obvious by proposition 2.2.7, since xRy.
($\Leftarrow$) follows from ($\Rightarrow$).

(ii) follows from (i).

$\square$

It follows that, at the points in the same cluster, exactly same formulas of the form $\Box\varphi$ and $\Diamond\varphi$ are true in a transitive model.

Definition 2.2.10 (Modal logic ($K_{\mathcal{ML}}$))**.** *We define the modal logic $K_{\mathcal{ML}}$ in the language $\mathcal{ML}$ as the set of all $\mathcal{ML}$-formulas that are valid in all modal Kripke frames, i.e.,*

$$K_{\mathcal{ML}} = \{\varphi \in \mathrm{For}\mathcal{ML} \mid \forall \mathfrak{F}\, [\mathfrak{F} \models \varphi]\}.$$

We drop the subscript $\mathcal{ML}$ and write, when understood, simply K. In section 3.3, we shall construct modal logics for various meaningful interpretations of $\Box$ by adding formulas to K which convey specific traits of these interpretations.

2.3 Truth-preserving Operations

There are some operations for models which do not change a validity of each formula. In this section, we introduce three operations: generation, reduction, and bulldozer.

Definition 2.3.1 ((Generated) subframe)**.** *A pair $\mathfrak{G} = \langle V, S\rangle$ of a non-empty set V and a relation S on it is called a subframe of a frame $\mathfrak{F} = \langle W, S\rangle$ (notation: $\mathfrak{G} \subseteq \mathfrak{F}$) if $V \subseteq W$ and S is the restriction of R to V ($S = R \restriction V$, in symbols), i.e., $S = R \cap V^2$. The subframe $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$ (noteation: $\mathfrak{G} \sqsubseteq \mathfrak{F}$) if V is an upward closed subset of W, i.e., $\forall x, y \in W\, [(x \in V$ and $xRy) \Rightarrow y \in V]$.*

Definition 2.3.2 ((Generated) submodel)**.** *A model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a submodel of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ (notation: $\mathfrak{N} \subseteq \mathfrak{M}$) if $\mathfrak{G} = \langle V, S\rangle$ is a subframe of $\mathfrak{F} = \langle W, R\rangle$ and $\forall p \in \mathrm{Var}\mathcal{ML}\, [\mathfrak{U}(p) = \mathfrak{V}(p) \cap V]$. In the case when $\mathfrak{G} \sqsubseteq \mathfrak{F}$ the model $\mathfrak{N}$ is called a generated submodel of $\mathfrak{M}$ (notation: $\mathfrak{N} \sqsubseteq \mathfrak{M}$).*

Theorem 2.3.3 (Generation)**.** *If $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a generated submodel of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ then, $\forall \varphi \in \mathrm{For}\mathcal{ML}\, \forall x \in \mathfrak{G}\, [(\mathfrak{N}, x) \models \varphi$ iff $(\mathfrak{M}, x) \models \varphi]$.*

Proof. Prove by induction on the construction of φ.

$\varphi \equiv p$

- ($\Rightarrow$) Fix any $x \in \mathfrak{G}$ and suppose that $(\mathfrak{N}, x) \models p$. Then we have $x \in \mathfrak{U}(p)$ and so, by the definition of a generated submodel, $x \in \mathfrak{V}(p)$. Hence $(\mathfrak{M}, x) \models p$.
- ($\Leftarrow$) Fix any $x \in \mathfrak{G}$ and suppose that $(\mathfrak{M}, x) \models p$. Then we have $x \in \mathfrak{V}(p)$ and so, since $x \in \mathfrak{G}$, $x \in \mathfrak{U}(p)$. Hence $(\mathfrak{N}, x) \models p$.

$\varphi \equiv \bot$

Obvious since neither $(\mathfrak{N}, x) \models \bot$ nor $(\mathfrak{M}, x) \models \bot$ holds.

$\varphi \equiv \psi \wedge \chi$

(⇒) Fix any $x \in \mathfrak{G}$ and suppose that $(\mathfrak{N}, x) \models \psi \wedge \chi$. Then we have $(\mathfrak{N}, x) \models \psi$ and $(\mathfrak{N}, x) \models \chi$. The induction hypothesis yields $(\mathfrak{M}, x) \models \psi$ and $(\mathfrak{M}, x) \models \chi$ and hence $(\mathfrak{M}, x) \models \psi \wedge \chi$.

(⇐) Similar to (⇒).

$\varphi \equiv \psi \vee \chi$

(⇒) Fix any $x \in \mathfrak{G}$ and suppose that $(\mathfrak{N}, x) \models \psi \vee \chi$. Then we have $(\mathfrak{N}, x) \models \psi$ or $(\mathfrak{N}, x) \models \chi$. The induction hypothesis yields $(\mathfrak{M}, x) \models \psi$ or $(\mathfrak{M}, x) \models \chi$ and hence $(\mathfrak{M}, x) \models \psi \vee \chi$.

(⇐) Similar to (⇒).

$\varphi \equiv \psi \to \chi$

(⇒) Fix any $x \in \mathfrak{G}$. Suppose that $(\mathfrak{N}, x) \models \psi \to \chi$ and $(\mathfrak{M}, x) \models \psi$. It suffices to show $(\mathfrak{M}, x) \models \chi$. By the assumption, we have $(\mathfrak{N}, x) \models \psi$ implies $(\mathfrak{N}, x) \models \chi$. By the induction hypothesis, $(\mathfrak{M}, x) \models \psi$ implies $(\mathfrak{N}, x) \models \psi$, and so $(\mathfrak{N}, x) \models \chi$. The induction hypothesis yields that $(\mathfrak{M}, x) \models \chi$.

(⇐) Similar to (⇒).

$\varphi \equiv \neg\psi$

(⇒) Fix any $x \in \mathfrak{G}$ and suppose that $(\mathfrak{N}, x) \models \neg\psi$. Then we have $(\mathfrak{N}, x) \not\models \psi$. The contraposition of the induction hypothesis yields $(\mathfrak{M}, x) \not\models \psi$ and hence $(\mathfrak{M}, x) \models \neg\psi$.

(⇐) Similar to (⇒).

$\varphi \equiv \Box\psi$

Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$.

(⇒) Suppose that $(\mathfrak{N}, x) \models \Box\psi$. By the assumption, we have $\forall z \in \mathfrak{G}\, [xSz \Rightarrow (\mathfrak{N}, z) \models \psi]$. Since $\mathfrak{N}$ is a generated submodel of $\mathfrak{M}$, we also have $y \in \mathfrak{G}$ and xSy, and so $(\mathfrak{N}, y) \models \psi$ for any $y \in \mathfrak{F}$ such that xRy. The induction hypothesis yields $(\mathfrak{M}, y) \models \psi$. Hence $(\mathfrak{M}, x) \models \Box\psi$.

(⇐) Fix any $x, y \in \mathfrak{G}$ such that xSy and suppose that $(\mathfrak{M}, x) \models \Box\psi$. Then we have $\forall z \in W\, [xRz \Rightarrow (\mathfrak{M}, z) \models \psi]$. Since $S = R \cap V^2$ and $V \subseteq W$, we have $(\mathfrak{M}, y) \models \psi$ and so, by the induction hypothesis, $(\mathfrak{N}, y) \models \psi$ for any y such that xSy. Therefore $(\mathfrak{N}, x) \models \Box\psi$.

□

Theorem 2.3.3 means that the truth-value of formulas at a point x are completely determined by the truth-value of their variables at the points in $x\uparrow$ and do not depend on other points in the model.

Corollary 2.3.4. *If $\mathfrak{G} \sqsubseteq \mathfrak{F}$ then, for every $x \in \mathfrak{G}$ and every formula φ,* *(i)* $(\mathfrak{G}, x) \models \varphi$ *iff* $(\mathfrak{F}, x) \models \varphi$*;*

(ii) $\mathfrak{F} \models \varphi$ *implies* $\mathfrak{G} \models \varphi$*.*

Corollary 2.3.5. $K = \{\varphi \in \mathrm{For}\mathcal{ML} \mid \mathfrak{F} \models \varphi$ *for all rooted frame* $\mathfrak{F}\}$*.*

Definition 2.3.6 (Reduction)**.** *Suppose we have two frames* $\mathfrak{F} = \langle W, R\rangle$ *and* $\mathfrak{G} = \langle V, S\rangle$*. A map* f *from* W *to* V *is called a reduction of* $\mathfrak{F}$ *to* $\mathfrak{G}$ *if the following conditions hold for every* $x, y \in W$*:*

(R1) xRy *implies* $f(x)Sf(y)$*;*

(R2) $f(x)Su$ *implies* $\exists y \in W[xRy$ *and* $(f(y) = u)]$*.*

In this case we also say that f *reduces* $\mathfrak{F}$ *to* $\mathfrak{G}$ *or* $\mathfrak{G}$ *is an* f*-reduct (or simply a reduct) of* $\mathfrak{F}$ *or* $\mathfrak{F}$ *is* f*-reducible (or simply reducible) to* $\mathfrak{G}$*. Such a map* f *is often called a pseudo-epimorphism or just a p-morphism.*

A reduction f *of* $\mathfrak{F}$ *to* $\mathfrak{G}$ *is called a reduction of a model* $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ *to a model* $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ *if* $\forall p \in \mathrm{Var}\mathcal{ML}\,[\mathfrak{V}(p) = f^{-1}(\mathfrak{U}(p))]$*, i.e.,* $\forall x \in \mathfrak{F}\,[(\mathfrak{M}, x) \models p$ *iff* $(\mathfrak{N}, f(x)) \models p]$*.*

Theorem 2.3.7 (Reduction)**.** *If* f *is a reduction of a model* $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ *to a model* $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ *then,*

$$\forall \varphi \in \mathrm{For}\mathcal{ML} \forall x \in \mathfrak{F}[(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, f(x)) \models \varphi].$$

Proof. Prove by induction on the construction of φ.

$\varphi \equiv p$

Obvious, since $\forall x \in \mathfrak{F}\,[(\mathfrak{M}, x) \models p$ iff $(\mathfrak{N}, f(x)) \models p]$ holds by the definition of reduction.

$\varphi \equiv \bot$

Obvious, since neither $(\mathfrak{M}, x) \models \bot$ nor $(\mathfrak{N}, f(x)) \models \bot$ holds.

$\varphi \equiv \psi \wedge \chi$

($\Rightarrow$) Fix any $x \in \mathfrak{F}$ and suppose that $(\mathfrak{M}, x) \models \psi \wedge \chi$. Then we have $(\mathfrak{M}, x) \models \psi$ and $(\mathfrak{M}, x) \models \chi$. The induction hypothesis yields $(\mathfrak{N}, f(x)) \models \psi$ and $(\mathfrak{N}, f(x)) \models \chi$ and hence $(\mathfrak{N}, f(x)) \models \psi \wedge \chi$.

($\Leftarrow$) Similar to ($\Rightarrow$).

$\varphi \equiv \psi \vee \chi$

($\Rightarrow$) Fix any $x \in \mathfrak{F}$ and suppose that $(\mathfrak{M}, x) \models \psi \vee \chi$. Then we have $(\mathfrak{M}, x) \models \psi$ or $(\mathfrak{M}, x) \models \chi$. The induction hypothesis yields $(\mathfrak{N}, f(x)) \models \psi$ or $(\mathfrak{N}, f(x)) \models \chi$ and hence $(\mathfrak{N}, f(x)) \models \psi \vee \chi$.

($\Leftarrow$) Similar to ($\Rightarrow$).

$\varphi \equiv \psi \to \chi$

($\Rightarrow$) Fix any $x \in \mathfrak{F}$. Suppose that $(\mathfrak{M},\, x) \models \psi \to \chi$ and $(\mathfrak{N},\, f(x)) \models \psi$. By the assumption, we have $(\mathfrak{M},\, x) \models \psi$ implies $(\mathfrak{M},\, x) \models \chi$. Since $(\mathfrak{N},\, f(x)) \models \psi$, the induction hypothesis yields $(\mathfrak{M},\, x) \models \psi$ and so $(\mathfrak{M},\, x) \models \chi$. Therefore, by the induction hypothesis, $(\mathfrak{N},\, f(x)) \models \chi$ and hence $(\mathfrak{N},\, f(x)) \models \psi \to \chi$.

($\Leftarrow$) Similar to ($\Rightarrow$).

$\varphi \equiv \neg\psi$

($\Rightarrow$) Fix any $x \in \mathfrak{F}$ and suppose that $(\mathfrak{M},\, x) \models \neg\psi$. Then we have $(\mathfrak{M},\, x) \not\models \psi$. The contraposition of the induction hypothesis yields $(\mathfrak{N},\, f(x)) \not\models \psi$ and hence $(\mathfrak{N},\, f(x)) \models \neg\psi$.

($\Leftarrow$) Similar to ($\Rightarrow$).

$\varphi \equiv \Box\psi$

Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$.

($\Rightarrow$) Fix any $x \in \mathfrak{F}$, and $u \in \mathfrak{G}$ such that $f(x)Su$. Suppose that $(\mathfrak{M},\, x) \models \Box\psi$. It suffices to show $(\mathfrak{N},\, u) \models \psi$. By the assumption, we have $\forall y \in W[xRy \Rightarrow (\mathfrak{M},\, y) \models \psi]$. By (R2) and $f(x)Su$, $\exists y \in W[xRy \text{ and } (f(y) = u)]$ and so $(\mathfrak{M},\, y) \models \psi$. The induction hypothesis yields $(\mathfrak{N},\, f(y)) \models \psi$. Hence $(\mathfrak{N},\, u) \models \psi$.

($\Leftarrow$) Fix any $x, y \in \mathfrak{F}$ such that xRy. Suppose that $(\mathfrak{N},\, f(x)) \models \Box\psi$. It suffices to show $(\mathfrak{M}, y) \models \psi$. By the assumption, we have $\forall u \in \mathfrak{G}[f(x)Su \Rightarrow (\mathfrak{N},\, u) \models \psi]$. By (R1) and xRy, we have $f(x)Sf(y)$ and so $(\mathfrak{N},\, f(y)) \models \psi$. The induction hypothesis yields $(\mathfrak{M},\, y) \models \psi$.

$\square$

2.4 Correspondence Theory

Let us find some characterizations of frames validating a number of important modal formulas we shall deal with in the sequel.

Proposition 2.4.1 (Reflexivity). *A frame $\mathfrak{F} = \langle W,\, R\rangle$ validates $\Box p \to p$ iff $\mathfrak{F}$ is reflexive.*

Proof. ($\Rightarrow$) Assume that $\mathfrak{F} \models \Box p \to p$ to show $\forall x \in W[\, xRx\,]$. Fix any $x \in W$. Define a valuation $\mathfrak{V}$ by taking $\mathfrak{V}(p) = \{y \,|\, xRy\}$. Then we have $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p$. By the assumption, we also have $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p \to p$, and so $\langle \mathfrak{F},\, \mathfrak{V}, x\rangle \models p$ holds. As we defined $\mathfrak{V}(p) = \{y \,|\, xRy\}$, $x \in \mathfrak{V}(p)$. Hence, $\forall x \in W[xRx]$.

($\Leftarrow$) Assume that $\forall x \in W[xRx]$ to show $\mathfrak{F} \models \Box p \to p$. Fix any point $x \in W$ and valuation $\mathfrak{V}$, and suppose that

$$
\begin{aligned}
&\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p \\
\Leftrightarrow &\forall y \in W[\, xRy \Rightarrow \langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models p\,]
\end{aligned}
$$

Since we have xRx by the assumption, $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models p$ holds. Hence, $\mathfrak{F} \models \Box p \to p$. $\Box$

Proposition 2.4.2 (Transitivity)**.** *A frame $\mathfrak{F} = \langle W,\, R\rangle$ validates $\Box p \to \Box\Box p$ iff $\mathfrak{F}$ is transitive.*

Proof. ($\Rightarrow$) Assume that $\mathfrak{F} \models \Box p \to \Box\Box p$ to show $\forall x,\, y,\, z \in W[\, xRy$ and $yRz \Rightarrow xRz\,]$. Fix any $x,\, y,\, z \in W$ such that xRy and yRz. Define a valuation $\mathfrak{V}$ by taking $\mathfrak{V}(p) = \{w \,|\, xRw\}$. Then we have $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p$ and so,

$$\begin{aligned}&\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Box p\\ \Leftrightarrow&\forall y \in W[\, xRy \Rightarrow \langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models \Box p\,]\end{aligned}$$

holds by the assumption. For any y such that xRy, we have

$$\begin{aligned}&\langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models \Box p\\ \Leftrightarrow&\forall z \in W[\, yRz \Rightarrow \langle \mathfrak{F},\, \mathfrak{V}, z\rangle \models p\,].\end{aligned}$$

We also have yRz, by the assumption, so that $\langle \mathfrak{F},\, \mathfrak{V}, z\rangle \models p$ holds. As we defined $\mathfrak{V}(p) = \{w \,|\, xRw\}$, $z \in \mathfrak{V}(p)$. Hence, $\forall x, y, z \in W[\, xRy$ and $yRz \Rightarrow xRz\,]$.

($\Leftarrow$) Assume that $\forall x, y, z \in W[\, xRy$ and $yRz \Rightarrow xRz\,]$. Fix any point $x \in W$ and valuation $\mathfrak{V}$, and suppose that

$$\begin{aligned}&\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p\\ \Leftrightarrow&\forall z \in W[\, xRy \Rightarrow \langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models p\,]\end{aligned}$$

Now it suffices to show $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Box p$. Fix any points $y,\, z \in W$ such that xRy and yRz. By the assumption, we have xRz and so $\langle \mathfrak{F},\, \mathfrak{V},\, z\rangle \models p$ holds. Therefore, $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Box p$. $\Box$

Proposition 2.4.3 (Symmetricity)**.** *A frame $\mathfrak{F} = \langle W,\, R\rangle$ validates $p \to \Box\Diamond p$ iff $\mathfrak{F}$ is symmetric.*

Proof. ($\Rightarrow$) Assume that $\mathfrak{F} \models p \to \Box\Diamond p$ to show $\forall x,\, y \in W[\, xRy \Rightarrow yRx\,]$. Fix any $x,\, y \in W$ such that xRy. Define a valuation $\mathfrak{V}$ by taking $\mathfrak{V}(p) = \{x\}$. Then we have $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box p$ and so,

$$\begin{aligned}&\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Diamond p\\ \Leftrightarrow&\forall y \in W[\, xRy \Rightarrow \langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models \Diamond p\,]\end{aligned}$$

holds by the assumption. Since xRy, we have

$$\begin{aligned}&\langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models \Diamond p\\ \Leftrightarrow&\exists z \in W[\, yRz \Rightarrow \langle \mathfrak{F},\, \mathfrak{V}, z\rangle \models p\,].\end{aligned}$$

As we defined $\mathfrak{V}(p) = \{x\}$, $z \in \mathfrak{V}(p)$, i.e., $z = x$. Hence, $\forall x,\, y \in W[\, xRy \Rightarrow yRx\,]$.

($\Leftarrow$) Assume that $\forall x,\, y \in W[\, xRy \Rightarrow yRx\,]$. Fix any point $x \in W$ and valuation $\mathfrak{V}$, and suppose that $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models p$. Now it suffices to show $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Diamond p$. Fix any points $y \in W$ such that xRy. By the assumption, we have yRx and so $\langle \mathfrak{F},\, \mathfrak{V},\, y\rangle \models \Diamond p$ holds. Therefore, $\langle \mathfrak{F},\, \mathfrak{V},\, x\rangle \models \Box\Diamond p$. $\Box$

Proposition 2.4.4 (Seriality). *A frame* $\mathfrak{F} = \langle W, R \rangle$ *validates* $\Box p \to \Diamond p$ *iff* $\mathfrak{F}$ *is serial.*

Proof. ($\Rightarrow$) Assume that $\mathfrak{F} \models \Box p \to \Diamond p$ to show $\forall x \in W \exists y \in W [\, xRy \,]$. Fix any $x \in W$. Define a valuation $\mathfrak{V}$ by taking $\mathfrak{V}(p) = \{y \mid xRy\}$. Then we have $\langle \mathfrak{F}, \mathfrak{V}, x \rangle \models \Box p$ and so,

$$\begin{aligned} &\langle \mathfrak{F}, \mathfrak{V}, x \rangle \models \Diamond p \\ \Leftrightarrow &\exists y \in W [\, xRy \text{ and } \langle \mathfrak{F}, \mathfrak{V}, y \rangle \models p \,] \end{aligned}$$

holds by the assumption. Hence, $\forall x \in W \exists y \in W [\, xRy \,]$.

($\Leftarrow$) Assume that $\forall x \in W \exists y \in W [\, xRy \,]$. Fix any point $x \in W$ and valuation $\mathfrak{V}$, and suppose that

$$\begin{aligned} &\langle \mathfrak{F}, \mathfrak{V}, x \rangle \models \Box p \\ \Leftrightarrow &\forall y \in W [\, xRy \Rightarrow \langle \mathfrak{F}, \mathfrak{V}, y \rangle \models p \,]. \end{aligned}$$

Now it suffices to show $\langle \mathfrak{F}, \mathfrak{V}, x \rangle \models \Diamond p$. Since $\mathfrak{F}$ is serial, there exists a point $y \in W$ such that xRy and so $\langle \mathfrak{F}, \mathfrak{V}, y \rangle \models p$ holds. Hence, $\langle \mathfrak{F}, \mathfrak{V}, x \rangle \models \Diamond p$. $\Box$

2.5 Hintikka Systems

In this section, we learn a kind of semantic tableau method, which is called Hintikka system. This system will not only provide us with a convenient tool for constructing countermodels but also help us providing the completeness theorem for the calculus K in next section.

Definition 2.5.1 (Disjoint saturated tableau). *A tableau in the language* $\mathcal{ML}$ *is any pair* $t = (\Gamma, \Delta)$ *of subsets of* $\mathrm{For}\mathcal{ML}$.
A tableau $t = (\Gamma, \Delta)$ *is saturated if, for all formulas* $\varphi, \psi \in \mathrm{For}\mathcal{ML}$,

($S1$) $(\varphi \land \psi) \in \Gamma$ *implies* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$,

($S2$) $(\varphi \land \psi) \in \Delta$ *implies* $\varphi \in \Delta$ *or* $\psi \in \Delta$,

($S3$) $(\varphi \lor \psi) \in \Gamma$ *implies* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$,

($S4$) $(\varphi \lor \psi) \in \Delta$ *implies* $\varphi \in \Delta$ *and* $\psi \in \Delta$,

($S5$) $(\varphi \to \psi) \in \Gamma$ *implies* $\varphi \in \Delta$ *or* $\psi \in \Gamma$,

($S6$) $(\varphi \to \psi) \in \Delta$ *implies* $\varphi \in \Gamma$ *and* $\psi \in \Delta$.

A tableau $t = (\Gamma, \Delta)$ *is disjoint if* $\Gamma \cap \Delta = \emptyset$ *and* $\bot \notin \Gamma$. *A tableau* $t' = (\Gamma', \Delta')$ *is a subtableau of* $t = (\Gamma, \Delta)$ *and denote as* $t' \subseteq t$ *if both* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$ *holds.*

Definition 2.5.2 (Hintikka system). *A Hintikka system in* K *is a pair* $\mathfrak{h} = \langle T, S \rangle$, *where* T *is a non-empty set of disjoint saturated tableaux and* S *a binary relation on* T *satisfying the following two conditions:*

(HS_M1) *if* $t = (\Gamma,\, \Delta)$*,* $t' = (\Gamma',\, \Delta')$ *and* tSt' *holds, then* $\varphi \in \Gamma'$ *holds for every* $\Box\varphi \in \Gamma$*;*

(HS_M2) *if* $t = (\Gamma,\, \Delta)$ *and* $\Box\varphi \in \Delta$ *holds, then there is* $t' = (\Gamma',\, \Delta')$ *in* T *such that* tSt' *and* $\varphi \in \Delta'$*.*

Say that $\mathfrak{h}$ *is a Hintikka system for a tableau* t *if* $t \subseteq t'$ *for some* t' *in* $\mathfrak{h}$*.*

Definition 2.5.3 (Realization)**.** *A tableau* $t = (\Gamma,\, \Delta)$ *is realized in (a point* x *of) a model* $\mathfrak{M}$ *if* $(\mathfrak{M},\, x) \models \varphi$ *for every* $\varphi \in \Gamma$*, and* $(\mathfrak{M},\, x) \not\models \varphi$ *for every* $\varphi \in \Delta$*.*
A tableau is called realizable in K *if it is realized in some model.*

Proposition 2.5.4. *A tableau* t *is realizable in* K *iff there is a Hintikka system for* t*.*

Proof. ($\Rightarrow$) Suppose that t is realizable in a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = \langle W,\, R\rangle$. Our goal is to show that there is a Hintikka system $\mathfrak{h}$ for t, i.e., $t \subseteq t'$ for some $t' \in \mathfrak{h}$. With each $x \in W$, we associate the tableau $t_x = (\Gamma_x,\, \Delta_x)$, where

$$\Gamma_x = \{\varphi \in \mathrm{For}\mathcal{ML} \mid (\mathfrak{M},\, x) \models \varphi\},$$
$$\Delta_x = \{\varphi \in \mathrm{For}\mathcal{ML} \mid (\mathfrak{M},\, x) \not\models \varphi\},$$

and define a partial order S on the set $T = \{t_x \mid x \in W\}$ by taking

$$t_x S t_y \Leftrightarrow xRy.$$

Now we show $\mathfrak{h} = \langle T,\, S\rangle$ is a Hintikka system for t.

- T is a non-empty set

  Obvious since we associate $t_x \in T$ with each $x \in W$, and $W \neq \emptyset$.

- t_x is disjoint

  Obvious since, with our definition of Γ_x and Δ_x, $\Gamma_x \cap \Delta_x = \emptyset$.

- t_x is saturated

  ($S1$) Suppose that $\psi \wedge \chi \in \Gamma_x$ to show $\psi \in \Gamma_x$ and $\chi \in \Gamma_x$. Since $\psi \wedge \chi \in \Gamma_x$, by the definition of Γ_x, we have

  $$\begin{aligned} &(\mathfrak{M},\, x) \models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{M},\, x) \models \psi \quad \textit{and} \quad (\mathfrak{M},\, x) \models \chi. \end{aligned}$$

  Hence $\psi \in \Gamma_x$ and $\chi \in \Gamma_x$.

  ($S2$) Suppose that $\psi \wedge \chi \in \Delta_x$ to show $\psi \in \Delta_x$ or $\chi \in \Delta_x$. Similarly, we have

  $$\begin{aligned} &(\mathfrak{M},\, x) \not\models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{M},\, x) \models \neg(\psi \wedge \chi) \\ \Leftrightarrow &(\mathfrak{M},\, x) \models \neg\psi \vee \neg\chi \\ \Leftrightarrow &(\mathfrak{M},\, x) \models \neg\psi \quad \textit{or} \quad (\mathfrak{M},\, x) \models \neg\chi \\ \Leftrightarrow &(\mathfrak{M},\, x) \not\models \psi \quad \textit{or} \quad (\mathfrak{M},\, x) \not\models \chi. \end{aligned}$$

  Hence $\psi \in \Delta_x$ or $\chi \in \Delta_x$.

($S3$) Suppose that $\psi \vee \chi \in \Gamma_x$ to show $\psi \in \Gamma_x$ or $\chi \in \Gamma_x$. Similarly, we have

$$\begin{aligned}
&(\mathfrak{M},\, x) \models \psi \vee \chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \psi \quad \textit{or} \quad (\mathfrak{M},\, x) \models \chi.
\end{aligned}$$

Hence $\psi \in \Gamma_x$ or $\chi \in \Gamma_x$.

($S4$) Suppose that $\psi \vee \chi \in \Delta_x$ to show $\psi \in \Delta_x$ and $\chi \in \Delta_x$. Similarly, we have

$$\begin{aligned}
&(\mathfrak{M},\, x) \not\models \psi \vee \chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \neg(\psi \vee \chi) \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \neg\psi \wedge \neg\chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \neg\psi \quad \textit{and} \quad (\mathfrak{M},\, x) \models \neg\chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \not\models \psi \quad \textit{and} \quad (\mathfrak{M},\, x) \not\models \chi.
\end{aligned}$$

Hence $\psi \in \Delta_x$ and $\chi \in \Delta_x$.

($S5$) Suppose that $\psi \to \chi \in \Gamma_x$ to show $\psi \in \Delta_x$ or $\chi \in \Gamma_x$. Similarly, we have

$$\begin{aligned}
&(\mathfrak{M},\, x) \models \psi \to \chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \psi \quad \textit{implies} \quad (\mathfrak{M},\, x) \models \chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \not\models \psi \quad \textit{or} \quad ((\mathfrak{M},\, x) \models \psi \quad \textit{and} \quad (\mathfrak{M},\, x) \models \chi).
\end{aligned}$$

Therefore we have $\psi \in \Delta_x$ or ($\psi \in \Gamma_x$ and $\chi \in \Gamma_x$) and hence $\psi \in \Delta_x$ or $\chi \in \Gamma_x$ holds.

($S6$) Suppose that $\psi \to \chi \in \Delta_x$ to show $\psi \in \Gamma_x$ and $\chi \in \Delta_x$. Similarly, we have

$$\begin{aligned}
&(\mathfrak{M},\, x) \not\models \psi \to \chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \neg(\psi \to \chi) \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \psi \wedge \neg\chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \psi \text{ and } (\mathfrak{M},\, x) \models \neg\chi \\
\Leftrightarrow &(\mathfrak{M},\, x) \models \psi \text{ and } (\mathfrak{M},\, x) \not\models \chi
\end{aligned}$$

Hence $\psi \in \Gamma_x$ and $\chi \in \Delta_x$ holds.

(HS_M1) Suppose that $t_x = (\Gamma_x,\, \Delta_x)$, $t_{x'} = (\Gamma_{x'},\, \Delta_{x'})$ and $t_x S t_{x'}$ to show $\forall \Box\psi \in \Gamma_x[\psi \in \Gamma_{x'}]$. Fix any $\Box\psi \in \Gamma_x$. Then, by the definition of Γ, we have

$$\begin{aligned}
&(\mathfrak{M},\, x) \models \Box\psi \\
\Leftrightarrow &\forall y \in W[xRy \Rightarrow (\mathfrak{M}, y) \models \psi].
\end{aligned}$$

Since $t_x S t_{x'}$, by the definition of S and T, xRx' holds. Therefore $(\mathfrak{M},\, x') \models \psi$ and so $\psi \in \Gamma_{x'}$. Hence $\psi \in \Gamma_{x'}$, for all formula in Γ_x of the form $\Box\psi$.

(HS_M2) Suppose that $t_x = (\Gamma_x,\, \Delta_x)$ and $\Box\psi \in \Delta_x$ to show that there is a tableau $t_{x'} = (\Gamma_{x'},\, \Delta_{x'})$ in T such that $t_x S t_{x'}$ and $\psi \in \Delta_{x'}$. Since $\Box\psi \in \Delta_x$, by the definition of Δ, we have

$$\begin{aligned}&(\mathfrak{M},\, x) \nvDash \Box\psi\\ \Leftrightarrow &\exists x' \in W[xRx' \text{ and } (\mathfrak{M},\, x') \nvDash \psi].\end{aligned}$$

Since xRx', there is a tableau $t_{x'} = (\Gamma_{x'},\, \Delta_{x'})$ in T such that $t_x S t_{x'}$. We also have $\psi \in \Delta_{x'}$ since $(\mathfrak{M},\, x') \nvDash \psi$.

- $t \subseteq t'$ for some $t' \in \mathfrak{h}$

  Since t is realizable in $\mathfrak{M}$, there is a tableau $t' \in T$ such that $t = t'$.

Hence $\mathfrak{h}$ is a Hintikka system for t.

($\Leftarrow$) Suppose that there is a Hintikka system $\mathfrak{h} = (T,\, S)$ for a tableau t to show that t is realizable in K. We will regard $\mathfrak{h}$ as a modal frame. Define a model $\mathfrak{M} = \langle \mathfrak{h},\, \mathfrak{V} \rangle$ on it by taking, for every variable p,

$$\mathfrak{V}(p) = \{u = (\Gamma,\, \Delta) \,|\, u \in T \quad and \quad p \in \Gamma\}.$$

We show that for all formula φ in For$\mathcal{ML}$ and for all tableau $u = (\Gamma, \Delta)$ in T

$$(\varphi \in \Gamma \text{ implies } (\mathfrak{M},\, u) \models \varphi) \text{ and } (\varphi \in \Delta \text{ implies } (\mathfrak{M},\, u) \nvDash \varphi)$$

by the induction on the composition of φ.

- Base case

  - $\varphi \equiv p$

    Suppose that $p \in \Gamma$. Then, by the definition of $\mathfrak{V}$,

    $$\begin{aligned}&u \in \mathfrak{V}(p)\\ \Leftrightarrow &(\mathfrak{M},\, u) \models p.\end{aligned}$$

    Suppose that $p \in \Delta$. Similarly,

    $$\begin{aligned}&u \notin \mathfrak{V}(p)\\ \Leftrightarrow &(\mathfrak{M},\, u) \nvDash p.\end{aligned}$$

  - $\varphi \equiv \bot$

    Since $\mathfrak{h}$ is a Hintikka system, $u \in \mathfrak{h}$ is disjoint and so $\bot \notin \Gamma$. So it suffices to show $(\mathfrak{M},\, u) \nvDash \bot$, which is obvious.

- Induction step

- $\varphi \equiv \psi \wedge \chi$

  Suppose that $\psi \wedge \chi \in \Gamma$. Our goal is to show $(\mathfrak{M},\, u) \models \psi \wedge \chi$. Since $\mathfrak{h}$ is a Hintikka system, $u \in \mathfrak{h}$ is saturated and so, by $(S1)$, $\psi \in \Gamma$ and $\chi \in \Gamma$. By the induction hypothesis, we have

  $$\begin{aligned}&(\mathfrak{M},\, u) \models \psi \quad \mathit{and} \quad (\mathfrak{M},\, u) \models \chi \\ \Leftrightarrow &(\mathfrak{M},\, u) \models \psi \wedge \chi.\end{aligned}$$

  Suppose that $\psi \wedge \chi \in \Delta$. Similarly, we have $\psi \in \Delta$ or $\chi \in \Delta$. By the induction hypothesis, we have

  $$\begin{aligned}&(\mathfrak{M},\, u) \not\models \psi \quad \mathit{or} \quad (\mathfrak{M},\, u) \not\models \chi \\ \Leftrightarrow &\mathit{not} \quad ((\mathfrak{M},\, u) \models \psi \quad \mathit{and} \quad (\mathfrak{M},\, u) \models \chi) \\ \Leftrightarrow &\mathit{not} \quad (\mathfrak{M},\, u) \models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{M},\, u) \not\models \psi \wedge \chi.\end{aligned}$$

- $\varphi \equiv \psi \vee \chi$
- $\varphi \equiv \psi \to \chi$

  Similar to the case $\varphi \equiv \psi \wedge \chi$.
- $\varphi \equiv \Box\psi$

  Suppose that $\Box\psi \in \Gamma$. Our goal is to show

  $$\begin{aligned}&(\mathfrak{M},\, u) \models \Box\psi \\ \Leftrightarrow &\forall u' \in T[uSu' \Rightarrow (\mathfrak{M}, u') \models \psi].\end{aligned}$$

  Fix any $u' = (\Gamma',\, \Delta')$ in T such that uSu'. Since $\mathfrak{h}$ is a Hintikka system, (HS_M1) holds and so $\psi \in \Gamma'$. By the induction hypothesis, we have $(\mathfrak{M},\, u') \models \psi$. Hence $\forall u' \in T[uSu' \Rightarrow (\mathfrak{M}, u') \models \psi]$.

  Suppose that $\Box\psi \in \Delta$. Our goal is to show

  $$\begin{aligned}&(\mathfrak{M},\, u) \not\models \Box\psi \\ \Leftrightarrow &\exists u' \in T[uSu' \quad \mathit{and} \quad (\mathfrak{M},\, u') \not\models \psi].\end{aligned}$$

  Since $\mathfrak{h}$ is a Hintikka system, (HS_M2) holds and so there is $u' = (\Gamma',\, \Delta')$ in T such that uSu' and $\psi \in \Gamma'$. By the induction hypothesis, we have $(\mathfrak{M},\, u') \not\models \psi$. Hence $\exists u' \in T[uSu' \quad \mathit{and} \quad (\mathfrak{M},\, u') \not\models \psi]$.

$\square$

Corollary 2.5.5. *If $\mathfrak{h}$ is a Hintikka system for $(\emptyset,\, \{\varphi\})$ then $\mathfrak{h} \not\models \varphi$.*

Proof. Suppose that $\mathfrak{h} = (T, S)$ is a Hintikka system for $(\emptyset, \{\varphi\})$. Our goal is to show $\mathfrak{h} \nvDash \varphi$, i.e., $\exists t \in T \exists \mathfrak{V}\colon \mathrm{Var}\mathcal{ML} \to \mathcal{P}(T)[(\mathfrak{h}, \mathfrak{V}, t) \nvDash \varphi]$. By the assumption, there is a tableau $t' = (\Gamma', \Delta')$ in T such that $\emptyset \subseteq \Gamma'$ and $\{\varphi\} \subseteq \Delta'$. With the same argument of $(\Leftarrow)$ of proposition 2.5.4, it is obvious that $(\mathfrak{h}, \mathfrak{V}, t') \nvDash \varphi$, where $\mathfrak{V}(p) = \{u = (\Gamma, \Delta) \mid u \in T \textit{ and } p \in \Gamma\}$. $\square$

Theorem 2.5.6. *A tableau t is realizable in K iff there is a Hintikka system for t containing at most $2^{|\Sigma|}$ tableaux, where Σ is the set of all subformula of the formulas in t.*

Proof. $(\rightarrow)$ Suppose that t is realizable in $\mathfrak{M} = \{\mathfrak{F}, \mathfrak{V}\}$. For every point $x \in W$, we form a tableau $t_x = (\Gamma_x, \Delta_x)$ by taking

$$\Gamma_x = \{\varphi \in \Sigma \mid (\mathfrak{M}, x) \models \varphi\},$$
$$\Delta_x = \{\varphi \in \Sigma \mid (\mathfrak{M}, x) \nvDash \varphi\}.$$

Let $\mathfrak{h} = \langle T, S\rangle$, where $T = \{t_x \mid x \in W\}$ and, for every $t_x = (\Gamma_x, \Delta_x)$ and $t_y = (\Gamma_y, \Delta_y)$ in T,

$$t_x S t_y \Leftrightarrow (\Box\varphi \in \Gamma_x \text{ implies } \varphi \in \Gamma_y, \text{ for any formula of the form } \Box\varphi \in \Sigma).$$

With the same argument of the proof of proposition 2.5.4, it is obvious that T is a nonempty set and, for every $t_x \in T$, t_x is a disjoint saturated tableau. Also, this definition guarantees that (HS_M1) is satisfied. So it remains to see that (HS_M2) also holds to show $\mathfrak{h}$ is a Hintikka system for t. Suppose that $t_x = (\Gamma_x, \Delta_x)$ and $\Box\varphi \in \Delta_x$. Our goal is to show that there is $t_y = (\Gamma_y, \Delta_y)$ in T such that $t_x S t_y$ and $\varphi \in \Delta_y$. By the assumption and the definition of Δ, we have

$$(\mathfrak{M}, x) \nvDash \Box\varphi$$
$$\Leftrightarrow \exists y \in W[xRy \textit{ and } (\mathfrak{M}, y) \nvDash \varphi].$$

On the other hand, consider any $\Box\psi \in \Sigma$ such that $\Box\psi \in \Gamma_x$. By the definition of Γ, we have

$$(\mathfrak{M}, x) \models \Box\psi$$
$$\Leftrightarrow \forall y \in W[xRy \Rightarrow (\mathfrak{M}, y) \models \psi].$$

Since xRy, $(\mathfrak{M}, y) \models \psi$ holds and so we have $\psi \in \Gamma_y$. Therefore, by the definition of S, we have $t_x S t_y$. Also, since $(\mathfrak{M}, y) \nvDash \phi$, we have $\phi \in \Delta_y$. Hence (HS_M2) holds and so $\mathfrak{h}$ is a Hintikka system for t. Since the number of formulas in each tableau in T is $|\Sigma|$ and T is not a multi-set, i.e., there is no duplicate tableau in T, it is clear that $|T| \leq |\Sigma|$.

$(\Leftarrow)$ follows from proposition 2.5.4. $\square$

Corollary 2.5.7.

1. *For every formula $\varphi \notin K$ there is a rooted frame refuting φ and containing at most $2^{|Sub\varphi|}$ points.*

2. *Every $\varphi \notin K$ is refuted in some finite intransitive tree.*

Proof.

1. Let $\varphi \notin K$ be a modal formula. Then, by corollary **??**, there is a rooted frame $\mathfrak{F} = \langle W, R\rangle$ such that $\mathfrak{F} \not\models \varphi$, i.e.,

$$\forall\,\mathfrak{V}\colon \mathrm{Var}\mathcal{ML} \to \mathcal{P}(W)\forall\, x \in W[(\mathfrak{F}, \mathfrak{V}, x) \not\models \varphi].$$

   Therefore, a tableau $t = (\emptyset, \{\varphi\})$ is realizable in K, and hence, by theorem 2.5.6, there is a Hintikka system $\mathfrak{h} = \langle T, S\rangle$ for t containing at most $2^{|\Sigma|}$ tableaux, where $T = \{t_x \mid x \in W\}$, $t_x S t_y \Leftrightarrow xRy$, and Σ is the set of all subformulas of the formulas in t. By corollary 2.5.5, we have $\mathfrak{h} \not\models \varphi$. Since $\mathfrak{F}$ is a rooted frame, by the definition of $\mathfrak{h}$, $\mathfrak{h}$ is also a rooted frame. Since $|T| \leq 2^{|\Sigma|}$, and Σ is, in this case, the subset of all subformulas of φ, $\mathfrak{h}$ contains at most $2^{|Sub\varphi|}$ points.

2. Let $\varphi \notin K$ be a modal formula. By corollary 2.5.7.1, there is a rooted frame $\mathfrak{h}$ refuting φ. By theorem **??**, there is an intransitive tree $\mathfrak{F} = \langle W, R\rangle$ which is reducible to $\mathfrak{h}$ and, by theorem **??**, refutes φ. Though $\mathfrak{F}$ may be infinite, every point in it has finitely many successors. Suppose that $\mathrm{md}(\varphi) = n$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is a model such that $(\mathfrak{M}, x) \not\models \varphi$ for some points x. By proposition **??**, the submodel $\mathfrak{N}$ of $\mathfrak{M}$, induced by the set $x\uparrow^0 \cup \ldots \cup x\uparrow^n$, also refutes φ, and $\mathfrak{N}$ is based upon a finite intransitive tree.

□

Corollary 2.5.8. $K = \{\varphi \in \mathrm{For}\mathcal{ML} \mid \mathfrak{F} \models \varphi$ *for all finite intransitive trees* $\mathfrak{F}\}$.

Proof. Follows from corollary 2.5.7.2. □

Chapter 3

Calculus K

The modal propositional calculus K in the language $\mathcal{ML}$ is sound and complete with respect to the possible world semantics. In this section we see the calculus, and the soundness and completeness theorems of not only logic K but also a few more modal logics.

3.1 Axioms and Inference Rules

The axioms and inference rules of calculus K is really similar to the Hilbert calculus in classical logic; just added an axiom $(A11)$ and an inference rule (RN) (see below).

Axioms

(A1) $p_0 \to (p_1 \to p_0)$

(A2) $(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$

(A3) $p_0 \wedge p_1 \to p_0$

(A4) $p_0 \wedge p_1 \to p_1$

(A5) $p_0 \to (p_1 \to p_0 \wedge p_1)$

(A6) $p_0 \to p_0 \vee p_1$

(A7) $p_1 \to p_0 \vee p_1$

(A8) $(p_0 \to p_2) \to ((p_1 \to p_2) \to (p_0 \vee p_1 \to p_2))$

(A9) $\perp \to p_0$

(A10) $p_0 \vee (p_0 \to \perp)$

(A11) $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$

Inference Rules

- Modus Ponens (MP): given formulas φ and $\varphi \to \psi$, we obtain ψ
- Substitution (Subst): given a formula φ, we obtain φs, where

$s: \quad \mathrm{Var}\mathcal{ML} \to \mathrm{For}\mathcal{ML}$

- $ps := s(p)$
- $\bot s := \bot$
- $(\psi \odot \chi)s := \psi s \odot \chi s, \quad \odot \in \{\wedge, \vee, \to\}$
- $(\Box\psi)s := \Box(\psi s)$

- Necessitation (RN): given a formula φ, we infer $\Box\varphi$

There are lots of well-known valid formulas in classical logic, listed in table 3.1, which we often use in this section. Some formulas in this table overlap with the axioms. Note that the formulas in this table are valid classically: some formulas are not valid in intuitionistic or intermidiate logics. In modal logics handled in this thesis, however, all of the formulas listed here are valid since these logics are based on classical logic.

Before proving the soundness and completeness theorems, let us define some important notions and prepare some lemmas.

Definition 3.1.1 (Derivation)**.** *We say that a sequence $\varphi_1, \ldots, \varphi_n$ of formulas is a derivation of formula φ if*

- $\varphi_n = \varphi$
- *for every $1 \leq i \leq n$, φ_i satisfies one of the following;*
  - *φ_i is an axiom;*
  - $\exists j, j' < i[\varphi_j \equiv \varphi_{j'} \to \varphi_i]$*;*
  - $\exists j < i[\varphi_i \equiv \Box\varphi_j]$*.*

We denote by $\vdash_K \varphi$ if a formula φ is derivable in K.

Lemma 3.1.2. $\vdash_K \varphi \to \psi \quad \Rightarrow \quad \vdash_K \Box\varphi \to \Box\psi$

Proof.

(1) $\varphi \to \psi$ (given)

(2) $\Box(\varphi \to \psi)$ (RN)

(3) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (A11)

(4) $\Box\varphi \to \Box\psi$ ((2), (3), MP)

$\Box$

Lemma 3.1.3. $\vdash_K \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$

Proof.

(1) $\varphi \wedge \psi \to \varphi$ (A3)

Table 3.1: A list of classically valid formulas

Formula	Name
$p \wedge p \leftrightarrow p$, $p \vee p \leftrightarrow p$	The laws of idempotency
$p \wedge q \leftrightarrow q \wedge p$, $p \vee q \leftrightarrow q \vee p$	The laws of commutativity
$p \wedge \bot \leftrightarrow \bot$, $p \wedge \top \leftrightarrow p$	
$p \vee \bot \leftrightarrow p$, $p \vee \top \leftrightarrow \top$	
$\bot \to p$, $p \to \top$	
$p \wedge \neg p \to q$	Duns Scotus' law
$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$	The law of associativity
$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$	The law of associativity
$(p \wedge q) \vee q \leftrightarrow q$, $p \wedge (p \vee q) \leftrightarrow p$	The laws of absorption
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	The law of distributivity
$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	The law of distributivity
$p \to (q \to p)$	The law of simplification
$(p \to q) \to ((q \to r) \to (p \to r))$	The law of syllologism
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	Frege's law
$p \wedge q \to p$, $p \to p \vee q$	
$(p \vee q) \wedge (p \vee \neg q) \leftrightarrow p$	
$p \to (q \to p \wedge q)$	The law of adjunction
$(p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$	The law of importation and exportation
$(p \to q) \to ((p \to r) \to (p \to q \wedge r))$	
$(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$	
$(p \to q) \wedge (p' \to q') \to (p \vee p' \to q \vee q')$	
$(p \to q) \wedge (p' \to q') \to (p \wedge p' \to q \wedge q')$	
$(p \to r) \to ((q \to r) \to (p \vee q \to r))$	
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$, $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	De Morgan's laws
$(p \to q) \leftrightarrow \neg p \vee q$	
$(p \to q) \leftrightarrow \neg(p \wedge \neg q)$	
$((p \to q) \to p) \to p$	Pierce's law
$p \vee \neg p$	The law of the excluded middle
$(p \to q) \leftrightarrow (\neg q \to \neg p)$	The law of contraposition
$p \leftrightarrow \neg\neg p$	The law of double negation
$(p \wedge q) \vee (p \wedge \neg q) \leftrightarrow p$	

(2) $\Box(\varphi \wedge \psi) \to \Box\varphi$ ((1), lemma 3.1.2)

(3) $\varphi \wedge \psi \to \psi$ (A4)

(4) $\Box(\varphi \wedge \psi) \to \Box\psi$ ((3), lemma 3.1.2)

(5) $\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$ ((2), (4))

(6) $\varphi \to (\psi \to \varphi \wedge \psi)$ (A5)

(7) $\Box\varphi \to \Box(\psi \to \varphi \wedge \psi)$ ((6), lemma 3.1.2)

(8) $\Box(\psi \to \varphi \wedge \psi) \to (\Box\psi \to \Box(\varphi \wedge \psi))$ (A11)

(9) $\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))$ ((7), (8), The law of syllogism)

(10) $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ ((9), The law of importation and exportation)

(11) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$ ((5), (10))

$\Box$

The deduction theorem, which is well-known theorem in the field of mathematical logic, should not hold for K if we want K to be sound with respect to the Kripke semantics as it was formulated for classical logic and intuitionistic logic. So, we have to formulate for modal logic. Before formulation, we need to expand a derivation in K, which could deal with assumptions, and define dependency of a formula.

Definition 3.1.4 (Derivation from a set of assumptions)**.** *Let Γ be a set of formulas. A sequence $\varphi_1, \ldots, \varphi_n$ of formulas is called a derivation of φ from the set of assumptions Γ if:*

- $\varphi_n = \varphi$;
- *for every $1 \leq i \leq n$, φ_i satisfies one of the following:*
  - φ_i *is an axiom;*
  - $\varphi_i \in \Gamma$;
  - $\exists j, j' < i[\varphi_j \equiv \varphi_{j'} \to \varphi_i]$;
  - $\exists j < i[\varphi_i \equiv \Box\varphi_j]$.

We denote by $\Gamma \vdash_K \varphi$ if a formula φ is derivable from a set Γ of assumptions in K.

Definition 3.1.5 (Dependency)**.** *Let $\varphi_1, \ldots, \varphi_n$ be a derivation from assumptions. Say that a formula φ_k depends on a formula φ_i in this derivation if one of the following holds:*

- $k = i$;
- $\exists j, j' < k.[\varphi_j \equiv \varphi_{j'} \to \varphi_k$ *and* $(\varphi_j$ *or* $\varphi_{j'}$ *depends on* $\varphi_i)]$;

- $\exists j < k.[\varphi_k \equiv \Box\varphi_j \quad \textit{and} \quad (\varphi_j \textit{ depends on } \varphi_i)]$.

Theorem 3.1.6 (Deduction theorem for K)**.** *Suppose that* $\Gamma, \psi \vdash_K \varphi$ *and there exists a derivation of* φ *from the assumptions* $\Gamma \cup \{\psi\}$ *in which RN is applied to formulas depending on* ψ $m \geq 0$ *times. Then*

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^m\psi \to \varphi$$

Proof. Let $\varphi_1, \ldots, \varphi_n$ be a derivation of $\varphi = \varphi_n$ from $\Gamma \cup \{\psi\}$, in which RN is applied to formulas depending on ψ m times. We show by induction on $1 \leq i \leq n$ that

$$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^l\psi \to \varphi_i, \quad (*)$$

where l is the number of applications of RN to formulas depend on ψ in the derivation $\Gamma_1 \vdash \varphi_1, \ldots, \Gamma_i \vdash \varphi_i$.

- φ_i is an axiom

  Then, the sequence

  (1) φ_i (axiom)
  (2) $\varphi_i \to (\psi \to \varphi_i)$ (A1)
  (3) $\psi \to \varphi_i$ ((1), (2), MP)
  (4) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \psi$ (A3)
  (5) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \varphi_i$ ((3), (4), The law of syllogism)

  is a derivation of $(*)$.

- φ_i is a formula in Γ_i

  Then, the sequence

  (1) φ_i ($\because \varphi_i \in \Gamma$)
  (2) $\varphi_i \to (\psi \to \varphi_i)$ (A1)
  (3) $\psi \to \varphi_i$ ((1), (2), MP)
  (4) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \psi$ (A3)
  (5) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \varphi_i$ ((3), (4), The law of syllogism)

  is a derivation of $(*)$.

- $\varphi_i \equiv \psi$

Then, the sequence

$$
\begin{array}{ll}
(1) & \varphi_i \to \varphi_i \vee \varphi_i \quad \text{(A6)} \\
(2) & \varphi_i \vee \varphi_i \to \varphi_i \quad \text{(The law of idempotency)} \\
(3) & \psi \to \varphi_i \quad ((1),\ (2),\ \text{MP},\ \varphi_i \equiv \psi) \\
(4) & \psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \psi \quad \text{(A3)} \\
(5) & \psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \varphi_i \quad ((3),\ (4),\ \text{The law of syllogism})
\end{array}
$$

is a derivation of $(*)$.

- φ_i is obtained from $\varphi_k \equiv \varphi_j \to \varphi_i$ and φ_j by MP

Suppose that RN is applied to formulas depending on ψ in $\varphi_i, \ldots, \varphi_k$ and $\varphi_i, \ldots, \varphi_j$ l_1 and l_2 times, respectively. Then, by the induction hypothesis, we have

$$
\begin{aligned}
\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to (\varphi_j \to \varphi_i), \\
\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_2}\psi \to \varphi_j,
\end{aligned}
$$

and so, we obtain

$$
\begin{aligned}
\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l}\psi \to (\varphi_j \to \varphi_i), \\
\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l}\psi \to \varphi_j,
\end{aligned}
$$

since

$$
\begin{array}{ll}
(1) & \Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to (\varphi_j \to \varphi_i) \quad \text{(given)} \\
(2) & (\Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi) \wedge (\Box^{l_1+1}\psi \wedge \ldots \wedge \Box^{l}\psi) \to (\Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi) \quad (\text{A3},\ l_1 \leq l) \\
(3) & \Box^0\psi \wedge \ldots \wedge \Box^{l}\psi \to (\varphi_j \to \varphi_i) \quad ((1),\ (2),\ \text{MP}) \\
(4) & \Box^0\psi \wedge \ldots \wedge \Box^{l_2}\psi \to \varphi_j \quad \text{(given)} \\
(5) & (\Box^0\psi \wedge \ldots \wedge \Box^{l_2}\psi) \wedge (\Box^{l_2+1}\psi \wedge \ldots \wedge \Box^{l}\psi) \to (\Box^0\psi \wedge \ldots \wedge \Box^{l_2}\psi) \quad (\text{A3},\ l_2 \leq l) \\
(6) & \Box^0\psi \wedge \ldots \wedge \Box^{l}\psi \to \varphi_j \quad ((4),\ (5),\ \text{MP}).
\end{array}
$$

Let us denote $\Box^0\psi \wedge \ldots \wedge \Box^l\psi$ by $\bigwedge_{1 \leq i \leq l} \Box^i\psi$. Then we get the following derivation, which derives $(*)$.

$$
\begin{array}{ll}
(7) & (\bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to (\varphi_j \to \varphi_i)) \to ((\bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to \varphi_j) \to (\bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to \varphi_i)) \quad \text{(A2)} \\
(8) & (\bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to \varphi_j) \to (\bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to \varphi_i) \quad ((3),\ (7),\ \text{MP}) \\
(9) & \bigwedge\limits_{1 \leq i \leq l} \Box^i\psi \to \varphi_i \quad ((6),\ (8),\ \text{MP}).
\end{array}
$$

- φ_i is obtained from φ_j by RN

  (i) φ_j does not depend on ψ

  Then, the sequence

  (1) $\Box\varphi_j$ (given)
  (2) $\Box\varphi_j \to (\psi \to \Box\varphi_j)$ (A1)
  (3) $\psi \to \Box\varphi_j$ ((1), (2), MP)
  (4) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \psi$ (A3)
  (5) $\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^l\psi) \to \Box\varphi_j$ ((3), (4), The law of syllogism)

  (ii) φ_j depends on ψ

  Suppose that RN is applied $l_1 < l$ times to formulas depending on ψ in $\varphi_1, \ldots, \varphi_j$. By the induction hypothesis, we have

  $$\Gamma \vdash_K \Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to \varphi_j$$

  and so,

  (1) $\Box^0\psi \wedge \ldots \wedge \Box^{l_1}\psi \to \varphi_j$
  (2) $\Box^1\psi \wedge \ldots \wedge \Box^{l_1+1}\psi \to \Box\varphi_j$ ((1), lemma 3.1.2)
  (3) $\Box^0\psi \wedge (\Box^1\psi \wedge \ldots \wedge \Box^{l_1+1}\psi) \to \Box^1\psi \wedge \ldots \wedge \Box^{l_1+1}\psi$ (A4)
  (4) $\Box^0\psi \wedge \ldots \wedge \Box^{l_1+1}\psi \to \Box\varphi_j$ ((2), (3), The law of syllogism).

  We get $(*)$ by repeating (2)-(4).

$\Box$

Corollary 3.1.7. *Suppose that $\Gamma, \psi \vdash_K \varphi$ and there exists a derivation of φ from the assumptions $\Gamma \cup \{\psi\}$ in which RN is not applied to formulas depending on ψ. Then $\Gamma \vdash_K \psi \to \varphi$.*

In the sequel, we will distinguish a derivation into two categories:

- a derivation in which RN is applied exceptionally to formulas that depend only on an axioms (derivability will be denoted by $\vdash$);
- a derivation without this restriction (derivability will be denoted by $\vdash^*$).

Now we are ready to prove the soundness and completeness theorems.

3.2 Soundness and Completeness

Theorem 3.2.1. $\vdash_K \varphi$ *iff* $\mathfrak{F} \models \varphi$ *for all frames* $\mathfrak{F}$

Proof. ($\Rightarrow$) Let $\varphi_1, \ldots, \varphi_n$ be a derivation of a formula $\varphi = \varphi_n$. Suppose that $\vdash_K \varphi$. Fix any frame $\mathfrak{F}$. We show $\mathfrak{F} \models \varphi$ by induction on the length of derivation.

(A1) $\varphi_i \equiv \psi \to (\chi \to \psi)$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$. Now it suffices to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \chi \to \psi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \chi \quad \textit{implies} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \psi, \end{aligned}$$

which is obvious.

(A2) $\varphi_i \equiv (\psi \to (\chi \to \sigma)) \to ((\psi \to \chi) \to (\psi \to \sigma))$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that

$$(\mathfrak{F}, \mathfrak{V}, x) \models \psi \to (\chi \to \sigma), \tag{3.1}$$

$$(\mathfrak{F}, \mathfrak{V}, x) \models \psi \to \chi, \tag{3.2}$$

$$(\mathfrak{F}, \mathfrak{V}, x) \models \psi. \tag{3.3}$$

Now it suffices to show $(\mathfrak{F}, \mathfrak{V}, x) \models \sigma$. By (2) and (3), we have $(\mathfrak{F}, \mathfrak{V}, x) \models \chi$. We also have $(\mathfrak{F}, \mathfrak{V}, x) \models \chi \to \sigma$ by (1) and (3). Hence we have $(\mathfrak{F}, \mathfrak{V}, x) \models \sigma$.

(A3) $\varphi_i \equiv \psi \wedge \chi \to \psi$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{and} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi \end{aligned}$$

Now it suffices to show $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$, which is obvious.

(A4) $\varphi_i \equiv \psi \wedge \chi \to \chi$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{and} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi \end{aligned}$$

Now it suffices to show $(\mathfrak{F}, \mathfrak{V}, x) \models \chi$, which is obvious.

(A5) $\varphi_i \equiv \psi \to (\chi \to \psi \wedge \chi)$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$ and $(\mathfrak{F}, \mathfrak{V}, x) \models \chi$. Now it suffices to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \wedge \chi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{and} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi, \end{aligned}$$

which is obvious.

(A6) $\varphi_i \equiv \psi \to \psi \vee \chi$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$. Now it suffices to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \vee \chi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{or} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi, \end{aligned}$$

which is obvious.

(A7) $\varphi_i \equiv \chi \to \psi \vee \chi$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that $(\mathfrak{F}, \mathfrak{V}, x) \models \chi$. Now it suffices to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \vee \chi \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{or} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi, \end{aligned}$$

which is obvious.

(A8) $\varphi_i \equiv (\psi \to \sigma) \to ((\chi \to \sigma) \to (\psi \vee \chi \to \sigma))$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that

$$(\mathfrak{F}, \mathfrak{V}, x) \models \psi \to \sigma, \tag{3.4}$$

$$(\mathfrak{F}, \mathfrak{V}, x) \models \chi \to \sigma, \tag{3.5}$$

$$(\mathfrak{F}, \mathfrak{V}, x) \models \psi \vee \chi \Leftrightarrow (\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{or} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \chi \tag{3.6}$$

Now it suffices to show $(\mathfrak{F}, \mathfrak{V}, x) \models \sigma$. There are two cases: if $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$, we obtain it by (4); otherwise we obtain it by (5).

(A9) $\varphi_i \equiv \bot \to \psi$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Then it is obvious since $(\mathfrak{F}, \mathfrak{V}, x) \not\models \bot$.

(A10) $\varphi_i \equiv \psi \vee (\psi \to \bot)$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Our goal is to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \vee (\psi \to \bot) \\ \Leftrightarrow &(\mathfrak{F}, \mathfrak{V}, x) \models \psi \quad \textit{or} \quad (\mathfrak{F}, \mathfrak{V}, x) \models \psi \to \bot. \end{aligned}$$

There are two cases: $(\mathfrak{F}, \mathfrak{V}, x) \models \psi$ or $(\mathfrak{F}, \mathfrak{V}, x) \not\models \psi$. Both cases are trivial.

(A11) $\varphi_i \equiv \Box(\psi \to \chi) \to (\Box\psi \to \Box\chi)$

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Suppose that

$$(\mathfrak{F}, \mathfrak{V}, x) \models \Box(\psi \to \chi) \Leftrightarrow \forall y \in \mathfrak{F}[xRy \Rightarrow (\mathfrak{F}, \mathfrak{V}, y) \models \psi \to \chi], \tag{3.7}$$

$$(\mathfrak{F}, \mathfrak{V}, x) \models \Box\psi \Leftrightarrow \forall y \in \mathfrak{F}[xRy \Rightarrow (\mathfrak{F}, \mathfrak{V}, y) \models \psi]. \tag{3.8}$$

Now it suffices to show

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \Box\chi \\ \Leftrightarrow &\forall y \in \mathfrak{F}[xRy \Rightarrow (\mathfrak{F}, \mathfrak{V}, y) \models \chi]. \end{aligned}$$

Fix any $y \in \mathfrak{F}$ such that xRy. By (7) and (8), we have $(\mathfrak{F}, \mathfrak{V}, y) \models \psi \to \chi$ and $(\mathfrak{F}, \mathfrak{V}, y) \models \psi$ and so $(\mathfrak{F}, \mathfrak{V}, y) \models \chi$ holds. Hence $(\mathfrak{F}, \mathfrak{V}, x) \models \Box\chi$.

φ_i is obtained from $\varphi_k \equiv \varphi_j \to \varphi_i$ and φ_j by MP

By the induction hypothesis, we have $\mathfrak{F} \models \varphi_j \to \varphi_i$ and $\mathfrak{F} \models \varphi_j$ and so obviously $\mathfrak{F} \models \varphi_i$ holds.

$\varphi_i \equiv \Box\varphi_j$ is obtained from φ_j by RN

Fix any point $x \in \mathfrak{F}$ and valuation $\mathfrak{V}$ on $\mathfrak{F}$. Our goal is to show that

$$\begin{aligned} &(\mathfrak{F}, \mathfrak{V}, x) \models \Box\varphi_j \\ \Leftrightarrow &\forall y \in \mathfrak{F}[xRy \Rightarrow (\mathfrak{F}, \mathfrak{V}, y) \models \varphi_j]. \end{aligned}$$

Fix any $y \in \mathfrak{F}$ such that xRy. The induction hypothesis yields $\mathfrak{F} \models \varphi_j$ and so $(\mathfrak{F}, \mathfrak{V}, y) \models \varphi_j$ holds. Hence $(\mathfrak{F}, \mathfrak{V}, x) \models \Box\varphi_j$.

($\Leftarrow$) First we show

$$\nvdash_K \varphi \Rightarrow (\text{There is a Hintikka system } \mathfrak{h} \text{ for the tableau } (\emptyset, \{\varphi\})).$$

Then, by corollary 2.5.5, we have $\mathfrak{h} \nvDash \varphi$. Before proving this side, we need to define two terms: consistent and maximal.

Definition 3.2.2 (Consistent tableau)**.** *A tableau $t = (\Gamma, \Delta)$ is consistent in K if $\forall\Delta' \subseteq \Delta[\Gamma \nvdash_K \bigvee_{\varphi_i \in \Delta'} \varphi_i]$.*

Definition 3.2.3 (Maximal tableau)**.** *A tableau $t = (\Gamma, \Delta)$ is maximal (relative to φ) if $\Gamma \cup \Delta = \mathrm{Sub}\varphi$.*

Now suppose that $\nvdash_K \varphi$. Let $\varphi_1, \ldots, \varphi_n$ be a list of all formulas in Subφ. Define a sequence of tableaux $t_0 = (\Gamma_0, \Delta_0), \ldots, t_n = (\Gamma_n, \Delta_n)$ by taking

$$\begin{aligned} t_0 &= (\emptyset, \{\varphi\}), \\ t_{i+1} &= \begin{cases} (\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\}) & ((\Gamma_i, \Delta_i \cup \{\varphi_{i+1}\}) \text{ is consistent}) \\ (\Gamma_i \cup \{\varphi_{i+1}\}, \Delta_i) & (\text{otherwise}). \end{cases} \end{aligned}$$

Note that $\Gamma_n \cup \Delta_n = \mathrm{Sub}\varphi$. Let us show that t_i is consistent for $0 \leq i \leq n$ by induction on i.

$(i = 0)$

Obvious by the assumption.

$(i = k + 1)$

$(t_{k+1} = (\Gamma_k, \Delta_k \cup \{\varphi_{k+1}\}))$

Trivial.

$(t_{k+1} = (\Gamma_k \cup \{\varphi_{k+1}\}, \Delta_k))$

Assume for contradiction that $(\Gamma_k \cup \{\varphi_{k+1}\}, \Delta_k)$ is not consistent. Then there is a subset $\Delta' = \{\psi_1, \ldots, \psi_m\}$ of Δ_k such that

$$\Gamma_k, \varphi_{k+1} \vdash_K \psi_1 \vee \cdots \vee \psi_m.$$

At the same time, there is a subset $\Delta'' = \{\chi_1, \ldots, \chi_n\}$ of Δ_k such that

$$\Gamma_k \vdash_K \chi_1 \vee \cdots \vee \chi_n \vee \varphi_{k+1},$$

since $(\Gamma_k, \Delta_k \cup \{\varphi_{k+1}\})$ is not consistent. Therefore, there is a subset $\Delta = \{\sigma_1, \ldots, \sigma_l\}$ of Δ_k such that $\Delta', \Delta'' \subseteq \Delta$ and

$$\Gamma_k, \varphi_{k+1} \vdash_K \sigma_1 \vee \cdots \vee \sigma_l,$$
$$\Gamma_k \vdash_K \sigma_1 \vee \cdots \vee \sigma_l \vee \varphi_{k+1}.$$

By corollary 3.1.7, we have $\Gamma_k \vdash_K \varphi_{k+1} \to \sigma_1 \vee \cdots \vee \sigma_l$. However, we could obtain $\Gamma_k \vdash_K \sigma_1 \vee \cdots \vee \sigma_l$, which is contrary to the consistency of t_k, by the following derivation:

(1) $\bigvee_{1 \le j \le l} \sigma_j \vee \varphi_{k+1}$ (given)

(2) $\varphi_{k+1} \to \bigvee_{1 \le j \le l} \sigma_j$ (given)

(3) $(\varphi_{k+1} \to \bigvee_{1 \le j \le l} \sigma_j) \to ((\bigvee_{1 \le j \le l} \sigma_j \to \bigvee_{1 \le j \le l} \sigma_j) \to (\bigvee_{1 \le j \le l} \sigma_j \vee \varphi_{k+1} \to \bigvee_{1 \le j \le l} \sigma_j))$ (A8)

(4) $(\bigvee_{1 \le j \le l} \sigma_j \to \bigvee_{1 \le j \le l} \sigma_j) \to (\bigvee_{1 \le j \le l} \sigma_j \vee \varphi_{k+1} \to \bigvee_{1 \le j \le l} \sigma_j)$ ((2), (3), MP)

(5) $\bigvee_{1 \le j \le l} \sigma_j \to \bigvee_{1 \le j \le l} \sigma_j \vee \bigvee_{1 \le j \le l} \sigma_j$ (A6)

(6) $\bigvee_{1 \le j \le l} \sigma_j \vee \bigvee_{1 \le j \le l} \sigma_j \to \bigvee_{1 \le j \le l} \sigma_j$ (The law of idempotency)

(7) $\bigvee_{1 \le j \le l} \sigma_j \to \bigvee_{1 \le j \le l} \sigma_j$ ((5), (6), The law of syllologism)

(8) $\bigvee_{1 \le j \le l} \sigma_j \vee \varphi_{k+1} \to \bigvee_{1 \le j \le l} \sigma_j$ ((4), (7), MP)

(9) $\bigvee_{1 \le j \le l} \sigma_j$ ((1), (8), MP)

Now we show that the tableau t_n is disjoint and saturated.

(disjoint)

Assume for contradiction that t_n is not disjoint. Then, there is a formula ψ such that $\psi \in \Gamma_n$ and $\psi \in \Delta_n$. Let $\Gamma'_n \equiv \Gamma_n \smallsetminus \{\psi\}$. It is obvious that $\Gamma_n \cup \{\psi\} \vdash_K \psi$. However, since t_n is consistent and $\psi \in \Delta_n$, we have $\Gamma'_n \cup \{\psi\} \nvdash_K \psi$.

(saturated)

(S1)

Suppose that $\psi \wedge \chi \in \Gamma_n$ and, for contradiction, $\psi \in \Delta_n$. Then, the sequence

$$
\begin{array}{lll}
(1) & \psi \wedge \chi & (\because \psi \wedge \chi \in \Gamma_n) \\
(2) & \psi \wedge \chi \to \psi & (A3) \\
(3) & \psi & ((1), (2), MP)
\end{array}
$$

is a derivation of $\Gamma_n \vdash_K \psi$. However, since t_n is consistent and $\psi \in \Delta_n$, $\Gamma_n \nvdash_K \psi$. Hence $\psi \notin \Delta_n$ and so $\psi \in \Gamma_n$ as t_n is disjoint.

(S2)

Suppose that $\psi \wedge \chi \in \Delta_n$ and, for contradiction, ψ, $\chi \in \Gamma_n$. Then, the sequence

$$
\begin{array}{lll}
(1) & \psi & (\because \psi \in \Gamma_n) \\
(2) & \chi & (\because \chi \in \Gamma_n) \\
(3) & \psi \to (\chi \to \psi \wedge \chi) & (A5) \\
(4) & \chi \to \psi \wedge \chi & ((1), (3), MP) \\
(5) & \psi \wedge \chi & ((2), (4), MP)
\end{array}
$$

is a derivation of $\Gamma_n \vdash_K \psi \wedge \chi$. However, since t_n is consistent and $\psi \wedge \chi \in \Delta_n$, $\Gamma_n \nvdash_K \psi \wedge \chi$. Hence $\psi, \chi \notin \Gamma_n$ and so $\psi \in \Delta_n$ or $\chi \in \Delta_n$ as t_n is disjoint.

(S3)

Suppose that $\psi \vee \chi \in \Gamma_n$ and, for contradiction, ψ, $\chi \in \Delta_n$. Then, since t_n is consistent, $\Gamma_n \nvdash_K \psi \vee \chi$ holds. However, it is obvious that $\Gamma_n \vdash_K \psi \vee \chi$ since $\psi \vee \chi \in \Gamma_n$. Hence $\psi, \chi \notin \Delta_n$ and so $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$ as t_n is disjoint.

(S4)

Suppose that $\psi \vee \chi \in \Delta_n$ and, for contradiction, $\psi \in \Gamma_n$. Then, the sequence

$$
\begin{array}{lll}
(1) & \psi & (\because \psi \in \Gamma_n) \\
(2) & \psi \to \psi \vee \chi & (A6) \\
(3) & \psi \vee \chi & ((1), (2), MP)
\end{array}
$$

is a derivation of $\Gamma_n \vdash_K \psi \vee \chi$. However, since t_n is consistent and $\psi \vee \chi \in \Delta_n$, $\Gamma_n \nvdash_K \psi \vee \chi$. Hence $\psi \notin \Gamma_n$ and so $\psi \in \Delta_n$ as t_n is disjoint.

(S5)

Suppose that $\psi \to \chi \in \Gamma_n$ and, for contradiction, $\psi \in \Gamma_n$ and $\chi \in \Gamma_n$.

Then, the sequence

$$\begin{array}{lll} (1) & \psi \to \chi & (\because \psi \to \chi \in \Gamma_n) \\ (2) & \psi & (\because \psi \in \Gamma_n) \\ (3) & \chi & ((1),\ (2),\ \mathrm{MP}) \end{array}$$

is a derivation of $\Gamma_n \vdash_K \chi$. However, since t_n is consistent and $\chi \in \Delta_n$, $\Gamma_n \nvdash_K \chi$. Hence $\psi \notin \Gamma_n$ or $\chi \notin \Delta_n$, and so $\psi \in \Delta_n$ or $\chi \in \Gamma_n$ as t_n is disjoint.

(S6)

Suppose that $\psi \to \chi \in \Delta_n$. Assume two cases for contradiction.

($\chi \in \Gamma_n$)

Then the sequence

$$\begin{array}{lll} (1) & \chi & (\because \chi \in \Gamma_n) \\ (2) & \chi \to (\psi \to \chi) & (\mathrm{A1}) \\ (3) & \psi \to \chi & ((1),\ (2),\ \mathrm{MP}) \end{array}$$

is a derivation of $\Gamma_n \vdash_K \psi \to \chi$. However, since t_n is consistent and $\psi \to \chi \in \Delta_n$, $\Gamma_n \nvdash_K \psi \to \chi$.

($\psi \in \Delta_n$)

Since t_n is consistent, $\Gamma_n \nvdash_K \psi \vee (\psi \to \chi)$. Now take the set T of all maximal (relative to φ) consistent tableaux and define a binary relation S on T by taking, for every $t = (\gamma,\ \Delta)$ and $t' = (\Gamma',\ \Delta')$ in T,

$$tSt' \quad \text{iff} \quad \psi \in \Gamma' \text{ whenever } \Box\psi \in \Gamma.$$

Obviously, the condition (HS_M1) is satisfied. So it remains to verify that (HS_M2) also holds. Suppose that $t = (\Gamma,\ \Delta) \in T$ and $\Box\psi \in \Delta$. Our goal is to show that there is $t' = (\Gamma',\ \Delta')$ in T such that tSt' and $\psi \in \Delta'$. Consider $t' = (\Gamma',\ \{\psi\})$, where $\Gamma' = \{\chi \mid \Box\chi \in \Gamma\}$. We can denote $\Gamma' = \{\chi_1,\ \ldots,\ \chi_m\}$ without loss of generality. First we see that t' is consistent in K. Assume for contradiction that t' is not consistent. Then the sequence is a derivation of $\Box\psi$ from the set $\{\Box\chi_1,\ \ldots,\ \Box\chi_m\}$ of assumptions and so $\Gamma \vdash_K \Box\psi$ holds since $\{\Box\chi_1,\ \ldots,\ \Box\chi_m\} \subseteq \Gamma$. However, by the consistency of t and $\Box\psi \in \Delta$, we have $\Gamma \nvdash_K \Box\psi$. Thus t' is consistent and so we can construct the tableau t'_n in the same manner of constructing $t_n = (\Gamma'_n,\ \Delta'_n)$ where $t_0 = t'$. It is obvious that $t'_n \in T$, since T is the set of all maximal consistent tableaux. It is also clear that tSt'_n and $\psi \in \Delta'_n$ since $\Gamma' = \{\chi \mid \Box\chi \in \Gamma\}$ and $\Delta' = \{\psi\}$. Therefore, $\mathfrak{h} = \langle T,\ S\rangle$ is a Hintikka system for $(\emptyset,\ \{\varphi\})$, from which $\mathfrak{h} \nvDash \varphi$.

$\Box$

Corollary 3.2.4. $K = \{\varphi \in \mathrm{For}\mathcal{ML} \mid\ \vdash_K \varphi\}$.

Chapter 4

Algebraic Semantics

In this chapter, we see another semantics, i.e. algebraic semantics. As we mentioned before, there is a modal logic, like S, in which no Kripke frame could validate all formulas. So, it is valuable to know various kind of semantics on considering new logic.

4.1 Preliminaries

First of all, we introduce the basic algebraic notions and notations to be used in this section.

Definition 4.1.1 (N-ary operation). *Let A be a non-empty set and $n \in \mathcal{N}$. For $n \geq 1$, An n-ary operation on A is a map o from A^n into A; n is called the arity of the operation. 0-ary operation on A is an element in A.*

For example, let $x, y \in \mathcal{R}$. Then a function $f(x, y) = x + y$ is a 2-ary or binary operation on the set $\mathcal{R}$.

Definition 4.1.2 (Universal algebra). *Let A be a non-empty set and $o_1, \ldots, o_n$ be operations on A. A universal algebra or simply an algebra is $\mathfrak{A} = \langle A, o_1, \ldots, o_n \rangle$.*

We will mainly use an algebra of the form $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \leftrightarrow, \neg, \bot, \Box \rangle$, which is called $\mathcal{ML}$-algebra.

Definition 4.1.3 (Term). *Let $\mathfrak{A} = \langle A, o_1, \ldots, o_n \rangle$ be an algebra. Then, a term in $\mathfrak{A}$ is defined inductively as follows.*

1. *Any $\overline{a} \in A$ is a term.*
2. *Any variable in A is a term.*
3. *Let o_i be an m-ary operation and $\overline{t_1}, \ldots, \overline{t_m}$ be any terms in $\mathfrak{A}$. Then, $o_i(\overline{t_1}, \ldots, \overline{t_m})$ is a term.*

We denote a term with overline.

Clearly, any $\mathcal{ML}$-formula is a term in $\mathcal{ML}$-algebra if we consider propositional variables in $\mathcal{ML}$ as variable in the algebra. For $a_1, \ldots, a_n \in \mathfrak{A}$, we denote by $\overline{t}(a_1, \ldots, a_n)$ the result of applying the operation associated with $\overline{t}$ in $\mathfrak{A}$ to the arguments $a_1, \ldots, a_n$.

Definition 4.1.4 (Valuation). *A valuation $\mathfrak{V}$ in an $\mathcal{ML}$-algebra $\mathfrak{A}$ is a map from* Var$\mathcal{ML}$ *to the element of $\mathfrak{A}$.*

Definition 4.1.5 (Value). *Let $\mathfrak{A}$ be an $\mathcal{ML}$-algebra and $\varphi(p_1, \ldots, p_n) \in$* **For**$\mathcal{ML}$ *where $p_1, \ldots, p_n$ are the propositional variables occur in φ. Then $\mathfrak{V}(\varphi) = \varphi(\mathfrak{V}(p_1), \ldots, \mathfrak{V}(p_n))$ is the value of φ in $\mathfrak{A}$ under $\mathfrak{V}$.*

Next we introduce two expressions in algebra; identity and quasi-identity.

Definition 4.1.6 (Identity). *Let $\mathfrak{A}$ be an algebra and $\overline{t_1}$, $\overline{t_2}$ be terms in $\mathfrak{A}$. An expression of the form $\overline{t_1} = \overline{t_2}$ is called an identity. It is true in $\mathfrak{A}$ if $\mathfrak{V}(\varphi) = \mathfrak{V}(\psi)$ for all valuation $\mathfrak{V}$ in $\mathfrak{A}$.*

Definition 4.1.7 (Quasi-identity). *Let $\mathfrak{A}$ be an $\mathcal{ML}$-algebra and $\overline{t_0}, \ldots, \overline{t_n}, \overline{u_0}, \ldots, \overline{u_n}$ be terms in $\mathfrak{A}$. An expression of the form*

$$(\overline{t_1} = \overline{u_1}) \wedge \ldots \wedge (\overline{t_n} = \overline{u_n}) \to ((\overline{t_0} = \overline{u_0})$$

is called a quasi-identity. It is true in $\mathfrak{A}$ if for all valuation $\mathfrak{V}$ in $\mathfrak{A}$, $\mathfrak{V}(\overline{t_0}) = \mathfrak{V}(\overline{u_0})$ whenever $\forall 1 \leq i \leq n \mathfrak{V}(\overline{t_i}) = \mathfrak{V}(\overline{u_i})$.

Definition 4.1.8 (Matrix). *Let $\mathfrak{A} = \langle A, o_1, \ldots, o_n \rangle$ be an algebra and ∇ be a non-empty subset of A. The pair $\langle \mathfrak{A}, \nabla \rangle$ is called a matrix and ∇ its set of distinguished elements.*

If the algebra $\mathfrak{A}$ of a matrix $\langle \mathfrak{A}, \nabla \rangle$ is an $\mathcal{ML}$-algebra, then the matrix is called $\mathcal{ML}$-matrix.

Definition 4.1.9 (Validity). *An $\mathcal{ML}$-formula ϕ is said to be valid in an $\mathcal{ML}$-matrix $\langle \mathfrak{A}, \nabla \rangle$ if the value of φ is in ∇ under every valuation in $\mathfrak{A}$. We write $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ to mean that ϕ is valid in $\langle \mathfrak{A}, \nabla \rangle$.*

We shall often deal with $\mathcal{ML}$-matrix in which ∇ only contains one element $\top = \perp \to \perp$. In this case instead of $\langle \mathfrak{A}, \nabla \rangle \models \varphi$ we write $\mathfrak{A} \models \varphi$ and say that φ is valid in $\mathfrak{A}$.

Definition 4.1.10 (Characteristic). *We say a logic L is characterized by a class $\mathcal{C}$ of matrices (or $\mathcal{C}$ is characteristic for L) if L coincides with the set of formulas that are valid in all matrices in $\mathcal{C}$.*

Definition 4.1.11 (Finite). *Al algebra is finite if its universe is finite.*

Definition 4.1.12 (Degenerate). *Al algebra whose universe contains only one element is called degenerate. A matrix is degenerate if its set of distinguished element coincides its universe.*

Definition 4.1.13 (Similarity)**.** *Let* $\mathfrak{A} = \langle A, o_1, \ldots, o_n \rangle$, $\mathfrak{B} = \langle B, o'_1, \ldots, o'_m \rangle$ *be algebras.* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are said to be similar if* $n = m$ *and, for every* $i \in \{1, \ldots, n\}$, *the operations* o_i *and* o'_i *are of the same arity.*

As a rule, corresponding operations in similar algebras are denoted by the same symbols.

Definition 4.1.14 (Homomorphism)**.** *Suppose that* $\mathfrak{A} = \langle A, o_1, \ldots, o_n \rangle$ *and* $\mathfrak{B} = \langle B, o_1, \ldots, o_n \rangle$ *are similar algebras. A map* f *from* A *into* B *is called a homomorphism of* $\mathfrak{A}$ *in* $\mathfrak{B}$ *if* f *preserves the operations in the following sense: for every*

Chapter 5

Conclusion

A Modal logic K is an extension of classical logic Cl. Syntactically, the only difference is that one unary symbol $\Box$ is added to Cl. Kripke semantics is one of the well-known semantics of various modal logics, which could be understood intuitively without deep mathematical knowledge. The model of this semantics is a triple $\langle W, R, \mathfrak{V} \rangle$, where W is a non-empty set, R is an arbitrary binary relation on W, and $\mathfrak{V}$ is an arbitrary map from Var$\mathcal{ML}$ to $\mathcal{P}(W)$. There are a few operations on a model, i.e. generation, reduction, and bulldozer, which never change the truth-value of each formula. A pair $\langle W, R, \mathfrak{V} \rangle$ is called frame, and we can classify frames with their property, e.g. reflexive, transitive, symmetric etc., and the property of those classes are correspond to specific modal formula. Hintikka system is a great tool to construct a counter model.

There are several proof theory on modal logics. Calculus K is one of those, which is a Hilbert-style deduction system. The only difference of classical Hilbert-style system is that there is an additional axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and inference rule "given a formula φ, we infer $\Box\varphi$" in it. Calculus K is sound and complete with respect to Kripke semantics. On proving the completeness of some extension of logic K, canonical models and filtration are useful.

There are lots of semantics other than Kripke semantics: algebraic semantics is one of them. With respect to modal logic, an algebra $\langle A, \wedge, \vee, \to, \leftrightarrow, \neg, \bot, \Box \rangle$, which is called $\mathcal{ML}$-algebra, is often used, where A is a non-empty set and other symbols are operations on A.

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