Nonstandard second-order arithmetic and Riemann's mapping theorem

Yoshihiro Horihata*
and
Keita Yokoyama†

Abstract

In this paper, we introduce systems of nonstandard second-order arithmetic which are conservative extensions of systems of second-order arithmetic. Within these systems, we do reverse mathematics for nonstandard analysis, and we can import techniques of nonstandard analysis into analysis in weak systems of second-order arithmetic. Then, we apply nonstandard techniques to a version of Riemann's mapping theorem, and show several different versions of Riemann's mapping theorem.

1 Introduction

In Tanaka [13], we can find a model theoretic method to do nonstandard analysis in WKL_0 by means of overspill and standard part principle. Using this method, some popular arguments of nonstandard analysis can be carried out in WKL_0 (cf. [6, 11, 12]). Similarly, we can use more techniques of nonstandard analysis in ACA_0 and prove some theorems in ACA_0 [16]. (For systems WKL_0 and ACA_0, see [9].)

On the other hand, Keisler[4, 5] introduced some systems of nonstandard arithmetic which are the counterparts of WKL_0, ACA_0 and other main systems for Reverse Mathematics. Inspired by the question '*can we canonically reconstruct formal proofs within* ACA_0 *or* WKL_0 *from such nonstandard arguments?*' posed by Professor Sakae Fuchino, the second author introduced systems of nonstandard second-order arithmetic $\mathsf{ns\text{-}ACA}_0$ and $\mathsf{ns\text{-}WKL}_0$ corresponding to ACA_0 and WKL_0 [17]. Using these systems, one can conveniently carry out nonstandard arguments and can interpret nonstandard proofs into standard proofs in second-order arithmetic.

In this paper, we do nonstandard analysis in nonstandard second-order arithmetic using some nonstandard axioms such as the standard part principle or the

*Yonago National College of Technology, Hikona-cho 4448, Yonago-shi, Tottori 683-8502, JAPAN, E-mail: horihata@yonago-k.ac.jp

†School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa, 923-1292, JAPAN, E-mail: y-keita@jaist.ac.jp

transfer principle within the basic system of nonstandard second-order arithmetic. Then, we can get some nonstandard proofs in $\mathsf{ns\text{-}ACA_0}$ or $\mathsf{ns\text{-}WKL_0}$ for some standard theorems. Our next aim is to do Reverse Mathematics for nonstandard analysis. Although standard theorems never imply nonstandard axioms, we can find nonstandard counterparts of standard theorems. These nonstandard counterparts often require nonstandard axioms. Therefore, we can do Reverse Mathematics for some nonstandard counterparts of standard theorems.

We also apply these nonstandard arguments to reverse mathematics for analysis in second-order arithmetic. It is known that Riemann's mapping theorem is equivalent to $\mathsf{ACA_0}$ (see [16]). In this Reverse Mathematics phenomenon, $\mathsf{ACA_0}$ is exactly required if we consider Riemann's theorem for general open sets. However, some weaker versions of Riemann's mapping theorem for a restricted domain, e.g., a polygonal domain or a Jordan region (the interior of Jordan curve), are still important for complex analysis. (The referee of [16] pointed out that the importance of these versions of Riemann's mapping theorem, and he/she also mentioned that they should be weaker than the general version.) In this paper, we show that Riemann's mapping theorem for a polygonal domain is provable within $\mathsf{RCA_0}$, and that for a Jordan region is equivalent to $\mathsf{WKL_0}$. To prove the latter one, we will use nonstandard techniques which are available within $\mathsf{ns\text{-}WKL_0}$.

2 Nonstandard second-order arithmetic and nonstandard analysis

2.1 Systems of nonstandard second-order arithmetic

We first introduce the language of nonstandard second-order arithmetic.

Definition 2.1. The language of nonstandard second-order arithmetic $\mathcal{L}_2^*$ is defined by the following:

- standard number variables: $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots$,
- nonstandard number variables: $x^*, y^*, \ldots$,
- standard set variables: $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots$,
- nonstandard set variables: $X^*, Y^*, \ldots$,
- function and relation symbols: $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}, 0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*, \surd$.

Here, $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}$ denote "the standard structure" of second-order arithmetic, $0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*$ denote "the nonstandard structure" of second-order arithmetic and $\surd$ denote an embedding from the standard structure to the nonstandard structure.

The terms and formulas of the language of nonstandard second-order arithmetic are as follows. *Standard numerical terms* are built up from standard number variables and the constant symbols 0^{s} and 1^{s} by means of $+^{\mathrm{s}}$ and $\cdot^{\mathrm{s}}$. *nonstandard numerical terms* are built up from nonstandard number variables, the constant symbols 0^* and 1^* and $\surd(t^{\mathrm{s}})$ by means of $+^{\mathrm{s}}$ and $\cdot^{\mathrm{s}}$, where t^{s} is a numerical term. *Standard set terms* are standard set variables and *nonstandard set terms* are nonstandard set variables and $\surd(X^{\mathrm{s}})$ whenever X^{s} is a standard set term. *Atomic formulas* are $t_1^{\mathrm{s}} =^{\mathrm{s}} t_2^{\mathrm{s}}$, $t_1^{\mathrm{s}} <^{\mathrm{s}} t_2^{\mathrm{s}}$, $t_1^{\mathrm{s}} \in^{\mathrm{s}} X^{\mathrm{s}}$, $t_1^* =^* t_2^*$, $t_1^* <^* t_2^*$ and $t_1^* \in^* X^*$ where $t_1^{\mathrm{s}}, t_2^{\mathrm{s}}$ are standard numerical terms, t_1^*, t_2^* are nonstandard numerical terms, X^{s} is a standard set term and X^* is a nonstandard set term. *Formulas* are built up from atomic formulas by means of propositional connectives and quantifiers. A *sentence* is a formula without free variables.

Let φ be an $\mathcal{L}_2$-formula. We write φ^{s} for the $\mathcal{L}_2^*$-formula constructed by adding $^{\mathrm{s}}$ to all occurrences of bound variables and relations of φ. Similarly, we write φ^* for the $\mathcal{L}_2^*$-formula constructed by adding *. Identifying $\mathcal{L}_2$-formula φ with $\mathcal{L}_2^*$-formula φ^{s}, we will consider that nonstandard second-order arithmetic is an expansion of second-order arithmetic. We sometimes omit $^{\mathrm{s}}$ and * of relations. We write $t^{\mathrm{s}\surd}$ for $\surd(t^{\mathrm{s}})$ and $X^{\mathrm{s}\surd}$ for $\surd(X^{\mathrm{s}})$. We sometimes write $\vec{x}$ ($\vec{X}$) for a finite sequences of variables $x_1, \ldots, x_k$ ($X_1, \ldots, X_k$).

In this paper, we use M^{s} to indicate the range of standard number variables, M^* to indicate the range of nonstandard number variables, S^{s} to indicate the range of standard set variables and S^* to indicate the range of nonstandard set variables in the system of nonstandard second-order arithmetic. Moreover, we use $V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}})$ to indicate the range of standard variables and $V^* = (M^*, S^*)$ to indicate the range of nonstandard variables, and we say that "φ holds in V^{s}" (abbreviated $V^{\mathrm{s}} \models \varphi$) if φ^{s} holds and we say that "φ holds in V^*" (abbreviated $V^* \models \varphi$) if φ^* holds. We are not going to describe the semantics of the system by these V^{s} and V^* but these symbols are introduced just to make the argument more accessible.

We next introduce some typical axioms of nonstandard second-order arithmetic.

Definition 2.2.

- embeddingness (EMB):

$$\forall \vec{x^{\mathrm{s}}}\, \forall \vec{X}^{\mathrm{s}}\, (\varphi(\vec{x^{\mathrm{s}}}, \vec{X}^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\vec{x^{\mathrm{s}\surd}}, \vec{X^{\mathrm{s}\surd}})^*)$$

  where $\varphi(\vec{x}, \vec{X})$ is any atomic formula in $\mathcal{L}_2$ with exactly the displayed free variables.

- end extension (E):

$$\forall x^* \forall y^{\mathrm{s}}\, (x^* < y^{\mathrm{s}\surd} \to \exists z^{\mathrm{s}}\, (x^* = z^{\mathrm{s}\surd})).$$

- finite standard part principle (FST):

$$\forall X^*\, (\mathrm{card}(X^*) \in M^{\mathrm{s}} \to \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}}\, (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow x^{\mathrm{s}\surd} \in X^*)).$$

- standard part principle (ST):

$$\forall X^{*}\, \exists Y^{\mathrm{s}}\, \forall x^{\mathrm{s}}\, (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow x^{\mathrm{s}\checkmark} \in X^{*}).$$

- Σ^i_j transfer principle (Σ^i_j-TP):

$$\forall \vec{x^{\mathrm{s}}}\, \forall \vec{X}^{\mathrm{s}}\, (\varphi(\vec{x^{\mathrm{s}}}, \vec{X}^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\vec{x^{\mathrm{s}}\checkmark}, \vec{X^{\mathrm{s}}\checkmark})^{*})$$

  where $\varphi(\vec{x}, \vec{X})$ is any Σ^i_j-formula in $\mathcal{L}_2$ with exactly the displayed free variables.

Now, we define the basic system of nonstandard second-order arithmetic.

Definition 2.3 (the system ns-BASIC)**.** The axioms of ns-BASIC are the following:

(standard and nonstandard structure) $(\mathsf{RCA}_0)^{\mathrm{s}} \wedge (\mathsf{RCA}_0)^{*}$,

(nonstandard axioms) EMB, E, FST, Σ^0_0-TP.

Trivially, ns-BASIC is an extension of RCA_0. Actually, they have the same standard consequences.

Theorem 2.1 (conservativity)**.** ns-BASIC *is a conservative extension of* RCA_0*, i.e.,* ns-BASIC $\vdash \psi^{\mathrm{s}}$ *implies* $\mathsf{RCA}_0 \vdash \psi$ *for any sentence in* $\mathcal{L}_2$*.*

Proof. Straightforward direction from Tanaka's self-embedding theorem [13] and Harrington's theorem [9, Theorem IX.2.1]. □

Next, we consider a very weak version of saturation principle called overspill principle, which is a significant tool for nonstandard analysis.

Proposition 2.2. ns-BASIC *proves the following* $\Sigma^0_1 \cup \Pi^0_1$*-overspill principle.*

- $\Sigma^0_1 \cup \Pi^0_1$*-overspill:*

  $$\forall \vec{x^{*}}\, \forall \vec{X^{*}}\, (\forall y^{\mathrm{s}}\, \exists z^{\mathrm{s}}\, (z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(z^{\mathrm{s}\checkmark}, \vec{x^{*}}, \vec{X^{*}})^{*}) \to \exists y^{*}\, (\forall w^{\mathrm{s}}\, (y^{*} > w^{\mathrm{s}\checkmark}) \wedge \varphi(y^{*}, \vec{x^{*}}, \vec{X^{*}})^{*}))$$

  where $\varphi(y, \vec{x}, \vec{X})$ *is any* Σ^0_1 *or* Π^0_1*-formula in* $\mathcal{L}_2$ *with exactly the displayed free variables.*

The contraposition of overspill is sometimes referred as underspill.

Proof. Since ns-BASIC contains $(\mathsf{RCA}_0)^{*}$, $V^{*} = (M^{*}, S^{*})$ satisfies $\Sigma^0_1 \cup \Pi^0_1$-induction. Thus, the cut M^{s} is not Σ^0_1 or Π^0_1-definable with parameters from V^{*}. Assume $\forall y^{\mathrm{s}}\, \exists z^{\mathrm{s}}\, (z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(z^{\mathrm{s}\checkmark}, \vec{x^{*}}, \vec{X^{*}})^{*})$ and $\neg\exists y^{*}\, (\forall w^{\mathrm{s}}\, (y^{*} > w^{\mathrm{s}\checkmark}) \wedge \varphi(y^{*}, \vec{x^{*}}, \vec{X^{*}})^{*})$ for some $\vec{x^{*}}, \vec{X^{*}} \in V^{*}$ and for some Σ^0_1 or Π^0_1-formula φ. Then, $a^{*} \in \surd(M^{\mathrm{s}})$ if and only if $\varphi(a^{*}, \vec{x^{*}}, \vec{X^{*}})^{*}$. Hence, a cut $\surd(M^{\mathrm{s}})$ is Σ^0_1 or Π^0_1-definable in V^{*}, which is a contradiction. □

Within $\mathsf{ns\text{-}BASIC}$, a standard set A^{s} is said to be a *standard part* of a nonstandard set B^* (abbreviated $B^* \restriction V^{\mathrm{s}} = A^{\mathrm{s}}$) if $\forall x^{\mathrm{s}}\,(x^{\mathrm{s}} \in A^{\mathrm{s}} \leftrightarrow x^{\mathrm{s}\surd} \in B^*)$. By Σ^0_0-TP, we can show $\forall X^{\mathrm{s}}\,(X^{\mathrm{s}\surd} \restriction V^{\mathrm{s}} = X^{\mathrm{s}})$. Nonstandard sets A^* and B^* are said to be s-*equivalent* (abbreviated $A^* \equiv_{\mathrm{s}} B^*$) if $\exists x^*\,(\forall y^{\mathrm{s}}\,(y^{\mathrm{s}\surd} < x^*) \wedge \forall z^* < x^*\,(z^* \in A^* \leftrightarrow z^* \in B^*))$. We write $A^* \sqsubseteq_{\mathrm{s}} B^*$ if $\exists x^*\,(\forall y^{\mathrm{s}}\,(y^{\mathrm{s}\surd} < x^*) \wedge \forall z^*\,(z^* \in A^* \leftrightarrow z^* \in B^* \wedge z^* < x^*))$, *i.e.*, $A^* = B^* \cap x^*$ for some nonstandard x^*. (We usually identify number a with a set $\{x \mid x < a\}$.) We sometimes use these notations for definable (possibly external) subsets of M^*. Note that $A^* \equiv_{\mathrm{s}} B^*$ is equivalent to $A^* \restriction V^{\mathrm{s}} = B^* \restriction V^{\mathrm{s}}$, *i.e.*, $\forall x^{\mathrm{s}}\,(x^{\mathrm{s}\surd} \in A^* \leftrightarrow x^{\mathrm{s}\surd} \in B^*)$ by overspill, however, this may not true for external sets.

$\mathsf{ns\text{-}BASIC}$ is a base system to do nonstandard analysis. Within $\mathsf{ns\text{-}BASIC}$, both the standard structure V^{s} and the nonstandard structure V^* satisfy $\mathsf{RCA_0}$, thus, we can develop basic part of mathematics in both V^{s} and V^* as same as in $\mathsf{RCA_0}$. For example, we can define real numbers, open sets, continuous functions, complete separable metric spaces, etc. in both V^{s} and V^* (see [9, II]). We write $\mathbb{N}^{\mathrm{s}}$ for natural number system in V^{s}, $\mathbb{N}^*$ for natural number system in V^*, $\mathbb{Q}^{\mathrm{s}}$ for rational number system in V^{s}, $\mathbb{Q}^*$ for rational number system in V^*, $\mathbb{R}^{\mathrm{s}}$ for real number system in V^{s}, $\mathbb{R}^*$ for real number system in V^*, etc. We regard $\mathbb{N}^{\mathrm{s}} \subseteq \mathbb{N}^*$, $\mathbb{Q}^{\mathrm{s}} \subseteq \mathbb{Q}^*$, etc. by using the embedding $\surd$, and then we usually omit $\surd$ for number variables. In addition, we often omit superscripts $^{\mathrm{s}}$ and * for number variables, and we write $\forall x \in \mathbb{N}^{\mathrm{s}}$, $\exists x \in \mathbb{N}^{\mathrm{s}}$, $\forall x \in \mathbb{N}^*$ or $\exists x \in \mathbb{N}^*$ instead of $\forall x^{\mathrm{s}}$, $\exists x^{\mathrm{s}}$, $\forall x^*$ or $\exists x^*$, respectively. Note that we cannot regard S^{s} as a subset of S^*, thus, $\mathbb{R}^{\mathrm{s}}$ cannot be regarded as a subset of $\mathbb{R}^*$ either.

Next, we define $\mathsf{ns\text{-}WKL_0}$ and $\mathsf{ns\text{-}ACA_0}$.

Definition 2.4 (the system $\mathsf{ns\text{-}WKL_0}$)**.** The system $\mathsf{ns\text{-}WKL_0}$ consists of $\mathsf{ns\text{-}BASIC}$ plus ST.

Proposition 2.3. $\mathsf{ns\text{-}WKL_0}$ *proves* $(\mathsf{WKL_0})^{\mathrm{s}}$*, i.e.,* $\mathsf{ns\text{-}WKL_0}$ *is an extension of* $\mathsf{WKL_0}$*.*

Proof. We reason within $\mathsf{ns\text{-}WKL_0}$. Let T^{s} be an infinite binary tree in V^{s}. Then, by $\Sigma^0_1 \cup \Pi^0_1$-overspill, there exist $K^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and a function $f^* : K^* \to 2$ such that $\forall x \leq K^*\, f^* \restriction x = \langle f^*(i) \mid i < x\rangle \in T^{\mathrm{s}\surd}$ in V^*. By ST, $f^{\mathrm{s}} = f^* \restriction V^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to 2$ exists and $f^{\mathrm{s}}(a^{\mathrm{s}}) = f^*(a^{\mathrm{s}})$ for any $a^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}$. Thus, $\forall x \in \mathbb{N}^{\mathrm{s}}\, f^{\mathrm{s}} \restriction x = \langle f^{\mathrm{s}}(i) \mid i < x\rangle \in T^{\mathrm{s}}$ in V^{s}. This means that f^{s} is a path through T^{s}, hence $(\mathsf{WKL_0})^{\mathrm{s}}$ holds. $\square$

Theorem 2.4 (conservativity)**.** $\mathsf{ns\text{-}WKL_0}$ *is a conservative extension of* $\mathsf{WKL_0}$*, i.e.,* $\mathsf{ns\text{-}WKL_0} \vdash \psi^{\mathrm{s}}$ *implies* $\mathsf{WKL_0} \vdash \psi$ *for any sentence in* $\mathcal{L}_2$*. Moreover, we can transform a proof in* $\mathsf{ns\text{-}WKL_0}$ *into a proof in* $\mathsf{WKL_0}$ *effectively.*

Proof. See [17]. $\square$

The following choice axiom is useful to do nonstandard analysis.

Lemma 2.5 (Σ_0^*-choice)**.** *The following are equivalent over* $\mathsf{ns\text{-}BASIC}$.

1. $\mathsf{ns\text{-}WKL_0}$.
2. Σ_0^**-choice:*

$$\forall \vec{z}^{\,*}\, \forall \vec{Z}^{*}\, (\forall x^{\,\mathrm{s}}\, \exists y^{\,\mathrm{s}}\, \varphi(x^{\,\mathrm{s}\checkmark}, y^{\,\mathrm{s}\checkmark}, \vec{z}^{\,*}, \vec{Z}^{*})^{\,*} \to \exists f^{\,\mathrm{s}}\, \forall x^{\,\mathrm{s}}\, \varphi(x^{\,\mathrm{s}\checkmark}, f^{\,\mathrm{s}}(x^{\,\mathrm{s}})^{\checkmark}, \vec{z}^{\,*}, \vec{Z}^{*})^{\,*})$$

where $\varphi(x, y, \vec{z}, \vec{Z})$ *is any* Σ_0^0*-formula in* $\mathcal{L}_2$.

Proof. $2 \to 1$ is trivial. We show $1 \to 2$. Let $\forall x \in \mathbb{N}^{\mathrm{s}}\, \exists y \in \mathbb{N}^{\mathrm{s}}\, \varphi(x, y, \vec{z}, \vec{Z}^{*})^{\,*}$ for some $\vec{z}, \vec{Z}^{*} \in V^{*}$. By $\Sigma_1^0 \cup \Pi_1^0$-overspill, there exists $\omega \in \mathbb{N}^{*} \setminus \mathbb{N}^{\mathrm{s}}$ such that $\forall x < \omega\, \exists y \in \mathbb{N}^{*}\, \varphi(x, y, \vec{z}, \vec{Z}^{*})^{*}$. Define sequence $\alpha^{*} = \langle a(i) \mid i < \omega \rangle$ as $a(x) = \min\{y \mid \varphi(x, y, \vec{z}, \vec{Z}^{*})^{\,*}\}$. By ST, define $f^{\,\mathrm{s}} = \alpha^{*} \restriction V^{\mathrm{s}}$, then we can easily check that this $f^{\,\mathrm{s}}$ is the desired choice function. $\square$

Definition 2.5 (the system $\mathsf{ns\text{-}ACA_0}$)**.** The system $\mathsf{ns\text{-}ACA_0}$ consists of $\mathsf{ns\text{-}WKL_0}$ plus Σ_1^1-TP.

Proposition 2.6. $\mathsf{ns\text{-}WKL_0} + \Sigma_1^0$-TP*, as well as* $\mathsf{ns\text{-}ACA_0}$*, proves* $(\mathsf{ACA_0})^{\,\mathrm{s}}$*, i.e.,* $\mathsf{ns\text{-}ACA_0}$ *is an extension of* $\mathsf{ACA_0}$.

Proof. We reason within $\mathsf{ns\text{-}ACA_0}$. By [9, Theorem III.1.3], we only need to show $\forall f : \mathbb{N} \to \mathbb{N}\; \exists A\; \forall x\, (x \in A \leftrightarrow \exists n\, (f(n) = x))$ in V^{s}. Let $f^{\,\mathrm{s}}$ be a function from $\mathbb{N}^{\mathrm{s}}$ to $\mathbb{N}^{\mathrm{s}}$ in V^{s}. Take $K^{*} \in \mathbb{N}^{*} \setminus \mathbb{N}^{\mathrm{s}}$. By Σ_1^0 bounded comprehension in V^{*}, there exists A^{*} such that $(\forall x \leq K^{*}\, (x \in A^{*} \leftrightarrow \exists n\, (f^{\,\mathrm{s}\checkmark}(n) = x)))^{\,*}$. Thus, $(a^{\mathrm{s}\checkmark} \in A^{*} \leftrightarrow (\exists n\, (f^{\,\mathrm{s}\checkmark}(n) = a^{\mathrm{s}}))^{*})$ for any $a^{\mathrm{s}} \in \mathbb{N}^{\mathrm{s}}$. By Σ_1^0-TP, $(\exists n\, (f^{\,\mathrm{s}\checkmark}(n) = a^{\mathrm{s}}))^{*} \leftrightarrow (\exists n\, (f^{\,\mathrm{s}}(n) = a^{\mathrm{s}}))^{\mathrm{s}}$. Hence, $(a^{\mathrm{s}} \in A^{*} \restriction V^{\mathrm{s}}) \leftrightarrow (a^{\mathrm{s}\checkmark} \in A^{*}) \leftrightarrow (\exists n\, (f^{\,\mathrm{s}}(n) = a^{\mathrm{s}}))^{\mathrm{s}}$. This means that $(\forall x \in \mathbb{N}^{\mathrm{s}}\, (x \in A^{*} \restriction V^{\mathrm{s}} \leftrightarrow \exists n\, (f^{\,\mathrm{s}}(n) = x)))$ in V^{s}, hence $(\mathsf{ACA_0})^{\mathrm{s}}$ holds. $\square$

Theorem 2.7 (conservativity)**.** $\mathsf{ns\text{-}ACA_0}$ *is a conservative extension of* $\mathsf{ACA_0}$*, i.e.,* $\mathsf{ns\text{-}ACA_0} \vdash \psi^{\,\mathrm{s}}$ *implies* $\mathsf{ACA_0} \vdash \psi$ *for any sentence in* $\mathcal{L}_2$*. Moreover, we can transform a proof in* $\mathsf{ns\text{-}ACA_0}$ *into a proof in* $\mathsf{ACA_0}$ *effectively.*

Proof. See [17]. $\square$

Since $\mathsf{ns\text{-}ACA_0}$ and $\mathsf{ns\text{-}WKL_0} + \Sigma_1^0$-TP have the same standard part $\mathsf{ACA_0}$, we usually use $\mathsf{ns\text{-}ACA_0}$ instead of $\mathsf{ns\text{-}WKL_0} + \Sigma_1^0$-TP. Within $\mathsf{ns\text{-}ACA_0}$, V^{*} satisfies Σ_0^1-induction by $(\mathsf{ACA_0})^{\mathrm{s}}$ and Σ_1^1-TP. Thus, Σ_0^1-overspill is available within $\mathsf{ns\text{-}ACA_0}$ as same as Proposition 2.2. On the other hand, the standard part of $\mathsf{ns\text{-}BASIC} + \Sigma_n^0$-TP is strictly weaker than $\mathsf{ACA_0}$.

Proposition 2.8. $\mathsf{ns\text{-}BASIC} + \Sigma_n^0$-TP *is a conservative extension of* $\mathsf{RCA_0} +$ Σ_n^0*-bounding. Here,* Σ_n^0*-bounding is the axiom scheme of the form* $\forall u\, (\forall x < u\, \exists y\, \varphi(x, y) \to \exists v\, \forall x < u\, \exists y < v\, \varphi(x, y))$ *for any* Σ_n^0*-formulas.*

Proof. See [14]. $\square$

Remark 2.9. In [4], Keisler adopted the finiteness principle, which is equivalent to FST in our formulation, as one of the base axiom. So, let us consider a slightly stronger base system $\mathsf{ns\text{-}BASIC}^+ = \mathsf{ns\text{-}BASIC} + \mathrm{FST}$. Then, $\mathsf{ns\text{-}BASIC}^+$ is again a conservative extension of $\mathsf{RCA_0}$, and $\mathsf{ns\text{-}WKL_0}$ preserves since ST implies FST. However, the standard part of $\mathsf{ns\text{-}BASIC}^+ + \Sigma^0_n\text{-TP}$ turns to be $\mathsf{RCA_0} + \Sigma^0_{n+1}$-induction. (See [14].) We won't use FST for nonstandard analysis in this paper, but Sanders showed that it is useful for nonstandard analysis in a different formulation in [7, 8]. (In [7], he introduce an axiom "ext", which is again equivalent to FST or finiteness principle.)

2.2 Reverse Mathematics for nonstandard analysis

In this section, we do nonstandard analysis in systems of nonstandard second-order arithmetic and do some Reverse Mathematics for nonstandard analysis. Our aim is to give some nonstandard characterizations for several notions of analysis and find nonstandard counterparts of standard theorems. Then, we will apply nonstandard characterizations to standard analysis in second-order arithmetic using conservation results.

2.2.1 The standard part of a real

We define the standard part of a real number.

Definition 2.6 (standard part)**.** The following definition is made in $\mathsf{ns\text{-}BASIC}$. Let $\alpha^* = \langle a(i) \mid i \in \mathbb{N}^* \rangle \in \mathbb{R}^*$ in V^* and $\beta^{\mathrm{s}} = \langle b(i) \mid i \in \mathbb{N}^{\mathrm{s}} \rangle \in \mathbb{R}^{\mathrm{s}}$ in V^{s}. Then, β^{s} is said to be the *standard part* of α^* (abbreviated $\mathrm{st}(\alpha^*) = \beta^{\mathrm{s}}$) if

$$\forall i \in \mathbb{N}^{\mathrm{s}}\ V^* \models |a(i) - b(i)| \leq 2^{-i+1}.$$

We sometimes write $\mathrm{st}(\alpha^*) \in \mathbb{R}^{\mathrm{s}}$ if $\exists \gamma^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\, \mathrm{st}(\alpha^*) = \gamma^{\mathrm{s}}$, and say that α^* is *near standard*.

Similarly to the definition of standard parts, we write $\mathrm{st}(\alpha^*) \leq \beta^{\mathrm{s}}$ if

$$\forall i \in \mathbb{N}^{\mathrm{s}}\ V^* \models a(i) \leq b(i) + 2^{-i+1}.$$

Note that we can define $\mathrm{st}(\alpha^*) \leq \beta^{\mathrm{s}}$ even if the standard part of α^* does not exist in $\mathbb{R}^{\mathrm{s}}$. We write $\alpha_1^* \approx \alpha_2^*$ if $\mathrm{st}(\alpha_1^* - \alpha_2^*) = 0$.

By $\Sigma^0_1 \cup \Pi^0_1$-overspill, we can easily show that

$$\forall \alpha^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\ \exists b^* \in \mathbb{Q}^*\ \ \mathrm{st}(b^*) = \alpha^{\mathrm{s}}.$$

The following theorem is the first example on Reverse Mathematics for nonstandard analysis.

Theorem 2.10. *The following are equivalent over* $\mathsf{ns\text{-}BASIC}$.

1. $\mathsf{ns\text{-}WKL_0}$.

2. *For any* $\alpha^* \in \mathbb{R}^*$,

$$\exists K \in \mathbb{N}^{\mathrm{s}}\ |\alpha^*| < K \to \exists \beta^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}\ \mathrm{st}(\alpha^*) = \beta^{\mathrm{s}}.$$

Proof. We first show $1 \to 2$. We reason within $\mathsf{ns\text{-}WKL_0}$. Let $\alpha^* \in \mathbb{R}^*$ and let $K \in \mathbb{N}^{\mathrm{s}}$ such that $|\alpha^*| < K$ in V^*. By $(\mathsf{RCA_0})^*$, we can find $\beta^* = \langle b(i) \mid i \in \mathbb{N}^* \rangle \in \mathbb{R}^*$ such that $\alpha^* = \beta^*$ and $\forall i \in \mathbb{N}^*\ b(i) \in \{j/2^i \mid j \in \mathbb{Z}^* \wedge -2^i K \leq j \leq 2^i K\}$ in V^*. Then, $\forall i \in \mathbb{N}^{\mathrm{s}}\ b(i) \in \mathbb{Q}^{\mathrm{s}}$. Thus, by ST, $\gamma^{\mathrm{s}} = \beta \upharpoonright V^{\mathrm{s}} = \langle b(i) \mid i \in \mathbb{N}^{\mathrm{s}} \rangle$ exists. By Σ^0_0-TP, we can show that $\gamma^{\mathrm{s}} \in \mathbb{R}^{\mathrm{s}}$ in V^{s} and $\mathrm{st}(\alpha^*) = \gamma^{\mathrm{s}}$.

For the converse, we only need to show 2 implies ST. We reason within $\mathsf{ns\text{-}BASIC}$. Let A^* be a nonstandard set and let χ_{A^*} be a characteristic function of A^* in V^*. Define

$$\alpha^* = \sum_{i \in \mathbb{N}^*} \frac{\chi_{A^*}(i)}{4^i}.$$

Then, $|\alpha^{\mathrm{s}}| \leq 1^{\mathrm{s}}$. By 2, take $\beta^{\mathrm{s}} = \mathrm{st}(\alpha^*) \in \mathbb{R}^{\mathrm{s}}$. Define $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to 2$ as

$$h^{\mathrm{s}}(n+1) = \begin{cases} 1 & \text{if } \beta^{\mathrm{s}} > \sum_{i \leq n} \frac{h^{\mathrm{s}}(i)}{4^i} + \frac{1}{2 \cdot 4^{n+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\beta^{\mathrm{s}} \neq \sum_{i \leq n}(h^{\mathrm{s}}(i)/4^i) + (1/(2 \cdot 4^{n+1}))$, thus this h^{s} can be constructed by $(\mathsf{RCA_0})^{\mathrm{s}}$. Define $B^{\mathrm{s}} = \{i \mid h^{\mathrm{s}}(i) = 1\}$, then we can easily check that $B^{\mathrm{s}} = A^* \upharpoonright V^{\mathrm{s}}$. This completes the proof. $\square$

Note that Definition 2.6 and Theorem 2.10 can be easily generalized into the Euclidean space $(\mathbb{R}^*)^n$ and $(\mathbb{R}^{\mathrm{s}})^n$.

2.2.2 Complete separable metric space and Heine/Borel compactness

Next, we consider complete separable metric spaces. First, we review the definition of complete separable metric spaces in second order arithmetic. Within $\mathsf{RCA_0}$, let $A \subseteq \mathbb{N}$. A pre-distance d on A is a function $d : A \times A \times \mathbb{N} \to \mathbb{Q}$ such that $d(a, b, i) \geq 0$, $d(a, b) = \langle d(a, b, i) \mid i \in \mathbb{N} \rangle \in \mathbb{R}$, $d(a, a) = 0$, $d(a, b) = d(b, a)$ and $d(a, c) \leq d(a, b) + d(b, c)$ for any $a, b, c \in A$. A pair $\langle A, d \rangle$ is said to be a (code for a) *complete separable metric space* if d is a pre-distance on A. A sequence $x = \langle x(i) \in A \mid i \in \mathbb{N} \rangle$ is said to be a *point* of $\langle A, d \rangle$ (abbreviated $x \in \hat{A}$) if $\forall i, j \in \mathbb{N}\ d(x(i), x(i+j)) \leq 2^{-i}$. For $x, y \in \hat{A}$, we define $d(x, y) = \lim_{i \to \infty} d(x(i), y(i))$, and we write $x =^{\hat{A}} y$ if $d(x, y) = 0$ (we usually omit the superscript $^{\hat{A}}$). We identify $a \in A$ with $\langle a \mid i \in \mathbb{N} \rangle$, and consider $A \subseteq \hat{A}$. Note that $\mathbb{R}^n$ is a complete separable metric space $\langle \mathbb{Q}^n, d_{\mathbb{R}^n} \rangle$ where $d_{\mathbb{R}^n}(a, b) = \|a - b\|$.

A function $p : A \times A \to \mathbb{Q}$ is said to be an *n-pre-distance* on A if for any $a, b, c \in A$, $p(a, b) \geq 0$, $p(a, a) \leq 2^{-n+1}$, $|p(a, b) - p(b, a)| \leq 2^{-n+1}$ and $p(a, c) \leq p(a, b) + p(b, c) + 2^{-n+2}$. If d is a pre-distance on A, then $d_k = d(\cdot, \cdot, k) : A \times A \to \mathbb{Q}$ is a k-pre-distance on A.

We say that a complete separable metric space $\langle A, d\rangle$ is *totally bounded* if for any $\varepsilon > 0$ there exists $\langle h(i) \in A \mid i \leq k\rangle$ such that $\forall x \in \hat{A}\, \exists i \leq k\ d(h(i), x) < \varepsilon$. Moreover, $\langle A, d\rangle$ is said to be *effectively totally bounded* if there exists an infinite sequence of finite sequences $\langle\langle h(i,j) \in A \mid i \leq p(j)\rangle \mid j \in \mathbb{N}\rangle$ such that $\forall x \in \hat{A}\ \forall j \in \mathbb{N}\ \exists i \leq p(j)\ d(h(i,j), x) < 2^{-j}$. For example, n-cube $[0,1]^n = \langle(\mathbb{Q} \cap [0,1])^n, d_{\mathbb{R}^n}\rangle$ is effectively totally bounded. We say that $\langle A, d\rangle$ is *Heine/Borel compact* if every open cover of $\hat{A}$ has a finite subcover, *i.e.*, for any sequence $\langle(a(i), r(i)) \in A \times \mathbb{Q}^+ \mid i \in \mathbb{N}\rangle$ such that $\forall x \in \hat{A}\ \exists i \in \mathbb{N}\ d(a(i), x) < r(i)$, there exists $k \in \mathbb{N}$ such that $\forall x \in \hat{A}\ \exists i < k\, d(a(i), x) < r(i)$, and we say that $\langle A, d\rangle$ is *sequentially Heine/Borel compact* if for any sequence of open covers $\langle\langle(a(i,j), r(r,j)) \in A \times \mathbb{Q}^+ \mid i \in \mathbb{N}\rangle \mid j \in \mathbb{N}\rangle$, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\langle(a(i,j), r(r,j)) \mid i < f(j)\rangle$ is an open cover for any $j \in \mathbb{N}$. For complete separable metric spaces in RCA_0, see also [9, II.5].

From now on, we argue within ns-BASIC. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space in V^{s}, and let ν^* be an ω-pre-distance on a set X^* for some $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ in V^*. Then, $\langle X^*, \nu^*\rangle$ is said to be a *nonstandard expansion* of $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ if $A^{\mathrm{s}} = X^* \upharpoonright V^{\mathrm{s}}$ and $d^{\mathrm{s}}(a,b) = \mathrm{st}(\nu^*(a,b))$ for any $a, b \in A^{\mathrm{s}}$. Let $x \in X^*$ and $y^{\mathrm{s}} = \langle y^{\mathrm{s}}(i) \mid i \in \mathbb{N}^{\mathrm{s}}\rangle \in \hat{A}^{\mathrm{s}}$. Then, y^{s} is said to be the *standard part* of x (abbreviated $\mathrm{st}(x) = y^{\mathrm{s}}$) if

$$\forall i \in \mathbb{N}^{\mathrm{s}}\ V^* \models \nu^*(x, y^{\mathrm{s}}(i)) \leq 2^{-i+1}.$$

Similarly, we write $\nu^*(x, y^{\mathrm{s}}) \leq \alpha^{\mathrm{s}}$ for $\alpha^{\mathrm{s}} = \langle a(i) \mid i \in \mathbb{N}^{\mathrm{s}}\rangle \in \mathbb{R}^{\mathrm{s}}$ if $\forall i \in \mathbb{N}^{\mathrm{s}}\ V^* \models \nu^*(x, y^{\mathrm{s}}(i)) \leq a(i) + 2^{-i+2}$. We write $x_1 \approx x_2$ if $\mathrm{st}(\nu^*(x_1, x_2)) = 0$. We say that $\langle X_1^*, \nu_1^*\rangle$ and $\langle X_2^*, \nu_2^*\rangle$ are s-*equivalent* (abbreviated $\langle X_1^*, \nu_1^*\rangle \equiv_{\mathrm{s}} \langle X_2^*, \nu_2^*\rangle$ or just $X_1^* \equiv_{\mathrm{s}} X_2^*$) if there exists $\langle Y^*, \nu_Y^*\rangle$ where ν_Y^* is a ω-pre-distance for some $\omega \in \mathbb{N}^* \setminus \mathbb{N}$ such that $Y^* \sqsubseteq_{\mathrm{s}} X_i^*$ and $\forall x, y \in Y^*\ |\nu_i^*(x,y) - \nu_Y^*(x,y)| < 2^{-\omega+2}$. (Recall that $A^* \sqsubseteq_{\mathrm{s}} B^*$ if $A^* = B^* \cap \omega_0$ for some $\omega_0 \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.) Note that if $\langle X^*, \nu^*\rangle$ is a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ and $Y^* \sqsubseteq_{\mathrm{s}} X^*$, then $\langle Y^*, \nu^* \upharpoonright Y^*\rangle$ is again a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$.

Proposition 2.11. *The following are provable in* ns-BASIC.

1. *Every complete separable metric space in V^{s} has a nonstandard expansion, and it is unique up to* s-*equivalence.*
2. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space in V^{s}, and let $\langle X^*, \nu^*\rangle$ be a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$. Then, for any $y^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}$, there exists $x \in X^*$ such that* $\mathrm{st}(x) = y^{\mathrm{s}}$.

Proof. By $\Sigma_1^0 \cup \Pi_1^0$-overspill if $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ is a complete separable metric space, $\langle A^{\mathrm{s}\checkmark} \cap \omega, (d^{\mathrm{s}\checkmark})_\omega\rangle$ is a nonstandard expansion for some $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. (Apply overspill to the assertion "d_i^{s} is an i-pre-distance on $A^{\mathrm{s}} \cap i$".) Thus, we have 1. We can prove the uniqueness similarly. Let $y^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}$. Applying $\Sigma_1^0 \cup \Pi_1^0$-overspill to the assertion "$\exists x \in X^*\ \nu^*(y^{\mathrm{s}}(i), x) < 2^{-i+1}$", we have 2 easily. □

Next, we will give nonstandard characterizations of some notions of general topology. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space in V^{s}, and let

$\langle X^*, \nu^* \rangle$ be a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$. We define the near standard set $\mathrm{Nst}(X^*)$, the approachable set $\mathrm{App}(X^*)$ and the limited set $\mathrm{Lim}(X^*)$ as follows:

$$\begin{aligned}
\mathrm{Nst}(X^*) &:= \{x \in X^* \mid \exists y^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}\ \mathrm{st}(x) = y^{\mathrm{s}}\}, \\
\mathrm{App}(X^*) &:= \{x \in X^* \mid \forall n \in \mathbb{N}^{\mathrm{s}}\ \exists a \in A^{\mathrm{s}}\ \nu^*(x, a) < 2^{-n}\}, \\
\mathrm{Lim}(X^*) &:= \{x \in X^* \mid \exists n \in \mathbb{N}^{\mathrm{s}}\ \exists a \in A^{\mathrm{s}}\ \nu^*(x, a) < n\}.
\end{aligned}$$

Note that any of $\mathrm{Nst}(X^*)$, $\mathrm{App}(X^*)$ or $\mathrm{Lim}(X^*)$ might not be a set in V^*. We sometimes say that a set $D^* \subseteq X^*$ is s-*bounded* if it is a subset of $\mathrm{Lim}(X^*)$. For the original idea of $\mathrm{App}(X^*)$, $\mathrm{Nst}(X^*)$ and $\mathrm{Lim}(X^*)$, see Goldblatt[2].

Proposition 2.12. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space in V^{s}, and $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Then, the following are provable within* ns-BASIC.

1. *If $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is effectively totally bounded, then, $X^* \equiv_{\mathrm{s}} \mathrm{App}(X^*)$. Conversely, if $X^* \equiv_{\mathrm{s}} \mathrm{App}(X^*)$, then, $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is totally bounded.*
2. *If $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$, then, $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is Heine/Borel compact.*
3. *$\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is bounded if and only if $X^* \equiv_{\mathrm{s}} \mathrm{Lim}(X^*)$.*

Moreover, the following is provable within ns-BASIC $+ \Sigma^0_1$-TP.

4. *$\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is totally bounded if and only if $X^* \equiv_{\mathrm{s}} \mathrm{App}(X^*)$.*

Proof. Both of 1, 2 and 3 are easy consequences of $\Sigma^0_1 \cup \Pi^0_1$-overspill, and 4 is again easy consequences of $\Sigma^0_1 \cup \Pi^0_1$-overspill $+ \Sigma^0_1$-TP. □

Note that $X^* \equiv_{\mathrm{s}} \mathrm{App}(X^*)$ (resp. $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$, $X^* \equiv_{\mathrm{s}} \mathrm{Lim}(X^*)$) means that $X^* = \mathrm{App}(X^*)$ (resp. $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$, $X^* \equiv_{\mathrm{s}} \mathrm{Lim}(X^*)$) holds up to s-equivalence, *i.e.*, there exists $\bar{X}^* \equiv_{\mathrm{s}} X^*$ such that $\bar{X}^* = \mathrm{App}(\bar{X}^*)$ (resp. $\bar{X}^* = \mathrm{Nst}(\bar{X}^*)$, $\bar{X}^* = \mathrm{Lim}(\bar{X}^*)$).

In the usual nonstandard analysis, $\mathrm{App}(X^*) = \mathrm{Nst}(X^*)$ holds for any nonstandard expansion of a complete metric space (see, e.g., [2]). To prove this, ns-WKL_0 is required.

Theorem 2.13. *The following are equivalent over* ns-BASIC.

1. ns-WKL_0.
2. *If $\langle X^*, \nu^* \rangle$ is a nonstandard expansion of a complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$, then, $\mathrm{App}(X^*) = \mathrm{Nst}(X^*)$.*
3. *If $\langle X^*, \nu^* \rangle$ is a nonstandard expansion of a complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$, then, $\mathrm{App}(X^*) \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$.*
4. *If $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ is an effectively totally bounded complete separable metric space and $\langle X^*, \nu^* \rangle$ is its nonstandard expansion, then, $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$.*

5. *If* $\langle X^*, \nu^* \rangle$ *is a nonstandard expansion of the closed unit interval* $[0, 1]$*, then,* $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$*, where,* $[0, 1] = \langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ *for* $A^{\mathrm{s}} = \{q \in \mathbb{Q}^{\mathrm{s}} \mid 0 \leq q \leq 1\}$ *and* $d^{\mathrm{s}}(p, q) = |q - p|$.

Proof. We first show $1 \to 2$. We reason within $\mathsf{ns\text{-}WKL_0}$. Let $\langle X^*, \nu^* \rangle$ be a nonstandard expansion of a complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ in V^{s}. $\mathrm{App}(X^*) \supseteq \mathrm{Nst}(X^*)$ is trivial, so we will show that $\mathrm{App}(X^*) \subseteq \mathrm{Nst}(X^*)$. Let $x \in \mathrm{App}(X^*)$. Then, by the definition of $\mathrm{App}(X^*)$, for any $i \in \mathbb{N}^{\mathrm{s}}$, there exists $a \in A^{\mathrm{s}}$ such that $\nu^*(x, a) < 2^{-i-2}$. Thus, by Σ^*_0-choice (Lemma 2.5), we can take a sequence $y^{\mathrm{s}} = \langle a(i) \in A^{\mathrm{s}} \mid i \in \mathbb{N}^{\mathrm{s}} \rangle \in V^{\mathrm{s}}$ such that $\forall i \in \mathbb{N}^{\mathrm{s}} \nu^*(x, a(i)) < 2^{-i-2}$. We can easily check that y^{s} is a point of $\hat{A}^{\mathrm{s}}$, and $\mathrm{st}(x) = y^{\mathrm{s}}$. This completes the proof of $1 \to 2$.

$2 \to 3$ and $4 \to 5$ are trivial. $3 \to 4$ is trivial from Proposition 2.12.1.

To prove $5 \to 1$, we show 5 implies ST. (This proof is essentially the same as the proof $2 \to 1$ of Theorem 2.10.) We reason within $\mathsf{ns\text{-}BASIC}$. Let Z^* be a nonstandard set and let $\chi^*_{Z^*}$ be a characteristic function of Z^* in V^*. Let $A^{\mathrm{s}} = \{q \in \mathbb{Q}^{\mathrm{s}} \mid 0 \leq q \leq 1\}$ and $d^{\mathrm{s}}(p, q) = |q - p|$. Then, by 5, there exists a nonstandard expansion $\langle X^*, \nu^* \rangle$ in V^* such that $X^* = \mathrm{Nst}(X^*)$. Since $\forall m \in \mathbb{N}^{\mathrm{s}}\ \{j4^{-m} \mid 0 \leq j \leq 4^m\} \subseteq A^{\mathrm{s}} \subseteq X^*$, there exists $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $\{j4^{-\omega} \mid 0 \leq j \leq 4^{\omega}\} \subseteq X^*$ by $\Sigma^0_1 \cup \Pi^0_1$-overspill. Then, $x_{Z^*} = \sum_{j \leq \omega} \chi^*_{X^*}(j) 4^{-j} \in X^*$ and $\mathrm{st}(x_{Z^*}) \in \hat{A}^{\mathrm{s}}$ exists. Hence we can show that $Z^* \upharpoonright V^{\mathrm{s}}$ exists as in the proof of Theorem 2.10. This completes the proof. □

Corollary 2.14. *Let* $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ *be a complete separable metric space in* V^{s} *and* $\langle X^*, \nu^* \rangle$ *be its nonstandard expansion. The following are equivalent over* $\mathsf{ns\text{-}WKL_0}$.

1. $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ *is effectively totally bounded.*

2. $X^* \equiv_{\mathrm{s}} \mathrm{App}(X^*)$.

3. $X^* \equiv_{\mathrm{s}} \mathrm{Nst}(X^*)$.

4. $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ *is sequentially Heine/Borel compact.*

Proof. We have already seen $1 \to 2 \to 3$ in Proposition 2.12 and Theorem 2.13, and $4 \to 1$ is trivial.

We show $3 \to 4$. Let $\langle \langle (a(i, j), r(r, j)) \in A^{\mathrm{s}} \times \mathbb{Q}^{\mathrm{s}+} \mid i \in \mathbb{N}^{\mathrm{s}} \rangle \mid j \in \mathbb{N}^{\mathrm{s}} \rangle$ be a sequence of open covers of $\hat{A}^{\mathrm{s}}$, and let $\langle X^*, \nu^* \rangle$ be a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ such that $X^* = \mathrm{Nst}(X^*)$. Without loss of generality, we can assume that X^* is bounded, *i.e.*, $\max X^* = K \in \mathbb{N}^*$ exists. By Proposition 2.12.2, for any $j \in \mathbb{N}^{\mathrm{s}}$, there exists $k_j \in \mathbb{N}^{\mathrm{s}}$ such that $\langle (a(i, j), r(r, j)) \mid i < k_j \rangle$ covers $\hat{A}^{\mathrm{s}}$. Then, we can easily show that $\forall x \leq K\ (x \in X^* \to \exists i < k_j\ \nu^*(x, a(i, j)) < r(i, j))$. Thus, by Σ^*_0-choice, there exists $f^{\mathrm{s}} \in V^{\mathrm{s}}$ such that $\forall j \in \mathbb{N}^{\mathrm{s}}\ (\forall x \leq K\ (x \in X^* \to \exists i < f(j)\ \nu^*(x, a(i, j)) < r(i, j)))$. Hence, $\langle (a(i, j), r(r, j)) \mid i < f^{\mathrm{s}}(j) \rangle$ covers $\hat{A}^{\mathrm{s}}$ for any $j \in \mathbb{N}^{\mathrm{s}}$. □

Remark 2.15. Total boundedness or Heine/Borel compactness does not imply effective total boundedness within $\mathsf{ns\text{-}WKL_0}$ or $\mathsf{WKL_0}$. In fact, it requires $\mathsf{ACA_0}$. (We can prove that $\mathsf{ACA_0}$ is necessary as follows: for given 1-1 function $f : \mathbb{N} \to$

$\mathbb{N}$, consider $\langle \mathbb{N}, d_f \rangle$ where $d_f(i,j) = |1/f(i-1) - 1/f(j-1)|$ for $i, j > 0$ and $d_f(i,0) = 1/f(i-1)$ for $i > 0$.)

2.2.3 Continuous functions

First, we recall the definition of continuous functions within RCA_0. Let $\langle A, d_A \rangle$, $\langle B, d_B \rangle$ be complete separable metric spaces. A *(code for a) continuous function* f from $\hat{A}$ to $\hat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ B \times \mathbb{Q}^+$ which satisfies the following three conditions and the domain condition:

(*i*) if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d_B(b,b') \leq s + s'$;

(*ii*) if $(a,r)\Phi(b,s)$ and $d_A(a',a) + r' < r$, then $(a',r')\Phi(b,s)$;

(dom) for any $x \in \hat{A}$ and for any $\varepsilon > 0$ there exists $(m,a,r,b,s) \in \Phi$ such that $d_A(x,a) < r$ and $s < \varepsilon$,

where $(a,r)\Phi(b,s)$ is an abbreviation for $\exists m((m,a,r,b,s) \in \Phi)$. We define the value $f(x)$ to be the unique $y \in \hat{B}$ such that $d_B(y,b) < s$ for all $(a,r)\Phi(b,s)$ with $d_A(x,a) < r$. The existence of $f(x)$ is provable in RCA_0. A continuous function $f : \hat{A} \to \hat{B}$ is said to be *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall x, y \in \hat{A}(d_A(x,y) < \delta \to d_B(f(x), f(y)) < \varepsilon)$, and $f : \hat{A} \to \hat{B}$ is said to be *effectively uniformly continuous* if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N} \forall x, y \in \hat{A}(d_A(x,y) < 2^{-h(n)} \to d_B(f(x), f(y)) < 2^{-n})$. (This h is said to be a modulus of uniform continuity for f.)

From now on, we argue within $\mathsf{ns\text{-}BASIC}$. We first define the nonstandard extension of continuous functions. In nonstandard analysis, a continuous function f^{s} from $\mathbb{R}^{\mathrm{s}}$ to $\mathbb{R}^{\mathrm{s}}$ can be approximated by a (partial) function F^* from $\mathbb{Q}^*$ to $\mathbb{Q}^*$. Note that F^* does not need to be a continuous function (within the nonstandard universe). In the following definition, we will generalize this idea.

Definition 2.7. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A \rangle$, $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B \rangle$ be complete separable metric spaces, and let $\langle X^*, \nu^*_X \rangle$, $\langle Y^*, \nu^*_Y \rangle$ be their nonstandard expansions. Then, the following definitions are made in $\mathsf{ns\text{-}BASIC}$.

1. A partial function $F^* : \subseteq X^* \to Y^*$ is said to be a *pre-extension* of a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$, or f^{s} is said to be a *pre-standard part* of F^* if $\mathrm{dom}(F^*) \sqsubseteq_{\mathrm{s}} X^*$ and for any $x \in \mathrm{Nst}(X^*) \cap \mathrm{dom}(F^*)$, $F^*(x) \in \mathrm{Nst}(Y^*)$ and $\mathrm{st}(F^*(x)) = f^{\mathrm{s}}(\mathrm{st}(x))$.

2. A partial function $F^* : \subseteq X^* \to Y^*$ is said to be an *extension* of a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$, or f^{s} is said to be the *standard part* of F^* if $\mathrm{dom}(F^*) \sqsubseteq_{\mathrm{s}} X^*$ and for any $x \in \mathrm{App}(X^*) \cap \mathrm{dom}(F^*)$ and for any $n \in \mathbb{N}^{\mathrm{s}}$, there exist $m \in \mathbb{N}^{\mathrm{s}}$ such that for any $a^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}$, $\nu^*_X(a^{\mathrm{s}}, x) < 2^{-m} \to \nu^*_Y(f^{\mathrm{s}}(a^{\mathrm{s}}), F^*(x)) < 2^{-n}$.

3. A partial function $F^* : \subseteq X^* \to Y^*$ is said to be s-*continuous* if for any $x, y \in \mathrm{App}(X^*)$, $x \approx y \to F^*(x) \approx F^*(y)$.

By the definition of (pre-)extension, a (pre-)extension $F^* : \subseteq X^* \to Y^*$ of a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ is a total function on $\langle \mathrm{dom}(F^*), \nu^*_X \restriction \mathrm{dom}(F^*)\rangle$ which is s-equivalent to X^*. Thus, we can consider that a (pre-)extension is a total function up to s-equivalence. Moreover, we can easily check that extension is unique up to infinitesimals on $\mathrm{App}(X^*)$, *i.e.*, if two partial functions F^* and G^* are extensions of the same continuous function, then, $F^*(x) \approx G^*(x)$ for any $x \in \mathrm{dom}(F^*) \cap \mathrm{dom}(G^*) \cap \mathrm{App}(X^*)$.

Proposition 2.16. *The following is provable within* ns-BASIC*. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A\rangle$, $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B\rangle$ be complete separable metric spaces, and let $\langle X^*, \nu^*_X\rangle$, $\langle Y^*, \nu^*_Y\rangle$ be their nonstandard expansions. Let $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ be a continuous function. Then, a partial function $F^* : \subseteq X^* \to Y^*$ of f^{s} is an extension of f^{s} if and only if it is a pre-extension of f^{s} and* s*-continuous.*

Proof. By overspill, a partial function $F^* : \subseteq X^* \to Y^*$ is s-continuous if and only if for any $x \in \mathrm{App}(X^*)$ and for any $n \in \mathbb{N}^{\mathrm{s}}$ there exists $m \in \mathbb{N}^{\mathrm{s}}$ such that $\forall y \in X^*(\nu^*_X(x, y) < 2^{-m} \to \nu^*_Y(F^*(x), F^*(y)) < 2^{-n})$. Similarly, F^* is a pre-standard part of f^{s} if and only if for any $x \in \mathrm{Nst}(X^*)$ and for any $n \in \mathbb{N}^{\mathrm{s}}$ there exists $m \in \mathbb{N}^{\mathrm{s}}$ such that $\forall y \in X^*(\nu^*_X(x, y) < 2^{-m} \to \nu^*_Y(f^{\mathrm{s}}(\mathrm{st}(x)), F^*(y)) < 2^{-n})$. We can easily prove the desired equivalence from these. □

Now we show the existence of extensions of continuous functions. The existence of pre-extensions is provable within ns-BASIC, however, to show the uniqueness of pre-extension, ns-$\mathsf{WKL_0}$ is required.

Proposition 2.17 (existence of pre-extensions of continuous functions)**.** *The following is provable within* ns-BASIC*. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A\rangle$, $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B\rangle$ be complete separable metric spaces, and let $\langle X^*, \nu^*_X\rangle$, $\langle Y^*, \nu^*_Y\rangle$ be their nonstandard expansions. Let $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ be a continuous function. Then, there exists a pre-extension $F^* : \subseteq X^* \to Y^*$ of f^{s}.*

Proof. We reason within ns-BASIC. Let Φ^{s} be a code for $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$. By conditions (i) and (ii) for a code for a continuous function, $\forall (n, a, r, b, s), (n', a', r', b', s') \in \Phi^{\mathrm{s}}(d^{\mathrm{s}}_A(a, a') + r' < r \to d^{\mathrm{s}}_B(b, b') \le s + s')$. Thus, for any $i \in \mathbb{N}^{\mathrm{s}}$,

$$\begin{aligned} (\dagger_i) \quad & \forall (n, a, r, b, s), (n', a', r', b', s') \in \Phi^{\mathrm{s}\checkmark} \cap i \\ & \qquad (\nu^*_X(a, a') + r' + 2^{-i} < r \to \nu^*_B(b, b') \le s + s' + 2^{-i}) \\ & \wedge \Phi^{\mathrm{s}\checkmark} \cap i \subseteq \mathbb{N}^* \times X^* \times \mathbb{Q}^{*+} \times Y^* \times \mathbb{Q}^{*+}. \end{aligned}$$

Applying $\Sigma^0_1 \cup \Pi^0_1$-overspill, we can find $\omega_0 \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ which satisfies $(\dagger_{\omega_0})$. Let $\Psi^* = \Phi^{\mathrm{s}\checkmark} \cap \omega_0 (\sqsubseteq_{\mathrm{s}} \Phi^{\mathrm{s}\checkmark})$. Then, by conditions (dom) and (ii), for any $i \in \mathbb{N}^{\mathrm{s}}$,

$$(\dagger\dagger_i) \quad \forall x \in X^* \cap i\ \exists (n, a, r, b, s) \in \Psi^*(\nu^*_X(x, a) < r + 2^{-i-1} < 2^{-i} \wedge s < 2^{-i}).$$

Again by $\Sigma^0_1 \cup \Pi^0_1$-overspill, we can find $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ which enjoys $(\dagger\dagger_\omega)$. Define a partial function $F^* : \subseteq X^* \to Y^*$ of f^{s} such that $\mathrm{dom}(F^*) = X^* \cap \omega$ as follows:

$$F_0^*(x) = \min\{(n', a', r', b', s') \in \Psi^* \mid \nu_X^*(x, a') < r' + 2^{-\omega-1} < 2^{-\omega} \wedge s' < 2^{-\omega}\}),$$
$$F^*(x) = y \leftrightarrow \exists n \in N^* \exists a \in X^* \exists r \in \mathbb{Q}^{*+} \exists s \in \mathbb{Q}^{*+}((n, a, r, y, s) = F_0^*(x)).$$

We will check that this F^* is a pre-extension of f^{s}. Let $x \in \mathrm{dom}(F^*) \cap \mathrm{Nst}(X^*)$, $\mathrm{st}(x) = c^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}$ and $F_0^*(x) = (n, a, r, y, s) \in \Psi^*$. Let $m \in \mathbb{N}^{\mathrm{s}}$. Then, there exists $(n_0, a_0, r_0, b_0, s_0) \in \Phi^{\mathrm{s}}$ such that $d_A^{\mathrm{s}}(c^{\mathrm{s}}, a_0) < r_0$ and $d_B^{\mathrm{s}}(f^{\mathrm{s}}(c^{\mathrm{s}}), b_0) \le s_0 < 2^{-m-1}$. Then, $\nu_X^*(a_0, a) + r + 2^{-\omega_0} < r_0$ since $\mathrm{st}(\nu_X^*(x, c^{\mathrm{s}}) + \nu_X^*(x, a) + r + 2^{-\omega_0}) = 0$. Thus, by $(\dagger_{\omega_0})$, $\nu_Y^*(f^{\mathrm{s}}(c^{\mathrm{s}}), y) < 2^{-m}$. This means that $\mathrm{st}(y) = f^{\mathrm{s}}(c^{\mathrm{s}})$. □

Theorem 2.18 (extensions of continuous functions)**.** *Let $\langle A^{\mathrm{s}}, d_A^{\mathrm{s}}\rangle$, $\langle B^{\mathrm{s}}, d_B^{\mathrm{s}}\rangle$ be complete separable metric spaces, and let $\langle X^*, \nu_X^*\rangle$, $\langle Y^*, \nu_Y^*\rangle$ be their nonstandard expansions. Then, the following are equivalent over* ns-BASIC.

1. $\mathsf{ns\text{-}WKL_0}$.
2. *Every pre-extension $F^* : X^* \to Y^*$ of a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ is* s*-continuous, thus it is an extension of f^{s}.*
3. *Pre-extension is unique up to infinitesimals, i.e., if two partial functions $F^* : X^* \to Y^*$ and $G^* : X^* \to Y^*$ are pre-extensions of the same continuous function, then, $F^*(x) \approx G^*(x)$ for any $x \in \mathrm{dom}(F^*) \cap \mathrm{dom}(G^*) \cap \mathrm{App}(X^*)$.*
4. *If a partial function $F^* : X^* \to Y^*$ is* s*-continuous and $F^*(\mathrm{dom}(F^*) \cap \mathrm{App}(X^*)) \subseteq \mathrm{App}(Y^*)$, then, F^* has a standard part.*

Proof. $1 \to 2$ is a straightforward direction of Theorem 2.13, and $2 \to 3$ is trivial. We first show $1 \to 4$. Let $F^* : X^* \to Y^*$ is s-continuous and $F^*(\mathrm{dom}(F^*) \cap \mathrm{App}(X^*)) \subseteq \mathrm{App}(Y^*)$. Define $\Phi^* \subseteq \mathbb{N}^* \times X^* \times \mathbb{Q}^{*+} \times Y^* \times \mathbb{Q}^{*+}$ as $(n, a, r, b, s) \in \Phi^* \leftrightarrow \forall x \in X^*(\nu_X^*(a, x) < r \to \nu_Y^*(F^*(x), b) < s)$. Then, we can easily check that $\Phi^* \restriction V^{\mathrm{s}}$ is a code for a continuous function in V^{s} and a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ coded by $\Phi^* \restriction V^{\mathrm{s}}$ is the standard part of F^*. Take $x_0 \in X^* = Y^*$ such that $x_0 \approx \alpha^*$. Then, $x_0 \in \mathrm{App}(X^*) \setminus \mathrm{Nst}(X^*) = \mathrm{App}(Y^*) \setminus \mathrm{Nst}(Y^*)$.

Finally, we show $\neg 1 \to \neg 3, \neg 4$. We assume $\neg 1$. Then, by Theorem 2.10, there exist $\alpha^* \in \mathbb{R}^*$ and $K \in \mathbb{N}^{\mathrm{s}}$ such that $|\alpha^*| < K$ and $\mathrm{st}(\alpha^*) \notin \mathbb{R}^{\mathrm{s}}$. Without loss of generality, we may assume that $0 \le \alpha^* \le 1$. Let $\langle A^{\mathrm{s}}, d_A^{\mathrm{s}}\rangle = \langle B^{\mathrm{s}}, d_B^{\mathrm{s}}\rangle = [0,1]^{\mathrm{s}}$, and let $\langle X^*, \nu_X^*\rangle = \langle Y^*, \nu_Y^*\rangle = \langle \{i/2^\omega \mid 0 \le i \le 2^\omega\}, |\cdot|\rangle$ for some $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. Then, X^* and Y^* are nonstandard expansions of $[0,1]^{\mathrm{s}}$. Define F^*, G^*, H^* from X^* to Y^* as

$$F^*(x) = 0, \qquad G^* = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}, \qquad H^*(x) = x_0.$$

Both of F^* and G^* are pre-extensions of zero function, but $F^*(x_0) \not\approx G^*(x_0)$, thus, $\neg 3$. H^* is s-continuous and $H^*(X^*) \subseteq \mathrm{App}(Y^*)$, but H^* does not have a standard part, thus, $\neg 4$. □

Next, we will give a nonstandard characterization of the uniform continuity.

Proposition 2.19 (characterization of uniform continuity). *Let $\langle A^{\mathrm{s}}, d_A^{\mathrm{s}}\rangle$, $\langle B^{\mathrm{s}}, d_B^{\mathrm{s}}\rangle$ be complete separable metric spaces, and let $\langle X^*, \nu_X^*\rangle$, $\langle Y^*, \nu_Y^*\rangle$ be their nonstandard expansions. Let $F^* : X^* \to Y^*$ be a pre-extension of a continuous function $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$. Then, the following are provable within* ns-BASIC.

1. *If f^{s} is effectively uniformly continuous, then there exists $Z^* \sqsubseteq_{\mathrm{s}} \mathrm{dom}(F^*)$ such that $\forall x, y \in Z^*(x \approx y \to F^*(x) \approx F^*(y))$. Particularly, every effectively uniformly continuous function has an extension.*

2. *Conversely, if there exists $Z^* \sqsubseteq_{\mathrm{s}} \mathrm{dom}(F^*)$ such that $\forall x, y \in Z^*(x \approx y \to F^*(x) \approx F^*(y))$, then, f^{s} effectively uniformly continuous.*

Moreover, the following is provable within ns-$\mathsf{WKL_0}$

3. *f^{s} is effectively uniformly continuous if and only if there exists $Z^* \sqsubseteq_{\mathrm{s}} \mathrm{dom}(F^*)$ such that $\forall x, y \in Z^*(x \approx y \to F^*(x) \approx F^*(y))$.*

Proof. 1 and 2 are easily proved by overspill. We can prove 3 by Σ_0^*-choice as in the proof of Corollary 2.14. □

Remark 2.20. Uniform continuity does not imply effective uniform continuity within ns-$\mathsf{WKL_0}$ or $\mathsf{WKL_0}$. In fact, it requires $\mathsf{ACA_0}$. (We can prove that $\mathsf{ACA_0}$ is necessary as follows: for given 1-1 function $f : \mathbb{N} \to \mathbb{N}$, consider a continuous function g from $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ to $\{0\} \cup \{1/f(n) \mid n \in \mathbb{N}\}$ defined as $g(0) = 0$ and $g(1/n) = 1/f(n)$.)

Next, we consider continuous functions on $\mathbb{R}^n$. A (partial) continuous function $f : \subseteq\mathbb{R}^n \to \mathbb{R}^m$ is said to be *piecewise linear* if for any bounded closed subset $D \subseteq \mathbb{R}^n$, there exists a finite (closed) cover $\{D_i\}_{i \le k}$ of D such that $f \restriction D_i$ is a linear function for any $i \le k$. Let $\mathbb{Q}^n[k] := \{a/2^k \in \mathbb{Q}^n \mid a \in \mathbb{Z}^n\}$.

Now, we argue within ns-BASIC. Our aim is to give an infinitesimal approximation of a continuous function by a hyperfinite piecewise linear function. Here, we only consider this on $[0,1]^n$, but generalization is not difficult.

Definition 2.8. The following definitions are made in ns-BASIC.

1. A continuous function $F^* : ([0,1]^*)^n \to (\mathbb{R}^*)^m$ is said to be a *pre-extension* of a continuous function $f^{\mathrm{s}} : ([0,1]^{\mathrm{s}})^n \to (\mathbb{R}^{\mathrm{s}})^m$, or f^{s} is said to be a *pre-standard part* of F^* if $\mathrm{st}(F^*(x^*)) = f^{\mathrm{s}}(\mathrm{st}(x^*))$ for any near standard $x^* \in ([0,1]^*)^n$.

2. A continuous function $F^* : ([0,1]^*)^n \to (\mathbb{R}^*)^m$ is said to be an *extension* of a continuous function $f^{\mathrm{s}} : [0,1]^{\mathrm{s}n} \to (\mathbb{R}^{\mathrm{s}})^m$, or f^{s} is said to be the *standard part* of F^* if for any $x^* \in ([0,1]^*)^n$ and for any $i \in \mathbb{N}^{\mathrm{s}}$, there exist $j \in \mathbb{N}^{\mathrm{s}}$ such that for any $y^{\mathrm{s}} \in [0,1]^{\mathrm{s}n}$, $d_{\mathbb{R}^n}(x^*, y^{\mathrm{s}}) < 2^{-j} \to d_{\mathbb{R}^m}(F^*(x^*), f^{\mathrm{s}}(y^{\mathrm{s}})) < 2^{-i}$.

3. A continuous function $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$ is said to be s-*continuous* if for any $x^*, y^* \in ([0,1]^*)^n$, $x^* \approx y^* \to F^*(x^*) \approx F^*(y^*)$.

We can prove the following proposition as in the nonstandard extension of a continuous function on a complete separable metric space.

Proposition 2.21. *The following is provable within* ns-BASIC. *Let $f^{\mathrm{s}}: ([0,1]^{\mathrm{s}})^n \to (\mathbb{R}^{\mathrm{s}})^m$ be a continuous function. Then, a continuous function $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$ is an extension of f^{s} if and only if it is a pre-extension of f^{s} and* s-*continuous.*

Now we give an infinitesimal approximation for a continuous function by a hyperfinite piecewise-linear function.

Proposition 2.22. *The following is provable within* ns-BASIC. *Let $f^{\mathrm{s}}: ([0,1]^{\mathrm{s}})^n \to (\mathbb{R}^{\mathrm{s}})^m$ be a continuous function. Then, there exists a pre-extension $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$ of f^{s} which is a linear function in V^*. Moreover, parameters for F^* can be taken from $\mathbb{Q}^*$.*

Proof. By Proposition 2.17, there exists a pre-extension $F_0^*: (\mathbb{Q}^*)^n[\omega_1] \cap ([0,1]^*)^n \to (\mathbb{Q}^*)^m[\omega_2]$ of f^{s} for some $\omega_1, \omega_2 \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. Then, we can extend F_0^* into a piecewise linear function $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$. □

Theorem 2.23 (Approximation by a hyperfinite piecewise linear function)**.** *The following are equivalent over* ns-BASIC*.*

1. $\mathsf{ns\text{-}WKL_0}$.
2. *Every continuous function $f^{\mathrm{s}}: ([0,1]^{\mathrm{s}})^n \to (\mathbb{R}^{\mathrm{s}})^m$ has an extension $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$ which is a piecewise-linear function. Moreover, parameters for F^* can be taken from $\mathbb{Q}^*$.*
3. *If a continuous function $F^*: ([0,1]^*)^n \to (\mathbb{R}^*)^m$ is* s-*continuous and* s-*bounded, i.e. there exists $K \in \mathbb{N}^{\mathrm{s}}$ such that $\|F^*(x^*)\| < K$ for any $x^* \in ([0,1]^*)^n$, then F^* has a standard part.*

Proof. Similar to the proof of Theorem 2.18. □

Moreover, we can prove the nonstandard Weierstraß approximation theorem, *i.e.*, a continuous function $f^{\mathrm{s}}: [0,1]^{\mathrm{s}} \to \mathbb{R}^{\mathrm{s}}$ can be infinitesimally approximated by a nonstandard polynomial $F^*: [0,1]^* \to \mathbb{R}^*$ such that every coefficient is taken from $\mathbb{Q}^*$. This is a consequence of the Weierstraß approximation theorem in $\mathsf{WKL_0}$ plus overspill.

2.2.4 Riemann integral

In this subsection, we will approximate the Riemann integral by a hyperfinite Riemann sum. Let $I^n[i] = \mathbb{Q}^n[i] \cap [0,1)^n$. For a function $F: I^n[i] \to \mathbb{Q}$, define $S_i[F] := \sum_{x \in I^n[i]} F(x)/2^{in}$. Within ns-BASIC, I^n_ω is a nonstandard expansion of $([0,1]^{\mathrm{s}})^n$ if $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.

Proposition 2.24. *The following is provable within* ns-BASIC*. Let $f^{\mathrm{s}} : ([0,1]^{\mathrm{s}})^n \to \mathbb{R}$ be a continuous function, and let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. If a function $F^* : I^n_\omega \to \mathbb{Q}^*$ is an extension of f^{s}, then, f^{s} is Riemann integrable and*

$$\int_{([0,1]^{\mathrm{s}})^n} f\,dx = \mathrm{st}(S_\omega[F^*]).$$

Proof. Easy imitation of the usual proof of nonstandard analysis. □

Theorem 2.25. *The following are equivalent over* ns-BASIC*.*

1. $\mathsf{ns\text{-}WKL_0}$.
2. *Let $f^{\mathrm{s}} : ([0,1]^{\mathrm{s}})^n \to \mathbb{R}$ be a continuous function, and let $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. If a function $F^* : I^n_\omega \to \mathbb{Q}^*$ is a pre-extension of f^{s}, then, f^{s} is Riemann integrable and*
$$\int_{([0,1]^{\mathrm{s}})^n} f\,dx = \mathrm{st}(S_\omega[F^*]).$$

Proof. $1 \to 2$ is a straightforward direction from Theorem 2.18 and Proposition 2.24. We show $\neg 1 \to \neg 2$. Assuming $\neg 1$, there exist $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $x_0 \in I^n_\omega$ such that x_0 is not near standard as in the proof of Theorem 2.18. Define $F^* : I^n_\omega \to \mathbb{Q}^*$ as $F^*(x_0) = 2^{\omega n}$ and $F^*(x) = 0$ if $x \neq x_0$. Then, F^* is a pre-extension of the zero function on $([0,1]^{\mathrm{s}})^n$, but

$$\int_{([0,1]^{\mathrm{s}})^n} 0\,dx \neq \mathrm{st}(S_\omega[F^*]) = 1.$$

□

Remark 2.26. In the Theorem 2.25, the item 2 is equivalent to a weaker system $\mathsf{ns\text{-}WWKL_0}$ if f^{s} is bounded. See [10].

2.2.5 Open and closed sets

In this part, we will show how to deal with open and closed sets within nonstandard second-order arithmetic.

We will argue within ns-BASIC. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space, and let $\langle X^*, \nu^*\rangle$ be its nonstandard expansion. For a subset $D^* \subseteq X^*$, we define subsets of $\hat{A}^{\mathrm{s}}$ as follows:

$$\begin{aligned}\mathrm{st}(D^*) :=& \{y^{\mathrm{s}} \in \hat{A}^{\mathrm{s}} \mid \exists x \in D^* \ \mathrm{st}(x) = y^{\mathrm{s}}\},\\ \mathrm{st}^i(D^*) :=& \{y^{\mathrm{s}} \in \hat{A}^{\mathrm{s}} \mid \exists n \in \mathbb{N}^{\mathrm{s}}\ \forall x \in X^*(\nu^*(x, y^{\mathrm{s}}) < 2^{-n} \to x \in D^*)\}.\end{aligned}$$

Proposition 2.27. *The following is provable within* ns-BASIC*. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space, and let $\langle X^*, \nu^*\rangle$ be its nonstandard expansion. Let $D^* \subseteq X^*$. Then, $y^{\mathrm{s}} \in \mathrm{st}(D^*)$ if and only if $y^{\mathrm{s}} \notin \mathrm{st}^i(X^* \setminus D^*)$.*

Proof. If $y^{\mathrm{s}} \in \mathrm{st}^i(X^* \setminus D^*)$ and $\mathrm{st}(x) = y^{\mathrm{s}}$, then $x \in X^* \setminus D^*$. Thus, $y^{\mathrm{s}} \in \mathrm{st}(D^*)$ implies $y^{\mathrm{s}} \notin \mathrm{st}^i(X^* \setminus D^*)$. We will show that $y^{\mathrm{s}} \notin \mathrm{st}^i(X^* \setminus D^*)$ implies $y^{\mathrm{s}} \in \mathrm{st}(D^*)$. Let $y^{\mathrm{s}} \notin \mathrm{st}^i(X^* \setminus D^*)$. Take $z \in X^*$ such that $\mathrm{st}(z) = y^{\mathrm{s}}$. Then, for any $n \in \mathbb{N}^{\mathrm{s}}$, there exists $x \in D^*$ such that $\nu^*(x, z) < 2^{-n}$. Thus, by overspill, there exists $x_0 \in D^*$ such that $x_0 \approx z$. Then, $\mathrm{st}(x_0) = \mathrm{st}(z) = y^{\mathrm{s}}$, which means that $y^{\mathrm{s}} \in \mathrm{st}(D^*)$. □

Theorem 2.28. *The following are equivalent over* $\mathsf{ns\text{-}BASIC}$.

1. $\mathsf{ns\text{-}WKL_0}$.

2. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space, and let $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let $D^* \subseteq X^*$. Then, for any $D^* \subseteq X^*$, $\mathrm{st}^i(D^*)$ is an open subset of $\hat{A}^{\mathrm{s}}$, i.e., there exists an open code $U^{\mathrm{s}} \subseteq \mathbb{N}^{\mathrm{s}} \times A^{\mathrm{s}} \times \mathbb{Q}^{\mathrm{s}+}$ such that $y^{\mathrm{s}} \in \mathrm{st}^i(D^*) \leftrightarrow y^{\mathrm{s}} \in U^{\mathrm{s}}$.*

3. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space, and let $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let $D^* \subseteq X^*$. Then, for any $D^* \subseteq X^*$, $\mathrm{st}(D^*)$ is a closed subset of $\hat{A}^{\mathrm{s}}$, i.e., there exists an open code $U^{\mathrm{s}} \subseteq \mathbb{N}^{\mathrm{s}} \times A^{\mathrm{s}} \times \mathbb{Q}^{\mathrm{s}+}$ such that $y^{\mathrm{s}} \in \mathrm{st}(D^*) \leftrightarrow y^{\mathrm{s}} \notin U^{\mathrm{s}}$.*

Proof. $2 \leftrightarrow 3$ is trivial from Proposition 2.27. We first show $1 \to 2$. We argue within $\mathsf{ns\text{-}WKL_0}$. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space, and let $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let $D^* \subseteq X^*$. Take $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ Define $U_0^* \subseteq \mathbb{N}^* \times X^* \times \mathbb{Q}^{*+}$ as $(n, a, r) \in U_0^* \leftrightarrow n = 1 \wedge \forall x \in X^* \cap \omega(\nu^*(a, x) < r \to x \in D^*)$. Then, $U^{\mathrm{s}} = U_0^* \restriction V^{\mathrm{s}}$ is (a code for) an open subset of $\hat{A}^{\mathrm{s}}$. We can easily check that $x^{\mathrm{s}} \in U^{\mathrm{s}} \leftrightarrow x^{\mathrm{s}} \in \mathrm{st}^i(D^*)$.

To show $\neg 1 \to \neg 2$, let $\hat{A}^{\mathrm{s}} = [0,1]^{\mathrm{s}}$ and $X^* = I[\omega] = \mathbb{Q}^*[\omega] \cap [0,1)^{*n}$. By $\neg 1$, there exists $x_0 \in X^*$ such that x_0 is not near standard as in the proof of Theorem 2.18. Then, clearly $0 < \mathrm{st}(x_0) < 1$ in V^{s}. Define D_0^* and D_1^* as $D_0^* = \{x \in X^* \mid x < x_0\}$ and $D_1^* = X^* \setminus D_0^*$. If both of $\mathrm{st}^i(D_0^*)$ and $\mathrm{st}^i(D_1^*)$ are open sets, $[0,1]^{\mathrm{s}}$ is divided into two non-empty open sets, which contradicts the fact that $[0,1]^{\mathrm{s}}$ is connected (in V^{s}). (Note that $\mathsf{RCA_0}$ proves that the unit interval is connected.) This completes the proof. □

For nonstandard approximation for an open or closed set, we need $\mathsf{ns\text{-}ACA_0}$.

Proposition 2.29. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space, and let $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Then, the following are provable within* $\mathsf{ns\text{-}ACA_0}$.

1. *For any open subset $U^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$, there exists $D^* \subseteq X^*$ such that $U^{\mathrm{s}} = \mathrm{st}^i(D^*)$.*

2. *For any closed subset $C^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$, there exists $D^* \subseteq X^*$ such that $C^{\mathrm{s}} = \mathrm{st}(D^*)$.*

Proof. We only need to prove 2 by Proposition 2.27. We argue within $\mathsf{ns\text{-}ACA_0}$. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric spaces, and let $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let U^{s} be (a code for) an open subset of $\hat{A}^{\mathrm{s}}$. By Σ^1_1-TP, $\langle A^{\mathrm{s}\surd}, d^{\mathrm{s}\surd}\rangle$ is a complete separable metric space and $U^{\mathrm{s}\surd}$ is its open subset in V^*. Then, by $\Sigma^0_1 \cup \Pi^0_1$-overspill, there exists $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $A^{\mathrm{s}\surd} \cap \omega = X^* \cap \omega$ and $d^{\mathrm{s}\surd} \approx \nu^*$ on $X^* \cap \omega$. Define D^* as $x \in D^*$ if and only if $x \in X^* \cap \omega$ and $x \notin U^{\mathrm{s}\surd}$ as an element of $A^{\mathrm{s}\surd}$. We show $\mathrm{st}(D^*) = \hat{A}^{\mathrm{s}} \setminus U^{\mathrm{s}}$. Note that $\nu^*(x^{\mathrm{s}}, a) < r \leftrightarrow d^{\mathrm{s}\surd}(x^{\mathrm{s}\surd}, a) < r$ for any $a \in X^* \cap \omega$ and $r \in \mathbb{Q}^{\mathrm{s}+}$. If $x^{\mathrm{s}} \in \mathrm{st}(D^*)$, then $x^{\mathrm{s}\surd} \notin U^{\mathrm{s}\surd}$, thus $x^{\mathrm{s}} \notin U^{\mathrm{s}}$ by the transfer principle. If $x^{\mathrm{s}} \notin U^{\mathrm{s}}$, there exists $y \in X^* \cap \omega$ such that $\nu^*(x^{\mathrm{s}}, y) < 2^{-\omega'}$ and $y \notin U^{\mathrm{s}\surd}$ for some $\omega' \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ by overspill, thus $x^{\mathrm{s}} \in \mathrm{st}(D^*)$. This completes the proof. □

Next, we consider totally boundedness and compactness for closed sets. We can define the notion *effectively totally bounded*, *totally bounded*, *Heine/Borel compact* and *sequentially Heine/Borel compact* for a closed set similar to the definition for $\hat{A}$ within RCA_0. Then, similarly to a complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$, we can prove the following.

Proposition 2.30. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space in V^{s}, and $\langle X^*, \nu^*\rangle$ be its nonstandard expansion. Let $C^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$ be a closed set. Then, the following are provable within* ns-BASIC.

1. *If C^{s} is effectively totally bounded, then, there exists a (V^*-)finite set $D^* \subseteq \mathrm{App}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$. Conversely, if there exists a (V^*-)finite set $D^* \subseteq \mathrm{App}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$, then, C^{s} is totally bounded.*

2. *If there exists a (V^*-)finite set $D^* \subseteq \mathrm{Nst}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$, then, C^{s} is Heine/Borel compact.*

The following is provable within $\mathsf{ns\text{-}WKL_0}$.

3. *The following are equivalent:*
   - *C^{s} is effectively totally bounded,*
   - *C^{s} is sequentially Heine/Borel compact,*
   - *there exists a (V^*-)finite set $D^* \subseteq \mathrm{App}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$,*
   - *there exists a (V^*-)finite set $D^* \subseteq \mathrm{Nst}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$.*

The following is provable within ns-BASIC $+ \Sigma^0_1$-TP.

4. *C^{s} is totally bounded if and only if there exists a (V^*-)finite set $D^* \subseteq \mathrm{App}(X^*)$ such that $C^{\mathrm{s}} \subseteq \mathrm{st}(D^*)$.*

Here, we can see that $\mathrm{st}(D^*)$ for a V^*-finite set $D^* \subseteq \mathrm{App}(X^*)$ plays an important role. Actually, it defines a "strongly compact-like" set. Within RCA_0, let $\langle A, d\rangle$ be a complete separable metric space. Then, a sequence of finite sequences $S = \langle\langle h(i,j) \in A \mid j < p(i)\rangle \mid i \in \mathbb{N}\rangle$ is said to be a (code for) a *strongly totally bounded closed set* (*stbc-set*, in short) if $p(i) \leq p(i+k)$ and $d(h(i,l), h(i+k,l)) \leq 2^{-i}$ for any $i, k \in \mathbb{N}$ and $l < p(i)$. For $x \in \hat{A}$, define

$x \in S \leftrightarrow \forall i \in \mathbb{N}\ \exists j < p(i)\ d(x, h(i,j)) \leq 2^{-i+1}$. Then, a closed set $C \subseteq \hat{A}$ is said to be *effectively totally bounded* if it is a subset of stbc-set. Note that, we can easily prove within RCA_0 that each stbc-set is a closed set which is separably closed, *i.e.*, a closure of a countable sequence, and located, *i.e.*, a continuous function $f(x) = d(x, S)$ exists. Conversely, we can prove within WKL_0 that a closed set which is separably closed and effectively totally bounded is a stbc-set. (It is still open that whether this is provable within RCA_0 or not.)

Now, we argue within $\mathsf{ns\text{-}BASIC}$. We can see that stbc-set can be characterized by a V^*-finite subset of $\mathrm{App}(X^*)$.

Proposition 2.31. *The following are provable within* $\mathsf{ns\text{-}BASIC}$. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space in V^{s}, and $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let $S^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$ be a stbc-set. Then, there exists a (V^*-)finite set $D^* \subseteq \mathrm{App}(X^*)$ such that $S^{\mathrm{s}} = \mathrm{st}(D^*)$.*

Proof. Let $S^{\mathrm{s}} = \langle \langle h^{\mathrm{s}}(i,j) \in A^{\mathrm{s}} \mid j < p^{\mathrm{s}}(i) \rangle \mid i \in \mathbb{N}^{\mathrm{s}} \rangle$ be a stbc-set of $\hat{A}^{\mathrm{s}}$. Then, by overspill, there exists $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $p^{\mathrm{s}\checkmark}(i) \leq p^{\mathrm{s}\checkmark}(j)$ and $\nu^*(h^{\mathrm{s}\checkmark}(i,l), h^{\mathrm{s}\checkmark}(j,l)) \leq 2^{-i}$ for any $i < j \leq \omega$ and $l < p^{\mathrm{s}\checkmark}(i)$. Let $D^* = \{h^{\mathrm{s}\checkmark}(\omega, l) \mid l < p^{\mathrm{s}\checkmark}(\omega)\}$. Then, clearly, $D^* \subseteq \mathrm{App}(X^*)$ and $\mathrm{st}(D^*) \subseteq S^{\mathrm{s}}$. Let $x^{\mathrm{s}} = \langle x^{\mathrm{s}}(i) \mid i \in \mathbb{N}^{\mathrm{s}} \rangle \in S^{\mathrm{s}}$. Then, there exists $\omega' \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $l < p^{\mathrm{s}\checkmark}(\omega')$ such that $\omega' < \omega$ and $\nu^*(x^{\mathrm{s}\checkmark}(\omega'), h^{\mathrm{s}\checkmark}(\omega', l)) \leq 2^{-\omega'+2}$. Thus, $x^{\mathrm{s}} \approx h^{\mathrm{s}\checkmark}(\omega', l) \approx h^{\mathrm{s}\checkmark}(\omega, l)$. This means $\mathrm{st}(D^*) \supseteq S^{\mathrm{s}}$, and this completes the proof. $\square$

For the converse, we need $\mathsf{ns\text{-}WKL}_0$ again.

Theorem 2.32. *The following are equivalent over* $\mathsf{ns\text{-}BASIC}$.

1. $\mathsf{ns\text{-}WKL}_0$.
2. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ be a complete separable metric space in V^{s}, and $\langle X^*, \nu^* \rangle$ be its nonstandard expansion. Let $D^* \subseteq \mathrm{App}(X^*)$ be a (V^*-)finite set. Then, there exists a stbc-set $S^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$ such that $S^{\mathrm{s}} = \mathrm{st}(D^*)$.*

Proof. We can prove $2 \to 1$ exactly same as the proof of $2 \to 1$ of Theorem 2.28. We show $1 \to 2$. Let $D^* \subseteq \mathrm{App}(X^*)$ be a (V^*-)finite set. Then, for any $i \in \mathbb{N}^{\mathrm{s}}$, $\forall x \in D^* \exists a \in A^{\mathrm{s}} \nu^*(x, a) < 2^{-i}$. Thus, by underspill, $\forall x \in D^* \exists a \in A^{\mathrm{s}} \cap M \nu^*(x, a) < 2^{-i}$ for some $M_i \in \mathbb{N}^{\mathrm{s}}$. Now, by using Σ^*_0-choice, we can easily construct a sequence $\langle \sigma_i \in A^{\mathrm{s}<\mathbb{N}^{\mathrm{s}}} \mid i \in \mathbb{N}^{\mathrm{s}} \rangle$ which satisfies the following:

- $\forall x \in D^* \exists l < \mathrm{lh}(\sigma_i) \nu^*(x, \sigma_i(l)) < 2^{-i+1}$ for any $i \in \mathbb{N}^{\mathrm{s}}$,
- $\forall l < \mathrm{lh}(\sigma_i) \exists x \in D^* \nu^*(x, \sigma_i(l)) < 2^{-i+1}$ for any $i \in \mathbb{N}^{\mathrm{s}}$,
- $\nu^*(\sigma_i(l), \sigma_j(l)) < 2^{-i}$ for any $i < j \in \mathbb{N}^{\mathrm{s}}$ and $l < \mathrm{lh}(\sigma_i)$.

We can easily check that this sequence codes a stbc-set $S^{\mathrm{s}} = \mathrm{st}(D^*)$. $\square$

We can also give some nonstandard characterization for some other properties of open or closed sets. In section 4, we will consider Jordan regions and simply connected open sets in $\mathbb{C}$ to prove several versions of Riemann's mapping theorem.

2.2.6 Transfer principle and sequential compactness

In this part, we consider sequential compactness using the transfer principle. We argue within $\mathsf{ns\text{-}BASIC} + \Sigma^1_1$-TP. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space in V^{s}. Then, by the transfer principle, $\langle A^{\mathrm{s}\surd}, d^{\mathrm{s}\surd}\rangle$ is a complete separable metric space in V^*. Let $X^* = A^{\mathrm{s}\surd}$ and $\nu^* = d^{\mathrm{s}\surd}$, then $\langle X^*, \nu^*_\omega\rangle$ is a nonstandard expansion of $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ for any $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$. Moreover, $\hat{A}^{\mathrm{s}\surd} = \hat{X}^*$ itself can be considered as a nonstandard expansion of $\hat{A}^{\mathrm{s}}$ since for any $x^* \in \hat{X}^*$ there exists $y \in X^*$ such that $\nu^*(x^*, y) \approx 0$. We can also define Lim, App and Nst for $\hat{A}^{\mathrm{s}\surd}$ similarly to X^*. Let $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$ be a sequence on $\hat{A}^{\mathrm{s}}$. Then, $h^{\mathrm{s}\surd}$ is a nonstandard sequence on $\hat{A}^{\mathrm{s}\surd}$, *i.e.*, $h^{\mathrm{s}\surd} : \mathbb{N}^* \to \hat{A}^{\mathrm{s}\surd}$. Note that $(\surd(h^{\mathrm{s}}))(i) = \surd(h^{\mathrm{s}}(i))$ for any $i \in \mathbb{N}^{\mathrm{s}}$, so we just write $h^{\mathrm{s}\surd}(i)$ for $\surd(h^{\mathrm{s}}(i))$.

Proposition 2.33 (Nonstandard characterization for some properties of sequences)**.** *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space. Then, the following are provable within* $\mathsf{ns\text{-}BASIC} + \Sigma^1_1$*-TP.*

1. *A sequence $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$ is a Cauchy sequence if and only if for any $i, j \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, $h^{\mathrm{s}\surd}(i) \approx h^{\mathrm{s}\surd}(j) \in \mathrm{App}(\hat{A}^{\mathrm{s}\surd})$.*

2. *A sequence $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$ is convergent if and only if for any $i, j \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, $h^{\mathrm{s}\surd}(i) \approx h^{\mathrm{s}\surd}(j) \in \mathrm{Nst}(\hat{A}^{\mathrm{s}\surd})$.*

3. *A sequence $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$ has a convergent subsequence if and only if there exists $i \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ such that $h^{\mathrm{s}\surd}(i) \in \mathrm{Nst}(\hat{A}^{\mathrm{s}\surd})$.*

4. *For any sequence $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$, a (possibly external) subset $\mathrm{st}(\{h^{\mathrm{s}\surd}(i) \mid i \in \mathbb{N}^*\}) \subseteq \hat{A}^{\mathrm{s}}$ is a closure of $\{h^{\mathrm{s}}(i) \mid i \in \mathbb{N}^{\mathrm{s}}\}$.*

   More generally, for any (nonstandard) finite sequence $\sigma : \omega_0 \to X^*$ on some nonstandard expansion $\langle X^*, \nu^*\rangle$, a subset $\mathrm{st}(\sigma(i) \mid i < \omega\}) \subseteq \hat{A}^{\mathrm{s}}$ is a closure of $\{\mathrm{st}(\sigma(i)) \mid i \in \mathbb{N}^{\mathrm{s}}\}$ for some $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.

Proof. Easy imitation of the usual proof of nonstandard analysis. □

Corollary 2.34. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}\rangle$ be a complete separable metric space. Then, the following are provable within* $\mathsf{ns\text{-}ACA_0}$*.*

1. *A sequence $h^{\mathrm{s}} : \mathbb{N}^{\mathrm{s}} \to \hat{A}^{\mathrm{s}}$ is convergent if and only if it is a Cauchy sequence.*

2. *$\hat{A}^{\mathrm{s}}$ is totally bounded if and only if it is sequentially compact, i.e., every sequence on $\hat{A}^{\mathrm{s}}$ has a convergent subsequence.*

3. *Every separably closed set $S \subseteq \hat{A}^{\mathrm{s}}$ (the closure of a sequence on $\hat{A}^{\mathrm{s}}$) is a closed set, i.e., $\hat{A}^{\mathrm{s}} \setminus S$ has an open code.*

Proof. Straightforward directions from Theorem 2.13 plus Proposition 2.33. □

We finally consider some properties of a sequence of continuous functions within $\mathsf{ns\text{-}BASIC} + \Sigma^1_1$-TP. Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A\rangle$ and $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B\rangle$ be complete separable metric spaces and $f^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ be a continuous function in V^{s}. Then, by the transfer principle, $f^{\mathrm{s}\checkmark} : \hat{A}^{\mathrm{s}\checkmark} \to \hat{B}^{\mathrm{s}\checkmark}$ is a continuous function in V^*. For a continuous function $g^* : \hat{A}^{\mathrm{s}\checkmark} \to \hat{B}^{\mathrm{s}\checkmark}$, we can define s-continuity and the (pre-)standard part for g^* similarly to Definition 2.7.

Proposition 2.35. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A\rangle$ and $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B\rangle$ be complete separable metric spaces. Then, the following are provable within* $\mathsf{ns\text{-}BASIC} + \Sigma^1_1$-TP.

1. *A sequence of continuous functions $\langle f^{\mathrm{s}}_n : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}} \mid n \in \mathbb{N}^{\mathrm{s}}\rangle$ is uniformly convergent on any stbc-set if and only if for any $i, j \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, $\mathrm{st}(f^{\mathrm{s}\checkmark}_i)$ and $\mathrm{st}(f^{\mathrm{s}\checkmark}_j)$ exist and $\mathrm{st}(f^{\mathrm{s}\checkmark}_i) = \mathrm{st}(f^{\mathrm{s}\checkmark}_j)$.*

2. *A sequence of continuous functions $\langle f^{\mathrm{s}}_n : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}} \mid n \in \mathbb{N}^{\mathrm{s}}\rangle$ is uniformly bounded at any $x \in A^{\mathrm{s}}$, i.e., $\{f^{\mathrm{s}}_n(x) \mid n \in \mathbb{N}^{\mathrm{s}}\}$ is bounded in $\hat{A}^{\mathrm{s}}$ for any $x \in A^{\mathrm{s}}$, if and only if for any $i \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $x \in A^{\mathrm{s}\checkmark}$, $f^{\mathrm{s}\checkmark}_i(x) \in \mathrm{Lim}(\hat{A}^{\mathrm{s}\checkmark})$.*

3. *A sequence of continuous functions $\langle f^{\mathrm{s}}_n : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}} \mid n \in \mathbb{N}^{\mathrm{s}}\rangle$ is uniformly totally bounded at any $x \in A^{\mathrm{s}}$, i.e., $\{f^{\mathrm{s}}_n(x) \mid n \in \mathbb{N}^{\mathrm{s}}\}$ is totally bounded in $\hat{A}^{\mathrm{s}}$ for any $x \in A^{\mathrm{s}}$, if and only if for any $i \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $x \in A^{\mathrm{s}\checkmark}$, $f^{\mathrm{s}\checkmark}_i(x) \in \mathrm{App}(\hat{A}^{\mathrm{s}\checkmark})$.*

4. *A sequence of continuous functions $\langle f^{\mathrm{s}}_n : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}} \mid n \in \mathbb{N}^{\mathrm{s}}\rangle$ is equicontinuous if and only if for any $i \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $x, y \in \mathrm{App}(A^{\mathrm{s}\checkmark})$, $x \approx y$ implies $f^{\mathrm{s}\checkmark}_i(x) = f^{\mathrm{s}\checkmark}_i(y)$.*

Proof. As in the usual nonstandard analysis. □

These nonstandard characterizations straightforwardly imply (a generalized version of) the Arzelà/Ascoli theorem. Within $\mathsf{RCA_0}$, a complete separable metric space $\langle A, d\rangle$ is said to be *locally totally bounded* if for any $x \in \hat{A}$ there exists $r > 0$ such that an open ball $B(a; r)$ is totally bounded, and $\langle A, d\rangle$ is said to be *effectively locally totally bounded* if there exists a sequence $\langle (a_i, r_i) \in A \times \mathbb{Q}^+ \mid i \in \mathbb{N}\rangle$ such that each of open ball $B(a; r)$ is totally bounded and $\bigcup_{i\in\mathbb{N}} B(a; r) = \hat{A}$.

Corollary 2.36 (the Arzelà/Ascoli theorem)**.** *The following is provable within* $\mathsf{ns\text{-}ACA_0}$. *Let $\langle A^{\mathrm{s}}, d^{\mathrm{s}}_A\rangle$ and $\langle B^{\mathrm{s}}, d^{\mathrm{s}}_B\rangle$ be complete separable metric spaces. Then, for any sequence of continuous functions $\langle f^{\mathrm{s}}_n : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}} \mid n \in \mathbb{N}^{\mathrm{s}}\rangle$ which is equicontinuous and uniformly totally bounded at any $x \in A^{\mathrm{s}}$, there exists a continuous function $g^{\mathrm{s}} : \hat{A}^{\mathrm{s}} \to \hat{B}^{\mathrm{s}}$ such that for any stbc-set $S^{\mathrm{s}} \subseteq \hat{A}^{\mathrm{s}}$ there exists a subsequence $\langle f^{\mathrm{s}}_{n_i} \mid i \in \mathbb{N}^{\mathrm{s}}\rangle$ which uniformly converges to g^{s} on S^{s}. In particular, if $\hat{A}^{\mathrm{s}}$ is effectively locally totally bounded, then there exists a subsequence which uniformly converges to g^{s} on any stbc-set, and if $\hat{A}^{\mathrm{s}}$ is totally bounded, then there exists a subsequence which uniformly converges to g^{s} on $\hat{A}^{\mathrm{s}}$.*

Proof. Straightforward direction from Proposition 2.35 and Theorem 2.18. □

Remark 2.37. It is still open whether the use of the transfer principles in Propositions 2.33 and 2.35 are exactly needed or not. Note that in [8, 7], Sanders showed that the transfer principle is equivalent to many nonstandard versions of mathematical theorems within a different formulation.

2.3 Some applications to analysis in weak second-order arithmetic

Combining nonstandard analysis within nonstandard second-order arithmetic with conservation results, we can easily apply nonstandard methods into reverse mathematics for analysis within second-order arithmetic. Actually, we can do analysis in second-order arithmetic by a "uniform approach", which is a combination of "finite versions of theorems provable in RCA_0" plus "nonstandard approximations or characterizations". The following theorems are just easy examples of this argument.

Proposition 2.38. *The following are provable within* RCA_0.

1. *Every piecewise-linear continuous function on* $[0,1]^n$ *has a maximal.*
2. *Every piecewise-linear continuous function from* $[0,1]^n$ *to itself has a fixed point.*
3. *For any piecewise-linear continuous function on* $\mathbb{R}^n$, $\int_{[0,1]^n} f\,dx$ *exists.*
4. *The Jordan curve theorem for broken-line curve: for any piecewise-linear Jordan curve* $J : [0,1] \to \mathbb{R}^2$, *the open set* $\mathbb{R}^2 \setminus \mathrm{Im}(J)$ *is divided into a disjoint union of two open sets, the interior and the exterior of* J.

Proof. Each of them is provable by a finite method within RCA_0 or just an almost trivial statement. □

Theorem 2.39. *The following are provable within* WKL_0.

1. *Every continuous function on* $[0,1]^n$ *has a maximal.*
2. *Every continuous function from* $[0,1]^n$ *to itself has a fixed point.*
3. *For any continuous function on* $\mathbb{R}^n$, $\int_{[0,1]^n} f\,dx$ *exists.*
4. *The Jordan curve theorem: for any Jordan curve* $J : [0,1] \to \mathbb{R}^2$, *the open set* $\mathbb{R}^2 \setminus \mathrm{Im}(J)$ *is divided into a disjoint union of two open sets, the interior and the exterior of* J.

Proof. Just the combination of basic results in RCA_0 (Proposition 2.38) plus nonstandard methods in the previous section plus conservation (Theorem 2.4). □

Theorem 2.40. *The following are provable within* ACA_0.

1. *Every continuous function on a bounded closed set $D \subseteq \mathbb{R}^n$ has a maximal.*

2. *For any continuous function on $\mathbb{R}^n$ and for any bounded closed set $D \subseteq \mathbb{R}^n$, $\int_D f\,dx$ exists.*

3. *The Arzelà/Ascoli theorem.*

Proof. Just the combination of basic results in RCA_0 (Proposition 2.38) plus nonstandard methods in the previous section plus conservation (Theorem 2.7). □

We can prove several versions of Riemann's mapping theorem in this way, too. In the next section, we will prove an RCA_0 version of Riemann's mapping theorem as the base case.

3 Riemann's mapping theorem for polygonal domains

In this section, we will do complex analysis within RCA_0, and prove Riemann's mapping theorem for polygonal domains within RCA_0.

3.1 Complex analysis in weak second-order arithmetic

In this section, we study some basic parts of complex analysis in weak second-order arithmetic.

We first define the complex number system and holomorphic functions.

Definition 3.1 (The complex number system)**.** The following definitions are made in RCA_0. We identify a complex number, an element of $\mathbb{C}$, with an element of $\mathbb{R}^2$, and define $+_{\mathbb{C}}$, $\cdot_{\mathbb{C}}$ and $|\cdot|_{\mathbb{C}}$ by:

$$\begin{aligned}(x_1, y_1) +_{\mathbb{C}} (x_2, y_2) &= (x_1 + x_2, y_1 + y_2);\\ (x_1, y_1) \cdot_{\mathbb{C}} (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1);\\ |(x, y)|_{\mathbb{C}} &= \|(x, y)\|_{\mathbb{R}^2} = \sqrt{x^2 + y^2}.\end{aligned}$$

We usually omit the subscript $\mathbb{C}$. We write $(0, 1) = i$ and $(x, y) = x + iy = z$, where $x, y \in \mathbb{R}$ and $z \in \mathbb{C}$. A *continuous (partial) function* from $\mathbb{C}$ to $\mathbb{C}$ is a continuous (partial) function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 3.2 (Holomorphic functions)**.** The following definition is made in RCA_0. Let D be an open subset of $\mathbb{C}$, and let f and f' be continuous functions from D to $\mathbb{C}$. Then a pair (f, f') is said to be *holomorphic* if

$$\forall z \in D \ \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f'(z).$$

Informally, we write f for a holomorphic function (f, f').

Next, we define line integrals. Let α, β, γ be elements of $\mathbb{C}$ and let r be a positive real number. Then we define

$$
\begin{aligned}
[\alpha, \beta] &:= \{\alpha + (\beta - \alpha)x \in \mathbb{C} \mid 0 \leq x \leq 1\}, \\
B(\alpha; r) &:= \{z \in \mathbb{C} \mid |z - \alpha| < r\}, \\
\overline{B(\alpha; r)} &:= \{z \in \mathbb{C} \mid |z - \alpha| \leq r\}, \\
\partial B(\alpha; r) &:= \{z \in \mathbb{C} \mid |z - \alpha| = r\}, \\
\Delta(r) &:= B(0; r), \\
\triangle\alpha\beta\gamma &:= \{\alpha x_1 + \beta x_2 + \gamma x_3 \in \mathbb{C} \mid x_1 + x_2 + x_3 = 1 \wedge 0 \leq x_1, x_2, x_3 \leq 1\}, \\
\partial\triangle\alpha\beta\gamma &:= [\alpha, \beta] \cup [\beta, \gamma] \cup [\gamma, \alpha].
\end{aligned}
$$

Definition 3.3 (Arc and line)**.** The following definitions are made in RCA_0. A continuous function $\gamma : [0,1] \to \mathbb{C}$ is said to be an *arc.* A function $\gamma : [0,1] \to \mathbb{C}$ is a *line* if

$$\forall t \in [0,1] \gamma(t) = \gamma(0) + t(\gamma(1) - \gamma(0)).$$

A function $\gamma : [0,1] \to \mathbb{C}$ is an *arc of circle* if

$$\forall t \in [0,1] \gamma(t) = z + re^{iat}$$

for some $z \in \mathbb{C}$ and $r, a \in \mathbb{R}$.

Definition 3.4 (Line integral)**.** Let D be an open or closed subset of $\mathbb{C}$, and let f be a continuous function from D to $\mathbb{C}$. The following definitions are made in RCA_0.

1. Let γ be a continuous function from $[0,1]$ to D. Then, we define $\int_\gamma f(z)\,dz$, the *line integral* of f along γ, as

   $$\int_\gamma f(z)\,dz = \lim_{|\Delta| \to 0} S_\gamma^\Delta(f)$$

   if this limit exists. Here, Δ is a partition of $[0,1]$, *i.e.* $\Delta = \{0 = x_0 \leq \xi_1 \leq x_1 \leq \cdots \leq \xi_n \leq x_n = 1\}$, $S_\gamma^\Delta(f) = \sum_{k=1}^n f(\gamma(\xi_k))(\gamma(x_k) - \gamma(x_{k-1}))$ and $|\Delta| = \max\{x_k - x_{k-1} \mid 1 \leq k \leq n\}$.

2. If $[a,b] \subseteq D$, we define $\gamma(t) = a + (b-a)t$ and define $\int_{[a,b]} f(z)\,dz$ as

   $$\int_{[a,b]} f(z)\,dz = \int_\gamma f(z)\,dz.$$

3. If $\triangle abc \subseteq D$, we define $\int_{\partial\triangle abc} f(z)\,dz$ as

   $$\int_{\partial\triangle abc} f(z)\,dz = \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz.$$

Definition 3.5 (Effectively integrable)**.** The following definitions are made in RCA_0. Let f be a continuous function from $D \subseteq \mathbb{C}$ to $\mathbb{C}$, and let $\gamma : [0,1] \to D$ be an arc. A *modulus of integrability* along γ for f is a function h_γ from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all partitions Δ_1, Δ_2 of $[0,1] \subseteq \mathbb{R}$, if $|\Delta_1|, |\Delta_2| < 2^{-h_\gamma(n)}$ then $|S_\gamma^{\Delta_1}(f) - S_\gamma^{\Delta_2}(f)| < 2^{-n+1}$. Here, $S_\gamma^{\Delta}(f)$ is the Riemann sum of f for partition Δ of $[0,1]$ along γ. We say that f is *effectively integrable* on D when for every $\gamma : [0,1] \to D$ such that γ is a line or an arc of a circle, we can find a modulus of integrability along γ.

Theorem 3.1. *The following assertions are equivalent over* RCA_0.

1. WKL_0.
2. *Every continuous function on an open set $D \subseteq \mathbb{C}$ is effectively integrable.*

Next, we review basic theorems for holomorphic functions contained in [15].

Lemma 3.2. *The following is provable in RCA_0. Let D be an open subset of $\mathbb{C}$, and let f be a holomorphic function from D to $\mathbb{C}$. If f is effectively integrable on D, then, for all $a, b, c \in D$ such that $\triangle abc \subseteq D$,*

$$\int_{\partial \triangle abc} f(z)\, dz = 0.$$

Proof. We can imitate the usual proof of Cauchy's integral theorem within RCA_0. For details, see [15]. □

By this lemma, we can apply Cauchy's integral theorem to a holomorphic function 'locally', *i.e.*, we can find a neighborhood of each point in the domain where Cauchy's theorem holds. Thus, we can imitate the usual proof of Taylor's theorem locally within RCA_0. However, Cauchy's integral theorem as itself is not provable in RCA_0 (see Theorem 3.3).

Theorem 3.3 (Cauchy's integral theorem)**.** *The following assertions are equivalent over* RCA_0.

1. WKL_0.
2. *Cauchy's integral theorem for triangles: if f is a holomorphic function on an open subset $D \subseteq \mathbb{C}$, then for all $\triangle abc \subseteq D$, $\int_{\partial \triangle abc} f(z)\, dz$ exists and*

$$\int_{\partial \triangle abc} f(z)\, dz = 0.$$

Proof. $1 \to 2$ is straightforward from Theorem 3.1 and Lemma 3.2. For $2 \to 1$, see [15]. □

Definition 3.6 (path, semi-polygon)**.** The following definitions are made in RCA_0. A *path* is a finite sequence of functions $\gamma = \langle \gamma_1, \cdots, \gamma_n \rangle$ where $\gamma_i : [(i-1)/n, i/n] \to \mathbb{C}$ $(1 \leq i \leq n)$ is a line or an arc of a circle with $\gamma_i(i/n) = \gamma_{i+1}(i/n)$ for all $1 \leq i \leq n$. We write $\gamma(t) := \gamma_i(t)$ if $t \in [(i-1)/n, i/n]$. A *semi-polygon* is a path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma_1(0) = \gamma_n(1)$. A semi-polygon γ is said to be *simple* if $\gamma(t) \neq \gamma(s)$ for all $0 \leq t < s < 1$.

Lemma 3.4. *The following is provable in* RCA_0*. Let* γ *be a semi-polygon in* $\mathbb{C}$*. Thereby, there exist two open sets called exterior and interior of* γ *and a closed set called the image of* γ*.*

Proof. Let $\varphi(z)$ (or $\psi(z)$) be a Σ^0_1-formula which represents the following:

- $z \notin \{\gamma(t) \mid t \in [0,1]\}$;
- there exists a $0 < \theta < \pi/2, \theta \in \mathbb{Q}$ such that the half-line $l(z,\theta) = \{w \in \mathbb{C} \mid \arg(w-z) = \theta\}$ is not tangent to γ and the cardinality of $l(z,\theta) \cap \{\gamma(t) \mid t \in [0,1]\}$ is even (or odd).

Then, we can find open sets U_1, U_2 such that $z \in U_1 \leftrightarrow \varphi(z)$ and $z \in U_2 \leftrightarrow \psi(z)$. U_1 is said to be the *exterior* of γ and U_2 is said to be the *interior* of γ, denoted by $\mathrm{Int}(\gamma)$. The *image* of γ is a closed set $\mathbb{C} \setminus (U_1 \cup U_2)$, denoted by $\mathrm{Im}(\gamma)$. □

We next study some theorems within RCA_0 using effective integrability.

Lemma 3.5. *The following assertions are provable in* RCA_0*.*

1. *If* $f : \Delta(r) \to \mathbb{C}$ *is an effectively integrable holomorphic function, there exists a sequence* $\{\alpha_i\}_{i \in \mathbb{N}}$ *such that* $f(z) = \sum_{k \in \mathbb{N}} \alpha_k z^k$ *for all* $z \in \Delta(r)$*.*
2. *(maximum value principle) Let* f *be an effectively integrable holomorphic function on an open subset* $D \subseteq \mathbb{C}$*, and let* $\overline{B(a;r)} \subseteq D$*. Then,* $\sup\{|f(z)| \mid z \in \overline{B(a;r)}\} = \sup\{|f(z)| \mid |z-a| = r\}$*.*
3. *(Schwarz' lemma) Let* $f : \Delta(1) \to \Delta(1)$ *be an effectively integrable holomorphic function such that* $f(0) = 0$*. Then,* $|f(z)| \leq |z|$ *and* $|f'(0)| \leq 1$*. Moreover, if* $|f(z)| = |z|$ *for some* $z \in \Delta(1)$*, then there exists* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$ *and* $f(z) = \lambda z$ *for all* $z \in \Delta(1)$*.*

Proof. By Lemma 3.2, we can use Cauchy's integral formula in RCA_0 if f is an effectively integrable holomorphic function. Then, we can easily prove 1 and 2 by imitating the usual proofs (see e.g. [1]) within RCA_0. □

Using line integrability in WKL_0, we can restate the above theorems.

Theorem 3.6. *The following assertions are provable in* WKL_0*.*

1. *If* $f : \Delta(r) \to \mathbb{C}$ *is a holomorphic function, there exists a sequence* $\{\alpha_i\}_{i \in \mathbb{N}}$ *such that* $f(z) = \sum_{k \in \mathbb{N}} \alpha_k z^k$ *for all* $z \in \Delta(r)$*.*

2. *(maximum value principle) Let f be a holomorphic function on an open subset $D \subseteq \mathbb{C}$, and let $\overline{B(a;r)} \subseteq D$. Then,* $\sup\{|f(z)| \mid z \in \overline{B(a;r)}\} = \sup\{|f(z)| \mid |z-a| = r\}$.

3. *(Schwarz' lemma) Let $f : \Delta(1) \to \Delta(1)$ be a holomorphic function such that $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $f(z) = \lambda z$ if $|f(z)| = |z|$ for some $z \in \Delta(1)$.*

Proof. Straightforward from Lemma 3.5 and Theorem 3.1. □

3.2 Weak Riemann's mapping theorem

In this section, we prove a weak version of Riemann's mapping theorem within RCA_0. Before this, we introduce a notion of weak version of effectively uniform continuity. Put $\overline{\mathrm{Int}(\gamma)} = \mathrm{Int}(\gamma) \cup \mathrm{Im}(\gamma)$.

Definition 3.7 (effectively uniformly continuous funcition)**.** The following definitions are made in RCA_0. Let f be a continuous function from $D \subseteq \mathbb{C}$ to $\mathbb{C}$, and let $D_0 \subseteq D$. A *modulus of uniform continuity* on D_0 for f is a function h_{D_0} from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $|f(z) - f(w)| < 2^{-n+1}$ if $|z - w| < 2^{-h_{D_0}(n)}$ for all $z, w \in D_0$. We say that f is *semi-effectively uniformly continuous* on D if for every semi-polygon $\gamma : [0,1] \to D$ such that $\overline{\mathrm{Int}(\gamma)} \subseteq D$, we can find a modulus of uniform continuity on $\overline{\mathrm{Int}(\gamma)}$ for f.

Note that we can easily prove within RCA_0 that f is effectively integrable on D if f is semi-effectively uniformly continuous.

In the proof of weak Riemann's mapping theorem, we construct an intended biholomorphic function approximately. In the construction, we focus attention on the differential coefficient of each function at 0 and we use the value as an indicator for the construction. Next lemma gives an upper bound for the differential coefficient.

Lemma 3.7. *The following is provable in RCA_0. Let $g : D \to D' \subseteq \Delta(1)$ be a semi-effectively uniformly continuous holomorphic function such that $g(0) = 0$. Let $D \supseteq \Delta(r)$. Then, $|g'(0)| \leq 1/r$.*

Proof. Easy rescaling of Schwarz' lemma (Lemma 3.5.3). □

Next, we define linear transformation.

Definition 3.8 (linear transformation)**.** The following definition is made in RCA_0. A *linear transformation* is a biholomorphic function on $\mathbb{C}$ which can be represented as

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Note that if $a = d = 1$ and $b = \bar{c} \in \Delta(1)$, then f is a biholomorphic function from $\Delta(1)$ to $\Delta(1)$.

The next lemma is an RCA_0 version of the square root principle.

Lemma 3.8. *The following is provable in* RCA_0*. Let* $D \subseteq \mathbb{C}$ *be a simply connected open subset, and let* f *be a semi-effectively uniformly continuous function on* D *such that* f *is holomorphic on* D*. If* $f(z) \neq 0$ *for all* $z \in D$*, then there exists a continuous function* g *such that* $g^2(z) = f(z)$*,* g *is holomorphic on* D *and* g *is semi-effectively uniformly continuous on* D*.*

Proof. Easy modification of the proof of the square root principle within WKL_0 [16, Lemma 4.16]. □

Let $0 \in D \subseteq \Delta(1)$ be a simply connected open set. By Lemma 3.8, for given $\alpha \in \Delta(1) \setminus D$, we can define $\eta^0_\alpha : D \to \Delta(1)$ and $\eta^1_\alpha : \Delta(1) \to \Delta(1)$ as follows:

$$\eta^0_\alpha(z) = \sqrt{(z-\alpha)/(1-\bar{\alpha}z)};$$
$$\eta^1_\alpha(z) = (z-\beta)/(1-\bar{\beta}z), \text{where}\, \beta = \eta^0_\alpha(0) = \sqrt{-\alpha}.$$

The next lemma is another technical lemma for the weak Riemann's mapping theorem. The lemma shows that the differential coefficient of the biholomorphic function f at 0 increase if one composes the functions, η^0_α and η^1_α, to f. It plays a key role for our constructions of biholomorphic functions.

Lemma 3.9. *The following is provable in* RCA_0*. Let* D *be a simply connected open subset of* $\mathbb{C}$ *and* $g : D \to D' \subsetneq \Delta(1)$ *be a semi-effectively uniformly continuous biholomorphic function such that* $g(0) = 0$*, and let* $\alpha \in \Delta(1) \setminus D'$*. Define a biholomorphic function* $h : D \to h(D) \subseteq \Delta(1)$ *as*

$$h(z) := \eta^1_\alpha(\eta^0_\alpha(g(z))).$$

Then, $h(0) = 0$ *and* $|h'(0)| > (1 + d^2/2)|g'(0)|$ *where* $d = 1 - |\beta| = 1 - \sqrt{|\alpha|}$*.*

Proof. We reason within RCA_0. By the definition of h,

$$h'(0) = \frac{1}{1-|\beta|^2}\frac{1-|\alpha|^2}{2\beta}g'(0).$$

Then,

$$|h'(0)| = \frac{|\beta|^2+1}{2|\beta|}|g'(0)| = \frac{d^2-2d+2}{2-2d}|g'(0)| = \left(1 + \frac{d^2}{2}\frac{1}{1-d}\right)|g'(0)| > \left(1+\frac{d^2}{2}\right)|g'(0)|.$$

It is clear that $h(D) \subseteq \Delta(1)$, h is biholomorphic and $h(0) = 0$. This completes the proof. □

Now, we show the following weak version of Riemann's mapping theorem for semi-polygons.

Theorem 3.10 (weak Riemann's mapping theorem)**.** *The following is provable in* RCA_0*. Let* $\gamma = \langle \gamma_1, \cdots, \gamma_l \rangle$ *be a simple semi-polygon on* $\mathbb{C}$*. Let* D *be the interior of* γ *and choose* $c_0 \in D$*. Then,* D *is conformally equivalent to the unit open ball* $\Delta(1)$*, i.e., there exists a biholomorphic uniformly continuous function* $f : D \to \Delta(1)$ *such that* $f(c_0) = 0$*.*

Proof. By applying a linear transformation, we may assume that $D \subseteq \Delta(1)$ and $c_0 = 0$. This theorem is proved by the following two steps:
(Step.1) Let $r_k := 1 - 2^{-2k}$ for all $k \in \mathbb{N}$ and let $D_0 := D$. Then, we construct a sequence $\langle \tilde{f}_i \mid i \in \mathbb{N} \rangle$ of functions and a sequence $\langle D_i \mid i \in \mathbb{N} \rangle$ of open sets which satisfy the following condition:

- $\tilde{f}_i : D_i \to \tilde{f}_i(D_i)$ is semi-effectively uniformly continuous biholomorphic, and $D_{i+1} := \tilde{f}_i(D_i) \supseteq \overline{\Delta(r_{i+1})}$.

(Step.2) Define a sequence $\langle f_i : D_0 \to D_i \mid i \in \mathbb{N} \rangle$ as $f_i := \tilde{f}_{i-1} \circ \cdots \circ \tilde{f}_0$. Then we prove that $\langle f_i \mid i \in \mathbb{N} \rangle$ is a uniformly convergent sequence on any compact subsets of $D_0 (= D)$. Thus $f := \lim_{i \to \infty} f_i$ exists and is the required biholomorphic function from D to $\Delta(1)$.

In what follows, we prove these steps.

(Step.1)
For each $\alpha \in \Delta(1)$, define ψ_α as follows:

$$\begin{aligned}
&\eta^0_\alpha(z) = \sqrt{(z - \alpha)/(1 - \bar{\alpha} z)};\\
&\eta^1_\alpha(z) = (z - \beta)/(1 - \bar{\beta} z) \text{ where } \beta := \sqrt{-\alpha};\\
&\psi_\alpha(z) = \eta^1_\alpha \circ \eta^0_\alpha(z).
\end{aligned}$$

We will construct sequences $\langle m_i \in \mathbb{N} \mid i \in \mathbb{N} \rangle$ and $\langle \langle \alpha_{ij} \in \mathbb{Q}^2 \cap \Delta(1) \mid j < m_i \rangle \mid i \in \mathbb{N} \rangle$ accompanying with $\langle D_i \mid i \in \mathbb{N} \rangle$ and $\langle f_{ij} \mid j < m_i, i \in \mathbb{N} \rangle$. Idea of the construction is (here, $\varepsilon_i = 2^{-4i}$):

For given $\langle m_i \mid i < I \rangle$ and $\langle \langle \alpha_{ij} \in \Delta(1) \cap \mathbb{Q}^2 \mid j < m_i \rangle \mid i < I \rangle^\smile \langle \langle \alpha_{Ij} \mid j < J \rangle \rangle$, we define sequences $\langle D_i \mid i \leq I \rangle$, $\langle \tilde{f}_i : D_i \to \Delta(1) \mid i < I \rangle$, $\langle \langle f_{ij} : D_i \to \Delta(1) \mid j \leq m_i \rangle \mid i < I \rangle$ and $\langle f_{Ij} : D_I \to \Delta(1) \mid j \leq J \rangle$ as follows: for each $i < I$,

$$\begin{aligned}
f_{00} &:= id_{D_0}\\
f_{ij} &:= \psi_{\alpha_{i,j-1}} \circ \cdots \circ \psi_{\alpha_{i0}} \circ f_{i0} \text{ if } m_i > 0 \wedge 0 < j \leq m_i\\
\tilde{f}_i &:= f_{i,m_i} |f'_{i,m_i}(0)| / f'_{i,m_i}(0)\\
D_{i+1} &:= \tilde{f}_i(D_i)\\
f_{i+1,0} &:= id_{D_{i+1}}\\
f_{Ij} &:= \psi_{\alpha_{I,j-1}} \circ \cdots \circ \psi_{\alpha_{I0}} \circ f_{I0} \text{ if } J > 0 \wedge 0 < j \leq J\\
F_{ij} &:= f_{ij} \circ \tilde{f}_{i-1} \circ \cdots \circ \tilde{f}_0 : D_0 \to \Delta(1) \text{ if } 0 \leq i < I \wedge j \leq m_i\\
F_{Ij} &:= f_{Ij} \circ \tilde{f}_{I-1} \circ \cdots \circ \tilde{f}_0 : D_0 \to \Delta(1) \text{ if } j \leq J.
\end{aligned}$$

In general, if a function f has a modulus of uniform continuity, then the composition of f and a square root, and the composition of f and a linear transformation

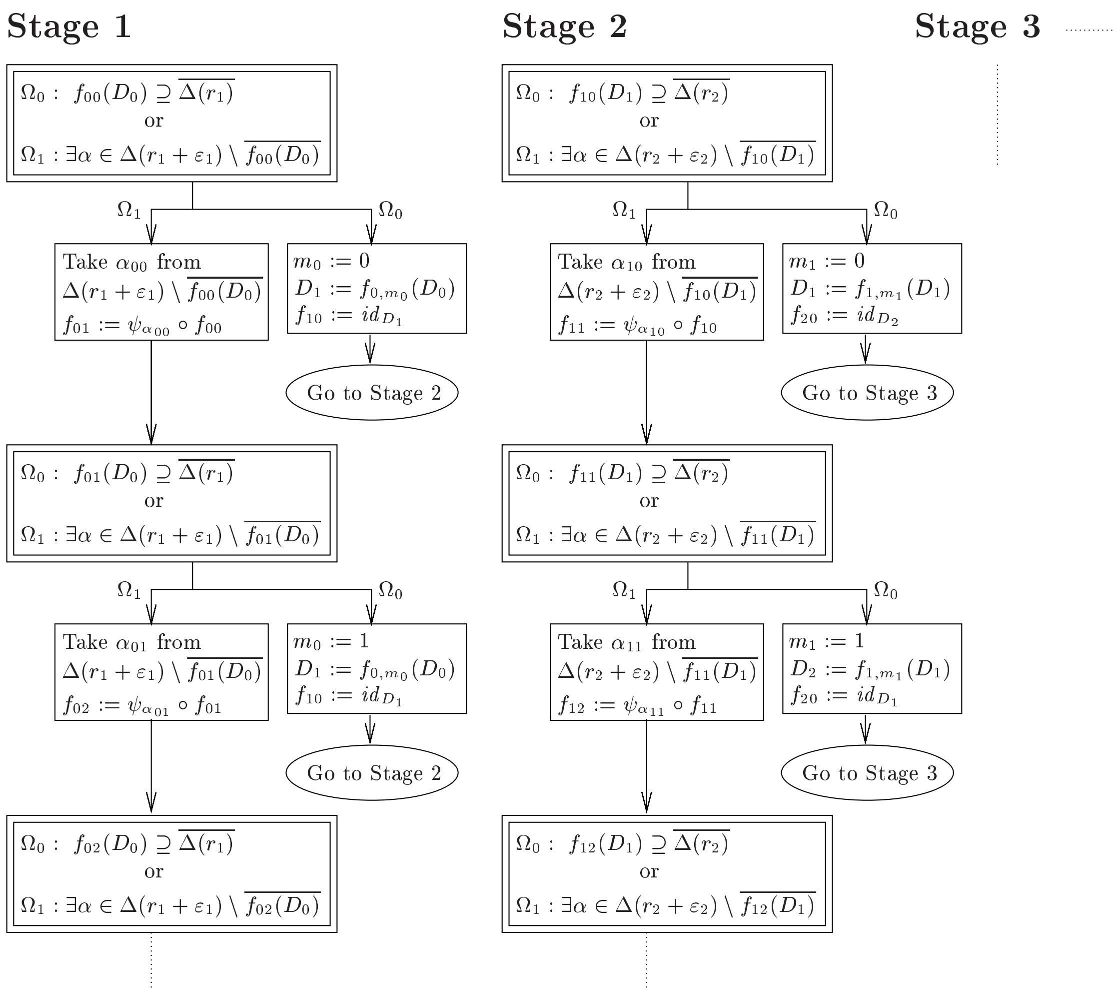

Figure 1: Skech of the construction

also have a modulus of uniform continuity. In fact, we can find the modulus of uniform continuity for the compositions by modifying the modulus of uniform continuity for f. Thus we can easily check that f_{ij} and $\tilde{f}_i$ are semi-effectively uniformly continuous. Note that $\tilde{f}_i'(0) \in \mathbb{R}$.

We require that these sequences; $\langle D_i \mid i \leq I \rangle$, $\langle \tilde{f}_i \mid i < I \rangle$, $\langle \langle f_{ij} \mid j \leq m_i \rangle \mid i < I \rangle$, $\langle f_{Ij} \mid j \leq J \rangle$ and $\langle \langle F_{ij} \mid j \leq m_i \rangle \mid 0 < i < I \rangle^\frown \langle \langle F_{Ij} \mid j \leq J \rangle \rangle$ satisfy the following conditions:

- $\forall i < I \, \Omega_0(F_{i,m_i}, r_{i+1})$;
- $\forall i < I \, \forall j < m_i \, \Omega_1(F_{ij}, r_{i+1}, \alpha_{ij})$;
- If $J > 0$ then $\forall j < J \, \Omega_1(F_{Ij}, r_{I+1}, \alpha_{Ij})$.

Here $\Omega_0(F, r)$ and $\Omega_1(F, r, \alpha)$ are the following formulas:

$$\Omega_0(F, r) \equiv r > 0 \wedge F(D_0) \supseteq \overline{\Delta(r)},$$
$$\Omega_1(F, r, \alpha) \equiv r > 0 \wedge \alpha \in \mathbb{Q}^2 \cap (\Delta(1) \setminus \overline{F(D_0)}) \wedge |\alpha| < r + (1-r)^2.$$

For these formulas, we prove the following assertions:

(A) For each $F = F_{ij}$ and $r = r_k$ and $\alpha \in \mathbb{Q}^2$, Ω_0 and Ω_1 are Σ^0_1-formula.

(B) For each $i \geq 0$ and j, $\Omega_0(F_{ij}, r_{i+1})$ holds or we can effectively choose $\alpha \in \mathbb{Q}^2$ which satisfies $\Omega_1(F_{ij}, r_{i+1}, \alpha)$.

Proof of (A).

First, we prove that $\Omega_0 \in \Sigma^0_1$. By the construction of F, we can easily expand the domain of F as $\mathrm{dom}(F) \supseteq D_0 \cup \mathrm{Im}(\gamma)$ $(= \overline{D_0})$. By the formula $\Psi(k, M)$, we represent

$$(F \circ \gamma)([0,1]) \subseteq \bigcup_{L \in \mathbb{N}, L < 2^M} B((F \circ \gamma)(L/2^M); 2^{-k+1}).$$

Then, since $F : D_0 \to F(D_0)$ is semi-effectively uniformly continuous on D_0,

$$\forall k \, \exists M \, \Psi(k, M).$$

Since $\Psi(k, M)$ is Σ^0_1, $\Psi(k, M) \equiv \exists q \, \Psi_0(k, M, q)$ for some Σ^0_0-formula Ψ_0. Thus,

$$\forall k \, \exists p \, \Psi_0(k, (p)_0, (p)_1).$$

Thus we can define a sequence $\langle M_k \in \mathbb{N} \mid k \in \mathbb{N} \rangle$ as

$$M_k := (\mu p \, \Psi_0(k, (p)_0, (p)_1))_0.$$

Then the following relation holds:

$$\Omega_0(F, r) \leftrightarrow r > 0 \wedge \exists k \, \forall l < 2^{M_k} \, |(F \circ \varphi \circ \gamma)(l/2^{M_k})| > 2^{-k+1} + r.$$

Since the right side of this relation is expressed by Σ_1^0-formula, $\Omega_0(F, r)$ is a Σ_1^0-formula.

Next, we prove that $\Omega_1 \in \Sigma_1^0$. Since $F : D_0 \to F(D_0)$ is biholomorphic on D_0, F is a closed mapping. Hence $F(\overline{D_0})$ is closed set. Thus $\overline{F(D_0)} = F(\overline{D_0})$ holds and $\Delta(1) \setminus F(\overline{D_0})$ is an open set. Therefore $\Omega_1(F, r, \alpha)$ is a Σ_1^0-formula.

Note that the formulas Ω_0 and Ω_1 represent the following:

$$\Omega_0(F_{ij}, r_{i+1}) \leftrightarrow f_{ij}(D_i) \supseteq \overline{\Delta(r_{i+1})},$$
$$\Omega_1(F_{ij}, r_{i+1}, \alpha) \leftrightarrow \alpha \in \mathbb{Q}^2 \cap (\Delta(1) \setminus \overline{f_{ij}(D_i)}) \wedge |\alpha| < r_{i+1} + 2^{-4i-4}.$$

Proof of (B).

We can prove that for each i and j, either $\Omega_0(F_{ij}, r_{i+1})$ or $\exists\alpha\, \Omega_1(F_{ij}, r_{i+1}, \alpha)$. By (A), write $\Omega_0(F_{ij}, r_{i+1}) \equiv \exists p\, \Theta_0(F_{ij}, r_{i+1}, p)$ and $\Omega_1(F_{ij}, r_{i+1}, \alpha) \equiv \exists q\, \Theta_1(F_{ij}, r_{i+1}, \alpha, q)$ where Θ_0 and Θ_1 are Σ_0^0-formulas. Then we can effectively choose $p \in \mathbb{N}$ such that $\Theta_0(F_{ij}, r_{i+1}, p)$ holds or effectively choose $(q, \alpha) \in \mathbb{N} \times \mathbb{Q}^2$ such that $\Theta_1(F_{ij}, r_{i+1}, \alpha, q)$ holds. Therefore, (B) holds.

Then we continue the construction of the sequences as follows: Checking $\Omega_0(F_{IJ}, r_{I+1}) \vee \exists\alpha \in \Delta(1) \cap \mathbb{Q}^2\ \Omega_1(F_{IJ}, r_{I+1}, \alpha)$, then, by (B) one of the following happen.

- If $\Omega_0(F_{IJ}, r_{I+1})$ holds, then let $m_I := J$.
- If we find $\alpha \in \Delta(1) \cap \mathbb{Q}^2$ such that $\Omega_1(F_{IJ}, r_{I+1}, \alpha)$, then let $\alpha_{I,J} := \alpha$ and $f_{I,J+1} := \psi_{\alpha_{I,J}} \circ f_{IJ}$.

To complete this construction, we have to prove that for each $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ which satisfies $\neg\exists\alpha\, \Omega_1(F_{ij}, r_{i+1}, \alpha)$. Let $i \in \mathbb{N}$ and assume that f_{ik} is defined for some $k \in \mathbb{N}$. By the definition of D_i, $D_i \supseteq \overline{\Delta(r_{i+1})}$ holds. This fact implies $|f'_{ik}(0)| \le 1/r_{i+1}$ by Lemma 3.7. By Lemma 3.9, $|f'_{ik}(0)| > (1 + 2^{-i})^k$ holds. Hence there exists $k_0 \in \mathbb{N}$ such that $1/r_{i+1} < (1 + 2^{-i})^{k_0}$ holds. Thus there exists $j \le k_0$ such that $\neg\exists\alpha\, \Omega_1(F_{ij}, r_{i+1}, \alpha)$ holds. Therefore $\forall i\, \exists j\, \neg\exists\alpha\, \Omega_1(F_{ij}, r_{i+1}, \alpha)$ holds. Hence this construction is well-defined.
(Step.2)
In this step, we construct a semi-effectively uniformly continuous biholomorphic function $f : D \to \Delta(1)$ by using the sequence $\langle \tilde{f}_i \mid i \in \mathbb{N} \rangle$, which is constructed in the Step 1.

Let $f_i = \tilde{f}_{i-1} \circ \cdots \circ \tilde{f}_0$. Then, each f_i is a biholomorphic function from D_0 to $D_i (\supseteq \overline{\Delta(r_{i+1})})$.

We will prove that

$$(\dagger)\quad \forall N\, \forall n \ge N\, \forall m > n\, |f_m(z) - f_n(z)| \le 2^{-n+4} \text{ on } f_N^{-1}(\Delta(r_N^3 \delta_N)) \subseteq D_0,$$

where $\delta_N := 1 - 2^{-N}$. This statement implies that the sequence $\langle f_i \mid i \in \mathbb{N} \rangle$ uniformly converges on compact subsets of D_0. To prove $(\dagger)$, it suffices to show that

(‡) $\forall N\, \forall n \geq N\, \forall m > n\, |h_{m,n}(z) - z| \leq 2^{-n+4}$ on $\Delta(r_n r_m \delta_N) \subseteq D_n$,

where $h_{m,n} := \tilde{f}_{m-1} \circ \cdots \circ \tilde{f}_n : D_n \to D_m$ is a biholomorphism.

Proof of (‡)→(†).

> Choose $z \in f_N^{-1}(\Delta(r_N^3 \delta_N))$. It suffices to prove that $f_n(z) \in \Delta(r_n r_m \delta_N)$. In fact, if this statement holds, then $|h_{m,n}(f_n(z)) - f_n(z)| \leq 2^{-n+4}$ holds by (‡). Thus $|f_m(z) - f_n(z)| \leq 2^{-n+4}$ holds, which implies (†).
>
> We now prove that $f_n(z) \in \Delta(r_n r_m \delta_N)$ holds. By Schwarz' lemma (Lemma 3.5.3) for $h_{n,N}$, $|f_n(z)| = |h_{n,N}(f_N(z))| \leq |f_N(z)| r_N^{-1}$. Since $z \in f_N^{-1}(\Delta(r_N^3 \delta_N))$ implies $f_N(z) \in \Delta(r_N^3 \delta_N)$, $|f_N(z)| r_N^{-1} \leq r_N^2 \delta_N \leq r_n r_m \delta_N$. Thus $f_n(z) \in \Delta(r_n r_m \delta_N)$ holds and this completes the proof of (‡) → (†).

Proof of (‡).

> Let N, n and m be such that $N \in \mathbb{N}, n \geq N$ and $m > n$. We first analyze behavior of the function $h_{m,n}/z$. Note that f_i and $h_{m,n}$ are semi-effectively uniformly continuous and, by the definition of $\tilde{f}_i$, each $f_i'(0)$ is a real number, and hence $h_{m,n}'(0) \in \mathbb{R}$. By Schwarz' lemma for $h_{m,n}$, $|h_{m,n}(z)/z| \leq r_n^{-1}$ on $\Delta(r_n)$. Hence $h_{m,n}(z) \in \Delta(r_m)$ if $z \in \Delta(r_n r_m)$. Again by Schwarz' lemma for $h_{m,n}^{-1} : D_m (\supseteq \overline{\Delta(r_m)}) \to D_n$, $|h_{m,n}^{-1}(w)/w| \leq r_m^{-1}$ on $\Delta(r_m)$. Thus, $r_m \leq |h_{m,n}(z)/z| \leq r_n^{-1}$ for all $z \in \Delta(r_n r_m)$.
>
> Define a function g from $\Delta(r_n r_m)$ to $\{z \in \Delta(1) \mid r_m \leq |z| \leq r_n^{-1}\}$ as $g(z) := h_{m,n}(z)/z$. Define a function $\hat{g}$ from $\Delta(1)$ to $\Delta(1) \setminus \Delta(r_n r_m)$ as $\hat{g}(\tilde{z}) := g(r_n r_m \tilde{z}) r_n$. Then $\hat{g}$ is a semi-effectively uniformly continuous biholomorphic function from $\Delta(1)$ to $\hat{g}(\Delta(1)) (\subseteq \Delta(1) \setminus \Delta(r_n r_m))$. Since $h_{m,n}$ is holomorphic on D_n ($\supseteq \Delta(r_n)$), by Lemma 3.5.1, $h_{m,n}$ has a Taylor expansion: $h_{m,n} = \sum_{j=0}^{\infty} a_j z^j$ on $\Delta(r_n)$ where $a_j \in \mathbb{C}$ for each $j \in \mathbb{N}$. Since $h_{m,n}(0) = 0$, $a_0 = 0$ and hence $g(z) = h_{m,n}(z)/z = \sum_{j=1}^{\infty} a_j z^{j-1}$. Since $h_{m,n}'(0) = a_1 \in \mathbb{R}$, $g(0) = a_1 \in \mathbb{R}$ and also $\hat{g}(0) = g(0) r_n \in \mathbb{R}$. Put $\psi(\tilde{z}) := (\hat{g}(\tilde{z}) - \hat{g}(0))/(1 - \hat{g}(0)\hat{g}(\tilde{z}))$. Then ψ is a semi-effectively uniformly continuous biholomorphic function from $\Delta(1)$ to $\psi(\Delta(1))$ with $\psi(0) = 0$. Then by Schwarz' lemma for ψ, $|\psi(\tilde{z})/\tilde{z}| \leq 1$ holds.
>
> Put $G(\tilde{z}) := \hat{g}(\tilde{z}) - \hat{g}(0)$. Then, $|G(\tilde{z})| = |\psi(\tilde{z})| \cdot |1 - \hat{g}(0)\hat{g}(\tilde{z})| \leq |\tilde{z}| \cdot |1 - \hat{g}(0)\hat{g}(\tilde{z})| = |\tilde{z}| \cdot |1 - \hat{g}(0)(\hat{g}(0) + G(\tilde{z}))| \leq |\tilde{z}| \cdot |1 - \hat{g}(0)^2 - |G(\tilde{z})|| \leq 1 - \hat{g}(0)^2 + |\tilde{z}| \cdot |G(\tilde{z})|$. Hence $|1 - |\tilde{z}|| \cdot |G(\tilde{z})| \leq 1 - \hat{g}(0)^2$. If $\tilde{z} \in \partial\Delta(\delta_N)$, $|G(\tilde{z})| \leq (1 - \hat{g}(0)^2) 2^n$ holds. Thus by the maximum value principle (Lemma 3.5.2), $|G(\tilde{z})| \leq (1 - \hat{g}(0)^2) 2^n$ for all $\tilde{z} \in \Delta(\delta_N)$.
>
> In order to prove $|h_{m,n}(z) - z| \leq 2^{-n+4}$ on $\Delta(r_n r_m \delta_N)$, fix $z \in \Delta(r_n r_m \delta_N)$. Then, $z = \tilde{z} r_n r_m$ for some $\tilde{z} \in \Delta(\delta_N)$. Then, $|G(r_n^{-1} r_m^{-1} z)| \leq 1 - |\hat{g}(0)|^2 2^n$. Since $\hat{g}(0) = g(0) r_n$, $|\hat{g}(0)| \geq r_n r_m$. Thus
>
> $$G(r_n^{-1} r_m^{-1} z) \leq (1 - r_n^2 r_m^2) 2^n. \tag{1}$$

Since $r_n^2 = (1-2^{-2n})^2 \geq 1-2^{-2n+1}$, $r_n^2 r_m^2 \geq (1-2^{-2n+1})(1-2^{-2m+1}) \geq 1-2^{-2n+2}$. By (1), $|G(r_n^{-1}r_m^{-1}z)| \leq 2^{-n+2}$ holds. Since $G(r_n^{-1}r_m^{-1}z) = \hat{g}(r_n^{-1}r_m^{-1}z) - \hat{g}(0) = r_n(g(z)-g(0))$,

$$|g(z)-g(0)| \leq 2^{-n+2}r_n^{-1} \leq 2^{-n+3}. \tag{2}$$

Since $g(0) = h'(0) \in \mathbb{R}$, $r_m - 1 \leq g(0)-1 \leq r_n^{-1}-1$ holds. Therefore, $-2^{-2m} \leq g(0)-1 \leq 2^{-2n}/(1-2^{-2n})$. By $-2^{-n+1} \leq -2^{-2m}$ and $2^{-2n}/(1-2^{-2n}) \leq 2^{-2n}/(2^{-n-1}) = 2^{-n+1}$, $|g(0)-1| \leq 2^{-n+1}$. Then, by (2), $|g(z)-1| \leq |g(z)-g(0)|+|g(0)-1| \leq 2^{-n+3}+2^{-n+1} \leq 2^{-n+4}$. Hence $|h_{m,n}(z)/z-1| \leq 2^{-n+4}$ *i.e.* $|h_{m,n}(z)-z| \leq 2^{-n+4}|z| \leq 2^{-n+4}$ on $z \in \Delta(r_n r_m \delta_N)$. This completes the proof of (‡).

Therefore, by (†), we have proved that $\langle f_i \mid i \in \mathbb{N}\rangle$ uniformly converges on compact subsets of $D_0(=D)$. Hence, we have a semi-effectively uniformly continuous biholomorphic function $f = \lim_{i\to\infty} f_i$ from D to $\Delta(1)$.

This completes the proof of this theorem. □

Remark 3.11. Moreover, we conjecture that we can prove in $\mathsf{RCA_0}$ that the f, which is in the above theorem, can be expanded into a homeomorphism $\bar{f} : \overline{D} \to \overline{\Delta(1)}$ and $\bar{f}$ has a modulus of uniform continuity on $\overline{D}$. But this remains an open question.

4 Nonstandard proofs for stronger versions of Riemann's mapping theorem

In this section, we will show that Riemann's mapping theorem for a Jordan region is equivalent to $\mathsf{WKL_0}$ by using nonstandard arithmetic introduced in Section 2.

We first consider the standard part of a holomorphic function. We will argue within nonstandard second-order arithmetic. Before this, we prepare the following lemma, whose proof is available in [3].

Lemma 4.1 (Cauchy's estimate)**.** *The following is provable in* $\mathsf{WWKL_0}$*. Let f be a holomorphic function on an open subset $D \subseteq \mathbb{C}$. Then, for all $\alpha \in D, r > 0$ and $M > 0$, if $\overline{B(\alpha;r)} \subseteq D$ and $|f| \leq M$ on $B(\alpha;r)$, then for all $k \in \mathbb{N}$,*

$$|f^{(k)}(\alpha)| \leq \frac{Mk!}{r^k}.$$

The next lemma shows that the standard part of an s-bounded nonstandard holomorphic function is a holomorphic function.

Lemma 4.2. *The following is provable within* $\mathsf{ns\text{-}WKL_0}$*. Let U_0^* and U_1^* are* s*-bounded open sets in $\mathbb{C}^*$, i.e., there exist $M, K \in \mathbb{N}^{\mathrm{s}}$ such that $U_0^* \subseteq B(0;M)$ and $U_1^* \subseteq B(0;K)$. Let $F^* : U_0^* \to U_1^*$ be an effectively integrable holomorphic function, and let $W^{\mathrm{s}} = U_0^* \restriction V^{\mathrm{s}}$ be a non-empty open set in V^{s}. Then, the standard part* $\mathrm{st}(F^*)$ *of F^* is a holomorphic function on W^{s}.*

Proof. Let U_0^* and U_1^* are s-bounded open sets in $\mathbb{C}^*$, and $F^* : U_0^* \to U_1^*$ be a holomorphic function in V^*, and let $W^{\mathrm{s}} = U_0^* \restriction V^{\mathrm{s}}$ be non-empty. We will show that either of F^* and $F^{*\prime}$ has a standard part on W^{s}. Let $(a, r) \in \mathbb{N}^{\mathrm{s}} \times \mathbb{Q}^{\mathrm{s}}$ such that $\overline{B(a;r)} \subseteq W^{\mathrm{s}}$. Since $|F^*| \leq K$ on $B(a; r/2)$, by Lemma 4.1, $F^{*(i)}(z^*) \leq K \cdot k! \cdot (2/r)^k$ for all $z^* \in B(a; r/2)$ and for all $k \in \mathbb{N}^*$. Hence F^* is s-bounded on $B(a;r)$, and F^* is s-continuous on $B(a;r)$ since $F^{*\prime}$ is s-bounded. Therefore, by Theorem 2.23, there exists a standard part of F^* on W^{s}, namely $f^{\mathrm{s}} = \mathrm{st}(F^*)$. Similarly, there exists a standard part of $F^{*\prime}$ on W^{s}, and we can easily check that $f^{\mathrm{s}\prime} = \mathrm{st}(F^{*\prime})$. This means that f^{s} is holomorphic on W^{s}, and this completes the proof. $\square$

A continuous function $J : [0,1] \to \mathbb{C}$ is said to be a *Jordan curve* if $J(x) = J(y) \leftrightarrow |x - y| \in \{0, 1\}$. Recall that a Jordan curve is said to be a polygon if it is a piecewise linear function. Given a Jordan curve J, we define the interior $\mathrm{Int}(J)$, the exterior $\mathrm{Ext}(J)$ and the image $\mathrm{Im}(J)$ of J as follows.

$$
\begin{aligned}
\mathrm{Im}(J) &:= \{z \in \mathbb{C} \mid \exists t \in [0,1]\ z = J(t)\},\\
\mathrm{Int}(J) &:= \{z \in \mathbb{C} \setminus \mathrm{Im}(J) \mid \text{every continuous function } h : [0, \infty) \to \mathbb{C} \setminus \mathrm{Im}(J) \text{ such that}\\
&\qquad h(0) = z \text{ is bounded}\},\\
\mathrm{Ext}(J) &:= \{z \in \mathbb{C} \setminus \mathrm{Im}(J) \mid \text{there exists a continuous function } h : [0, \infty) \to \mathbb{C} \setminus \mathrm{Im}(J)\\
&\qquad \text{such that } h(0) = z \text{ and } \lim_{t\to\infty} \|h(t)\| = \infty\}.
\end{aligned}
$$

A *Jordan region* is the interior $\mathrm{Int}(J)$ of a Jordan curve. Note that these definitions coincide with those for polygons in Lemma 3.4.

Lemma 4.3. *The following is provable within* $\mathsf{ns\text{-}WKL_0}$*. Let* $J^{\mathrm{s}} : [0,1]^{\mathrm{s}} \to \mathbb{C}^{\mathrm{s}}$ *be a Jordan curve. Then, there exists a nonstandard polygon* $P^* : [0,1]^* \to \mathbb{C}^*$ *and (a code for) an open set* $U^* \subseteq \mathbb{C}^*$ *which satisfy the following:*

- $\mathrm{st}(P^*) = J^{\mathrm{s}}$,
- $U^* = \mathrm{Int}(P^*)$,
- $\mathrm{Int}(J^{\mathrm{s}}) = \mathrm{st}^i(\mathrm{Int}(P^*) \cap \mathbb{Q}^2[\omega]) = U^* \restriction V^{\mathrm{s}}$ *for some* $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.

Moreover, we can prove that $\mathrm{Int}(J^{\mathrm{s}}) \neq \emptyset$.

Proof. We will argue within $\mathsf{ns\text{-}WKL_0}$. Let $J^{\mathrm{s}} : [0,1]^{\mathrm{s}} \to \mathbb{C}^{\mathrm{s}}$ be a Jordan curve in V^{s}. By Theorem 2.23, there exists a (hyperfinite) broken line $P^* : [0,1]^* \to \mathbb{C}^*$ such that $\mathrm{st}(P^*) = J^{\mathrm{s}}$ and $P^*(0) = P^*(1)$. By $\mathrm{I}\Sigma^0_1$ in V^*, we can easily reduce P^* into an injective broken line (polygon) whose standard part is J^{s}. Define an open set U^* in V^* as $(n, a, r) \in U^* \leftrightarrow a \in \mathrm{Int}(P^*) \wedge d(a, P^*) > r$. Then, we can easily check that $U^* = \mathrm{Int}(P^*)$ and $U^* \restriction V^{\mathrm{s}} = \mathrm{Int}(J^{\mathrm{s}})$. We can also show that $\mathrm{Int}(J^{\mathrm{s}}) \neq \emptyset$ by modifying usual nonstandard argument for the Jordan curve theorem (see [6]). $\square$

Theorem 4.4. *The following assertions are equivalent over* $\mathsf{RCA_0}$.

1. $\mathsf{WKL_0}$.

2. *For any Jordan curve J,* $\mathrm{Int}(J)$ *is a non-empty bounded connected open set.*

3. *Every Jordan region is conformally equivalent to $B(0;1)$.*

Proof. For the proof of $2 \to 1$, see [6]. $3 \to 2$ is trivial. We show $1 \to 3$. We will argue within $\mathsf{ns\text{-}WKL_0}$. Let $J^{\mathrm{s}} : [0,1] \to \mathbb{C}^{\mathrm{s}}$ be a Jordan curve in V^{s}. By Lemma 4.3, there exist a (hyperfinite) polygon $P^* : [0,1]^* \to \mathbb{C}^*$ and $U^* = \mathrm{Int}(P^*)$ such that $\mathrm{st}(P^*) = J^{\mathrm{s}}$, $U^* = \mathrm{Int}(P^*)$ and $U^* \restriction V^{\mathrm{s}} = \mathrm{Int}(J^{\mathrm{s}})$. Then, applying Riemann's mapping theorem for polygonal region (Theorem 3.10) for $\mathrm{Int}(P^*)$ in V^*, we can obtain holomorphic functions $H_0^* : U^* \to B(0;1)$ and $H_1^* : B(0;1) \to U^*$ in V^* such that $H_0^* \circ H_1^* = \mathrm{id}_{B(0;1)}$ and $H_1^* \circ H_0^* = \mathrm{id}_{U^*}$. By Lemma 4.2, both of $h_0^{\mathrm{s}} = \mathrm{st}(H_0^*) : \mathrm{Int}(J^{\mathrm{s}}) \to B(0;1)$ and $h_1^{\mathrm{s}} = \mathrm{st}(H_1^*) : B(0;1) \to \mathrm{Int}(J^{\mathrm{s}})$ are holomorphic functions in V^{s}. We can easily check that $h_0^{\mathrm{s}} \circ h_1^{\mathrm{s}} = \mathrm{id}_{B(0;1)}$ and $h_1^{\mathrm{s}} \circ h_0^{\mathrm{s}} = \mathrm{id}_{\mathrm{Int}(J^{\mathrm{s}})}$. This means that $\mathrm{Int}(J^{\mathrm{s}})$ is conformally equivalent to the open unit disk. By the conservation theorem (Theorem 2.4), this is provable within $\mathsf{WKL_0}$. □

Next, we prove the general version of Riemann's mapping theorem within $\mathsf{ACA_0}$ by using the polygonal version plus a nonstandard method.

Definition 4.1 (simply connected)**.** The following definitions are made in $\mathsf{RCA_0}$. Let D be an open subset of $\mathbb{C}$.

1. D is said to be *path connected* if for all $\alpha, \beta \in D$ there exists a path from α to β in D.

2. D is said to be *simply connected* if D is path connected and for all semi-polygon γ in D, the interior of γ is included in D.

Lemma 4.5. *The following is provable within* $\mathsf{ns\text{-}ACA_0}$*. Let $U^{\mathrm{s}} \subseteq \mathbb{R}^{\mathrm{s}2}$ be a bounded simply connected open set. Then, there exists a nonstandard polygon $P^* : [0,1]^* \to \mathbb{R}^{*2}$ and (a code for) an open set $W^* \subseteq \mathbb{R}^{*2}$ which satisfy the following:*

- $W^* = \mathrm{Int}(P^*)$,
- $U^{\mathrm{s}} = \mathrm{st}^i(\mathrm{Int}(P^*) \cap \mathbb{Q}^2[\omega]) = W^* \restriction V^{\mathrm{s}}$ *for some* $\omega \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$.

Proof. We will first show a sublemma within $\mathsf{WKL_0}$. Let $L(n, \mathbb{R}^2) = \{[k2^{-n}, (k+1)2^{-n}] \times [l2^{-n}, (l+1)2^{-n}] \mid k, l \in \mathbb{Z}\}$, and let $L_{n,\mathbb{R}^2} = \{A_0 \cup \cdots \cup A_{k-1} \mid k \in \mathbb{N}, A_i \in L(n, \mathbb{R}^2)\}$. For a bounded open set $U \subseteq \mathbb{R}^2$, define $U \restriction L(n, \mathbb{R}^2) \in L_{n,\mathbb{R}^2}$ as $U \restriction L(n, \mathbb{R}^2) = \bigcup\{A \in L(n, \mathbb{R}^2) \mid A \subseteq U\}$.

Sublemma. *The following is provable within* $\mathsf{WKL_0}$*. Let $U \subseteq \mathbb{R}^2$ be a bounded simply connected open set. Then, for any $n \in \mathbb{N}$ there exists $A \in L_{m,\mathbb{R}^2}$ for some $m \geq n$ such that $U \restriction L(n, \mathbb{R}^2) \subseteq A \subseteq U$ and A is simply connected.*

Proof of Sublemma. We will prove this by the number of connected components of $U \restriction L(n, \mathbb{R}^2)$. Let $U \restriction L(n, \mathbb{R}^2) = A_0 \sqcup \cdots \sqcup A_{k-1}$ where each of A_i is a connected component of $U \restriction L(n, \mathbb{R}^2)$. Note that each of A_i is simply connected since U is simply connected. If $k = 1$, then A_0 is desired. Otherwise, take a broken-line γ which connects A_0 and A_1, and find a simply connected set $B \in L_{m,\mathbb{R}^2}$ for some $m \geq n$ such that $A_0 \cup A_1 \cup \gamma \subseteq B \subseteq U$ and $B \cap A_i = \emptyset$ for any $i \geq 2$. Then, we have $U \restriction L(n, \mathbb{R}^2) \subseteq B \sqcup A_2 \sqcup \cdots \sqcup A_{k-1} \subseteq U$ and each of connected components is simply connected. We can repeat this process as long as the number of connected components is one. $\square$

Now, we argue within $\mathsf{ns\text{-}ACA_0}$. Let $U^{\mathrm{s}} \subseteq \mathbb{R}^{\mathrm{s}2}$. Then, by Σ^1_1-TP, $U^{\mathrm{s}\checkmark}$ is an open set in $\mathbb{R}^{*2}$ and $B(a; r) \subseteq U^{\mathrm{s}}$ in V^{s} if and only if $B(a; r) \subseteq U^{\mathrm{s}\checkmark}$ in V^* for any $a \in \mathbb{Q}^{\mathrm{s}2}$ and $r \in \mathbb{Q}^{\mathrm{s}+}$. By the sublemma and overspill, there exist $\omega, \omega' \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$ and $A \in L_{\omega'\mathbb{R}^2}$ such that $U^{\mathrm{s}\checkmark} \restriction L(\omega, \mathbb{R}^2) \subseteq A \subseteq U^{\mathrm{s}\checkmark}$ and A is simply connected. Then, $\partial A \subseteq U^{\mathrm{s}\checkmark}$ is a polygonal domain. Take a polygon $P^* : [0, 1]^* \to \mathbb{R}^{*2}$ such that $\mathrm{Im}(P^*) = \partial A$, and take (a code for) an open set $W^* = \mathrm{Int}(P^*)$ as in the proof of Lemma 4.3. Then, we can easily prove that P^* and W^* satisfy the desired conditions. $\square$

Now we can give a simpler proof of the full version of Riemann's mapping theorem.

Theorem 4.6 (Theorem 4.13 of [16])**.** *The following assertions are equivalent over* $\mathsf{WKL_0}$.

1. $\mathsf{ACA_0}$.
2. *Every simply connected open subset* $U \subsetneq \mathbb{C}$ *is conformally equivalent to* $B(0; 1)$.

Proof. We will prove $1 \to 2$. See [16] for $2 \to 1$. We argue within $\mathsf{ns\text{-}ACA_0}$. Let $U^{\mathrm{s}} \subsetneq \mathbb{C}^{\mathrm{s}}$ be a simply connected open set. Within $\mathsf{WKL_0}$, we can prove that a simply connected open set is conformally equivalent to a bounded open set as usual (see [16]), so we can assume that U^{s} is bounded without loss of generality. Then, by Lemma 4.5, there exists a (hyperfinite) polygon P^* such that $\mathrm{Int}(P^*) \restriction V^{\mathrm{s}} = U^{\mathrm{s}}$. Applying Riemann's mapping theorem for polygonal region (Theorem 3.10) for $\mathrm{Int}(P^*)$ in V^* and taking the standard part, we can obtain a conformal map $h^{\mathrm{s}} : U^{\mathrm{s}} \to B(0; 1)$ as in the proof of Theorem 4.4. By the conservation theorem (Theorem 2.7), this is provable within $\mathsf{ACA_0}$. $\square$

Within $\mathsf{RCA_0}$, we can easily prove that an (effectively integrable) conformal mapping for Riemann's mapping theorem is unique up to rotation if a base point is fixed (see [16]). Thus, in conclusion, we have proved the three versions of Riemann's mapping theorem.

- $\mathsf{RCA_0}$ proves for any polygonal region $U \subseteq \mathbb{C}$ with a base point $x \in U$, there exists a unique (up to rotation) conformal mapping $h : U \to B(0; 1)$ such that $h(x) = 0$.

- $\mathsf{WKL_0}$ proves for any Jordan region $U \subseteq \mathbb{C}$ with a base point $x \in U$, there exists a unique (up to rotation) conformal mapping $h : U \to B(0;1)$ such that $h(x) = 0$.
- $\mathsf{ACA_0}$ proves for any simply connected open set $U \subsetneq \mathbb{C}$ with a base point $x \in U$, there exists a unique (up to rotation) conformal mapping $h : U \to B(0;1)$ such that $h(x) = 0$.

This is an example of "uniform approach" to analysis explained in Section 2.3.

Questions

Question 1. *Find a version of Riemann's mapping theorem which corresponds to the subsystem* $\mathsf{WWKL_0}$ *of second-order arithmetic.*

For the system $\mathsf{WWKL_0}$, see [18].

Question 2. *Can* $\mathsf{RCA_0}$ *prove the weak version of Caratheodory's theorem ? Here, the weak version of Caratheodory's theorem is for any given simple semi-polygon D and a biholomorphic $f : D \to \Delta(1)$, one can expand f into a homeomorphism $\overline{f} : \overline{D} \to \overline{\Delta(1)}$.*

This version of Caratheodory's theorem can be used to prove Picard's little theorem.

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