Research on the Common Developments of Plural Cuboids

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In 1525, the German painter and thinker Albrecht Dürer published his masterwork on geometry "On Teaching Measurement with a Compass and Straightedge", which opened an area with a lot of open problems. Lubiw and O'Rourke started investigating polygons that can fold into polyhedra in 1996. In 2007, Demaine and O'Rourke published a scholarly book, which is about geometric folding algorithms.

In this paper, we study common developments that can fold into two or more incongruent orthogonal convex polyhedra, that is *boxes*. Those common developments are obtained by cutting along their unit lines. These developments are simple polygons that consist of unit squares of four connected so that the boxes and their corresponding developments have the same surface area. In searching for common developments that fold into two or more incongruent orthogonal boxes, we start from a relative simple task: finding two or more incongruent orthogonal boxes whose surface areas are equal but sizes are different.

A development with a hole or an overlap is not what we want to talk about in this paper, for they cannot fold into a box with the same surface area. Also, even if a development has a cut inside, which cannot fold another box using the cut. Therefore, we only consider developments with no cuts inside.

The smallest surface area that different orthogonal boxes can appear is 22 which admits to fold into two boxes of size $1\times1\times5$ and $1\times2\times3$. All common developments of these two boxes have been already known. The next smallest integer N such that surface area N can fold into two different boxes is 30. Matsui tried to list the common developments of two different boxes of sizes $1\times1\times7$ and $1\times3\times3$ of surface area 30, but failed due to the limitation of computational power. We filled this gap and research further.

By a new algorithm on a supercomputer, we first enumerate all common developments of boxes of size $1\times1\times7$ and $1\times3\times3$. Starting from the development of surface area 1, repeat adding one new unit square to all possible places of the development, which generates many new developments. In each step, these new developments will get a mount check for whether they can fit in the box or not. The algorithm checks all the ways from 1 to 30.

Previous research stopped at surface area 22 (858803412 developments) due to the memory overflow caused by too many developments generated. Every step we always generate all possible developments for the surface area, and go to the next step, just like a breadth-first search for the surface area. To decrease the amount of developments, we use a hybrid search method combing breadth-first search and depth-first search. At step 16, all 7486799 developments are divided into 75 small groups, then they are proceeded in a multi-threaded environment (Cray XC30). In this environment, process starts from step 17 to step 30, every thread starts from 1 development, which ensures that memory is not exhausted. Finally, we obtained the number of common developments of boxes of size $1\times1\times7$ and $1\times3\times3$, which is 1076.

We had already known another polygon that can fold into boxes of size $1\times1\times7$ and $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$, whose surface area is also 30. So next we try to find a polygon of surface area 30 that folds into above three different boxes. It is a great improvement from previous known one that the smallest surface area that folds into three different boxes is greater than 500. To achieve that we propose a new efficient algorithm. It determines if a polygon P that can fold into two boxes of size $1\times1\times7$ and $1\times3\times3$ can also fold into box of size $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$, as follows.

Obviously, the development of the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$ cannot fold along the edge of unit squares, which means the previous method of check the development can fit the box or not is no more available. We have two algorithms for the approach of checking the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$, one algorithm is checking for each P the vertex of the cube on the development boundary. Since the surface of cube is also square, an angle of 26.6 degrees exist between its edge and the unit square's edge. Besides, a vertex is made of three planes, which means every vertex on the development should be surrounded by 3 squares. Therefore, we need to check whether there is a development fit such requirements: all vertices of the cube are on the boundary of development, constituting a cross network with 26.6 degrees to the unit square's edge, squares around all vertices should be exactly 24. The second algorithm is checking the positional relationships of each unit square on the development. Every polyhedron, including cube, as long as it has a development made of a certain kind of basic shape, the positional relationships of each unit shape does not change with different development. In this point, if in any development, its unit squares have the same positional relationships even on the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$ we can identify if it can fold into that cube. The first algorithm can not consider the dislocation of center square, which actually makes a hole and an overlap on the cube. Therefore, the second algorithm is more feasible in this paper.

Our new positional relationship algorithm checked whether each of 1076 common developments of boxes of surface area 30 of size $1\times1\times7$, $1\times3\times3$ can fold into the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$. As a result, nine out of 1076 developments can fold into the cube. While eight developments have only one way of folding into the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$, the other one development has two different ways of folding into the cube of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$, for there is an angle of 26.6 degrees to the unit square's edge clockwise and anti-clockwise. That is, the last development can actually fold into 4 boxes: $1\times1\times7$, $1\times3\times3$ and 2 types of $\sqrt{5}\times\sqrt{5}\times\sqrt{5}$.