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Algebraic Structures of Operational Logics in Physical and Automata Experiments

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Contents

1 Introduction **2**

2 Orthomodular law and Greechie diagram **6**
2.1 Poset and orthomodular law 6
2.2 Loop lemma 9
2.3 Greechie diagram 12

3 Physical experiments and Operational logics **14**
3.1 Manuals of experiments 14
3.2 Operational logics 17
3.3 Examples 21

4 Automata experiments **26**
4.1 Moore and Mealy automata 26
4.2 State decision problem of automata 30
4.3 Operational logics of automata experiments 32
4.4 Origin of Moore's uncertainty 33

5 Concluding remarks **42**
5.1 Features of uncertainty 42
5.2 Quantum physical experiments 42

Chapter 1

Introduction

It had been argued many times from Newton's time whether a light is a particle or a wave, but, the wave theory had been the mainstream until the end of 19th century because of the Huygens' theory of interference and the triumph of Maxwell's electromagnetics. In the beginning of this century, however, this argument appeared again as a very important problem of physics.

It's origin was the spectral distribution problem of blackbody radiation. About spectral distribution of lights emitted from heated blackbodies, there had been formulas obtained from Maxwell's electromagnetics, but they agree with the observational results only in the region of the short and long wave length, and there had not been obtained such a formula that agrees with the whole region of wave length, by using classical physics(Newton's mechanics and Maxwell's electromagnetics).

One more difficult problem was to explain the stability of materials. It was known such a picture from atomic experiments that an atom consists of a heavy, positive charged nucleus and negative charged electrons which surround the the nucleus like planets. In this picture, however, electrons emit electromagnetic waves because of their accelerated motions, then they lose their kinetic energies and fall in to the nucleus. This means that every material collapses in a short time, contradicting our experiences. Moreover, in this picture, the spectra of the electromagnetic waves emitted by the electrons have to distribute continuously, but experiments show their discrete distributions.

In order to solve these problems, in 1900 M.Planck introduced the quantum hypothesis:

energy of electromagnetic radiation is emitted and absorbed only in the forms of discrete quanta.

This is the start of quantum mechanics.

On the other hand, from electron diffraction experiments on crystals, it appeared that every particle like an electron also shows features of a wave. Nowadays, it is known that every material has a *wave−particle duality*, i.e., a particle whose energy E and momentum

$\mathbf{p}$ also has features of a wave whose angular frequency ω and wave number $\mathbf{k}$, and they are related

$$E = \hbar\omega,\ \mathbf{p} = \hbar\mathbf{k}.$$

The formulas above is called "Einstein - de Bloglie's relation" and $\hbar = 1.054 \times 10^{-27}$erg·sec is the "Planck's constant".

It is quantum mechanics that was born to treat phenomena which cannot be explained by classical physics. At first it was introduced as matrix mechanics by W.Heisenberg in 1925, and next, introduced as wave mechanics by E.Schrödinger in 1926. These two theories are quite different each other in their formalism, but afterword, J.von.Neumann gave an unified interpretation of them by using Hilbert space theory. According to it, quantum mechanics has the following axioms([10],[11],[13]).

Axiom 1
The state of a physical system is represented by an unit vector of a complex Hilbert space H. This vector is called "state vector". Provided that state vectors $\psi \in H$ and $\alpha\psi \in H$ represent the same state for an arbitrary complex number α with $|\alpha| = 1$.

Axiom 2
A physical quantity — sometimes called "observable" — is represented by a self adjoint operator on H.

Axiom 3
For a state vector ψ and an observable A, the expected value of A is given by the inner product $< \psi, \hat{A}\psi >$ of two vector ψ and $\hat{A}\psi$, where $\hat{A}$ is the self adjoint operator that corresponds to the observable A.

In addition to these axioms, the foundation of quantum mechanics consists of some physical principles, say, the law of the time development of ψ or the method to construct Hamiltonian of Schrödinger equation, but we shall not discuss such details.

What is important here is that observables are represented by operators on Hilbert space and these operators do not commute in general, i.e., if $\hat{A}$ and $\hat{B}$ are such operators, then in general

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}.$$

It follows that $< \psi, \hat{A}\hat{B}\psi > \neq < \psi, \hat{B}\hat{A}\psi >$, and this means that experimental results(informations about a physical system) depend on the order of the experiment of A and that of B.

In fact, about some particle, say an single electron, the operator $\hat{x}$ of coordinate x and the operator $\hat{p}$ of coordinate p satisfies $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$, and this relation yields that for standard deviations $\Delta x, \Delta p$ of x and p

$$\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar. \tag{1.1}$$

Namely, for this pair of physical quantities, there exists a "limit" such that we cannot know more precisely about the physical system. This is an example of the well-known "Heisenberg's uncertainty principle".

Meanwhile, a similar uncertainty is found in a kind of experiment on automata, i.e., the order dependency of experiments appears and there exists a "limit" concerning what we can know about automata.

"The state decision problem" is such an example of automata experiments. Let us consider that an automaton with an output function is contained in a black box and we will apply input words and observe its output words. At any time, the automaton is in one of its states, and the state decision problem is to determine by experiment — observing input-output behavior — which state the automaton currently in. An important property in this experiment is that if we apply some input words, then the automaton transits from the current state to another one. Therefore, the experimental results(informations about the automaton) are depend on the order of experiments. Moreover, the state decision problem is not always solvable, i.e., there exist a "limit" for the informations about the current state. In 1956 E.F.Moore pointed out these facts quite analogous to quantum physical experiments, and afterword, in 1971 J.H.Conway called them "Moore's uncertainty principle".

In this thesis, we take examples from physical experiments and automata ones, and discuss algebraically the relations of informations obtained by experiments. Such a study about quantum physical experiments are well-known as quantum logics. If we start from the axioms above and construct quantum logics as algebras of the self adjoint operators on Hilbert space, the logics appears to be orthomodular lattices([1],[7]).

Similarly, there exists a way for the state decision problem, which start from partitions of a state set of an automaton and make use of Hilbert space theory. Such a study is called "automaton logics" or "partition logics"([2],[4]).

But we do not take such a "top-down" style. Instead, we take a "bottom-up" style i.e., we begin with to investigate the features of experiments and then unify them like as physicists make a theory from a collection of physical experiments. A merit of our style is in a possibility to treat uniformly all kinds of experiments or observations which contains not only physical experiments, but also automata experiments and so on.

Now, in Chapter 2, we will give a basic notion of an orthomodular poset etc., and provide the proof of “loop lemma” and introduce “Greechie diagram”. These are useful methods to investigate properties of a system of Boolean algebras.

In Chapter 3, we take examples of physical experiments and introduce operational logics, which are methods to treat experiments set theoretically. Making use of Greechie diagram, we show that an operational logic is generally an orthoposet, and becomes an orthomodular poset or an orthomodular lattice under some conditions. We also shows examples of physical experiments corresponding to these algebras.

In Chapter 4, we will give definitions of automata with output functions and introduce procedures to minimize or equivalently transform automata. Next, we show examples of the state decision problems, and making use of operational logics, investigate algebraic structures of the problems. In addition, we discuss about the origin of Moore’s uncertainty in detail. Heisenberg’s uncertainty is one of the laws of nature. Contrary, automata are artificial objects, and therefore it is worth considering the origin of Moore’s uncertainty. We will give the definition of an “uncertainty of the state decision problem” and show several theorems which provide the relation between an uncertainty and a minimal property of automata and so on. Moreover, we will define a “degree of uncertainty”. By this definition, we can discuss quantitatively about Moore’s uncertainty and show an inequality analogous to the Heisenberg’s one(expression (1.1)).

Chapter 2

Orthomodular law and Greechie diagram

2.1 Poset and orthomodular law

A *partially ordered set*(*poset*) is a set P with a partial order $\leq$. A *bounded poset* P is a poset which has a least element 0 and a greatest element 1, such that $0 \leq a \leq 1$ holds for all $a \in P$.

For $a, b \in P$, the *supremum* of a and b(denoted by $a \vee b$) is an element satisfying the following conditions:
(i) $a \leq a \vee b$ and $b \leq a \vee b$,
(ii) for $c \in P$, if $a \leq c$ and $b \leq c$, then $a \vee b \leq c$.
Similarly, the $infimum$ of a and b(denoted by $a \wedge b$) is an element satisfying the following conditions:
(i) $a \wedge b \leq a$ and $a \wedge b \leq b$,
(ii) for $c \in P$, if $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$.

By definition, it is clear that $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$. The associativity also holds i.e., $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. The supremum and the infimum do not always exist in P for an arbitrary pair $a, b \in P$.

Definition 2.1.1 (orthocomplimentation, orthopoet)
Let $(P, \leq)$ be a bounded poset and $a, b \in P$. An *orthocomplimentation* on P is an unary operation $'$ on P satisfying
(i) if $a \leq b$, then $b' \leq a'$,
(ii) $a'' = a$,

(iii) the supremum $a \vee a'$ exists and $a \vee a' = 1$.
An *orthoposet* is a bounded poset with an orthocomplimentation.

Let P be an orthoposet and $a, b \in P$. Then the following proposition is fundamental.

Proposition 2.1.2
(i) If $a \vee b$ exists, $(a \vee b)' = a' \wedge b'$. (de Morgan's law)
(ii) $1' = 0$ and $0' = 1$.
(iii) $a \wedge a' = 0$.

[*Proof*]
(i) Since $a \leq a \vee b$, we have $(a \vee b)' \leq a'$(by Definition 2.1.1(i)). Similarly, we have $(a \vee b)' \leq b'$. If $c \leq a', b'$, then $a \leq c'$ and $b \leq c'$. Therefore $a \vee b \leq c'$ and $c \leq (a \vee b)'$.∎
(ii) $0 = 1' \wedge 0 = (1 \vee 0')' = 1'$ and by Definition 2.1.1(iii) we have $0' = 1'' = 1$.∎
(iii) By Definition 2.1.1(i) and (iii), $a \wedge a' = (a' \vee a)' = 1' = 0$.∎

The relation *orthogonal* $\perp$ for elements a, b of an orthoposet P is defined by $a \perp b$ if $a \leq b'$ holds.

Proposition 2.1.3
Let P be an orthoposet and $x \in P$. Then $x \perp x$ if and only if $x = 0$.

[*Proof*]
If $x \perp x$, then $x \leq x'$. Hence, we have $x = x \wedge x \leq x \wedge x' = 0$. Conversely, it is clear that $0 \leq 1 = 0'$ and $0 \perp 0$.∎

Definition 2.1.4 (orthomodular poset)
An *orthomodular poset(OMP)* P is an orthoposet P satisfying
(i) if $a \perp b$, then the supremum $a \vee b$ exists in P,
(ii) $a \leq b$ implies $b = a \vee (a' \wedge b)$. (orthomodular law)

Proposition 2.1.5
Let P be an OMP. If $a, b \in P$ and $a \perp b$, then $(a \vee b) \wedge a' = b$.

[*Proof*]
Since $a \perp b$, $a \leq b'$ holds, and by the orthomodular law, we have $b' = a \vee (a' \wedge b')$. Making use of de Morgan's law, we obtain $b = a' \wedge (a \vee b)$.∎

Proposition 2.1.6

Let P be an OMP. The orthomodular law is equivalent to each of the following conditions:
(i) if $a, b \in P$ and $a \perp b$, then $(a \vee b) \wedge b' = a$;
(ii) if $a, b \in P$ and $a \perp b$, then the assumption $a \vee b = 1$ implies $a = b'$;
(iii)if $a, b \in P$ and $a \leq b$, then the assumption $a' \wedge b = 0$ implies $a = b$.

$[Proof]$
(i) According to Proposition 2.1.5, the orthomodular law yields (i). Let us assume the validity of condition (i). Assume also that $a \leq b$. Then $a \perp b'$ and therefore, $(b' \vee a) \wedge a' = b'$(condition (i) holds for b' and a). Hence, we obtain $b = a \vee (b \wedge a')$, which is the orthomodular law.∎
(ii),(iii) We easily see that the conditions (ii) and (iii) are equivalent. If $a \leq b$, the orthomodular law yields $b = a \vee (a' \wedge b)$. Therefore, if $a' \wedge b = 0$, we obtain $a = b$, and condition (iii) holds.

Conversely, if $a \leq b$, $a' \wedge b$ exists. Since $a' \wedge b \leq b$, we have the inequality $a \vee (a' \wedge b) \leq b$. Thus
$(a \vee (a' \wedge b))' \wedge b = a' \wedge (a \vee b') \wedge b' = (a' \wedge b) \wedge (a \vee b') = 0.$
Therefore if condition (iii) holds, and we obtain $b = a \vee (a' \wedge b)$.∎

Definition 2.1.7 (lattice)

A *lattice* is a poset $(P, \leq)$ such that for any $a, b \in P$, the supremum $a \vee b$ and infimum $a \wedge b$ exist in$P$.

Definition 2.1.8 (orthomodular lattice)

An *orthomodular lattice*(OML) is an OMP which is also a lattice.

Definition 2.1.9 (Boolean algebra)

A *Boolean algebra* is a lattice L with orthocomplementation $'$ and satisfying
(i) for all $a \in L$, there exists a' in L,
(ii) for all $a, b, c \in L$, $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$ and $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. (distributive law)

Definition 2.1.10 (subalgebra, block)

A *subalgebra* M of an algebra P is a subset M of P which is closed under the operations $'$, $\vee$, $\wedge$ and which contains 0 and 1. A *block* of an algebra P is a maximal Boolean subalgebra of P.

2.2 Loop lemma

In this section we will introduce the "pasting" technique to construct an algebra from a system of Boolean algebras.

Definition 2.2.1 (atom, coatom)
Let L be an orthoposet. A nonzero element $a \in L$ is called an *atom* in L if the inequality $b \leq a$ for $b \in L$ implies ether $b = 0$ or $b = a$. Dually, an element $a \in L$ is called a *coatom* in L if a' is an atom of L.

Definition 2.2.2 (almost disjoint system)
Let $\mathcal{B}$ be a system of Boolean algebras. We say that $\mathcal{B}$ is *almost disjoint* if for any pair $A, B \in \mathcal{B}$ one of the following condition is satisfied:
(i) $A = B$;
(ii) $A \cap B = \{0, 1\}$;
(iii)$A \cap B = \{0, 1, x, x'\}$, where x is an atom in both Boolean algebras A and B, and moreover, $0 = 0_A = 0_B$, $1 = 1_A = 1_B$ and $x' = x'^A = x'^B$(here the indices indicate that the operations belong to the respective Boolean algebra).

Definition 2.2.3 (loop of order n)
Let $\mathcal{B}$ be an almost disjoint system of Boolean algebras. A finite sequence $\{B_0, B_1, ..., B_{n-1}\}$ of elements of $\mathcal{B}$ is called a *loop of order* n ($n \geq 3$) if the following conditions are satisfied (the indices are understood modulo n):
(i) for any $i \in \{0, 1, ..., n-1\}$ we have $B_i \cap B_{i+1} = \{0, 1, x_i, x_i'\}$, where x_i is an atom in both algebras B_i and B_{i+1};
(ii) if $j \notin \{i-1, i, i+1\}$, then $B_i \cap B_j = \{0, 1\}$;
(iii) we have $B_i \cap B_j \cap B_k = \{0, 1\}$ for distinct indices i, j, k.

Observe that every loop $\{B_0, B_1, ..., B_{n-1}\}$ uniquely determines a sequence of atoms $\{e_0, e_1, ..., e_{n-1}\}$, where e_i is an atom belonging to both B_{i-1} and $B_i (0 \leq i \leq n-1)$. Then we have the following lemma.

Lemma 2.2.4
Let $\mathcal{B}$ be an almost disjoint system of Boolean algebras and $\{B_0, B_1, ..., B_{n-1}\}$ be a loop of order n. Let e_i is an atom belonging to both B_{i-1} and B_i ($0 \leq i \leq n-1$ and the indices are understood modulo n). Then $e_i \perp e_{i+1}$.

[*Proof*]
Since $B_{i-1} \cap B_i = \{0, 1, e_i, e_i'\}$ and $B_i \cap B_{i+1} = \{0, 1, e_{i+1}, e_{i+1}'\}$, e_i and e_{i+1} are atoms of B_i. Since $B_{i-1} \cap B_i \cap B_{i+1} = \{0, 1\}$, we have $e_i \neq e_{i+1}$. Thus, $e_i \wedge e_{i+1} = 0$ and
$e_i \leq e_i \vee e_{i+1}' = (e_i \vee e_{i+1}') \wedge (e_{i+1} \vee e_{i+1}') = (e_i \wedge e_{i+1}) \vee e_{i+1}' = e_{i+1}'$.
Therefore we obtain $e_i \perp e_{i+1}$.∎

Proposition 2.2.5
Let $\mathcal{B}$ be an almost disjoint system of Boolean algebras. Put $L = \bigcup \mathcal{B}$ and define the relation $\leq$ as follows:
$x \leq y$ in L if and only if there exists a $B \in \mathcal{B}$ such that $x, y \in B$ and $x \leq_B y$.
Similarly, we define the operation $'$ in L by putting $x' = x'^B$, where $B \in \mathcal{B}$ and $x \in B$. Then $\leq$ is a partial order and the operation $'$ is an orthocomplementation.

[*Proof*]
We first consider the operation $'$. Since $\mathcal{B}$ is an almost disjoint system, we have the equality $1_A = 1_B$ for any $A, B \in \mathcal{B}$. Also $x'^A = x'^B$ for any $x \in A \cap B$. Thus, $'$ is an orthocomplimentation on L.
Next, let us check that the relation $\leq$ is a partial order.
(i)reflexivity
$x \leq x$ if and only if $x \leq_A x (x \in A)$, thus reflexivity is clearly valid.
(ii)transitivity
Suppose that $u \leq v$ and $v \leq w$. If there is a Boolean algebra $A \in \mathcal{B}$ such that $\{u, v, w\} \subseteq A$, then $u \leq w$, in view of the transitivity of the relation $\leq_A$. If this is not the case, then there exist two Boolean algebras $A, B \in \mathcal{B}$ such that $u \leq_A v$ and $v \leq_B w$(i.e., $v \in A \cap B$). If v is either 0 or an atom, then we obtain $u \in \{0, v\} \subseteq A$. If v is either 1 or a coatom, then we obtain $w \in \{1, v\} \subseteq B$. In both case $u \leq w$ is valid.
(iii)antisymmetry
Suppose that $x \leq_A y$ and $y \leq_B x$ for two elements $A, B \in \mathcal{B}$. If $A = B$, then $x = y$. If $A \neq B$, then $x, y \in A \cap B$. Moreover, if x or y equals to 0 or 1, then $x = y$. Suppose that $x, y \in A \cap B \backslash \{0, 1\}$. Then $\{x, y\} \subseteq \{z, z'\}$, where z is the only atom belonging to both A and B. Since z and z' are incomparable in any $B \in \mathcal{B}$, we obtain either $x = z = y$ or $x = z' = y$. In any case we have $x = y$, thus antisymmetry is valid.∎

Theorem 2.2.6 (loop lemma(Greechie))
Let $\mathcal{B}$ be an almost disjoint system of Boolean algebras. Put $L = \bigcup \mathcal{B}$. Then
(i) L is an OMP if and only if the system does not contain a loop of order 3;
(ii)L is an OML if and only if the system does not contain either a loop of order 3 or a loop of order 4.

[*Proof*]

We have shown above that L is an orthoposet.

(i) Let L be an OMP. Suppose that $\mathcal{B}$ contains a loop of order 3, say $\{B_0, B_1, B_2\}$. Let $\{e_0, e_1, e_2\}$ be the triple of atoms which correspond to this loop(i.e., e_i is the atom common to B_i, B_{i+1}, where $i \in \{0,1,2\}$ modulo 3). We have the inequality $e_0 \leq e_1'$, and therefore, $e_0 \vee e_1$ exists in L. Suppose that $e_0 \vee e_1 = 1$. Then $e_0 = e_1'$, and by Lemma 2.2.4 we have $e_1 \leq e_2'$, therefore $e_2 \leq e_0$. We have also $e_2 \leq e_0'$, thus, $e_2 = 0$, and this is contradiction. Suppose, on the other hand, that $e_0 \vee e_1 \neq 1$. Since $e_0 \leq e_2'$ and $e_1 \leq e_2'$, we obtain $e_0 \vee e_1 \leq e_2'$. This means that there exists a Boolean algebra $B \in \mathcal{B}$ such that $\{e_2', e_0 \vee e_1\} \subseteq B$. If $e_0 \vee e_1$ is a coatom, then we have $e_2' = e_0 \vee e_1$. This implies that $e_2 \in B_1$ and therefore, $e_2 \in B_0 \cap B_1 \cap B_2$, a contradiction with the definition of a loop in $\mathcal{B}$. If $e_0 \vee e_1$ is not a coatom, then by the definition of the almost disjoint system, B_1 is the only element of $\mathcal{B}$ which contains $e_0 \vee e_1$. Thus, B_1 has to contain e_2', too, because otherwise the elements $e_0 \vee e_1, e_2'$ are incomparable(i.e., $B = B_1$). Hence we obtain $e_2 \in B_1$ and we have shown that $\mathcal{B}$ cannot contain a loop of order 3.

Conversely, suppose now that $\mathcal{B}$ does not contain a loop of order 3. We need to show that L also fulfills the following conditions:

(1) if $a, b \in L$ and $a \perp b$, then $a \vee b$ exists in L,

(2) if $a, b \in L$ and $a \leq b$, then $b = a \vee (a' \wedge b)$.

Let us first check condition (1). If $a \perp b$, there exist a Boolean algebra $B \in \mathcal{B}$ which contains a and b. Suppose that there exist $c \in L$ such that $a \leq c, b \leq c$ and $c \notin B$. Then there exist $B_1, B_2 \in \mathcal{B}$ such that $\{a, c\} \subseteq B_1, \{b, c\} \subseteq B_2$. Since $\mathcal{B}$ is an almost disjoint system, we have $B \cap B_1 = \{0, 1, a, a'\}$, $B \cap B_2 = \{0, 1, b, b'\}$, $B_1 \cap B_2 = \{0, 1, c, c'\}$, and this yields $\{B, B_1, B_2\}$ is a loop of order 3, contradicting our assumption. Therefore a, b, c are elements of a single Boolean algebra B. Hence $a \vee_B b$ is the supremum of a and b in L. Condition (2) is clearly holds because by the discussion above, $a, b, a', a' \wedge b$ are elements of a single Boolean algebra.∎

(ii) By (i), no OML contains a loop of order 3. Suppose L contains a loop $\{B_0, B_1, B_2, B_3\}$ of order 4 with the atoms $e_0, e_3 \in B_0$, $e_0, e_1 \in B_1$, $e_1, e_2 \in B_2$, $e_2, e_3 \in B_3$. Obviously $e_0, e_2 \leq e_1', e_3'$ hold, and e_0 and e_2(e_1' and e_3') are incomparable. Therefore, if $e_0, e_2 \leq g \leq e_1', e_3'$ hold for some g in L, then g is not an atom or a coatom of L. Thus, there is exactly one Boolean algebra $A \in \mathcal{B}$ containing all these elements. In this case $\{0, 1, e_1, e_3, e_1', e_3'\} \subseteq A \cap B_0$ and this implies $A = B_0$ because $\mathcal{B}$ is an almost disjoint system. Then $B_0 \cap B_2$ contains e_2, contradicting $B_0 \cap B_2 = \{0, 1\}$. Therefore $e_0 \vee e_2$ does not exist in L.

Conversely, let the least order of the loops in L be at least 5. Then by (i), L is an OMP. Take two elements $a, b \in L$. We will show that the supremum $a \vee b$ exists in L. We may(and will) assume that $\{a, b\} \cap \{0, 1\} = \phi$ and $a \neq b$. If there is $B \in \mathcal{B}$ such that $\{a, b\} \subseteq B$, by (i), there exist $a \vee b = a \vee_B b$ in L. Suppose that the elements a, b do not belong to a single $B \in \mathcal{B}$. Let $c \in L$ be an element such that $a \leq c$ and $b \leq c$. Then c has to be a coatom or 1. Suppose that there are two distinct coatoms $c_1, c_2 \in L$ such that

$\{a, c_1\} \subseteq B_0$, $\{b, c_1\} \subseteq B_1$, $\{b, c_2\} \subseteq B_2$, $\{a, c_2\} \subseteq B_3$. It follows that $\{B_0, B_1, B_2, B_3\}$ is a loop of order 4, contradicting our assumption. Thus there are only two possibilities — either there exists exactly one coatom $c \in L$ such that $a \leq c$ and $b \leq c$, or there exists no such coatom. In the former case we obviously have $a \vee b = c$, and the latter case we have $a \vee b = 1$. We have shown that the supremum $a \vee b$ always exists and, therefore, L is a lattice.∎

2.3 Greechie diagram

Definition 2.3.1 (Greechie logic)
An algebra L is called a *Greechie logic* if the following conditions are satisfied:
(i) every element of L can be written as a supremum of at most countably many mutually orthogonal atoms in L;
(ii)the collection of all blocks in L forms an almost disjoint system.

There is an useful way of exhibiting the Greechie logics, by drawing their *Greechie diagrams*. A Greechie diagram consists of points and lines. The points represent the atoms in the logic and lines link the points belonging to a block.

For instance, Figure 2.1.1a is the Greechie diagram of the Boolean algebra exp$\{1, 2, 3\}$(of all subsets of the set $\{1, 2, 3\}$) and the same Boolean algebra would be represented by Figure 2.1.1b and 1c. Figure 2.1.2 and Figure 2.1.3 represents the Boolean algebra exp$\{1, 2, 3, 4, 5\}$ and exp$\mathbf{N}$ respectively. Figure 2.1.4 represents the Greechie logic exp$\{a, b, c\}\cup$exp$\{c, d, e\}$ where c is a common atom of two Boolean algebras.

Greechie diagrams allow us to detect the presence of the loops of order 3 or 4. A loop of order 3 shows up a "triangle" and a loop of order 4 as a "square". Figure 2.1.5 does not define an OMP. Figure 2.1.6 defines an OMP, but it is not a lattice.

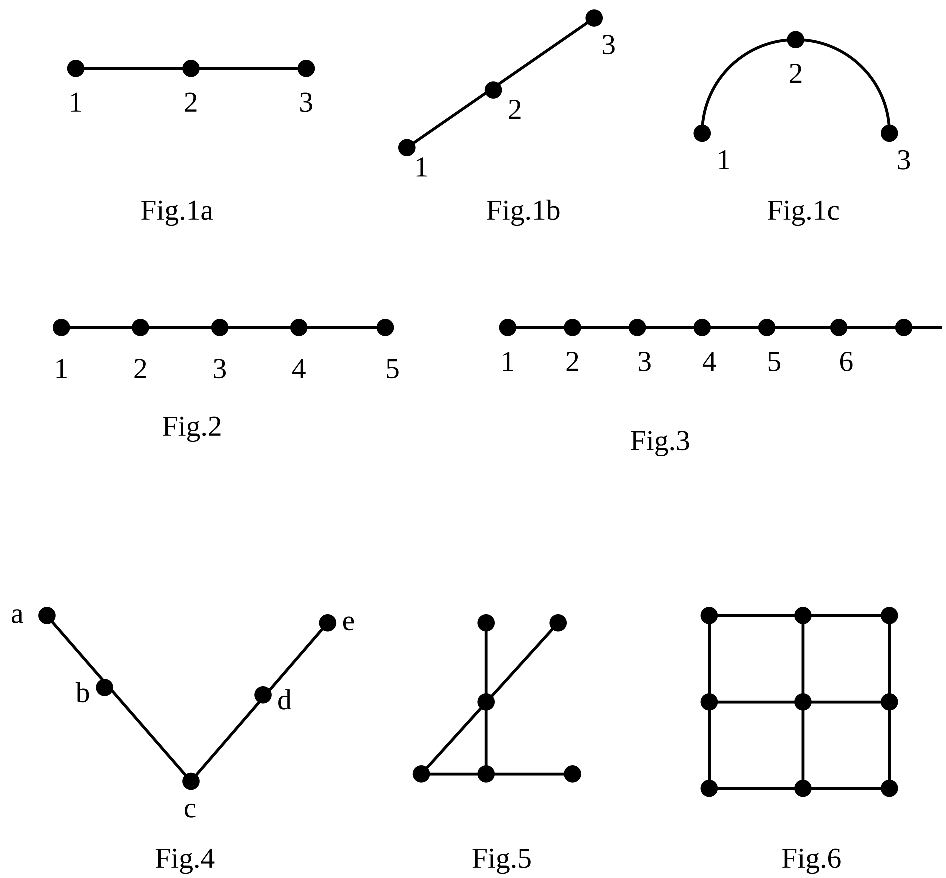

Figure 2.1: Examples of Greechie diagrams

Chapter 3

Physical experiments and Operational logics

3.1 Manuals of experiments

A "physical system" is anything on which we perform experiments. Examples of physical systems are planets, an atom, a magnetic fields, the entire universe, etc.

Let us consider a physical system consisting of a firefly in a box(see Figure 3.1). This box has a clear plastic window at the front and another one side. Suppose each window has thin vertical line drawn down the center to divide the window in half. Place a firefly in the box.

We shall consider two experiments on the system. Experiment E is: Look at the front window. The outcomes of E will be:

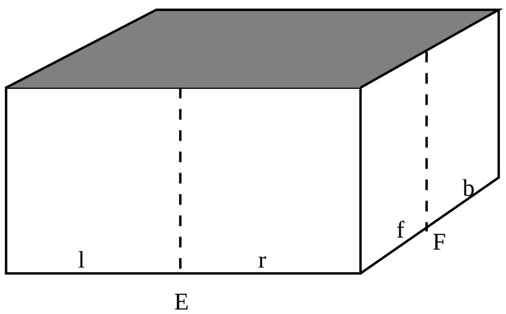

Figure 3.1: A physical system consisting of a box with two windows and a firefly inside.

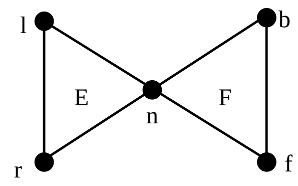

Figure 3.2: The "bow-tie" manual.

$$l = \text{see a light in the left half of the window,}$$
$$r = \text{see a light in the right half of the window,}$$
$$n = \text{see no light.}$$

Let us denote this experiment by $E = \{l, r, n\}$. A second experiment F is: Look at the side window. The outcomes of F will be:

$$f = \text{see a light in the front half of the window,}$$
$$r = \text{see a light in the back half of the window,}$$
$$n = \text{see no light.}$$

Let us denote this experiment by $F = \{f, b, n\}$.

In Figure 3.2 we show a diagram that illustrates our experiments and their outcomes. We can see the two experiments as two sets which contain a common element n. This gives us a key concept to treat physical experiments set theoretically.

Our approach to quantum logics is based on a *manual*, which is a set theoretical representation of laboratory experiments. Let us start with the following definitions.

Definition 3.1.1 (quasimanual, experiment, outcome, event)

(i) A *quasimanual* Q is a nonempty collection of nonempty sets called *experiments*. The members of the experiments are called *outcomes*. The set of all outcomes is denoted by X_Q.

(ii) An *event* in quasimanual Q is a subset of an experiment in Q.

We say we *test for event A* by performing an experiment that contains A. If we test for A and obtain an outcome in A, we say *event A occured*.

Definition 3.1.2 (orthogonal complements)

Suppose Q is a quasimanual.

(i) Two events A, B in Q are said to be *orthogonal*, denoted $A \perp B$, if they are disjoint subsets of a single experiment in Q. (For outcomes x and y of Q we write $x \perp y$ to mean $\{x\} \perp \{y\}$.)

(ii) If A, B are orthogonal events in Q and $A \cup B$ is an experiment in Q, then we say that A and B are *orthogonal complements* in Q. We denote this by $AocB$.

The relation $AocB$ means that if event A occurs then event B never occurs, and vice versa. In what follows, we use a symbol $\sqcup$ for disjoint union.

Definition 3.1.3 (manual)

A *manual* is a quasimanual M which satisfies the following conditions:

(i) If A, B, C, D are events in M with $AocB$, $BocC$, and $CocD$, then $A \perp D$;

(ii) If $E, F \in M$ and $E \subseteq F$, then $E = F$.

Property (ii) ensures that experiments are maximal events.

We shall discuss about the rationale for property (i) by referring to Figure 3.3. Suppose we test for event A by performing experiment E, and A occurs. Then we know that B did not occur. Thus, if we had performed experiment F, then C would have occurred; so if we had performed experiment G, event D could not occur. In summary, if we test for A, and A occurs, then testing for D would result in D not occurring. A similar reasoning shows that if D occurs when tested, then A cannot occur when tested at the same time. Therefore it is natural to require that there is a single experiment H that contains A and B, so that $A \perp D$ in M.

Definition 3.1.4 (operationally perspective)

If M is a manual, A, B, and C are events in M, and $AocB$ and $BocC$, then we say that A and C are *operationally perspective*, which we denote by $AopC$.

$AopC$ means that if A occurs then C occurs, and vice versa.

Lemma 3.1.5

If A is an event in M, then $AopA$.

[*Proof*]

Let E be an experiment which contains A. Then $AocE\backslash A$ and $E\backslash AocA$. Hence $AopA$.∎

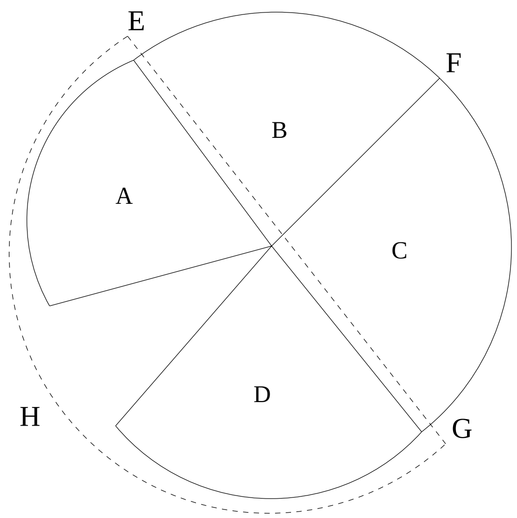

Figure 3.3: Property (i) of experiments

3.2 Operational logics

Definition 3.2.1 (implication)
If M is a manual and A and B are events in M, then we say A *implies* B, denoted by $A \leq_e B$, if and only if there is an event C with $C \perp A$ and $(A \sqcup C) op B$.

$A \leq_e B$ means that if A occurs, then B necessarily occurs.

Lemma 3.2.2
The relation $\leq_e$ is a preorder on the collection of events in M.

$[Proof]$
(i)reflexivity
$\phi \perp A$, $(A \sqcup \phi) oc E \backslash A$ and $E \backslash A oc A$. Hence $A \leq_e A$.
(ii)transitivity
Suppose that $A \leq_e B$ and $B \leq_e C$. Then there exist events U and V such that
$U \perp A$ and $(A \sqcup U) op B$,
$V \perp B$ and $(B \sqcup V) op C$.
These expressions mean, respectively, there exist events K and L such that
$(A \sqcup U) oc K$, $K oc B$ and
$(B \sqcup V) oc L$, $L oc C$.
Then $(V \sqcup L) oc B$, and by Definition 3.1.3(i), we have $(V \sqcup L) \perp (A \sqcup U)$. Therefore, A, U,

V and L are disjoint subsets of a single experiment E, and there exists an event W disjoint from these four events such that $A \sqcup U \sqcup V \sqcup L \sqcup W = E$. Then $(A \sqcup (U \sqcup V \sqcup W))ocL$ and since $LocC$, we have $(A \sqcup (U \sqcup V \sqcup W))opC$. Hence, $A \leq_e C$.∎

Lemma 3.2.3

Let A and B are events in a manual M. If $A \subseteq B$, then $A \leq_e B$.

[*Proof*]

$B \backslash A \perp A$ and $(A \sqcup B \backslash A)= BopB$. ∎

Definition 3.2.4 (logical equivalence)

If M is a manual and A and B are events in M, then we say A is *logically equivalent* to B, denoted $A \leftrightarrow B$, if and only if $A \leq_e B$ and $B \leq_e A$.

$A \leftrightarrow B$ means that B occurs whenever A occurs, and vice versa.

Lemma 3.2.5

The relation $\leftrightarrow$ is an equivalence relation on the set of events in a manual.

[*Proof*]

(i)reflexivity

Since $A \leq_e A$, $A \leftrightarrow A$ is satisfied.

(ii)transitivity

If $A \leftrightarrow B$ and $B \leftrightarrow C$, then $A \leq_e B$, $B \leq_e A$, $B \leq_e C$, and $C \leq_e B$. Since $\leq_e$ is a preorder, we have $A \leq_e C$ and $C \leq_e A$, hence $A \leftrightarrow C$.

(iii)symmetry

By definition, if $A \leftrightarrow B$, then $A \leq_e B$ and $B \leq_e A$. Hence $B \leftrightarrow A$.

Lemma 3.2.6

(i) If A,B are events in M, then $AopB$ if and only if $A \leftrightarrow B$.

(ii) If $E, F \in M$, then $E \leftrightarrow F$.

(iii) If A is an event in M and $E, F \in M$, with $A \subseteq E$ and $A \subseteq F$, then $E \backslash A \leftrightarrow F \backslash A$.

[*Proof*]

(i) If $AopB$, then $\phi \perp A, (A \sqcup \phi)opB$. Therefore $A \leq_e B$. Moreover, if $AopB$, there exists an event C such that $AocC$ and $CocB$. Then $BopA$ and $B \leq_e A$. Therefore if $AopB$, then $A \leftrightarrow B$.

Conversely, if $A \leq_e B$ and $B \leq_e C$, according to the proof of Lemma 3.2.2, there exist five disjoint sets A, U, V, L and W such that $(A \sqcup U)opB$, $(A \sqcup (U \sqcup V \sqcup W))ocL$ and $LocC$. If $A = C$, by Definition 3.1.3(ii), we obtain $(A \sqcup (U \sqcup V \sqcup W)) \sqcup L = A \sqcup L$. Hence $U = V = W = \phi$ and $AopB$.∎

(ii) If E and F are two events in a manual M, $Eoc\phi$ and ϕocF, hence $EopF$. By the part A of this lemma, we have $E \leftrightarrow F$.∎

(iii) Since $E\backslash AocA$ and $AocF\backslash A$, we have $E\backslash AopF\backslash A$. Therefore $E\backslash A \leftrightarrow F\backslash A$.∎

Definition 3.2.7 (operational logic)

Suppose M is a manual.

(i) If A is an event in M, then we define

$$[A]{:=}\{B|B \text{ is an event in } M \text{ and } A \leftrightarrow B\},$$

and we call $[A]$ the *logical proposition determined by* A. We say we *test for proposition* $[A]$ if we test for any events in M logically equivalent to A. An event used to test for $[A]$ confirms that $[A]$ *is true* (resp.*false*) if the event occurs (resp.does not occur). For outcome x we write $[x]$ for $[\{x\}]$.

(ii) We define $\Pi(M){:=}\{[A]|A \text{ is an event in } M\}$.

(iii) We define

$$[A] \leq [B] \text{ if and only if } A \leq_e B,$$

in which case we say $[A]$ *implies* $[B]$.

(iv) For $[A] \in \Pi(M)$, we define the *orthocomplement* of $[A]$ by $[A]' = [E\backslash A]$, where E is any experiment in M with $A \subseteq E$.

(v) If $[A], [B] \in \Pi(M)$, we say that they are *orthogonal*, denoted by $[A] \perp [B]$ if and only if $[A] \leq [B]'$.

(vi) The set $\Pi(M)$, together with the implication $\leq$ and orthocomplementation $'$ is called the *operational logic* of manual M.

If $[A] \leq [B]$, then a test for $[A]$ which confirms that $[A]$ is true at the same time confirms that $[B]$ is true. That is why we use the word "implies" to express the relation $[A] \leq [B]$.

Lemma 3.2.8

(i) $[E] = [F]$.

(ii) $[E\backslash A] = [F\backslash A]$.

[*Proof*]
These are immediately obtained by Lemma 3.2.6.

Theorem 3.2.9
(i) The relation $\leq$ *is a partial order on the set* $\Pi(M)$.
(ii) $[A] \leq [B]$ *if and only if* $[B]' \leq [A]'$.
(iii) $[A]'' = [A]$.

[*Proof*]
(i) Since $\leq_q$ is a preorder, we have to check only the antisymmetry of $\leq$. If $[A] \leq [B]$ and $[B] \leq [A]$, then $A \leq_e B$ and $B \leq A$. Hence $A \leftrightarrow B$ and by definition $[A] = [B]$.∎
(ii) By definition $[A] \leq [B]$ if and only if $A \leq_e B$. Then there exist experiments E and F such that $A \sqcup K \sqcup L = E$ and $L \sqcup B = F$, where A, B, K and L are mutually disjoint. Then $L = F \backslash B$, $(F \backslash B \sqcup K) oc A$ and $A oc E \backslash A$. Hence $(F \backslash B \sqcup K) op E \backslash A$ and then $F \backslash B \leq_e E \backslash A$. Therefore $[B]' \leq [A]'$.∎
(iii) Let E, F be two experiments. Suppose $A \subseteq E$ and $E \backslash A \subseteq F$. Then using Lemma 3.2.8(ii), $[A]'' = [F \backslash (E \backslash A)] = [E \backslash (E \backslash A)] = [A]$.∎

Lemma 3.2.10
$[A] \perp [B]$ *if and only if* $A \perp B$.

[*Proof*]
If $[A] \perp [B]$, then $A \leq_e E \backslash B$, where E is an experiment which contains B. Therefore there exist events K, L such that $(A \sqcup K) oc L$ and $L oc E \backslash B$, where A, B, K and L are mutually disjoint. Since $E \backslash B oc B$, we have $(A \sqcup K) \perp B$(by Definition 3.1.3(i)). Therefore $A \perp B$.
Conversely, if $A \perp B$, there exist an event C such that $C \perp A$ and $(A \sqcup C) oc B$. Since $B oc E \backslash B$, we have $(A \sqcup C) op E \backslash B$. Therefore $A \leq_e E \backslash B$.∎

Proposition 3.2.11
$\Pi(M)$ *has the least element* $[\phi]$ *and the greatest element* $[E]$.

[*Proof*]
Let A is an event in manual M. Then there exist some experiments that contain A. Let us choose one of them, say E. For ϕ and A, $A \perp \phi, (\phi \sqcup A) op A$. Hence $\phi \leq_e A$. For A and E, $E \backslash A \perp A, (A \sqcup E \backslash A) op E$. Hence $A \leq_e E$. Therefore $\phi \leq_e A \leq_e E$ and by definition $[\phi] \leq [A] \leq [E]$. By Lemma 3.2.8, all experiments are logically equivalent. Therefore $\Pi(M)$

has the least element $[\phi]$ and the greatest element $[E]$. ∎

We introduce a binary operation $\vee$ such that $[A] \vee [B]$ is a supremum of both $[A]$ and $[B]$ by partial order $\leq$. $[A] \vee [B]$ does not exist for every pair of elements. But following proposition is valid.

Proposition 3.2.12
$[A] \vee [A]' = [E]$.

[Proof]
If $[A] \leq [X]$ and $[A]' \leq [X]$, then $[X]' \leq [A] \leq [X]$. Hence $[X]' \perp [X]'$ and this means $E \backslash X$ and $E \backslash X$ is disjoint. Therefore $X = E$ and there exists only one element $[A] \vee [A]' = [E]$. ∎

By Theorem 3.2.9, Proposition 3.2.11 and Proposition 3.2.12, an operational logic is an orthoposet in general.

Theorem 3.2.13
If a manual M contains only one experiment E, $\leq_e$ is an inclusion and $'$ is a complementation in E. Moreover the operational logic $(\Pi(M), \leq, ')$ is a Boolean algebra.

[Proof]
It is obvious that $'$ is a complementation in E. If $A \leq_e B$, there exist events C and D such that $C \perp A$, $(A \sqcup C) oc D$ and $D oc B$. By assumption A, B, C, D are disjoint subsets of E. Hence $A \sqcup C = B$ and $A \subseteq B$. Therefore $\Pi(M)$ is the power set of E and $(\Pi(M), \leq, ')$ is a Boolean algebra. ∎

3.3 Examples

We can see a manual as a system of Boolean algebras. We shall introduce some examples of manuals related to certain cases of Greechie diagrams.

Example 3.3.1
A firefly is in a triangle box. These experiments consists a manual which is represented by the triangle Greechie diagram. The operational logic of this manual is merely an orthoposet. (See Figure 3.4 and 3.5.)

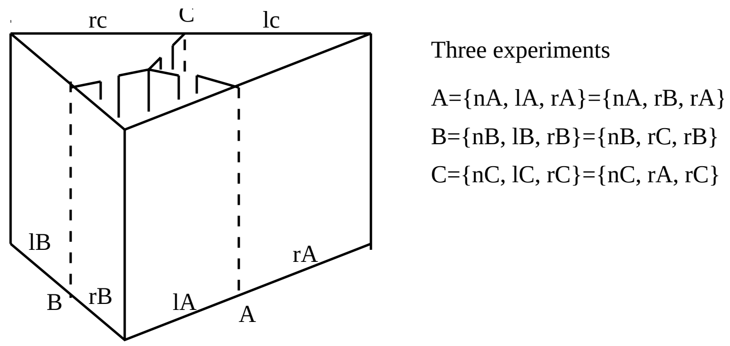

Figure 3.4: A firefly in a triangle chamber. There are three experiments A, B and C.

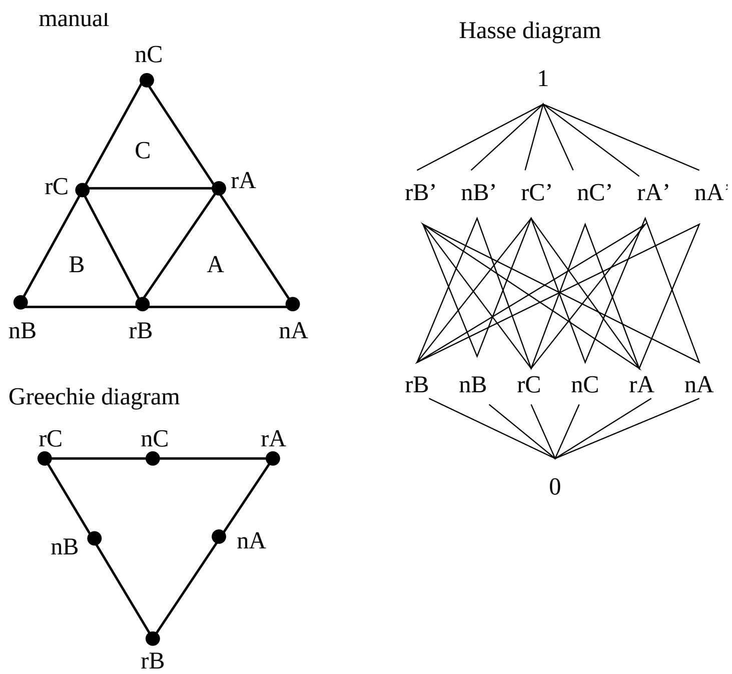

Figure 3.5: The manual, Greechie diagram and Hasse diagram of the triangle chamber experiment.

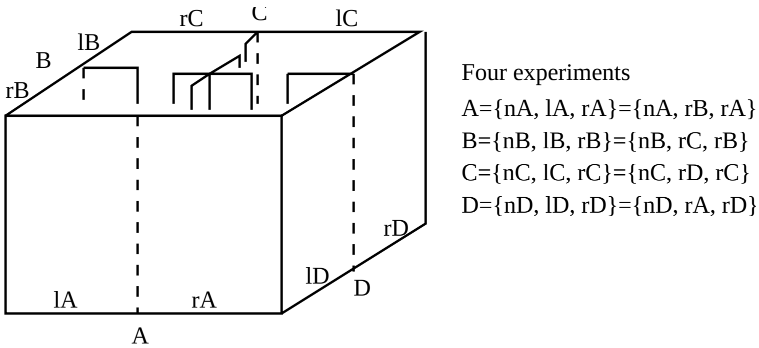

Figure 3.6: A firefly in a square chamber. There are four experiments A, B, C and D.

Example 3.3.2
A firefly is in a square box. These experiments consists a manual which is represented by the square Greechie diagram. The operational logic of this manual is an OMP, but not a lattice. (See Figure 3.6 and 3.7.)

Example 3.3.3
A firefly is in a pentagon box. These experiments consists a manual which is represented by the pentagon Greechie diagram. The operational logic of this manual is an OML. (See Figure 3.8 and 3.9.)

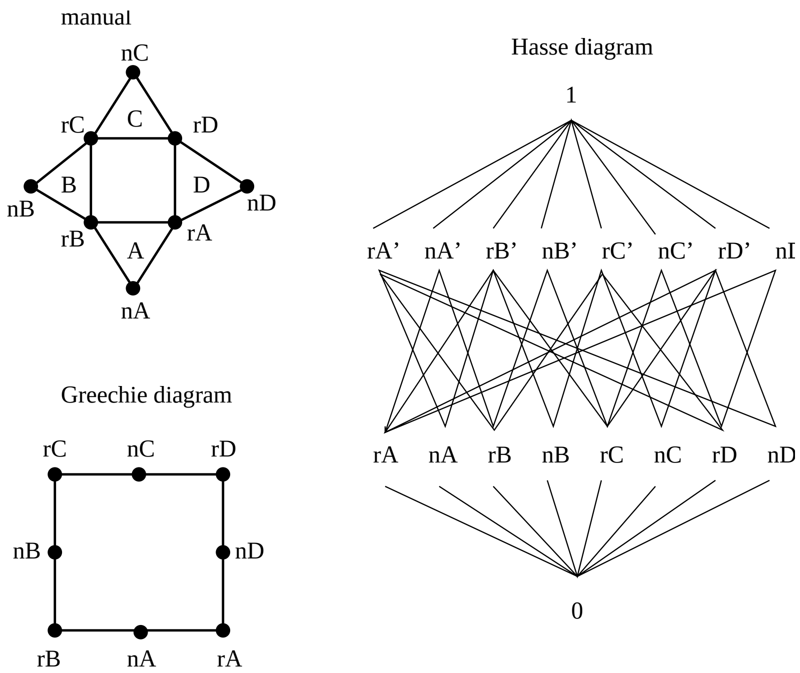

Figure 3.7: The manual, Greechie diagram and Hasse diagram of the square chamber experiment.

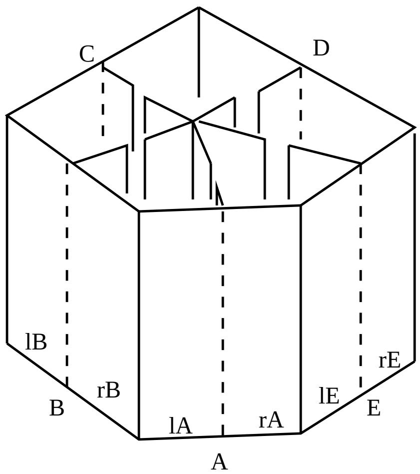

Figure 3.8: A firefly in a pentagon chamber. There are five experiments A, B, C, D and E.

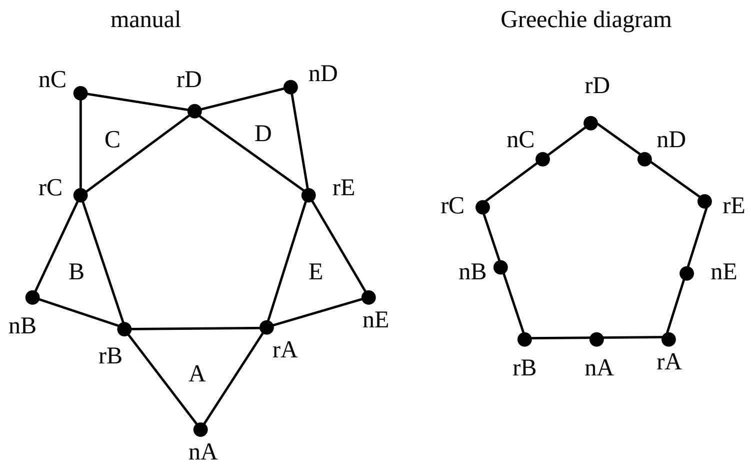

Figure 3.9: The manual and Greechie diagram of the pentagon chamber experiment.

Chapter 4

Automata experiments

4.1 Moore and Mealy automata

An *alphabet* is a finite nonempty set. The elements of an alphabet are called *symbols*. A *word* is a finite(possibly empty) sequence of symbols. Σ^* denotes the set of all words over an alphabet Σ. The *empty word* is denoted by ϵ.

Definition 4.1.1 (Moore Automaton)
A Moore Automaton is a five-tuple $M = (Q, \Sigma, \Delta, \delta, \lambda)$, where:
(i) Q is a finite set, called the state set.
(ii) Σ is an alphabet, called the input alphabet.
(iii) Δ is an alphabet, called the output alphabet.
(iv) δ is a mapping $Q \times \Sigma$ to Q, called the transition function.
(v) λ is a mapping Q to Δ , called the output function.

Let us sketch the appropriate picture informally. At any time, the automaton is in a state $q \in Q$, emitting the output $\lambda(q) \in \Delta$. If an input $a \in \Sigma$ is applied to the automaton, in the next discrete time step the automaton instantly assumes the state $p = \delta(q, a)$ and emits the output $\lambda(p)$.

Definition 4.1.2 (Mealy Automaton)
A Mealy Automaton is a five-tuple $M = (Q, \Sigma, \Delta, \delta, \lambda)$, where:
(i) Q is a finite set, called the state set.
(ii) Σ is an alphabet, called the input alphabet.
(iii) Δ is an alphabet, called the output alphabet.

(iv) δ is a mapping Q $\times$ Σ to Q, called the transition function.
(v) λ is a mapping Q $\times$ Σ to Δ , called the output function.

A Mealy automaton emits the output at the instant of the transition from one state to another. The output depends both on the previous state and on the input.

We define a mapping $\hat{\delta} : Q \times \Sigma^* \to Q$ as follows:
(i) $\hat{\delta}(q, \epsilon) := q$;
(ii) $\hat{\delta}(q, aw) := \hat{\delta}(\delta(q, a), w)$ for $w \in \Sigma^*, a \in \Sigma$.

We also extend the output function λ to a mapping $\hat{\lambda} : Q \times \Sigma^* \to \Delta^*$ as follows. Let $a_1, ..., a_n \in \Sigma$. We define:
For Moore automata,
(i) $\hat{\lambda}(q, \epsilon) := \lambda(q)$;
(ii) $\hat{\lambda}(q, aw) := \lambda(q)\hat{\lambda}(\delta(q, a), w)$.
(Explicitly, $\hat{\lambda}(q, a_1 \cdots a_n) = \lambda(q)\lambda(\delta(q, a_1))\lambda(\delta(q, a_1 a_2)) \cdots\cdots \lambda(\delta(q, a_1 \cdots a_n)$.)
For Mealy automata,
(i) $\hat{\lambda}(q, \epsilon) := \epsilon$;
(ii) $\hat{\lambda}(q, aw) := \lambda(q, a)\hat{\lambda}(\delta(q, a), w)$.
(Explicitly, $\hat{\lambda}(q, a_1 \cdots a_n) = \lambda(q, a_1)\lambda(\delta(q, a_1), a_2) \cdots\cdots \lambda(\delta(q, a_1 \cdots a_{n-1}), a_n)$.)

In the following, we use δ in place of $\hat{\delta}$(respectively λ in place of $\hat{\lambda}$) so far as we don't confuse them.

Definition 4.1.3 (equivalence of states)

Let p, q be any two states belonging to the state set Q. We say p *is equivalent to* q, denoted $p \equiv q$, if and only if

$$\forall w \in \Sigma^*, \lambda(p, w) = \lambda(q, w). \tag{4.1}$$

We define $[q] := \{p \in Q, p \equiv q\}$, and call it *equivalence class* of Q.We also define the quotient set of Q by $\equiv$ as $Q/\equiv := \{[q] \mid q \in Q\}$.

We shall define a somewhat weaker equivalence property as follows.

Definition 4.1.4 (w-equivalence of states)

Let p, q be any two states belonging to the state set Q. We say *p is $w-$equivalent to q*, denoted $p \equiv_w q$, if and only if, for an input word $w \in \Sigma^*$,

$$\lambda(p, w) = \lambda(q, w). \tag{4.2}$$

We define $[q]_w := \{p \in Q, p \equiv_w q\}$, and call it *$w-$equivalence class* of Q. We also define the quotient set of Q by $\equiv_w$ as $Q/\equiv_w := \{[q]_w \mid q \in Q\}$.

Lemma 4.1.5

The relations $\equiv_w$ and $\equiv$ are equivalence relations.

$[Proof]$
(i)symmetry: If $p \equiv_w q$, then $\lambda(q, w) = \lambda(p, w)$. Therefore $q \equiv_w p$.
(ii)reflexivity: Obviously $\lambda(q, w) = \lambda(q, w)$, hence, $q \equiv_w q$.
(iii)transitivity: If $p \equiv_w q$ and $q \equiv_w r$, then $\lambda(p, w) = \lambda(q, w) = \lambda(r, w)$, therefore $p \equiv_w r$.
For the relation $\equiv$, the proof is similar.∎

It is clear that if $p \equiv q$, then $p \equiv_w q$, and therefore $[q] \subseteq [q]_w$ for all $w \in \Sigma^*$.

Definition 4.1.6 (minimal automaton)

We call an automaton *minimal* if any two distinct states of the automaton are not equivalent, i.e.;

$$\forall p, q \in Q, p \neq q, \exists w \in \Sigma^*, \lambda(p, w) \neq \lambda(q, w). \tag{4.3}$$

Definition 4.1.7 (equivalence of automata)

Let $M_1 = (Q_1, \Sigma, \Delta, \delta_1, \lambda_1)$, $M_2 = (Q_2, \Sigma, \Delta, \delta_2, \lambda_2)$ be two automata of the same type(both are either Moore or Mealy automata).

A state $q_1 \in Q_1$ is said to be *equivalent* to a state $q_2 \in Q_2$ if and only if

$$\forall w \in \Sigma^*, \lambda_1(q_1, w) = \lambda_2(q_2, w). \tag{4.4}$$

The two automata M_1 and M_2 are said to be *equivalent* if for each state $q_1 \in Q_1$ there exists an equivalent state $q_2 \in Q_2$, and, conversely, for each state $q_2 \in Q_2$ there exists an equivalent state $q_1 \in Q_1$.

Theorem 4.1.8

Let $M = (Q, \Sigma, \Delta, \delta, \lambda)$ be a Moore or Mealy automaton. Then there exists a minimal automaton equivalent to M.

[*Proof*]

Put $M^m = (Q/\equiv, \Sigma, \Delta, \delta^m, \lambda^m)$. Define $\delta^m([q], a) = [\delta(q, a)]$ for all $a \in \Sigma$. If M is a Moore automaton, define $\lambda^m([q]) = \lambda(q)$. If M is a Mealy automaton, define $\lambda^m([q], a) = \lambda(q, a)$. According to the construction, M^m is minimal. Every state $q \in Q$ is equivalent to the state $[q] \in Q/\equiv$. Therefore, also M and M^m is equivalent.∎

Now, let M_1 be a Moore automaton and M_2 be a Mealy automaton. There can never be equivalence in the above sense between these automata because the output of a Moore automaton to the input $w \in \Sigma^*$ contains one more symbol than the output of the Mealy automaton. However, we may neglect the first output symbol of a Moore automaton $M = (Q, \Sigma, \Delta, \delta, \lambda)$ by using a reduced output function $\lambda' : Q \times \Sigma^* \to \Delta^*$ defined by

$\lambda'(q, a_1 \cdots a_n) = \lambda(\delta(q, a_1)) \cdots \lambda(\delta(q, a_1 \cdots a_n))$
(i.e.,$\lambda(q, w) = \lambda(q)\lambda'(q, w)$).

Then we have the following theorems, equating the Moore and Mealy automaton.

Theorem 4.1.9

Let $M_1 = (Q, \Sigma, \Delta, \delta_1, \lambda_1)$ be a Moore automaton. Then there exists a Mealy automaton M_2 equivalent to M_1.

[*Proof*]

Put $M_2 = (Q, \Sigma, \Delta, \delta_1, \lambda_2)$, where $\lambda_2(q, a) = \lambda_1(\delta(q, a))$ for any $q \in Q$ and any $a \in \Sigma$. Then the two automaton are equivalent. In fact, for an input word $a_1 a_2 \cdots a_n$, putting $\delta(q_{i-1}, a_i) = q_i (1 \leq i \leq n)$,

$\lambda_1(q_0, a_1 a_2 \cdots a_n) = \lambda_1(q_0)\lambda_1(q_1)\lambda_1(q_2) \cdots \lambda_1(q_n)$,
$\lambda_2(q_0, a_1 a_2 \cdots a_n) = \lambda_1(q_1)\lambda_1(q_2) \cdots \lambda_1(q_n)$,
hence $\lambda_1'(q_0, a_1 a_2 \cdots a_n) = \lambda_2(q_0, a_1 a_2 \cdots a_n)$. ∎

Theorem 4.1.10

Let $M_1 = (Q, \Sigma, \Delta, \delta_1, \lambda_1)$ be a Mealy automaton. Then there exists a Moore automaton M_2 equivalent to M_1.

[*Proof*]

Put $M_2 = (Q \times \Delta, \Sigma, \Delta, \delta_2, \lambda_2)$. Define $\delta_2((q, x), a) = (\delta_1(q, a), \lambda_1(q, a))$ and $\lambda_2((q, x)) = x$ for any $(q, x) \in Q \times \Delta$ and $a \in \Sigma$. Then, the states $q \in Q$ of M_1 and $(q, x) \in Q \times \Delta$, x arbitrary, of M_2 are equivalent. Therefore, also M_1 and M_2 are equivalent. In this case,

$\lambda_1(q_0, a_1 a_2 \cdots a_n) = \lambda_1(q_0, a_1)\lambda_1(q_1, a_2) \cdots \lambda_1(q_{n-1}, a_n)$,
$\lambda_2(q_0, a_1 a_2 \cdots a_n) = \lambda_2((q_0, x))\lambda_1(q_0, a_1)\lambda_1(q_1, a_2) \cdots \lambda_1(q_{n-1}, a_n)$,
hence $\lambda_1(q_0, a_1 a_2 \cdots a_n) = \lambda_2'(q_0, a_1 a_2 \cdots a_n)$. ∎

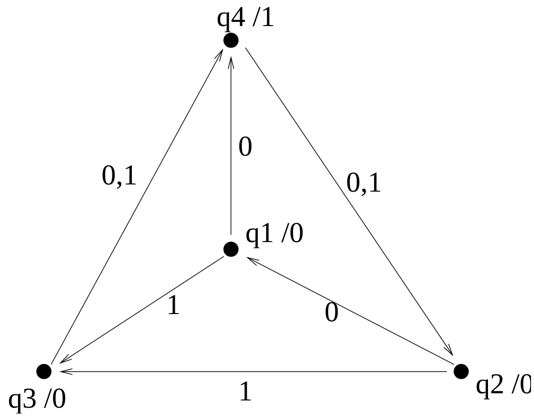

Figure 4.1: Moore's uncertainty automaton.

4.2 State decision problem of automata

In what follows we assume that we are dealing with a Moore or Mealy automaton, which is contained in a black box with input-output interface. Thus, we are only allowed to observe the input and output words associated with the box. We suppose, however, that we know the transition table of considering automaton(i.e., the five-tuple$(Q, \Sigma, \Delta, \delta, \lambda)$).

Let us consider the Moore automaton of Figure4.1. Every vertex is labeled by a pair (q/x), $q \in \{q_1, q_2, q_3, q_4\}(= Q)$, $x \in \{0, 1\}(= \Delta)$, and this means $\lambda(q) = x$. For example, if the current state is q_1 and we apply input 0, then the automaton emits 0 as a free output, and after the transition q_1 to q_4, it emits 1. Consequently we observe the output word 01.

This automaton is minimal because,
$\lambda(q_4, \epsilon) = 1 \neq \lambda(q_{1,2,3}, \epsilon) = 0$;
$\lambda(q_1, 0) = 01 \neq \lambda(q_2, 0) = 00$;
$\lambda(q_1, 1) = 00 \neq \lambda(q_3, 1) = 01$;
$\lambda(q_2, 0) = 00 \neq \lambda(q_3, 0) = 01$.

Suppose that we don't know which the current state q is. How we can know it? This is called "the state decision problem" of automata. To decide q, we shall perform some experiments on the automaton, i.e., apply some input words.

If the free output is 1, we can say $q = q_4$ so that we need not to perform any experiment. If the free output is 0, we can only know $q \in \{q_1, q_2, q_3\}$, and to know more detail, we have to perform some experiment.

Let us apply the input 0. If we observe the output 0, then $q = q_2$, and the problem is solved. If the output is 1, however, we cannot know more detail, because in either case of

input\	q1	q2	q3	q4
0	4	1	4	2
1	3	3	4	2

Transition Function

input\	q1	q2	q3	q4
0	a	b	a	c
1	a	a	b	c

Output Function

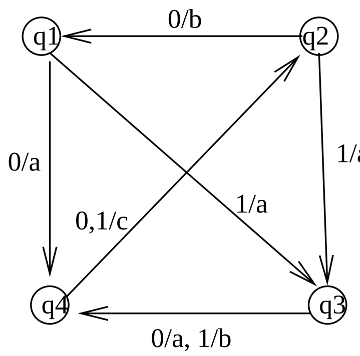

Figure 4.2: Mealy's uncertainty automaton.

$q = q_1$ or $q = q_3$, after applying the input 0, the current state transits to q_4 so that any consequent input gives us the same output words.

Any experiment which begins with the input 0(we shall call it "experiment $E(0)$") cannot distinguish q_1 from q_3. Similarly, any experiment which begins with the input 1("experiment $E(1)$") cannot distinguish q_1 from q_2. This phenomenon is called "Moore's uncertainty principle".

One more example is drawn in Figure4.2. This is a Mealy automaton which has the same type problem as the Moore automaton of Figure4.1. Every edge is labeled by a pair (u/x), where $u \in \{0,1\}(= \Sigma)$ and $x \in \{a,b,c\}(= \Delta)$. This automaton is minimal, and also in this case, experiment $E(0)$ cannot distinguish q_1 from q_3, and experiment $E(1)$ cannot distinguish q_1 from q_2. (Note that we do not say that these two automata above are equivalent. We only say that they have the same type problem about uncertainty.)

The types of experiments that we can perform are limited by the number of identical copies of the automaton we have available for investigation. Usually, when we are carrying out an experiment, we assume that only a single copy of the automaton is available. Such an experiment is called a *simple experiment*. On occasion, however, we have several identical

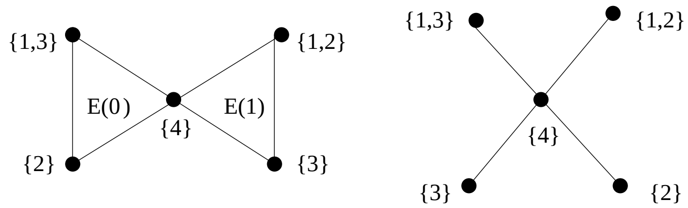

Figure 4.3: The "bow-tie" manual and Greechie diagram.

copies of the automaton or a single automaton with a "reset" button. Experiments that take advantage of the ability of effectively more than one copy of an automaton are called *multiple experiments*.

In the following, we shall discuss about only simple experiments in view of the correspondence to quantum physical experiments, because, in quantum physics, it is a fundamental condition that we cannot reset a physical system after performing any experiments.

4.3 Operational logics of automata experiments

In the above two examples, by the experiment $E(0)$, we can know $q \in \{q_1, q_3\}$ or $q \in \{q_2\}$ or $q \in \{q_4\}$. We shall denote this $E(0) = \{\{1,3\},\{2\},\{4\}\}$. Using the terms of operational logic, outcomes of experiment $E(0)$ are

$\{1,3\}$, $\{2\}$, $\{4\}$

and events of $E(0)$ are

ϕ, $\{1,3\}$, $\{2\}$, $\{4\}$, $\{\{1,3\},\{2\}\}$, $\{\{1,3\},\{4\}\}$, $\{\{1,3\},\{2\},\{4\}\}$.

Similarly $E(1) = \{\{1,2\},\{3\},\{4\}\}$,

its outcomes are

$\{1,2\}$, $\{3\}$, $\{4\}$

and its events are

ϕ, $\{1,2\}$, $\{3\}$, $\{4\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,2\},\{4\}\}$, $\{\{1,2\},\{3\},\{4\}\}$.

$E(0)$ and $E(1)$ contains a common element $\{4\}$, and we can write their "bow-tie" manual, Greechie diagram(see Figure4.3) and Hasse diagram(see Figure4.4). follows. In this case, the operational logic becomes an orthomodular lattice.

Let us introduce two more examples. Figure4.5 is a Mealy automaton with four states and three experiments

$E(0) = \{\{1, 2\}, \{3\}, \{4\}\}$,

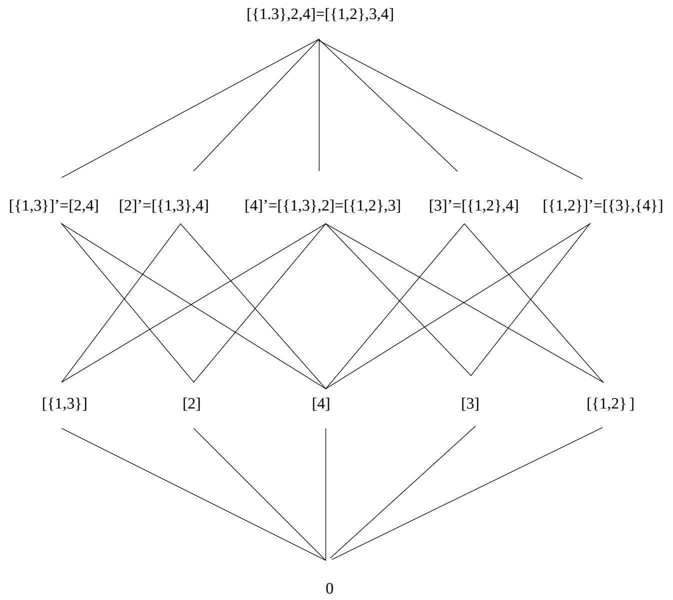

Figure 4.4: Hasse diagram.

$E(1) = \{\{1, 3\}, \{2\}, \{4\}\}$,
$E(2) = \{\{1, 4\}, \{2\}, \{3\}\}$.
The manual and Greechie diagram are drawn in Figure4.6, and the operational logic of these experiments becomes an orthoposet.

Figure4.7 is a Mealy automaton with five states and four experiments
$E(0) = \{\{1, 2, 3\}, \{4\}, \{5\}\}$,
$E(1) = \{\{1, 3, 4\}, \{2\}, \{5\}\}$,
$E(2) = \{\{1, 4, 5\}, \{2\}, \{3\}\}$,
$E(3) = \{\{1, 2, 5\}, \{3\}, \{4\}\}$.
The manual and Greechie diagram are drawn in Figure4.8, and the operational logic of these experiments becomes an orthomodular poset.

4.4 Origin of Moore's uncertainty

Heisenberg's uncertainty is one of the laws of nature and we have to accept it. But automata are artificial objects, and therefore it is worth considering the origin of Moore's

input	q1	q2	q3	q4
0	q1	q1	q1	q1
1	q2	q2	q2	q2
2	q3	q3	q4	q3

Transition Function

input	q1	q2	q3	q4
0	a	a	b	c
1	a	b	a	c
2	a	b	c	a

Output Function

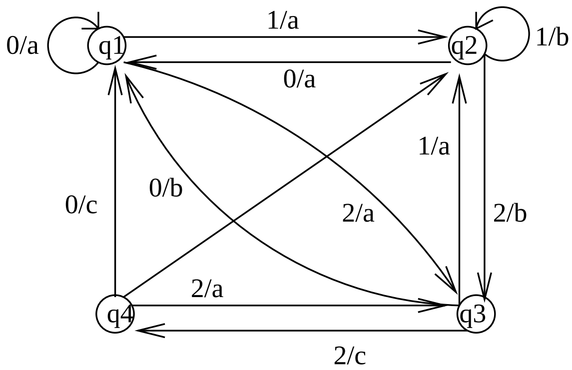

Figure 4.5: Mealy automaton with three experiments.

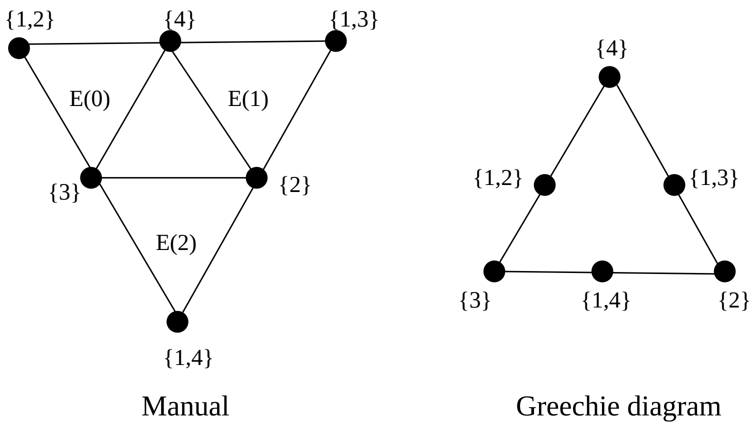

Figure 4.6: Manual and Greechie diagram of three experiments.

input	q1	q2	q3	q4	q5
0	q1	q1	q1	q5	q5
1	q2	q2	q2	q2	q2
2	q3	q3	q3	q3	q3
3	q4	q4	q4	q5	q4

Transition Function

input	q1	q2	q3	q4	q5
0	a	a	a	b	c
1	a	b	a	a	c
2	a	b	c	a	a
3	a	a	b	c	a

Output Function

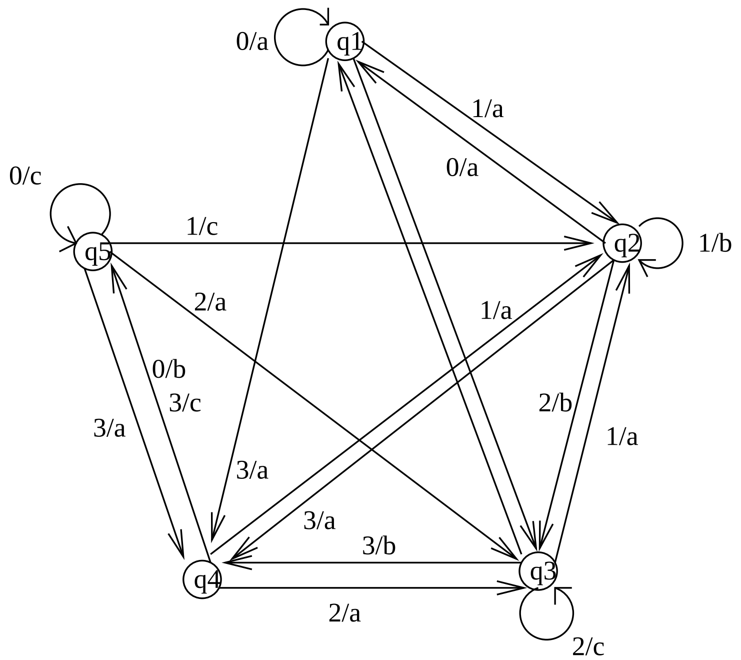

Figure 4.7: Mealy automaton with four experiments.

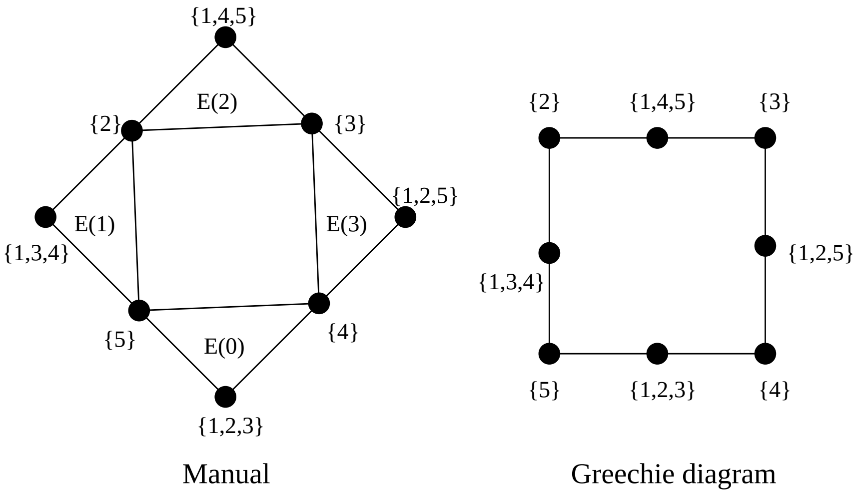

Figure 4.8: Manual and Greechie diagram of four experiments.

uncertainty.

According to the discussion in Section 4.3, it seems proper to define an uncertainty of the of the state decision problem as follows.

Definition 4.4.1 (uncertainty of the state decision problem)

For an automaton $M = (Q, \Sigma, \Delta, \delta, \lambda)$, we shall say *the state decision problem has an uncertainty*, or simply say *the automaton M is uncertain* if the following condition is satisfied.

$$\forall w \in \Sigma^*, \exists p, q \in Q, p \neq q, \lambda(p, w) = \lambda(q, w). \tag{4.5}$$

We shall say *the automaton is decidable* if it is not uncertain.

The negation of the condition(4.5) is

$$\exists w \in \Sigma^*, \forall p, q \in Q, p \neq q, \lambda(p, w) \neq \lambda(q, w). \tag{4.6}$$

This means there exists at least one word to distinguish all different states, therefore the state decision problem is solvable.

If the automaton is not minimal, there exist at least two equivalent states, say p, q, and $\forall w \in \Sigma^*, \lambda(p, w) = \lambda(q, w)$. Hence the condition(4.5) is satisfied and we have the following lemma.

Lemma 4.4.2

If the automaton is not minimal, then it is uncertain. In other words, if the automaton is decidable, then it is minimal.

For uncertain automata, we have following theorems.

Theorem 4.4.3

Let $M = (Q, \Sigma, \Delta, \delta, \lambda)$ be a minimal Moore automaton. Then M is uncertain if the following condition is satisfied.

$$\forall a \in \Sigma, \exists p, q \in Q, p \neq q, \lambda(p) = \lambda(q) \textit{ and } \delta(p, a) = \delta(q, a). \quad (4.7)$$

$[Proof]$

If $\lambda(p) = \lambda(q)$ and $\forall a \in \Sigma, \delta(p, a) = \delta(q, a)$, for arbitrary $w' \in \Sigma^*$,
$\lambda(p, aw')$
$= \lambda(p)\lambda(\delta(p, a), w')$
$= \lambda(q)\lambda(\delta(q, a), w')$
$= \lambda(q, aw')$.
Together with $\lambda(p, \epsilon) = \lambda(p) = \lambda(q) = \lambda(q, \epsilon)$, the condition (4.5) is satisfied.∎

Theorem 4.4.4

Let $M = (Q, \Sigma, \Delta, \delta, \lambda)$ be a minimal Mealy automaton. Then M is uncertain if the following condition is satisfied.

$$\forall a \in \Sigma, \exists p, q \in Q, p \neq q, \lambda(p, a) = \lambda(q, a) \textit{ and } \delta(p, a) = \delta(q, a). \quad (4.8)$$

$[Proof]$

If $\lambda(p, a) = \lambda(q, a)$ and $\forall a \in \Sigma, \delta(p, a) = \delta(q, a)$, for arbitrary $w' \in \Sigma^*$,
$\lambda(p, aw')$
$= \lambda(p, a)\lambda(\delta(p, a), w')$
$= \lambda(q, a)\lambda(\delta(q, a), w')$
$= \lambda(q, aw')$.
Together with $\lambda(p, \epsilon) = \lambda(q, \epsilon) = \epsilon$, the condition (4.5) is satisfied.∎

Theorem 4.4.5

Let M_1 be a minimal and uncertain automaton, and M_2 be an automaton equivalent to M_1. Then M_2 is also uncertain.

[*Proof*]
Let us put $M_1 = (Q_1, \Sigma, \Delta, \delta_1, \lambda_1)$ and $M_2 = (Q_2, \Sigma, \Delta, \delta_2, \lambda_2)$. If M_1 is uncertain, then

$$\forall w \in \Sigma^*, \exists p_1, q_1 \in Q_1, p_1 \neq q_1, \lambda_1(p_1, w) = \lambda_1(q_1, w).$$

By Definition 4.1.7, there exist $p_2, q_2 \in Q_2$ equivalent to p_1, q_1 respectively, and $\lambda_2(p_2, w) = \lambda_2(q_2, w)$. Since M_1 is minimal, $p_2 \neq Q_2$, and then the condition

$$\forall w \in \Sigma^*, \exists p_2, q_2 \in Q_2, p_2 \neq q_2, \lambda_2(p_2, w) = \lambda_2(q_2, w)$$

is satisfied. Therefore, M_2 is also uncertain.∎

Hence, if M_1 is minimal and uncertain automaton, then there exists no decidable automaton M_2 which is equivalent to M_1. (Intuitively speaking, we cannot "eliminate" the uncertainty of M_1.)
To be noted, the equivalence between a Moore automaton and a Mealy type one is defined by the reduced output function.

Let us discuss about a "degree of uncertainty" of an automaton. In the following, we use the symbol $|A|$ to denote the number of elements of the set A.

Definition 4.4.6 (degree of uncertainty)
Let $M = (Q, \Sigma, \Delta, \delta, \lambda)$ be an automaton. An input word $w \in \Sigma^*$ determines a w-equivalent class $[q]_w$, which is a subset of Q. We define

$$d(M, w) := \max_{q \in Q} |[q]_w| - 1 \tag{4.9}$$

and call it *the degree of uncertainty of M for w*. Moreover, we define

$$d(M) := \min_{w \in \Sigma^*} d(M, w) \tag{4.10}$$

and call it *the degree of uncertainty of the automaton M*.

For example, let us introduce a Mealy automaton M shown in Figure 4.9. This automaton is decidable because we can distinguish all three states by applying input word w_0 which is an arbitrary word beginning with 0. But, if we apply w_1(an arbitrary word beginning with 1), q_1 and q_3 becomes indistinguishable. For this automaton,

$$\begin{aligned} d(M, w_0) &= \max_{q \in \{q_1, q_2, q_3\}} |[q]_{w_0}| - 1 = 0, \\ d(M, w_1) &= \max_{q \in \{q_1, q_2, q_3\}} |[q]_{w_1}| - 1 = 1, \end{aligned}$$

input	q1	q2	q3
0	q2	q3	q1
1	q2	q3	q2

Transition Function

input	q1	q2	q3
0	a	b	c
1	a	b	a

Output Function

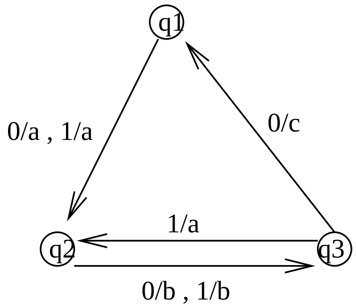

Figure 4.9: An example of a decidable Mealy automaton.

and the degree of uncertainty of this automaton is

$$d(M) = \min_{w_0, w_1 \in \Sigma^*} \{d(M, w_0), d(M, w_1)\} = 0.$$

Let M_1, M_2 be two automata. If $d(M_1) \leq d(M_2)$, we say M_2 *is more uncertain than* M_1, or M_1 *is less uncertain than* M_2.

If $d(M_1) = d(M_2)$, we say M_1 *and* M_2 *have the same uncertainty.*

Theorem 4.4.7

(i) Let M_1 *be an uncertain Moore automaton and* M_2 *be a Mealy automaton equivalent to* M_1*, made by the procedure in Theorem 4.1.9. Then* M_1 *and* M_2 *have the same uncertainty.*

(ii) Let M_1 *be an uncertain Mealy automaton and* M_2 *be a Moore automaton equivalent to* M_1*, made by the procedure in Theorem 4.1.10. Then* M_2 *is more uncertain than* M_1*.*

$[Proof]$

Let us put $M_1 = (Q_1, \Sigma, \Delta, \delta_1, \lambda_1)$ and $M_2 = (Q_2, \Sigma, \Delta, \delta_2, \lambda_2)$. If M_1 is uncertain

$$\forall w \in \Sigma^*, \exists p_1, q_1, p_1 \neq q_1, \lambda_1(p_1, w) = \lambda_1(q_1, w),$$

and by Definition 4.1.7, for each state $q_1 \in Q_1$, there exist an equivalent state $q_2 \in Q_2$.

(i) By the procedure in Theorem 4.1.9, $Q_1 = Q_2$ and the correspondence between the equivalent states $q_1 \in Q_1$ and $q_2 \in Q_2$ is bijective. Hence for all $w \in \Sigma^*$ and for all $q_1 \in Q_1, q_2 \in Q_2$, $|[q_1]_w| = |[q_2]_w|$, and we have $d(M_1) = d(M_2)$. Therefore M_1 and M_2 have the same uncertainty.∎

(ii) By the procedure in Theorem 4.1.10, for $q_1 \in Q_1$, the equivalent states of $Q_2 = Q_1 \times \Delta$ is (q, x), where x is an arbitrary symbol of Δ. Hence $|[q_2]_w| = |[q_1]_w| \times |\Delta| \geq |[q_1]_w|$ for all $w \in \Sigma^*$ and for all, $q_1 \in Q_1, q_2 \in Q_2$, and we have $d(M_1) \leq d(M_2)$. Therefore M_2 is more uncertain than M_1.∎

Theorem 4.4.8

Let $M = (Q, \Sigma, \Delta, \delta, \lambda)$ be a Moore or a Mealy automaton and M^m is a minimal automaton equivalent to M. Then M^m is less uncertain than M.

[*Proof*]

By Theorem 4.1.8, $M^m = (Q/\equiv, \Sigma, \Delta, \delta^m, \lambda^m)$ where $\lambda^m([q], w) = \lambda(q, w)$ for all $w \in \Sigma^*$, and every state $q \in Q$ is equivalent to the state $[q] \in Q/\equiv$. By definition,

$$\begin{aligned} d(M^m) &= \min_{w-in\Sigma^*} \max_{q \in Q} |[[q]]_w| - 1 \\ d(M) &= \min_{w-in\Sigma^*} \max_{q \in Q} |[q]_w| - 1. \end{aligned}$$

For $p, q \in Q$, we have

$$\begin{aligned} [p] \in [[q]]_w &\Leftrightarrow \lambda^m([p], w) = \lambda^m([p], w) \\ &\Leftrightarrow \lambda(p, w) = \lambda(q, w)) \\ &\Leftrightarrow p \in [q]_w. \end{aligned}$$

Thus, for $p' \in Q$, $[p] = [p']$ and $[p], [p'] \in [[q]]_w$ if and only if $p \equiv p'$ and $p, p' \in [q]_w$. Hence, we obtain $|[[q]]_w| = |[q]_w/\equiv|$, and this yields $|[[q]]_w| \leq |[q]_w|$ for all $q \in Q$ and all $w \in \Sigma^*$. Thus, we have $d(M^m) \leq d(M)$, and therefore M^m is less uncertain than M.∎

By this theorem, we can say in general that if we minimize an automaton, then the degree of uncertainty decreases.

For a degree of uncertainty, we have the following theorem.

Theorem 4.4.9

An automaton $M = (Q, \Sigma, \Delta, \delta, \lambda)$ is decidable if and only if $d(M) = 0$.

[*Proof*] If M is decidable, then there exists at least one input $w \in \Sigma^*$ such that

$$\forall p, q \in Q, p \neq q, \lambda(p, w) \neq \lambda(q, w). \text{ (Condition (4.6))}$$

Thus, for this word w, $Q/\equiv_w = Q$ and $|[q]_w| = 1$ for all $q \in Q$. Therefore $d(M) = 0$.

Conversely, if $d(M) = 0$, there exists at least one input word $w \in \Sigma^*$ such that $d(M, w) = 0$. Thus, by the definition of $d(M, w)$, we have $|[q]_w| = 1$ for all $q \in Q$, and this means that for this w,

$$\forall p, q \in Q, p \neq q, \lambda(p, w) \neq \lambda(q, w).$$

Therefore M is decidable.∎

Corollary 4.4.10

An automaton $M = (Q, \Sigma, \Delta, \delta, \lambda)$ *is uncertain if and only if* $d(M) \geq 1$.

[*Proof*]

For the proof, it is sufficient to take the negation of Theorem 4.4.9. If M is uncertain, $d(M) \neq 0$. By definition, $d(M) \geq 0$, and therefore we have $d(M) \geq 1$.∎

Chapter 5

Concluding remarks

5.1 Features of uncertainty

Let us discuss about the difference between the quantum physical uncertainty and the Moore's uncertainty.

In the former case, for example, the coordinate of an single electron has a distribution in the physical space with some probability, and does not have an unique value. This kind of uncertainty is called *ontological uncertainty.* [1]

In the latter case, the state of an automaton is just one of the state in Q, but we have no experiment to decide it. This kind of uncertainty is called *epistemic uncertainty*. Some people call it *undecidability* or *indeterminancy* and distinguish it from uncertainty.

For Moore's uncertainty principle, we obtained

$$d(M) \geq 1$$

if and only if the automaton M is uncertain. This is analogous to Heisenberg's uncertainty principle, and in the expression above, the constant "1" plays a similar roll to the Planck's constant $\hbar$(see expression (1.1)).

5.2 Quantum physical experiments

[1] A.Einstein disliked the existence of this uncertainty in the theory of quantum physics and said "God doesn't play a dice".

In Chapter 3, we proved that an operational logic is in general an orthoposet, and as such an example, making use of loop lemma(Theorem 2.2.6), we introduced the firefly experiment in a triangle chamber(Example 3.3.1).

Let us take this example again. This experiment is not a quantum physical one, in sense of that there exists "physically" possible experiments to know the more details about the considering physical system.

In the Example 3.3.1, the operational logic becomes an orthoposet because such a condition exists that we cannot perform three experiments $E(0)$, $E(1)$ and $E(2)$ simultaneously. But, this condition is physically eliminative by the following procedures.

(i) Posting three experimenters, one at each window, and assuming that they can communicate instantaneously.

or

(ii) Taking off the cover of the chamber and looking into the system from the top of it.

These procedures are physically possible, and if we do so, the experiment becomes a classical one. The experiment consists of four outcomes;

- n = see no light,
- r_A = see a light in a right half of window A (or a left half of window C),
- r_B = see a light in a right half of window B (or a left half of window A),
- r_C = see a light in a right half of window C (or a left half of window B).

The manual and the Greechie diagram of it are shown in Figure 5.1 and the operational logic is a Boolean algebra.

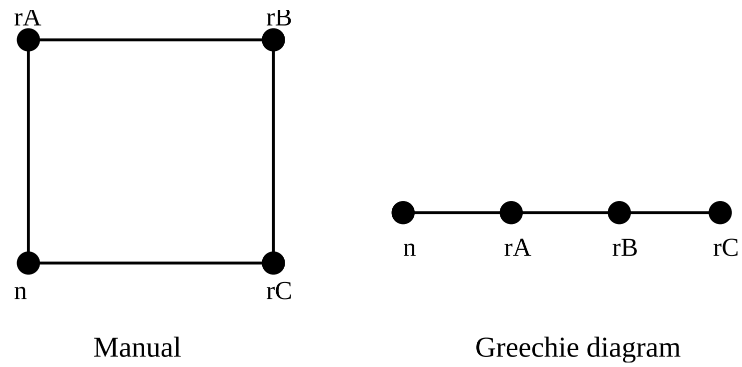

Figure 5.1: The classical manual and Greechie diagram of the triangle chamber experiment.

Thus, we have the following conclusion.

If quantum logics are OMLs, quantum physical experiments satisfies some conditions. Especially, for any combination of quantum physical experiments, their Greechie diagrams must not contain either a loop of order 3 or loop of order 4.

This seems a very difficult problem depending on quantum physics, and to prove it, we have to investigate the properties of all quantum physical experiments.

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