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Algebraic Structures of Operational Logics in Physical and Automata Experiments

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Quantum mechanics, which began with M.Planck's "quantum hypothesis" in 1900, has the following axioms.

Axiom 1
The state of a physical system is represented by an unit vector of a complex Hilbert space H. This vector is called "state vector". Provided that state vectors $\psi \in H$ and $\alpha\psi \in H$ represent the same state for an arbitrary complex number α with $|\alpha| = 1$.

Axiom 2
A physical quantity — sometimes called "observable" — is represented by a self adjoint operator on H.

Axiom 3
For a state vector ψ and an observable A, the expected value of A is given by the inner product $< \psi, \hat{A}\psi >$ of two vector ψ and $\hat{A}\psi$, where $\hat{A}$ is the self adjoint operator that corresponds to the observable A.

What is important here is that self adjoint operators do not commute in general, i.e., if $\hat{A}$ and $\hat{B}$ are such operators, then $< \psi, \hat{A}\hat{B}\psi > \neq < \psi, \hat{B}\hat{A}\psi >$. This means that experimental results depend on the order of the experiment of A and that of B. In fact, about an single electron, the operator $\hat{x}$ of coordinate x and the operator $\hat{p}$ of coordinate p satisfies $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$ ($\hbar = 1.054 \times 10^{-27}$erg·sec is the "Planck's constant"), and this relation yields that for standard deviations $\Delta x, \Delta p$ of x and p

$$\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar. \tag{1}$$

Namely, for this pair of physical quantities, there exists a "limit" such that we cannot know more precisely about the physical system. This is an example of the well-known "Heisenberg's uncertainty principle".

Meanwhile, a similar uncertainty is found in a kind of experiment on automata. "The state decision problem" is such an example of automata experiments. Let us consider that an automaton with an output function is contained in a black box and we will apply input words and observe its output words. At any time, the automaton is in one of its states, and the state decision problem is to determine by experiment — observing input-output behavior — which state the automaton currently in. An important property in this experiment is that if we apply some input words, then the automaton transits from the current state to another one. Therefore, the experimental results(informations about the automaton) are depend on the order of experiments. Moreover, the state decision problem is not always solvable, i.e., there exist a "limit" for the informations about the current state. In 1956 E.F.Moore pointed out these facts quite analogous to quantum physical experiments, and afterword, in 1971 J.H.Conway called them "Moore's uncertainty principle".

In this thesis, we take examples from physical experiments and automata ones, and discuss algebraically the relations of informations obtained by experiments. Such a study about quantum physical experiments are well-known as quantum logics. If we start from the axioms above and construct quantum logics as algebras of the self adjoint operators on Hilbert space, the logics appears to be orthomodular lattices([1],[3]).

Similarly, there exists a way for the state decision problem, which start from partitions of a state set of an automaton and make use of Hilbert space theory. Such a study is called "automaton logics" or "partition logics"([4],[5]).

But we do not take such a "top-down" style. Instead, we take a "bottom-up" style i.e., we begin with to investigate the features of experiments and then unify them like as physicists make a theory from a collection of physical experiments.

Now, in Chapter 2, we will give a basic notion of an orthomodular poset etc., and provide the proof of "loop lemma" and introduce "Greechie diagram". For a system of Boolean algebras $\mathcal{B} = \{B_0, B_1, ...B_{n-1}\}$, we define a partial order and an orthocomplementation on $L = \bigcup \mathcal{B}$ as follows;

- $x \leq y$ in L if and only if there exists a $B \in \mathcal{B}$ such that $x, y \in B$ and $x \leq_B y$.
- $x' = x'^B$, where $B \in \mathcal{B}$ and $x \in B$.

By this definition we can see $L = \bigcup \mathcal{B}$ as an algebra, and loop lemma is a method to investigate the algebraic property of L. Greechie diagram is also such a visual method.

In Chapter 3, taking examples of physical experiments, we define an *experiment* as a set which elements are called *outcomes*. We define a *manual* as a system of experiments satisfying a certain properties. A manual provides a whole property of experiments which we will perform.

We shall call a subset of a experiment in a manual *event* and it corresponds to an experimental result. By these definitions, we can treat physical experiments set theoretically.

Moreover, we define properly an orthocomplementation for each event and partial order between two events. These correspond to "negation" and "implication" respectively, i.e., for events A, B;

- $A' \Longleftrightarrow$ event A does not occur,
- $A \leq B \Longleftrightarrow$ if event A occurs, then event B occurs.

Thus, we can see a manual as an algebra and call it an *operational logic*. Making use of Greechie diagram, we show that an operational logic is generally an orthoposet, and becomes an orthomodular poset or an orthomodular lattice under some conditions. We also shows examples of physical experiments corresponding to these algebras.

In Chapter 4, we will give definitions of automata with output functions and introduce procedures to minimize or equivalently transform automata. Next, we show examples of the state decision problems, and making use of operational logics, investigate algebraic structures of the problems. We also give examples of the state decision problems corresponding respectively to orthoposet, orthomodular poset, or orthomodular lattice.

In addition, we discuss about the origin of Moore's uncertainty in detail. Heisenberg's uncertainty is one of the laws of nature and we have to accept it. Contrary, automata are artificial objects, and therefore it is worth considering the origin of Moore's uncertainty. We will give the definition of an "uncertainty of the state decision problem" as follows.

Definition (uncertainty of the state decision problem)

If an automaton satisfies the following condition, we shall say *the state decision problem has an uncertainty*, or simply say *the automaton is uncertain* .

- For any input words, there exist at least two distinct states, and they give the same output word.

We shall say *the automaton is decidable* if it is not uncertain.

By this definition, we can obtain the following theorems.

Lemma

If the automaton is not minimal, then it is uncertain. In other words, if the automaton is decidable, then it is minimal.

Theorem

An automaton is uncertain if for all input symbols, there exists at least two distinct states such that they gives the same output symbol and they transit to the same state.

Theorem
Let M_1 be a minimal and uncertain automaton, and M_2 be an automaton equivalent to M_1. Then M_2 is also uncertain.

Moreover, we define properly a "degree of uncertainty" and discuss quantitatively about Moore's uncertainty. If we denote a degree of uncertainty of an automaton M by $d(M)$, we can obtain the following theorems.

Theorem
An automaton M is decidable if and only if $d(M) = 0$.
Corollary
An automaton M is uncertain if and only if $d(M) \geq 1$.

The inequality in this corollary is analogous to the Heisenberg's one(expression (1).
We can show that for an automaton M_1, if we make an automaton M_2 equivalent to M_1, $d(M_1) = d(M_2)$ does not always holds, and $d(M_1) \leq d(M_2)$ and $d(M_1) \geq d(M_2)$ possibly occurs depending on the procedure to construct M_2. We can also show the following theorem.

Theorem
Let M is an automaton and M^m is a minimal automaton equivalent to M. Then $d(M^m) \leq d(M)$ holds.

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