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Description	

A Construction of Sound Semantic Linguistic Scales Using 4-Tuple Representation of Term Semantics

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Abstract

Data semantics plays a fundamental role in computer science, in general, and in computing with words, in particular. The semantics of words arises as a very sophisticated problem, since words being actually vague linguistic terms are pieces of information characterized by impreciseness, incompleteness, uncertainty and/or vagueness. The qualitative semantics and the quantitative semantics are two aspects of vague linguistic information, which are closely related. However, the qualitative semantics of linguistic terms, and even the qualitative semantics of the symbolic approaches, seem to be not elaborated explicitly in the literature up to now. In this study, we propose an interpretation of the inherent order-based semantics of terms through their qualitative semantics modeled by hedge algebra structures. The quantitative semantics of terms are developed based on the quantification of hedge algebras. With such an explicit approach, we propose two concepts of assessment scales for addressing decision problems: linguistic scales used for representing expert linguistic assessments and semantic linguistic scales based on 4-tuple linguistic representation model, which form a formalized structure useful for computing with words. An example of solving a simple multi-criteria decision problem is examined via a comparative study. We also analyze of the main advantages of the proposed approach.

Key words: Computing with words, fuzzy decision making, order-based semantics of terms, hedge algebras, 4-tuple linguistic representation model, semantic linguistic scales

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1 Introduction

Computing with words (CW) was firstly introduced by Zadeh [36] as a methodology for reasoning and computing with human-sourced information described in natural language, of which the idea was actually rooted from his previous work on linguistic variables, fuzzy constraints and fuzzy if-then rules [33–35]. During the last decade or so, CW has attracted considerable attention of the fuzzy set community. In order to establish a mechanism for automated reasoning, computing or decision-making with words, it is necessary to establish appropriate mathematical models for representing linguistic information and perceptions, which would be able to capture certain semantic characteristics of words that underline the way human beings reason or make decisions using natural language. This is really a challenging task because of the flexibility (e.g., context/culture dependent) of as well as fuzziness and uncertainty associated with semantic characteristics of words used in human reasoning. All of these make the process very difficult to model and represent the meaning of words and perceptions to carry out meaningful computing. This eventually makes CW comprehensive research area by being open to interpretations and different instantiations, as intensively discussed in [19].

So far, there have been numerous models developed for CW with applications to a spectrum of practical human-centric problems. From the perspective of modeling and reasoning application, Zadeh's seminal work [36] is the first one that provides a general framework for CW in which the use of fuzzy sets becomes crucial as they provide a means of modeling the fuzziness inherent in natural language utterances; while its computational mechanism is essentially based on the so-called extension principle in association with generalized rules of inference in fuzzy logic. As a matter of fact, Zadeh's general framework would be further developed and detailed for widely application to human-centered modeling and reasoning problems in practice. Interestingly also, in [14,15], Lawry proposed an alternative approach to CW based on mass assignment theory [2] and probability theory and provided a mechanism for reasoning with linguistic descriptions of imprecise probabilities that avoids the computational complexity problems incurred by applying the extension principle in Zadeh's theory of CW.

From the perspective of decision-making application, over the last decade much work have been done so as to develop CW approaches for solving decision problems involving vague and imprecise information. Typically, in decision-making applications CW is mainly involved with the problem of how to represent and aggregate linguistic information in decision making. In recognizing that "words mean different things to different people", Mendel [20] proposed to use type-2 fuzzy sets for modeling words in CW for assisting people in making subjective judgments. While most early methods for dealing with linguistic information in decision-making were making use of fuzzy sets as a means of modeling linguistic terms and the corresponding CW models were based on Zadeh's extension principle, e.g., [1,4,12]. Clearly, the computing results of these methods, classified as semantic models [26], are also fuzzy sets that in general do not match exactly pre-defined fuzzy sets of linguistic terms and, therefore, a linguistic approximation process must be applied to obtain linguistic recommendations for the problem at hand. Consequently, such linguistic approximations may cause loss of information and lack of precision in the final results as well. This has motivated Herrera and Martínez [9] to propose a so-called 2-tuple linguistic representation model as a tool for CW, which aims at overcoming the limitation of the loss of information caused by the process of linguistic approximation in fuzzy set-based approaches, Practically, the 2-tuple linguistic model has been also applied to decision making problem in various application areas, including group decision making, distributed intelligent agent systems, information filtering, information retrieval, and engineering management [17,18].

In 2-tuple linguistic representation model or its variations such as proportional 2-tuple representation model [32], which are classified as symbolic models [26], each set of the linguistic terms under consideration, denoted by $\mathcal{S} = \{s_0, \ldots, s_g\}$, is assumed to be strictly ordered, i.e., we have $s_0 < s_1 < \ldots < s_g$. In this study the set $\mathcal{S}$ is called a linguistic scale. Basically, symbolic models aimed to map a linguistic term set into an appropriate numerical scale and then computation for linguistic information aggregation is performed over this numerical scale so that many numerical aggregation operators available can be applied in a direct manner. Finally, the computing results will be converted back to linguistic 2-tuples in the corresponding models.

As we have observed, in the fuzzy-set-based approaches, the semantic representation of words makes them to become complicated in terms of underlying computations and, in addition, the ordering relationships between terms of the scale become blurred as well. While in the symbolic-model-based approaches, although they allow directly performing computations on the set of linguistic values in which only a totally ordered structure is assumed by mapping the set of linguistic values into suitable numerical scale, we might be losing much of the information we have purposely been keeping at the structural phase of linguistic decision problems. Note that the use of a linguistic approach is only necessary when the information in decision situations cannot be assessed precisely in a quantitative form (i.e., by numerical values). Moreover, note again that most linguistic scales used in the previous studies of linguistic decision analysis are assumed to be totally ordered. That is, one can recognize an order based on the qualitative semantics of terms, called inherent order-based semantics, which are directly associated with the string expression of terms regarded as their syntax. Obviously, this qualitative semantics of terms is present in any natural language. When experts provide their linguistic assessments, they focus on the ordering semantics of terms of comparison with some other terms of the linguistic scales. We may refer here to *qualitative linguistic scales*. Quantitative semantics of terms is required from computational standpoint and, therefore, it is desirable to establish formal linkages between the quantitative semantics with the corresponding qualitative semantics of terms. Such a connection between two semantics could help to have the linguistic computational models developed in a more interpretable and convincing fashion. However, it seems that this observation has been overlooked in the previously developed models for linguistic decision-making.

In light of the discussion presented above, the main objectives of this paper are as follows:

(i) To discuss linguistic scales based on the proposed concept of qualitative semantics of linguistic terms, namely the order-based semantics of terms. As the terms of a linguistic scale, even equipped with such a qualitative semantics, exhibit no computational (numeric) characteristics, the concept of (computational) semantic linguistic scales associated with given linguistic scales will be introduced.

(ii) Motivated by the above discussion, we first propose fundamental requirements for constructing legitimate linguistic scales so as to ensure the soundness of constructed semantic linguistic scales such as:
   - A solid identification of qualitative and quantitative semantics of terms, which should be related to their inherent order-based semantics present at a certain level;
   - A sufficient formal basis for constructing a semantic linguistic scale associated with a given qualitative linguistic scale to ensure that the constructed semantic linguistic scale and its respective (qualitative) linguistic scale exhibit a closed relationship;
   - A sufficient operational structure for the constructed semantic linguistic scale, e.g., its closeness with respect to some typical aggregation operations.

(iii) To propose the inherent order-based semantics of terms of a linguistic variable to form qualitative semantics of linguistic terms that is associated directly with linguistic terms regarded as their syntax. Here, hedge algebras introduced in [21,24], considered as one of the feasible models of order-based structure of term-domains, will be used as a mathematical basis to model the qualitative semantics of terms as well as their quantitative semantics.

(iv) To develop a 4-tuple semantic representation model, based on which we will construct a sound 4-tuple (computational) semantic linguistic scale that meets the proposed requirements and would hopefully be useful for applications.

The rest of this paper is organized as follows. Section 2 first discusses about the essential qualitative semantics of terms and the semantic linguistic scales that are useful from computational perspective, and then proposes some requirements for the construction of sound and legitimate semantic linguistic scales. In Section 3, the order-based qualitative semantics of terms will be discussed based on the basis of order-based structure of hedge algebras. Quantitative semantic aspects of linguistic terms previously developed will be reviewed to offer necessary background knowledge for examination of the (qualitative) linguistic scales and for construction of semantic linguistic scales in the proposed approach. The concept of a 4-tuple semantic representation model and 4-tuple semantic linguistic scales exploiting numeric quantitative semantics and interval-semantics of terms are introduced in Section 4. It is shown that for a given so-called superior-closed linguistic scale being considered as a term set generated from the primary terms, its associated 4-tuple semantic linguistic scales can be constructed, utilizing the quantitative semantics of terms presented in Section 3. Such scales can be considered as being sound since they meet all requirements introduced in Section 2. To show the advantages of the 4-tuple semantic linguistic scales, a simple multi-criteria decision problem is examined in Section 5. It is shown that, with the same linguistic assessments of an expert for both given superior-closed linguistic scales, one is a proper sub-scale of the other, the decision results produced in the 4-tuple approach are the same, i.e. they show to depend mainly on the semantics of the terms defined in the context of their associated semantic linguistic scales, while the decision results produced in the 2-tuple approach are different. Finally, some conclusions are given in the last section.

2 A concept of semantic linguistic scales and some essential requirements for their construction

When dealing with linguistic scales or, more generally, with CW, the following two aspects should be considered:

(i) the qualitative and quantitative semantics expressed by linguistic terms in a given linguistic scale $\mathcal{S}$, including the question of which qualitative semantics of terms induces the order of $\mathcal{S}$ and,

(ii) the computational model of the term semantics, which is useful for developing a computing mechanism for $\mathcal{S}$.

With regard to this viewpoint, it can be observed from the above analysis that the concept of linguistic scales is still not obvious and unified. In the following we will discuss these two aspects in more details.

2.1 A concept of semantic linguistic scales

Our starting position to define this concept is to consider each linguistic scale as a mechanism to generalize a numeric scale. For example, let us consider a numeric scale consisting of numbers in the interval [0, 10] to assess the mathematical ability of students on a basis of their grades obtained in mathematics examinations. Intuitively this numeric scale exhibits two main characteristics:

(i) The elements of the scale have their own semantics, using which teachers can express their assessments of students' examination.

(ii) This scale has its own computational structure that allows teachers to carry out certain operations such as aggregation. It is worth noting that this computational structure is *closed with respect to the usually considered aggregation operations*. In addition, in accordance with logical nature, these operations must act on the semantics of elements of the scale, which, in fact, are merely symbolic notation carrying their assigned semantics.

Now, in order to relate a numeric scale with its respective linguistic one, we may imagine how teacher gives an assessment, for instance, of 7.5 to a student's examination. Here, the teacher realizes this assessment in a fuzzy environment [12], and usually used to utilize vague concepts in his examination assessment process. Then he has to convert his linguistic assessments into corresponding numbers positioned on the numeric scale. That is, linguistic terms are present in the teacher's assessment process. In practice, at the end of a study-year, students are classified into groups, named by linguistic terms, in accordance with an aggregation of the results of their mathematical examination, by using a given quantitative-linguistic scale. For example, in Vietnamese colleges, such a scale can be defined as follows:

$$\begin{aligned}\{(Bad, [0.0, 3.5)), (Weak, [3.5, 5.0)), (Medium, [5.0, 6.5)),\\ (Fair, [6.5, 7.5)), (Good, [7.5, 8.5)),\\ (Very\ Good, [8.5, 9.5)), (Excellent, [9.5, 10.0])\}\end{aligned} \tag{1}$$

where intervals shown in (1), called in this study *interval-semantics* of terms, are user declaration of the quantitative semantics of linguistic terms of the scale. This means, for instance, that a student is classified into the group "Good" with respect to the mathematical ability if the aggregated result of all her/his mathematical examination assessments in the study-year falls into [7.5, 8.5), which is declared as the interval-semantics of "Good" as shown in (1). As such, the set in (1) gives an example of the semantic representation model for the concept of "semantic linguistic scales", which can be considered as a generalization of the numeric scale [0, 10], although it should be extended to include richer quantitative semantics of terms for computing with linguistic terms. Since the terms present in (1) are only labels, treated as the *syntax of* *terms*, an extension of the semantic representation model given in (1) must be adequate to guarantee that we can develop operations working usefully on the semantics of the linguistic scale, including the ordinary aggregation operations.

Under such an observation, it is natural to require that, for each linguistic scale, the so-called *semantic linguistic scale* associated with it that we will construct has to be characterized by the following two main properties:

Property 1. Linguistic scales should have a semantic representation model with a clear declaration of qualitative and quantitative semantics of linguistic terms of the scales. The qualitative semantics of terms is devoted to experts to express their linguistic assessments, while the quantitative semantics of terms is exploited to develop the computationally operational mechanism on the scales.

Property 2. Linguistic scales should be associated with their respective semantic linguistic scales constructed based on the proposed semantic representation model, which is equipped with an adequate computational structure so as to be able to perform certain operations, including aggregation operations, on it.

In addition, as we desire that the proposed linguistic computational model would exhibit a proper relationship between the qualitative semantics of linguistic terms and their suitable quantitative semantics, several requirements for constructing semantic linguistic scales should be granted as discussed in the following section.

2.2 Some essential requirements for construction of semantic linguistic scales

As aforementioned, the relationship between syntax, i.e. label representation, and semantics of the linguistic terms of a linguistic variable is a fundamental problem of linguistic decision analysis. However, it seems to be not easy to deal with this problem for imprecise linguistic information. Therefore, it becomes important to discuss what the actual semantics of this kind of information is.

Conventionally, fuzzy sets assigned to linguistic terms can be considered as their semantics based on the viewpoint of a generalization of crisp concepts. By this it is difficult to explain, for instance, what the semantics of linguistic hedges is. On the other hand, another viewpoint on the semantics of terms can ben observed when we follow the concept of semantics in formal logics. Normally, the meaning of a word or a phrase is a collection of objects or phenomena present in real world that the term or phrase points at. The question is at what points a vague term in a linguistic domain of an attribute, which comprises terms generated from primary terms (atoms) by using hedges. “Good” or “very good” do not indicate concrete items or phenomena in the real world, because we still do not know whether they indicate human or animals or other things. However, they carry a qualitative semantics and may be used to describe different properties of different objects. The presence of these terms in natural language aims to compare properties of distinct items. This semantics seems to be very crucial for human decision making as it will be discussed next.

The same argument can be applied to explain the presence of hedges in natural languages: their presence is due to the need of the comparison of alternatives in human decision making. So, every hedge aims to intensify vague terms to generate new terms, which are comparable with the original ones. For example, “very young” is comparable with “young”. Therefore, another characteristics of the term semantics is what that provides us a basis to identify the order relationship of terms in a term-domain.

This characteristic of the term-semantics seems to be more essential, when we observe that there are fundamental facts supporting it: natural language is a vehicle to cognize reality and offer a communication vehicle within the community. Subsequently, natural language is rich enough to fully describe phenomena of the real world serving for human activities. Life is composed of a series of decisions. The aim of decision making in human daily life is to choose an alternative which is better than others. Therefore, in natural languages there should be elements to describe preference of an alternative in comparison to other alternatives in question. Linguistic terms with their own semantics and, in particular, hedges are elements facilitating this process.

In turn, such feature shows to be an essential characteristic of the semantics of linguistic terms. As a consequence, we consider order-based semantics of terms as their natural intrinsic qualitative semantics.

To ensure the soundness of the linguistic scale, we introduce three intuitively appealing requirements for construction of semantic linguistic scales. We do not consider them as criteria, since it is not easy to define the proper semantics of vague linguistic terms. It depends on what the starting point of view is and there is no exact condition for this task in such an uncertain environment.

Requirement 1 *Linguistic representation models of linguistic scales should be developed based on a clearly declared qualitative and quantitative semantics of linguistic terms, which are related with their inherent order-based semantics as much as possible.*

This requirement seems to be natural as linguistic scales are ordered by the semantics of their terms. In addition, as discussed above, our starting point viewing the semantics of terms comes from the demand of decision-making tasks: Ranking the alternatives in question in accordance with certain criteria based on expert linguistic evaluations. This implies that establishing an appropriate formalized mechanism for comparison of the alternatives becomes highly relevant. Following this, the semantics of terms based on their inherent order relationships seems to be intrinsic to this research field and, thus, this requirement is necessary.

Requirement 2 *There should be a suitable formalized mechanism to construct, based on the declared qualitative and quantitative semantics of terms, a semantic linguistic scale characterized by* Property 1 *and* Property 2 *with computational characteristics useful for practical applications.*

Since the computational operations of linguistic scales should work on the quantitative semantics of terms and there is a closed relationship between the qualitative semantics and the quantitative semantics of each term, the above requirement is obvious. The inherent qualitative semantics of terms is essential and, in principle, it determines their quantitative semantics. Since, up to now, there did not exist a mathematical structure, except the structure of hedge algebras, to model the inherent qualitative semantics of terms, we have no formalized mechanism to relate the qualitative semantics of terms with their existing quantitative semantics examined in this field. The lack of this relationship implies the lack of basic criteria to decide, which linguistic scales are better than the others with respect to this requirement, and the lack of formalized basis to construct appropriate semantic linguistic scales for particular applications.

Requirement 3 *The semantic linguistic scale should bring necessary advantages to develop computational operations for developing decision-making methods, including aggregation operators, in particular. The constructed semantic linguistic scales should be* closed with respect to the developed aggregation operators.

To show the usefulness of the proposed requirements, in the next section we construct semantic linguistic scales based on hedge-algebra-structure-based semantics of term.

3 Order-based qualitative and quantitative semantics of linguistic terms

As discussed previously, it is observed that linguistic scales are always assumed to be totally ordered using the inherent term meaning that is recognized naturally by a human community. This suggests us to consider this term meaning as qualitative semantics of linguistic terms of a linguistic variable, which can be formulated in terms of an order relation on a term-set under consideration.

That is the semantics of a given term of a linguistic variable is determined by a collection of ordering relationships between this term and the other ones in a term-domain of the linguistic variable. Besides, in practice, a linguistic scale is always associated with a numeric scale, of which the values have semantic relations with terms of the linguistic scale. Any value of the numeric scale should be related to the semantics of terms of the linguistic scale in some way to a certain degree. For instance, the quantitative semantic representation of terms appearing in (1) is a way to specify that values of interval [7.5, 8.5) relate more preferably to the semantics of the term “Good” than to the others.

This section aims to establish a formal basis to relate the qualitative semantics of terms in linguistic scales to the values of the respective associated numeric scales, using appropriate representation of quantitative semantics of terms. There is a mathematical foundation for realizing this purpose, since, as it will be seen, every term-domain of an attribute can be considered as a subset of a hedge algebra under an isomorphic mapping preserving semantic order-based structure [13,14,16]. A computational semantic representation model of linguistic scales can be developed based on the quantification of hedge algebras with the concepts of fuzziness measure, fuzziness intervals, similarity intervals and numeric quantitative values of linguistic terms [14,15]. We will give a short overview of necessary concepts of hedge algebras and their quantification tasks. Regarding more details and a formal presentation the reader can refer to [14–16].

The motivation for development of hedge algebras is to discover semantic properties of vague linguistic terms by means of order relation that can be considered as the inherent semantic order relation on term-domains of linguistic variables. This approach to linguistic semantics seems to be, on an abstract level, compatible with the natural term semantics and quite sufficient for human fuzzy decision making tasks.

From this viewpoint, each term-domain of a linguistic variable $\mathcal{X}$ can be considered as an order-based structure, denoted by $\mathcal{AX} = (X, G, C, H, \leq)$, where $X = Dom(\mathcal{X})$ being the set of all terms of $\mathcal{X}$, $G = \{g, g'\}$ is the set of its primary terms, $C = \{\mathbf{0}, \mathbf{W}, \mathbf{1}\}$ is the set of specific terms, called constants, with $\mathbf{0}$ and $\mathbf{1}$ being the least and the greatest terms in the structure $Dom(\mathcal{X})$, respectively, and $\mathbf{W}$ being the neutral concept positioned in between the two primary terms, i.e. we have $\mathbf{0} \leq g \leq \mathbf{W} \leq g' \leq \mathbf{1}$, H is the set of hedges considered as unary operations on X and $\leq$ is an order relation on X. Then, $(X, \leq)$ becomes a poset.

We also assume that the set $Dom(\mathcal{X})$ is just a set generated from the primary terms by using hedges acting on them in concatenation, i.e. each term in $Dom(\mathcal{X})$ can be written in a string $h_n \dots h_1 c$, where $h_i \in H$ and $c \in G$. An example of a term in natural language is *very-rather-small*, which is a string consisting of two hedges *very* and *rather* in concatenation and a primary term *small*. So, the string representation of the elements of $\mathcal{AX}$ is identical with the string representation of term expressions in natural languages and, hence, we call its elements also terms for convenience.

Denote by $H(Y)$ the set of all terms generated from the terms in Y using the hedges in H. We will often use in the sequel the notations $H(G)$, i.e. $Y = G$, and $H(\{x\})$ will be written as $H(x)$, for short.

Hedge algebras were axiomatized in such a way that they can be regarded as the isomorphic images of the ordered-based structure of certain term-domains of linguistic variables. This requires that the axioms of hedge algebras should suitably be selected from the natural properties of $Dom(\mathcal{X})$ that can be formulated in terms of inherent semantic ordering relationships between linguistic terms. By this way, each element of X can be understood as conveying the qualitative semantics of the respective linguistic term of $\mathcal{X}$.

3.1 Some descriptions of qualitative term semantics

We intend to show that the qualitative semantics of linguistic terms can be formalized based on the inherent semantic order relation and, hence, also called the order-based semantics of linguistic terms. Indeed, let us consider an order-based structure $\mathcal{AX} = (X, G, C, H, \leq)$ of a linguistic variable $\mathcal{X}$, as described above. The starting point is that since the function of hedges is to intensify linguistic terms, for any hedge $h \in H$ and any term $x \in X$, hx and x should be comparable, i.e. the (order-based) effect of h acting on x is expressed by the fact that either $hx \geq x$ or $hx \leq x$. In the case that $kx \geq hx \geq x$ or $y \geq hy \geq ky$, for certain x or y, we say that the effect of k is greater than h, and write $k \geq h$. For example, in this sense, we have $Extremely \geq Very$. That is the semantics of hedges of $\mathcal{X}$ can be formulated in terms of the semantic order relation $\leq$.

The inherent order-based semantics of terms and the linguistic hedge lead to discovering many new notions and properties of linguistic term formulated in terms of $\leq$.

1) *Linguistic terms possessing their own "algebraic" sign.* We observe that the semantic order-based semantics of terms leads to the fact that terms and hedges have the so-called semantic tendencies that can be recognized as follows:

- In practice the primary terms $g, g' \in G$ are comparable and have different semantic tendencies recognized by different inequality directions when any hedge applies to them. For instance, *bad* $\leq$ *good* and while *very_bad* $\leq$ *bad* we have *very_good* $\geq$ *good*. We say as a convention that the greater primary term is of a *positive semantic tendency* and denoted by c^+, and the other is of a *negative semantic tendency* and denoted by c^-. For $x = c \in G$, the comparability of hc and c implies that H is partitioned into two sets: $H^+ = \{h \in H : hc^- \leq c^- \text{ or } hc^+ \geq c^+\}$, which consists of the hedges that *increase the both semantic tendencies* of the primary terms; $H^- = \{h \in H : hc^- \geq c^- \text{ or } hc^+ \leq c^+\}$, which consists of the hedges that *decrease these semantic tendencies*. This leads to the interesting concept that the primary terms and hedges have their own "algebraic" sign: $sign(c^+) = +1$, $sign(c^-) = -1$, $sign(h) = +1$, for $h \in H^+$, and $sign(h) = -1$, for $h \in H^-$.

- The comparability of khx and hx implies also that either k increases or decreases the effect of h. For instance, having $x \leq hx$ the inequalities $x \leq hx \leq khx$ state that k increases the effect of h and $x \leq khx \leq hx$ state that k decreases the effect of h. We shall write $sign(k, h) = +1$, for the former case, and $sign(k, h) = -1$, for the latter one. They are called the relative sign of k with respect to h. For example, it can be verified that $sign(V, L) = +1$ while $sign(L, V) = -1$.
- If $hh'x \neq h'x$, we say that the effect of h in the expression $hh'x$ is proper. A string $h_n \dots h_1 c$, where $h_i \in H$ and $c \in G$, is said to be a canonical string representation of x if $x = h_n \dots h_1 c$ and the effect of all h_i's is proper in this expression. It was proved that the canonical representation of x is unique, for any term x, and, hence, we may define the length of x, denoted by $|x|$, which is just the length of the string $h_m \dots h_1 c$ of x. Now, the sign of a linguistic term x, can be defined by

$$Sgn(x) = sign(h_m, h_{m-1}) \times \dots \times sign(h_2, h_1) \times sign(h_1) \times sign(c) \quad (2)$$

The meaning of Sgn is expressed as follows:

$$Sgn(hx) = +1 \Rightarrow hx \geq x \text{ and } Sgn(hx) = -1 \Rightarrow hx \leq x \quad (3)$$

For example, since

$$Sgn(VL\_true) = sign(V, L)sign(L)sign(true) = -1$$

we have $VL\_true \leq L\_true$.

2) *Semantic heredity – an essential meaning of linguistic hedges*. An essential property of hedges is the so-called *semantic heredity*, which states that the terms generated from a given term x by using hedges must inherit or contain the (genetic) core meaning of x own. This implies that hedges cannot change the essential meaning of terms expressed in terms of the *semantic order relation* $\leq$, i.e. it results in the following:

- If the meaning of hx and kx is expressed by the order relationship $hx \leq kx$, $h \neq k$, then any hedges h' and k' cannot change this semantic relationship, that is

$$hx \le kx \Rightarrow h'hx \le k'kx \tag{4}$$

- Similarly, if the meaning of x and hx is expressed by either $x \le hx$ or $hx \le x$, then

$$x \le hx \Rightarrow x \le h'hx \text{ or } hx \le x \Rightarrow h'hx \le x \tag{5}$$

Assuming that $H^- = \{h_0, h_{-1}, \ldots, h_{-q}\}$ and $H^+ = \{h_0, h_1, \ldots, h_p\}$, where $h_0 = I$, the artificial hedge identity, and $h_0 < h_{-1} < \ldots < h_{-q}$ and $h_0 < h_1 < \ldots < h_p$, the hedge heredity leads to the following:

- For $Sgn(h_p x) = -1$

$$H(h_p x) \le \ldots \le H(h_1 x) \le \{x\} \le H(h_{-1} x) \le \ldots \le H(h_{-q} x) \tag{6}$$

- For $Sgn(h_p x) = +1$

$$H(h_{-q} x) \le \ldots \le H(h_{-1} x) \le \{x\} \le H(h_1 x) \le \ldots \le H(h_p x) \tag{7}$$

with a note that $H(h_0 x) = H(Ix) = \{x\}$, by convention. In particular, we have

$$\{\mathbf{0}\} \le H(c^-) \le \{\mathbf{W}\} \le H(c^+) \le \{\mathbf{1}\} \tag{8}$$

- The sets $H(h_j x), j \in [-q, p]$, where $[-q, p] = \{j | -q \le j \le p\}$, constitute a partition of $H(x)$, i.e. they are disjoint and

$$H(x) = \bigcup_{j \in [-q,p]} H(h_j x) \tag{9}$$

These only such listed properties show already that term-domains of linguistic variables with such qualitative semantics of terms possess a rich order-based structure. Therefore, we may observe that hedge algebras are formalized structures of the qualitative semantics of term-domains, noting that the meaning of a term represented in a formalized structure carries much information than by a fuzzy set itself, in general.

3) *The comparison criterion of hedge algebra $\mathcal{AX}$*. The following criterion for comparing any two terms of $\mathcal{AX}$ can be proved in these formalized structures.

Let $x = h_m \ldots h_1 c$ and $y = k_n \ldots k_1 c$, $c \in G$, be the canonical representations of two terms x and y, respectively. Then, there exists a greatest integer $j \le \min\{m, n\}$ such that $h_i = k_i$, for all $i \le j$, and, moreover, putting $x_{|j} = h_j \ldots h_1 c$, called the j-suffix of x, we have

(i) $x < y$ iff $h_{j+1} x_{|j} < k_{j+1} x_{|j}$, where in the case $j = n$ or $j = m$, say $j = m$, we understand conventionally that $h_{m+1} = h_0 = I$ and for $j = n$, $k_{n+1} = h_0$, i.e. we have $h_{j+1} x_{|j} < x$.
(ii) $x = y$ iff $n = m = j$.

From now on we always assume that the hedge algebra under consideration is free, i.e. each term representation $h_n \dots h_1 c$ is a canonical and, hence, it defines a unique term.

In hedge algebra approach, the quantitative semantics of terms comprises several characteristics: numeric quantitative semantics, fuzziness measure, fuzziness intervals and similarity intervals of terms. They all have closed relationships and are completely determined by providing fuzziness measure values of the primary terms and hedges in question. Numeric semantic quantification of a term x is realized by construction of a mapping $f : X \to [0, 1]$, where unit interval $[0, 1]$ is the normalized domain of the universe of discourse of $\mathcal{X}$, which assigns a numeric value in $[0, 1]$ to x. This mapping has to satisfy the following conditions and is called a *semantically quantifying mapping* (SQM):

(i) It is one-to-one mapping and $f(X)$ is dense in $[0, 1]$, i.e. every value in the universe is approximated by numeric semantics of terms in X;
(ii) It preserves the order of X.

Given an SQM $f : X \to [0, 1]$, denote by $\mathfrak{I}(x)$ the least sub-interval of $[0, 1]$ containing the image $f(H(x))$, which is closed on the left and open on the right, except the case where the right endpoint is equal to 1. This definition ensures, by $(6) - (9)$, that

- The family $\{\mathfrak{I}(y) : y \in H(x)\}$ is a partition of $\mathfrak{I}(x)$, for every $x \in X$.
- The family $\{\mathfrak{I}(x) : x \in X_k\}$, where $X_k = \{x \in H(G) : |x| = k\}$, is a partition of $[0, 1]$. In particular, for $k = 1$, $\{\mathfrak{I}(c^-), \mathfrak{I}(c^+)\}$ is a partition of $[0, 1]$.
- The family $\{\mathfrak{I}(y) : y \in X_{k+1}\}$ is finer in the term of topology than $\{\mathfrak{I}(x) : x \in X_k\}$ and

$$\mathfrak{I}(x) = \cup\{\mathfrak{I}(y) : y \in X_{k+1}, y = hx, h \in H\} \tag{10}$$

Since $H(x)$ consists of the terms, which intuitively still contain a core meaning of x, it can be considered as a fuzziness model of x. Hence, the interval $\mathfrak{I}(x)$ is called the *fuzziness interval* of x and its length, $|\mathfrak{I}(x)|$, is interpreted as the *fuzziness measure* of term x, denoted by $fm(x)$, for every $x \in H(G)$. Semantically, $\mathfrak{I}(x)$ consists of numeric values that are compatible with the semantics of x to a degree indicated by $k = |x|$ and, hence, it is called also the $|x|$-fuzziness interval of x. The larger the length of x is, the more compatible the values of $\mathfrak{I}(x)$ with the semantics of x to a degree $|x|$ than with the semantics of any $y \neq x$.

When particular terms associated with these intervals are not necessary to be mentioned, the intervals in $\{\mathfrak{I}(x) : x \in X_k\}$ are called simply fuzziness interval of a degree k of X_k.

Assuming that the proportion $\frac{|\mathfrak{I}(hx)|}{|\mathfrak{I}(x)|}$ does not depend on a given term x, this quantity is called the fuzziness measure of the hedge h, denoted by $\mu(h)$.

Thus, the quantitative notions of fuzziness of terms and hedges, $fm(x)$ and $\mu(h)$, can be defined formally in this approach and we have the following properties:

$(fm1)$ $fm(\mathbf{0}) = fm(\mathbf{W}) = fm(\mathbf{1}) = 0, fm(c^-) + fm(c^+) = 1.$

$(fm2)$ $\sum_{j \in [-q,p]} fm(h_j x) = fm(x), x \in X$, and $\sum_{x \in X_k} fm(x) = 1.$

$(fm3)$ $fm(hx) = \mu(h)fm(x)$, for $hx \neq x, x \in X$, and, hence, $fm(y) = \mu(h_m)\ldots\mu(h_1)fm(c)$, where $y = h_m \ldots h_1 c$ is the canonical representation of y.

$(fm4)$ $\sum_{-q \leq i \leq -1} \mu(h_i) = \alpha$, and $\sum_{1 \leq i \leq p} \mu(h_i) = \beta$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Thus, the fuzziness measure of the terms and the hedges, one of the aforementioned characteristics of the quantitative term semantics, can be defined based on a formalization of the qualitative semantics of terms and they will play an important role in determining other characteristics of the quantitative term semantics. The quantities $fm(c^-)$ (or $fm(c^+)$) and $\mu(h), h \in H$, are the only parameters, called *fuzziness parameters* of $\mathcal{X}$, for calculating the characteristics of the quantitative semantics of terms. This may be compatible with the intuitive idea that fuzzy information should be characterized by their fuzziness.

3.3 SQMs induced by fuzziness measure

The SQM value of x to be defined comes with a meaning that it is the core value of the fuzziness interval $\mathfrak{I}(x)$, which is similar as the core of a fuzzy set. However, in this approach, one can compute the SQM-values of terms, when the fuzziness parameter values are given. To establish a formula for computing an SQM ν, for a given fuzziness measure fm of $\mathcal{AX}$, we look at the inequalities given in (6) and (7) and note that ν preserves the order relationships between terms. Then, we will see that $\nu(x)$ should be defined to assume the value lying in between fuzziness intervals $\mathfrak{I}(hx), h \in H^-$, and the ones $\mathfrak{I}(hx), h \in H^+$, i.e. in between $\mathfrak{I}(h_{-1}x)$ and $\mathfrak{I}(h_1 x)$. Then, ν-values can be calculated recursively as follows:

(SQM1) $\nu(\mathbf{W}) = \theta = fm(c^-)$, $\nu(c^-) = \theta - \alpha fm(c^-) = \beta fm(c^-)$, and $\nu(c^+) = \theta + \alpha fm(c^+)$.

(SQM2) $\nu(h_j x) = \nu(x) + Sgn(h_j x)\left(\left[\sum_{i=sign(j)}^{j} fm(h_i x)\right] - \omega(h_j x) fm(h_j x)\right)$
where $\omega(h_j x) = \frac{1}{2}\left[1 + Sgn(h_j x) Sgn(h_p h_j x)(\beta - \alpha)\right] \in \{\alpha, \beta\}$, for all $j \in [-q, p]$.

This shows that the concept of SQMs can be defined by the fuzziness concept of terms.

The SQM-values of terms represent *numeric quantitative semantics* of terms, a characteristic of the quantitative semantics of terms, which carry also much information about the meaning of the terms of $\mathcal{X}$, since this approach guarantees that they are most compatible with their respective terms than other values in the universe of discourse.

3.4 k-Similarity intervals of linguistic terms

A similarity relation [2,19], which is an equivalence relation on a universe of discourse, is a fuzzy concept that is useful to manipulate fuzzy data in fuzzy databases, but it has no relation with the inherent order-based semantics of terms. To link this term semantics with this concept, the similarity intervals of terms were defined in such a way that they are built up from fuzziness intervals of $X_{k'}$, for some k' and a given fuzziness measure fm of $\mathcal{X}$ [14].

The problem is formulated as follows: for a given fm of $\mathcal{X}$, a $k > 0$ and a finite set of terms $X_{(k)} = \{x \in X : |x| \leq k\}$, construct a set of intervals, $\{\mathfrak{T}_k(x) | x \in X_{(k)}\}$, of the normalized reference domain $[0, 1]$, called *k-similarity intervals*, that satisfy the following conditions:

- **(S1)** They form a partition of $[0, 1]$.
- **(S2)** $\mathfrak{T}_k(x)$ contains one and only one the value $\nu(x)$ of the SQM ν induced by fm and the values in $\mathfrak{T}_k(x)$ can be considered as being similar with $\nu(x)$ or, for convenience, compatible with the meaning of x to a degree k.

The condition (S1) guarantees that the set of k-similarity intervals of the terms in $X_{(k)}$ determines a similarity relation on $[0, 1]$ in the sense of Buckles and Petry [2]. A crucial and important difference here is the semantic property (S2), which states that the numeric values in each equivalence class of this similarity relation, i.e. in each k-similarity interval, are compatible with the semantics of a term in $X_{(k)}$ to a degree k.

This problem can be applied not only in the construction of semantic linguistic scales in decision making, but also in other fields such as fuzzy classification [15] and, possibly fuzzy databases.

The idea for solving this problem is as follows. The fuzziness intervals in $\{\mathfrak{I}(x) : x \in X'_k\}$ of degree k' of the terms of $X_{k'}, k' > k$, which in terms of topology are finer that the fuzziness intervals of degree k, will be utilized to be considered as elementary intervals for constructing k-similarity intervals, noting that they constitute a partition of $[0, 1]$. Since each such an interval comprises the values of $[0, 1]$ that are similar with each other and compatible with the semantics of its respective term in $X_{k'}$ to a degree k', they can be taken to form a neighbourhood basis for a topology. Similarly as generating open sets of a topology, k-similarity interval of each term x of $X_{(k)}$ will be constructed by taking the union of certain fuzziness intervals of $X_{k'}$that lie around the SQM-value $\nu(x)$. In this way, the values of each resulting interval can be considered as similar with each other and with $\nu(x)$ to a degree lower than k' but not less than k and, therefore, to the degree k. Thus, we construct k-similarity intervals of the terms in $X_{(k)}$ by partitioning the set $\{\mathfrak{I}(x) : x \in X_{k'}\}$ into clusters $C(x), x \in X_{(k)}$, that satisfy the following conditions:

(i) the clusters are disjoint and comprise consecutive fuzziness intervals,
(ii) each $C(x)$ contains at least two fuzziness intervals of degree k', whose common endpoint is $\nu(x)$, except for the clusters $C(\mathbf{0})$ and $C(\mathbf{1})$, they contain at least one fuzziness interval, one of the end-points of which is 0 or 1, respectively.

Such clusters $C(x)$ always exist for $k' \geq k + 1$, if $|H^-|$ and $|H^+| \geq 2$, and for $k' \geq k + 2$, if otherwise. Similar as in [15], for simplicity, in this study k' is selected to be the minimum, which satisfies the above stated conditions. Then, $\mathfrak{T}_k(x)$ is defined to be the set union of the intervals in $C(x), x \in X_{(k)}$. Clearly, these intervals satisfy conditions (S1) and (S2) and, hence, they are k-similarity intervals of $X_{(k)}$. For simplicity, to construct k-similarity intervals of $X_{(k)}$ in this study we consider only the case that $|H^-| = |H^+| = 1$, e.g. $H^- = \{R\}$ and $H^+ = \{V\}$, and, hence, $k' = k+2$. Then, the calculation of the similarity intervals of terms can be established by a simple formula. In fact, since the hedges R and V have different signs, for every term x, one of the terms Rx and Vx is smaller than x and the other is greater than x. Therefore, it can be seen that the term-set of $\mathcal{AX}$ that is generated from a primary term $c \in G$ can be represented as a full binary tree: (i) c is assigned to the root of the tree. (ii) The smaller term among Rx and Vx is assigned to the left child of the node x and the greater one is assigned to the right child of x.

The graph $Gr(\mathcal{AX})$ representing the term-set of $\mathcal{AX}$ comprises isolated nodes of the constants $C = \{\mathbf{0}, \mathbf{W}, \mathbf{1}\}$ and the two trees associated with $c \in G$ arranged from left to right in accordance with their order. It can be checked that this graph has the following properties:

Gr1) X_l is just the set of all terms of the same level l of the graph and $X_{(l)}$ is the set of all terms assigned to the nodes of depth not greater than l (refer to Figure 2 for illustration).

Gr2) By the properties of the heredity of hedges exhibited in (5) and (8), we observe recursively for any two consecutive terms x_i and x_{i+1} in $X_{(l)}, l > 0$ and $x_i < x_{i+1}$, that

(i) There exists exactly one of them is of the length l.

(ii) For $X_{(l)}, l > 0$, in between any two consecutive terms x_i and $x_{i+1}, x_i < x_{i+1}$, there is only one term $y \in X_{l+1}$, i.e. $x_i < y < x_{i+1}, y \in \{Rx, Vx\}$, for some $x \in X_l$.

Gr3) The terms assigned to its nodes are positioned from left to right in accordance with their linear order.

For $x \in X_{(k)}$, denote by x_L and x_R, respectively, the left and the right adjacent terms of x defined in $X_{(k)}$. By (ii) of Gr2), there are exactly two terms $x_{L,k+1}$ and $x_{R,k+1}$ in X_{k+1} satisfying the inequalities $x_L < x_{L,k+1} < x < x_{R,k+1} < x_R$. Similarly, again by (ii) of Gr2) applied to the set $X_{(k+1)}$, there are exactly four terms $y_{L,k+2}, z_{L,k+2}, u_{R,k+2}$ and $v_{R,k+2}$ in X_{k+2} satisfying $x_L < y_{L,k+2} < x_{L,k+1} < z_{L,k+2} < x < u_{R,k+2} < x_{R,k+1} < v_{R,k+2} < x_R$. Since the cluster $C(x)$ is defined based on the fuzziness intervals of X_{k+2}, it follows that $C(x) = \{\mathfrak{I}(z_{L,k+2}), \mathfrak{I}(u_{R,k+2})\}$. By the fact that the SQM ν preserves the order of the terms of $\mathcal{AX}$, these fuzziness intervals satisfy $\nu(x_{L,k+1}) \leq \mathfrak{I}(z_{L,k+2}) \leq \nu(x) \leq \mathfrak{I}(u_{R,k+2}) \leq \nu(x_{R,k+1})$ (refer to Figure 1). Therefore, we have

$$\begin{aligned} \mathfrak{T}_k(x) &= [\nu(x_{L,k+1}), \nu(x_{R,k+1})) \\ &= \bigcup\{\mathfrak{I}(y) : |y| = k+2, \mathfrak{I}(y) \subseteq [\nu(x_{L,k+1}), \nu(x)) \text{ or } \mathfrak{I}(y) \subseteq [\nu(x), \nu(x_{R,k+1}))\} \end{aligned} \tag{11}$$

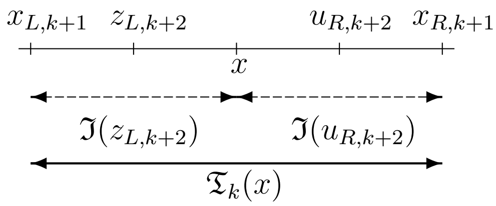

Fig. 1. The k-similarity interval of x

Remark 3.1 *We emphasize that all the four characteristics of the quantitative semantics of terms, the fuzziness measure fm, the fuzziness intervals, the k-similarity intervals and the fm-induced SQM ν, are completely determined, providing the values of the fuzziness parameters $fm(c^-), fm(c^+)$ and $\mu(h), h \in H$, of $\mathcal{X}$. Using the constraints given in $(fm1)$ and $(fm4)$, the necessary number of such fuzziness parameters is only $|H| + |G| - 2 = |H|$.*

4 4-Tuple representation of the semantics of linguistic terms and 4-tuple semantic linguistic scales

At the end of Subsection 3.4, it has been shown that we can associate every term with a similarity interval, whose values are similar with its semantics to a degree indicated by the maximal length of the terms in question. The similarity interval associated with a term can be interpreted in this study as its interval-semantics. Based on this we introduce a 4-tuple linguistic representation model for developing semantic linguistic scales that may meet the requirements discussed in Section 2 better than the existing linguistic scales mentioned previously. An attempt to develop a semantic linguistic scale based on a 3-tuple representation model was examined in [11], utilizing also order-based semantics of terms. In that study, the linguistic scales are assumed to be of the form $X_{(l)}$. In this study, the term-set T of every linguistic scale is assumed to be a subset of $X_{(l)}, T \subseteq X_{(l)}$, for some positive integer l. It is worth emphasizing that in our approach the semantics of a term in T may be different from the semantics of the same term in $X_{(l)}$, since its semantics depends on the presence of other terms. For instance, the semantics of "medium" in the linguistic scale $\{\mathbf{0}, bad, medium, good, \mathbf{1}\}$ is different from its semantics in the case we add a term "Rather bad" more to this scale. The meaning of "medium" in the latter scale is more specific than its meaning in the former one. So, we may use the concept of the *semantics of terms in the context of* T.

4.1 4-Tuple semantic linguistic representation model of linguistic scales

As mentioned previously, the semantics of terms is context-dependent, i.e. it depends on which are their adjacent terms appearing together with them in a given linguistic scales. Therefore, the semantics of each term in a linguistic scale should be determined in the context of its adjacent terms in the scale. From this viewpoint, we discuss fist about semantic properties that the term-sets of linguistic scales should have.

We always assume that the term-set T intuitively should have the properties that (i) $(y = hx \in T) \Rightarrow x \in T$, and (ii) $(h \leq k \ \& \ kx \in T) \Rightarrow hx \in T$. In fact, the first property describes a practical requirement of an application. For instance, assume that the term "very rather good" is required to use to describe a property of certain students that they are *very rather good*. Then, there are potentially also *rather good* students or *good* students in the same application. That is the terms "rather good" and "good" should also be required to present in T to describe the same property of certain application objects. In other words, the term "very rather good" may have its full meaning only when it is present together with the terms “rather good” and “good” in T. Similarly, Property (ii) can be explained as follows: if $h \leq k$, e.g. $h = rather$ and $k = little$, we have $little\_good \leq rather\_good \leq good$. In term of generation of terms using hedges, the term “rather good” possesses the meaning closer to the parent “good” than the term “little good” does and we interpret the former term as *a superior* to the latter. Semantically, when both hedges rather and little are needed to generate T, the presence of “rather good (student)” will make the meaning of “little good student” more fully. Thus, it is better that “little good” appears in T implies that “rather good” also does. A term-set with these both properties is called *superior-closed*. For example, the set $T_0 = \{\mathbf{0}, Very\_bad, bad, medium, Rather\_good, Very\_good, \mathbf{1}\} \subseteq X_{(2)} = T_0 \cup \{R\_bad, good\}$ is not superior-closed since it contains the terms $Rather\_good$ and $Very\_good$ but not the term *good*.

To construct the interval-semantics of such a superior-closed term-scale $T \subseteq X_{(l)} = \{x \in X : |x| \leq l\}$, we assume that the l-similarity intervals of $X_{(l)}$ have been constructed, which form a partition of $[0, 1]$. Since every $x \in X_{(l)}$ owns an l-similarity interval, for x’s $\notin T$ we lost their l-similarity intervals to cover the whole interval $[0, 1]$. To retake the lost intervals we may link their terms to certain terms in T, based on their semantic context and the superior-closed property of T. For instance, the meaning of *bad* appearing in the context of its adjacent terms V_bad and *medium* in T_0 above is less specific or more general than *bad* appearing together with the terms $Very\_bad$ and R_bad in $X_{(2)}$. The interval-semantics of R_bad that is absent in T_0 will be gotten back by determining the interval-semantics of *bad* in the context of it’s the adjacent terms V_bad and *medium* in T_0, noting that *bad* is superior to R_bad and, hence, to all terms, to which R_bad is superior. By the superior-closed property of T_0, these terns together with R_bad do not belong to T_0. Since *bad* is adjacent to V_bad in the context of the set $X_{(2)}$, the left part of the similarity interval of *bad* should be determined in the context of $X_{(2)}$. While since *bad* is adjacent to *medium* in the context of $X_{(1)} = X_1$, the right part of the similarity interval of *bad* should be determined in the context of X_1.

This suggests us the following way to define the interval-semantics of the terms in T.

Since the qualitative semantics of a term x in T depends on the presence of its left and right adjacent terms in T, the interval-semantics of $x, I(x) = [a, b)$, with $\nu(x) \in [a, b)$, will be defined based on the determination of its left subinterval $I_L(x) = [a, \nu(x)]$ and its right subinterval $I_R(x) = [\nu(x), b)$. These subintervals will be defined by utilizing the k-similarity intervals of $X_{(k)}$, for $k = 1, \ldots, l$. Since the specificity of a terms characterized by the number of the occurrences of hedges in the canonical representation of terms is an important characteristic in this definition, we suppose that T is of specificity of $l > 1$, i.e. there is a term $x \in T$ such that $|x| = l$. Given a set of the fuzziness parameter values of $\mathcal{X}$, the k-similarity interval are completely determined and we denote, for each $x \in X_{(k)}$, the left end-point of its similarity interval $\mathfrak{T}_k(x)$ by $lep\mathfrak{T}_k(x)$, where lep is an abbreviation of "left end-point", and put $L\mathfrak{T}_k(x) = [lep\mathfrak{T}_k(x), \nu(x)]$. Similarly, the corresponding notations for the right hand side of x are $rep\mathfrak{T}_k(x)$ and $R\mathfrak{T}_k(x) = [\nu(x), rep\mathfrak{T}_k(x))$, where rep is an abbreviation of "right end-point".

For each $x \in T$, to define $I_L(x)$ and $I_R(x)$ we must determine the left adjacent term x_L of x in T, utilizing k-similarity intervals. In this study, we suppose that the sets of hedges H^- and H^+ are singleton for simplification of the proofs of the next propositions. However, such sets of hedges H^- and H^+ are still sufficient for the construction of practical linguistic scales, since for $X_{(2)}$ we have already a term-set with $|X_{(2)}| = 9$. Moreover, if necessary, we may extend to consider a larger set $X_{(3)}$.

First we show some properties of the structure of the term-scale T.

Proposition 4.1 *Let $T \subseteq X_{(l)}$ be a superior-closed term-scale of $\mathcal{AX}$ with a specificity l. For any two adjacent terms x and x' of T, assuming $x' < x$, we have*

(1) If $x \in C = \{\mathbf{0}, \mathbf{W}, \mathbf{1}\}$, then $x' = h'_q \dots h'_1 c$, for some $c \in G$, and

$$c < h'_1 c < h'_2 h'_1 c < \dots < h'_q \dots h'_1 c = x'$$

If $x' \in C$, then $x = h_m \dots h_1 c$, for some $c \in G$, and

$$c > h_1 c > h_2 h_1 c > \dots > h_m h_{m-1} \dots h_1 c = x$$

(2) If $x, x' \notin C$, then $x, x' \in H(c)$, for some $c \in G$, and, assuming $x = h_m \dots h_1 c$ and $x' = h'_q \dots h'_1 c$, we have $q \neq m$ and

(L1) For $q < m$, we have (i) $x' = x_{|q} = h_q \dots h_1 c$, (ii) $x' < h_{q+1} x'$ and (iii) $x' < x = h_m \dots h_{q+1} x' < \dots < h_{q+2} h_{q+1} x' < h_{q+1} x'$.

(L2) For $q > m$, we have (i) $x = x'_{|m} = h_m \dots h_1 c$, (ii) $h'_{m+1} x < x$ and (iii) $h'_{m+1} x < h'_{m+2} h'_{m+1} x < \dots < h'_q \dots h'_{m+1} x = x'$.

Proof: First, we prove the first statement of 1). By (8), we have $\{\mathbf{0}\} \leq H(c^-) \leq \{\mathbf{W}\} \leq H(c^+) \leq \{\mathbf{1}\}$, which shows that $x' \in H(c)$, for some $c \in G$, i.e. it can be written in the form $x' = h'_q \dots h'_1 c$. Now, suppose to the contrary that for some $j, 0 \leq j \leq q$, we have $h'_j h'_{j-1} \dots h'_1 c \leq h'_{j-1} \dots h'_1 c$. From the proper effect of h'_j mentioned previously, it follows that $h'_j x'_{|j-1} < h'_{j-1} \dots h'_1 c = x'_{|j-1} < x$, which implies, by (6) or (7) and (8), that $x' \in H(h'_j x'_{|j-1}) < x'_{|j-1} < x \in C$, which contradict the fact that x' and x are adjacent terms in T, as $x'_{|j-1} \in T$, by the superior-closed property of T. Since the second statement can be proved analogously, the proof of 1) is complete. Now, we prove 2). Since $x' < x$, by the comparison criterion of hedge algebra $\mathcal{AX}$ given in Subsection 3.1, we have $(\exists j)\{j \leq \min(m, q), h'_i = h_i, i = 1, \dots, j$ and $h'_{j+1} x_{|j} < h_{j+1} x_{|j}\}$.

Suppose firstly that $j < \min(m, q)$. This implies that h'_{j+1} and h_{j+1} are different from $h_0 = I$, the identity hedge, by convention. Since H^- and H^+ are singletons, hedges h'_{j+1} and h_{j+1} have different signs and, hence, $h'_{j+1}x_{|j} < x_{|j} < h_{j+1}x_{|j}$, which result in $x' \in H(h'_{j+1}x_{|j}) < x_{|j} < H(h_{j+1}x_{|j}) \ni x$, by (6) or (7). From the condition $x \in T$ and the superior-closed property of T, it follows that $x_{|j} \in T$, which contradicts the assumption that x and x' are adjacent terms. Thus, we should have $j = \min(m, q)$. There are two possibilities:

- For $j = q < m$, we have $x' = x_{|q} = h_q \ldots h_1 c$, which implies that $x \in H(h_{q+1}x')$. By the comparison criterion of hedge algebras given in Subsection 3.1, from $x' < x$ one derives that $x' = h_0 x_{|q} < h_{q+1}x'$, i.e. (ii) of (L1) holds. Using a similar argument as in the proof of 1) above, the inequalities in (iii) can be proved and, hence, the proof of (L1) is complete.
- For $j = m < q$, we have $x = x'_{|m} = h'_q \ldots h'_{m+1}x'_{|m}$. Then, $h'_{m+1}x < h_0 x_{|m} = x$, i.e. (ii) of (L2), follows from $x' < x$, by the comparison criterion. The validity of the inequalities in (iii) of (L2) can easily be obtained in a similar way as presented above. ■

For illustration, consider the term $x = VLV\_bad$ of the linguistic scale $T \subseteq X_{(4)}$, whose terms generated from the primary term "bad" are shown in boldface in Figure 2. Consider $x' = VLV\_bad$ and $x = bad$, i.e. $q = 3$ and $m = 0$. It can be verified that (L2) is true in this case: (i) $x = bad = x'_{|0}$, (ii) $h'_1 x = V\_bad < x$ and (iii) $h'_1 x = V\_bad < LV\_bad < VLV\_bad = x'$.

Before proving the next two propositions, we show the validity of the following lemma.

Lemma 4.1 *For any x, y and z, where $x = h_m \ldots h_1 c$, $y = h'_q \ldots h'_1 c$ and $z = k_n \ldots k_1 c$, the conditions that $x < z < y$, $h_i = h'_i$, for $i = 1, \ldots, p \le \min(m, q)$, and $h_{p+1} \ne h'_{p+1}$ result in $k_i = h_i$, for $i = 1, \ldots, p$. That is, if x and y have a maximal common suffix $u = h_p \ldots h_1 c$ then u is also a suffix of z.*

Proof: Indeed, by the comparison criterion of $\mathcal{AX}$, it follows from $x < z$ that $(\exists j)\{j \le \min(m, n), h_i = k_i, i = 1, \ldots, j$ and $h_{j+1}x_{|j} < k_{j+1}x_{|j}\}$ and from $z < y$ that $(\exists j')\{j' \le \min(n, q), h'_i = k_i, i = 1, \ldots, j'$ and $k_{j'+1}x_{|j'} < h'_{j'+1}x_{|j'}\}$. It cannot happen that $j, j' < p$ and $j \ne j'$. In fact, for instance, if $j < j'$, then we obtain $h'_{j+1}x_{|j} = h_{j+1}x_{|j} < k_{j+1}x_{|j}$, which implies by (6) or (7) that $y \in H(h'_{j+1}x_{|j}) < H(k_{j+1}x_{|j}) \ni z$, a contradiction to the assumption made on y and z. Similarly, if $j' < j$, then $k_{j'+1}x_{|j'} < h'_{j'+1}x_{|j'} = h_{j'+1}x_{|j'}$, which implies that $z < x$, a contrary to the assumption. Now, assume that $j = j' < p$. Then, we have $h_{j+1}x_{|j} < k_{j+1}x_{|j} < h'_{j+1}x_{|j} = h_{j+1}x_{|j}$, again also a contradiction. Thus, we should have $j = j' = p$, i.e. u is also a suffix of z. ■

Proposition 4.2 *Let be given a superior-closed term-scale T of $\mathcal{AX}$ with a specificity l. Then,* *(i) For every $x \in T \setminus C$, if x_L (respectively x_R) is the left (respectively the right) adjacent term of x in T, then it is also the left (respectively the right) adjacent term of x in X_{p_L} (respectively, X_{p_R}), where $p_L = \max(|x_L|, |x|) \leq l$ (respectively $p_R = \max(|x_R|, |x|) \leq l$).*

(ii) In particular, if we have in addition that $|x| = l$, then x_L (respectively x_R) is also the left (respectively the right) adjacent term of x in $X_{(l)}$.

Proof: We will only prove the proposition for the left adjacent term x_L of x, since the proof for x_R can be obtained by an analogous argument.

As (ii) is a particular case of (i), we shall prove only (i).

- First we consider the case that $x_L \in C = \{\mathbf{0}, \mathbf{W}, \mathbf{1}\}$. Since x_L and x are adjacent terms in T, by 1) of Proposition 4.1, we have that $x = h_m \ldots h_1 c$ and $h_1 c < c$. Since $|x_L| = 1$, we have $p_L = m$. Suppose that there exists $z \in X_{p_L}$ satisfying the condition $x_L < z < x$. By (8) we have $\mathbf{0} < H(c^-) < \mathbf{W} < H(c^+) < \mathbf{1}$, from which and the condition $x_L \in C$, it follows that z and x belong to the same $H(c)$ and $x_L < H(c)$. Therefore, we may assume that $z = k_n \ldots k_1 c$. By the comparison criterion given in Subsection 3.1, it follows from $z < x$ that $(\exists j)\{j \leq \min(n, m), h_i = k_i, i = 1, \ldots, j$ and $k_{j+1}x_{|j} < h_{j+1}x_{|j}\}$. Since H^- and H^+ are singleton, hedges k_{j+1} and h_{j+1} have different signs and, hence, we have $k_{j+1}x_{|j} < x_{|j} < h_{j+1}x_{|j}$. By (6) or (7) we obtain $z \in H(k_{j+1}x_{|j}) < \{x_{|j}\} < H(h_{j+1}x_{|j}) \ni x$, which implies that $C \ni x_L < x_{|j} < x$, a contrary to the fact that x and x_L are adjacent terms in T, as $x_{|j} \in T$, by the superior-closed property of T. This contradiction shows that x_L is also the left adjacent of x in $X_{(p_L)}$, i.e. (i) is valid.
- Suppose that $x_L \notin C$. Since, by the definition of x_L, the adjacent terms x and x_L must belong to the same $H(c)$, we may assume that $x = h_m \ldots h_1 c$, $x_L = h'_q \ldots h'_1 c$ and $m \neq q$. Suppose the contrary that there exists $z = k_n \ldots k_1 c \in X_{(p_L)}$ satisfying $x_L < z < x$.

Assume first that $q < m$ and, hence, $p_L = m$. By (L1) of Proposition 4.1, $x_L = x_{|q} = h_q \ldots h_1 c$ and $x_L < h_{q+1}x_L$. Since $|z| \leq p_L$, we have $n \leq m = p_L$. As x_{vertq} is the maximal common suffix of x_L and x, by Lemma 4.1, $z_{|q} = x_{|q}$. By the comparison criterion of $\mathcal{AX}$, from $x_L = x_{|q} < z$ it follows that $x_L < k_{q+1}z_{|q} = k_{q+1}x_{|q}$. On the other hand, from $z < x$ it follows that $(\exists j')\{j' \leq \min(n, m), h_i = k_i, i = 1, \ldots, j'$ and $k_{j'+1}x_{|j'} < h_{j'+1}x_{|j'}\}$. Suppose that $j' < \min(n, m) = n$. Then, $k_{j'+1}$ and $h_{j'+1}$ are different from h_0, the identity hedge, and since H^- and H^+ are singleton, they have different sign and, hence, $k_{j'+1}x_{|j'} < x_{|j'} < h_{j'+1}x_{|j'}$. These inequalities result in $z \in H(k_{j'+1}x_{|j'}) < \{x_{|j'}\} < H(h_{j'+1}x_{|j'}) \ni x$, based on (6) or (7). That is, by the superior-closed property of T, there is $x_{|j'} \in T$ lying in between x_L and x, a contradiction to the definition of x_L. This shows that $j' = n$, i.e. $k_{j'+1} = k_{n+1} = h_0$ and $z = x_{|n} = h_0 x_{|n} < h_{n+1}x_{|n}$, which implies again by (6) or (7) that $z < H(h_{n+1}x_{|n})$. Since $x \in H(h_{n+1}x_{|n})$, we obtain $x_L < z = x_{|n} < x$, a contrary to the definition of x_L, because $x_{|n} \in T$.

Now, assume that $m < q$ and, hence, $p_L = q$. By (L2) of Proposition 4.1 we have $x_L = h'_q \dots h'_{m+1}x$ and $h'_{m+1}x < x$. Suppose that $x_L < z < x$, where $z = k_n \dots k_1 c \in X_{(p_L)}$, which implies that $n \leq q$. Since x is the maximal suffix of x and x_L, it follows from Lemma 4.1 that $z_{|m} = x = x_L|m$. From $x_L < z$ we infer that $(\exists j)\{j \leq \min(n, q), h'_i = k_i, i = 1, \dots, j$ and $h'_{j+1}x_{L|j} < k_{j+1}x_{L|j}\}$. If $j < \min(n, q)$, then $h'_{j+1} \neq h_0$ and $k_{j+1} \neq h_0$ and we have $h'_{j+1}x_{L|j} < x_{L|j} < k_{j+1}x_{L|j}$, since h'_{j+1} and k_{j+1} have different signs, by the singleton property of H^- and H^+. Applying (6) or (7), these inequalities result in $H(h'_{j+1}x_{L|j}) < x_{L|j} < H(k_{j+1}x_{L|j})$. Since $x_L \in H(h'_{j+1}x_{L|j})$ and $z \in H(k_{j+1}x_{L|j})$, it follows that $x_L < x_{L|j} < z < x$, which contradicts the definition of x_L in T, as $x_{L|j} \in T$, by the superior-closed property of T. Thus, we should have $j = \min(n, q) = n$ and $x_{L|n} = z$. Since $x_L \neq z$, we have $n < q$, $k_{n+1} = h_0$ and $h'_{n+1}x_{L|n} < h_0 x_{L|n} = x_{L|n} = z$. Hence, again by (6) or (7), $H(h'_{n+1}x_{L|n}) < x_{L|n}$. As $x_L \in H(h'_{n+1}x_{L|n})$, we obtain $x_L < x_{L|n} = z < x$, a contrary to the definition of x_L, since $x_{L|n} \in T$.

Since the assumption made on z always leads to a contradiction, it is shown that x_L is also the left adjacent term of x in $X_{(p_L)}$. ■

Corollary 4.1 *Proposition 4.2 is also valid for $x \in C$.*

Proof: Since $x \in C$ and x has the left adjacent term x_L in T, we infer that $x \neq \mathbf{0}$ and x_L should be in the form $x_L = h_m \dots h_1 c$, as $\mathbf{0} < H(c^-) < \mathbf{W} < H(c^+) < \mathbf{1}$. By the definition, the right adjacent of x_L in T is just x. Therefore, applying Proposition 4.2 to $x_L \in T \setminus C$, x is also the right adjacent of x_L in $X_{(p_R)} = X_{(p_L)}$ and, hence, x_L is also the left adjacent term of x in $X_{(p_L)}$.

By duality, the assertion is also true for the right adjacent term x_R of x in T. ■

Proposition 4.2 and the corollary shown above come with some practical relevance. As on the aforementioned discussion, the interval-semantics of a term x in T should be defined in the context with x_L and x_R and the k-similarity interval $\mathfrak{T}_k(x)$ of a term x in $X_{(k)}$ is interpreted as an interval-semantics of x defined in the context of $T = X_{(k)}$. Therefore, Proposition 4.2 and Corollary 4.1 provide a basis to determine the interval-semantics of x, $I(x)$, in the context of T. That is it is defined by the equality $I(x) = I_L(x) \cup I_R(x)$, where $I_L(x) = L\mathfrak{T}_{p_L}(x)$ and $I_R(x) = R\mathfrak{T}_{p_R}(x)$, remembering that $\mathfrak{T}k(x), x \in X_{(k)}$, can be computationally determined for given k and fuzziness parameter values of $\mathcal{AX}$.

For example, let us consider again the term-set T with the specificity 4 represented in Figure 2 and the term $x = V\_bad$. It can be seen that $x_L = \mathbf{0}$ and $x_R = LV\_bad$. Hence, $p_L = 2$, $p_R = 3$ and $I(V\_bad) = L\mathfrak{T}2(x) \cup R\mathfrak{T}_3(x)$.

However, for $y = LV\_bad$ we can see that $y_L = V\_bad$ and $y_R = VLV\_bad$ and, therefore, $I(LV\_bad) = L\mathfrak{T}_3(x) \cup R\mathfrak{T}_4(x)$.

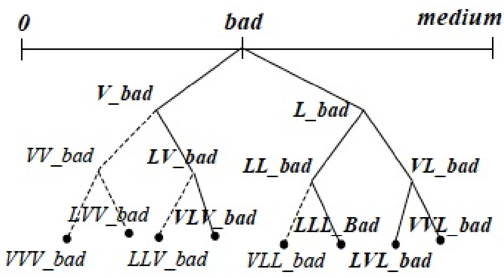

Fig. 2. The tree structure of $T \subseteq X_{(4)}$

This suggests us to introduce that following definition, which is correct based on Proposition 4.2 and Corollary 4.1.

Definition 4.1 Let be given the fuzziness parameter values of $\mathcal{AX}$. Then, for every $x \in T$, the interval-semantics of x in the context of T is defined to be the interval $I(x) = I_L(x) \cup I_R(x)$, where $I_L(x) = L\mathfrak{T}_{p_L}(x) = [lpt\mathfrak{T}_{p_L}(x), \nu(x)]$ with $p_L = \max(|x_L|, |x|)$ and $I_R(x) = R\mathfrak{T}_{p_R}(x) = [\nu(x), rpt\mathfrak{T}_{p_R}(x))$ with $p_R = \max(|x_R|, |x|)$. Obviously, the numeric quantitative semantics $\nu(x) \in I(x)$ and the similarity degree of the elements of $I(x)$ with the semantics of x can be indicated by $p = \min(p_L, p_R)$. Consequently, if both the terms x_L and x_R of x in T are also the left and the right adjacent terms of x in $X_{(k)}$, then $I(x) = \mathfrak{T}_k(x)$, where $k = p_L = p_R$.

For example, in the example given just above, the degree of similarity of $I(V\_bad) = 3$, while the one of $I(LV\_bad) = 4$.

Proposition 4.3 *The set of the interval-semantics* $I(x) = I_L(x) \cup I_R(x)$, $x \in T$, *defined as in Definition 4.1 forms a partition of* $[0, 1]$.

Proof: Since T is finite and totally ordered, we assume that $T = \{x_j : 0 \leq j \leq l\}$ and $x_0 = \mathbf{0} < x_1 < \ldots < x_l = \mathbf{1}$. By the definition, ν preserves the order relation, it follows that $0 = \nu(\mathbf{0}) < \nu(x_1) < \ldots < \nu(x_{l-1}) < \nu(\mathbf{1}) = 1$. Thus, to prove the proposition, it is sufficient to show that for every pair of consecutive terms x_j and x_{j+1} in T, $I_R(x_j) \cap I_L(x_{j+1}) = \emptyset$ and $I_R(x_j)$ and $I_L(x_{j+1})$ cover the interval $[\nu(x_j), \nu(x_{j+1})]$.

Indeed, obviously we have $x_{j,R} = x_{j+1}$ and $x_{j+1,L} = x_j$ in T and, therefore, $I_R(x_j) = R\mathfrak{T}_{p_R}(x_j)$ and $I_L(x_{j+1}) = L\mathfrak{T}_{p_L}(x_{j+1})$, where $p_R = \max(|x_{j,R}|, |x_j|) = \max(|x_{j+1}|, |x_j|) = \max(|x_{j+1}|, |x_{j+1,L}|) = p_L$, by Definition 4.1. Since the similarity intervals of the term in $X_{(p_L)}$ are pairwise disjoint, we have $\mathfrak{T}_{p_R}(x_{j+1}) \cap \mathfrak{T}_{p_L}(x_{j+1}) = \mathfrak{T}_{p_R}(x_j) \cap \mathfrak{T}_{p_L}(x_{j+1}) = \emptyset$ and, hence, it follows that $I_R(x_j) \cap I_L(x_{j+1}) = R\mathfrak{T}_{p_R}(x_j) \cap L\mathfrak{T}_{p_L}(x_{j+1}) = \emptyset$. Clearly, we have $I_R(x_j) \cup I_L(x_{j+1}) = [\nu(x_j), \nu(x_{j+1})]$, by Definition 4.1. ■

So, each term x of $\mathcal{X}$ can be associated with an interval $I(x)$ which represents an interval-semantics of x. This suggests us to introduce the following notion of 4-tuple semantics representation model of linguistic terms for a general case.

Definition 4.2 Let be given a linguistic variable $\mathcal{X}$ and its numeric reference domain U, *4-tuple semantic representation* of x, $x \in Dom(\mathcal{X})$, is a 4-tuple defined in the following form:

$$(x, I(x), Q(x), r_x) \tag{12}$$

where $I(x) \subseteq U$ and $Q(x) \in I(x)$ are declared to be the interval-semantics and the numeric quantitative semantics of the term x, respectively, and r_x is an arbitrary value in $I(x)$.

It is obvious that (12) can be considered as an extension of the linguistic representation of terms given in (1) that one has used in practice. In general, this 4-tuple semantic representation model of linguistic terms represent a compound relationship, which relates the quantitative semantics with the qualitative semantics of terms. The meaning of the components of (12) is that x carries a qualitative meaning, which is determined in a semantic order-based term-set. The second component represents an interval-semantics of x. The third represents its numeric quantitative semantics of x, considered as to be the value most compatible with x among the values of the interval-semantics of x or the core of this interval-semantics. The fourth is intended to represent a numeric assessment, which represents a user numeric assessment considered as his approximation of the semantics of x. Similarly as in the case of 2-tuple linguistic representation [8,9], the quantity $r_x - \nu(x)$ reflects how large difference of a numeric assessment from the core of the interval-semantics of x. That is (12) may bring useful information.

The concept of 4-tuple semantic representations of terms given in (12) permits a possibility to unify the ordinary numeric scales associated with the linguistic ones. This may be a potential demand of many practical applications. For example, when we are required to deal with a data warehouse of historical project bid data, the one part of which may include *numeric assessments* given by certain experts for a certain criterion, while the other part of which may include *linguistic assessments* for the same criterion given by other experts. It seems to be practical to imagine that in the case of a social investigation through social network, for flexible, we may design two parallel numeric and linguistic scales to permit people to express their opinion by choosing numeric or linguistic assessments.

In the case someone chooses a linguistic assessment, say x_j, his opinion is automatically represented in the system as the 4-tuple $(x_j, I(x_j), Q(x_j), Q(x_j))$, since $Q(x_j)$ and $I(x_j)$ have been declared in advance. In the case someone chooses a numeric assessment, say $r_0 \in [0, 1]$, a normalized interval of the numeric assessments, we will show below that this opinion will be automatically represented uniquely as the respective 4-tuple $(x(r_0), I(x(r_0), Q(x(r_0)), r_0)$.

4.2 4-Tuple semantic linguistic scales and their computation features

In this subsection we will show that semantic linguistic scales using the above semantic 4-tuple semantic representation model of terms will have a rich functionality. In this study, we distinguish between linguistic scales, whose carry their own order-based semantics the experts use to express their assessments, and their respective semantic linguistic scales expressed in terms of 4-tuple semantic representation model, whose 4-tuples form an underlying set of a rich computational structure. Since the terms in linguistic scales are elements of a hedge algebra associated with the linguistic variable in question, we introduce the following definition, noticing that a term x is more specific than a term y if $|x| \geq |y|$, i.e. the number of hedge occurrences in x is greater than the one in y. Thus, the length of terms characterizes their specificity level.

Definition 4.3 Given a hedge algebra $\mathcal{AX} = (X, G, C, H, \leq)$ associated with $\mathcal{X}$. A term-set $\mathcal{S} = \{s_0, \ldots, s_g\} \subseteq X$ is said to be a *superior-closed linguistic scale* with a specificity level l if it satisfies the following conditions:

(i) $\exists s \in S$ such that $|s| = l$ and $\mathcal{S}$ is totally ordered by $\leq$.
(ii) $c^-, c^+, \mathbf{0}, \mathbf{W}, \mathbf{1} \in \mathcal{S}$.
(iii) The term-set $\mathcal{S}$ is superior-closed.

It is different from the 2-tuple approach, in this study it is not required that the cardinality $|\mathcal{S}|$ is an odd number, since $\mathcal{S}$ will be associated with a semantic linguistic scale which is constructed based on only the qualitative semantics of linguistic terms themselves. Condition (i) is a natural requirement. (ii) is necessary, since they are elementary vague concepts for determining other terms in the context of their presence. (iii) is required by the previous discussion of the superior-closed notion.

As $\mathcal{S}$ is not a computational structure, a semantic linguistic scale with necessary computational features in company with $\mathcal{S}$ will be introduced based on 4-tuple semantic representation model of terms, called 4-tuple semantic linguistic scale.

Definition 4.4 A *4-tuple semantic linguistic scale* (4-tSLS) of a specificity level l associated with the given linguistic scale $\mathcal{S}$ is the set of 4-tuples, $\mathcal{S}_{Q,l} = \{(s, I_{\partial(s)}(s), Q(s), r_s) : s \in S, r_s \in I_{\partial(s)}(s)\}$, that satisfies the following conditions:

$(S1)$ $\mathcal{S}$ satisfies Definition 4.3.

($S2$) $I_{\partial(s)}(s)$ and $Q(s)$ are, respectively, the interval-semantics and numeric quantitative semantics of $s \in \mathcal{S}$, that are declared explicitly and uniquely defined in some way. The values of $I_{\partial(s)}(s)$, $s \in \mathcal{S}$, can be interpreted as being similar with s (i.e. with $Q(s)$), to a degree indicated by the numeric value $\partial(s)$. In addition, the interval-semantics of the terms in $\mathcal{S}$ form a partition of $[0, 1]$.

($S3$) For any $s, s' \in S$, $s \leq s' \Rightarrow I_{\partial(s)}(s) \leq I_{\partial(s')}(s')$

($S4$) $Q(s) \in I_{\partial(s)}(s), s \in S$.

The following proposition shows the existence of the 4-tuple semantic representation of any given numeric assessment and describes the relationship between numeric assessments and linguistic assessments utilizing the scale $\mathcal{S}_{Q,l}$.

Proposition 4.4 *Given a 4-tSLS $\mathcal{S}_{Q,l}$, every numeric assessment r is represented by a unique 4-tuple in $\mathcal{S}_{Q,l}$, denoted by $(t(r), I(t(r), Q(t(r)), r)$, and if $r \leq r'$, then their respective terms $t(r)$ and $t(r')$ satisfy the inequality $t(r) \leq t(r')$.*

Proof: Since, by (S2) of Definition 4.4, the interval semantics $I_{\partial(s)}(s), s \in S$, form a partition of $[0, 1]$, for any $r \in [0, 1]$, there exists only one term s such that $r \in I_{\partial(s)}(s)$. Moreover, its interval semantics $I_{\partial(s)}(s)$ and its numeric quantitative value $Q(s)$ are uniquely defined and, hence, the first statement of the proposition is valid.

Now, suppose that $r \leq r'$. In the case that the numeric assessments r and r' belong to the same interval-semantics $I(s)$ of a certain term s, we have clearly that $t(r) = t(r') = s$. In the case that $r \in I(t(r)), r' \in I(t(r'))$ and $I(t(r)) \neq I(t(r'))$, from $r \leq r'$ and the fact that $I(t(r))$ and $I(t(r'))$ are disjoint it follows that $I(t(r)) \leq I(t(r'))$ and, therefore, $t(r) \leq t(r')$ follows from (S3) of Definition 4.4. ■

Note again that Definition 4.4 and Proposition 4.4 have been examined in a general case. In the approach based on hedge-algebra-based linguistic semantics, we have the following.

Proposition 4.5 *Given a hedge algebra $\mathcal{AX} = (X, G, C, H, \leq)$ of a linguistic variable $\mathcal{X}$ and $\mathcal{S}$ is a superior-closed linguistic scale with specificity level l defined based on $\mathcal{AX}$. Then, for each given set of fuzziness parameter values of $\mathcal{AX}$, the set $\mathcal{S}_\nu = \{(s, I_{\partial(s)}(s), \nu(s), r_s) : s \in \mathcal{S}, r_s \in I_{\partial(s)}(s)\}$, where $I_{\partial(s)}(s)$ is defined as in Definition 4.1 and ν is the SQM induced by the fuzziness measure defined by the given fuzziness parameter values, satisfying the following primary properties:*

(i) $\mathcal{S}_\nu$ *is* 4-tuple semantic linguistic scale *associated with $\mathcal{S}$.*

(ii) *Every interval $I_{\partial(s)}(s)$ is defined and calculated based on the semantics of* *the terms of* $\mathcal{AX}$*:* $I_{\partial(s)}(s) = I_L(s) \cup I_R(s)$ *and*

$$I_{\partial(s)}(s) = \bigcup\{\mathfrak{I}(x) : x \in X_{l+2} \text{ and } \mathfrak{I}(x) \subseteq [\nu(s_{L,p_L+1}), \nu(s_{R,p_R+1})\} \qquad (13)$$

Proof: To prove (i), it is easily seen that (S2) of Definition 4.4 follows from Definition 4.1 and Propositions 4.3. The validity of (S3) follows from the fact that intervals $I_{\partial(s)}(s), s \in \mathcal{S}$, are disjoint and the SQM ν preserves the order relation of $\mathcal{S}$. Since (S1) and (S4) are clearly satisfied, $\mathcal{S}_\nu$ is a 4-tuple semantic linguistic scale, by Definition 4.4.

Now, we prove the statement (ii). Since $\mathcal{S}$ is a superior-closed linguistic scale with specificity of level l, we have $\mathcal{S} \subseteq X_{(l)}$ and, for every $s \in \mathcal{S}$, the interval-semantics $I_{\partial(s)}(s) = I_L(s) \cup I_R(s)$ defined by Definition 4.1, where $\partial(s) = \min(p_L, p_R)$, is well-defined based on Proposition 4.2 and Corollary 4.1. This shows that $I_{\partial(s)}(s)$ can be calculated based on the k-similarity intervals of terms and, hence, the fuzziness intervals of terms, which are two of the characteristics of the quantitative semantics of terms examined in Section 3. That is $I_{\partial(s)}(s)$ is defined and calculated based on the qualitative and quantitative semantics of terms of $\mathcal{AX}$, giving the fuzziness parameter values, by Remark 4.1.

By Definition 4.1, we have $I_L(s) = L\mathfrak{T}_{p_L}(s)$, where $p_L = \max(|s_L|, |s|) \leq l$ with s_L is the left adjacent term of s in $\mathcal{S}$. Denoting by s_{L,p_L+1} and s_{R,p_L+1} the left and the right adjacent terms of s in the term-set X_{p_L+1}, respectively, by formula (11),

$$\begin{aligned} \mathfrak{T}_{p_L}(s) = [\nu(s_{L,p_L+1}), \nu(s_{R,p_L+1})) \\ = \bigcup\{\mathfrak{I}(z) | z \in X_{p_L+2} \ \& \ \mathfrak{I}(z) \subseteq [\nu(s_{L,p_L+1}), \nu(s)) \text{ or } \mathfrak{I}(z) \subseteq [\nu(s), \nu(s_{R,p_L+1}))\} \end{aligned}$$

This implies that $I_L(s) = [\nu(s_{L,p_L+1}), \nu(s)) = \bigcup\{\mathfrak{I}(z) : z \in X_{p_L+2} \ \& \ \mathfrak{I}(z) \subseteq [\nu(s_{L,p_L+1}), \nu(s))\}$. Since $p_{L+2} \leq l+2$ and, by inductive derivation from (10), each fuzziness interval of X_{p_L+2} is the union of the fuzziness intervals of X_{l+2}, we infer that

$$I_L(s) = \bigcup\{\mathfrak{I}(z) : z \in X_{l+2} \ \& \ \mathfrak{I}(z) \subseteq [\nu(s_{L,p_L+1}), \nu(s))\}$$

Analogously, we have

$$\begin{aligned} I_R(s) = R\mathfrak{T}_{p_R}(s) = [\nu(s), \nu(s_{R,p_R+1})) \\ = \bigcup\{\mathfrak{I}(z) : z \in X_{l+2} \ \& \ \mathfrak{I}(z) \subseteq [\nu(s), \nu(s_{R,p_R+1}))\} \end{aligned}$$

Since $I(s) = I_L(s) \cup I_R(s)$, the formula (13) follows. ■

Before the examination of the computational structure of 4-tuple semantic linguistic scales we consider the following example.

Example 4.1 *Construction of a 4-Tuple semantic linguistic scale*: As mentioned previously, a practical linguistic scale used in schools of Vietnam is the following:

$$\{(Bad, [0.0, 3.5)), (Weak, [3.5, 5.0)), (Medium, [5.0, 6.5)), (Fair, [6.5, 7.5)), (Good, [7.5, 8.5)), (VeryGood, [8.5, 9.5)), (Excellent, [9.5, 10.0])\}$$

We show that this practical scale can be represented approximately as a 4-tuple semantic linguistic scale defined above. For illustration of the proposed method, the scale construction comprises following tasks:

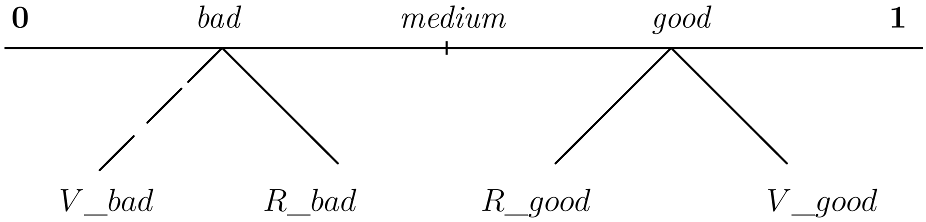

Fig. 3. The tree representing the structure of $\mathcal{S} \subseteq X_{(2)}$

(*i*) *Determining a hedge algebra for this scale*: Since the terms of linguistic scales examined in this study should be certain terms of a hedge algebra, we assume that $\mathcal{AX} = (X, G, C, H, \leq)$ is defined with $G = \{bad, good\}$, $H = \{R, V\}$. Let us consider the linguistic scale T, which is represented as a tree with term labels shown in boldface in Figure 3. Then, the linguistic terms present in (14) are converted to the terms of $\mathcal{AX}$ as follows:

$$Excellent \triangleq \mathbf{1},\ Very\ good \triangleq Very\ good,\ Good \triangleq good,$$
$$Fair \triangleq Rather\ good,\ Medium \triangleq \mathbf{W},\ Weak \triangleq Rather\ bad,\ Bad \triangleq \{bad, \mathbf{0}\}$$

where *Bad* defined by the set $\{bad, \mathbf{0}\}$ means that the concept *Bad* comprises two concepts *bad* and $\mathbf{0}$, since in accordance with his school education policy the user does not require to distinguish them. So,

$$\mathcal{S} = \{Bad, R\_bad, \mathbf{W}, R\_good, good, V\_good, \mathbf{1}\} \subseteq X_{(2)} \cup C$$

(*ii*) *The construction of 4-tuple semantic linguistic scale for the given linguistic scale* $\mathcal{S}$

The question is to determine a value of the fuzziness parameter so that intervals $I_{\partial(s)}(s)$, $s \in \mathcal{S}$, of the constructed 4-tuple semantics linguistic scale are more or less equal to the interval-semantics of the terms given in (14).

- *Construction of 4-tuple semantic linguistic scale for the given* $\mathcal{S}$

Since the specificity level of $\mathcal{S}$ is $l = 2$, by (13) it is required to consider fuzziness intervals of X_4, which consists of the following terms:

$$VVVc^-, RVVc RRVc^-, VRVc^-, VRRc^-, RRRc^-, RVRc^-, VVRc^-,$$
$$VVRc^+, RVRc^+, RRRc^+, VRRc^+, VRVc^+, RRVc^+, RVVc^+, VVVc^+$$

+ *Compute fuzziness intervals of terms in X_4 and the similarity intervals of $X_{(2)} \cup C$*

Since the interval-semantics of "Medium" given in (14) is [5.0, 6.5) of the numeric scale [0,10], the fuzziness parameter values are determined so that the left end point of $I_2(\mathbf{W})$ is 0.5 in the interval [0,1]. Thus, we should have the following equality, utilizing the constraints $(fm1)$ – $(fm4)$ imposed on the fuzziness measure fm given in Subsection 3.2:

$$fm(c^-) = 0.5/\mu^2(V)(1 - \mu(V))$$

which implies that $\mu(V) = 0.484$ and $fm(c^-) = 0.5687$. Then, we obtain the values of the fuzziness measure of the terms in X_4 as presented in Table 1 below.

Table 1

a) The fuzziness measure of terms in X_4 generated from c^-							
$VVVc^-$	$RVVc^-$	$RRVc^-$	$VRVc^-$	$VRRc^-$	$RRRc^-$	$RVRc^-$	$VVRc^-$
0.0645	0.0687	0.0733	0.0687	0.0733	0.0781	0.0733	0.0687
b) The fuzziness measure of terms in X_4 generated from c^+							
$VVRc^+$	$RVRc^+$	$RRRc^+$	$VRRc^+$	$VRVc^+$	$RRVc^+$	$RVVc^+$	$VVVc^+$
0.0521	0.0556	0.0592	0.0556	0.0521	0.0556	0.0521	0.0489
c) SQM-values in $[0, 10]$ of terms in the linguistic scale $\mathcal{S}$							
0	*bad*	*R_bad*	*medium*	*R_good*	*good*	*V_good*	**1**
0	2.75	4.27	5.79	6.77	7.91	8.99	10.0

Using these values, we can easily calculate the fuzziness intervals of terms of X_4 and the similarity of terms in $X_{(2)} \cup C$, (utilizing the fact that noting that they form a partition of [0,1]).

+ *Compute the interval-semantics of the terms in $\mathcal{S}$*

Now, we compute the actual intervals $I_2(s), s \in S$, and obtain the following, recalling that the maximum numeric mark is 10 and, hence, the interval-semantics and the SQM-values in the interval [0,1] are multiplied by 10.

– $I_1(Bad) = I_1(\mathbf{0}) \cup I_1(bad) = [0, 3.49)$, as $Bad = \{\mathbf{0}, bad\}$ and $I_R(bad) =$

$R\mathfrak{T}_2(bad)$ and, therefore, by formula (13) of Proposition 4.5, it follows that

$$I_1(Bad) = \mathfrak{T}_1(\mathbf{0}) \cup L\mathfrak{T}_1(bad) \cup R\mathfrak{T}_2(bad) = [0.0, lptI_1(Bad)) = [0, 3.49)$$

since from the values of the fuzziness measure of terms given in Table 1 we have

$$\begin{aligned}|I_1(Bad)| =& [fm(VVVc^-) + fm(LVVc^-) + fm(LLVc^-) \\ &+ fm(VLVc^-) + fm(VLLc^-)] \times 10.0 = 3.49\end{aligned}$$

– Since the interval-semantics of the remaining terms s in $\mathcal{S}$ are defined in the context of $X_{(2)}$, they are identical with their 2-similarity intervals. Similarly as above, by (13), we obtain

$I_2(Weak) = \mathfrak{T}_2(R\_bad) = [3.49, 5.0)$; $I_2(Medium) = \mathfrak{T}_2(\mathbf{W}) = [5.0, 6.21)$
$I_2(Fair) = \mathfrak{T}_2(R\_good) = [6.21, 7.36)$; $I_2(Good) = \mathfrak{T}_2(good) = [7.36, 8.43)$
$I_2(Very\ good) = \mathfrak{T}_2(V\_good) = [8.43, 9.51)$; $I_2(Excellent) = \mathfrak{T}_2(\mathbf{1}) = [9.51, 10.0]$

Thus, the 4-tuple semantic linguistic scale associated with the given linguistic scale comprises the following 4-tupe-marks:

$$\begin{aligned}
&(Bad, [0, 3.49), 2.75, r_1), r_1 \in I_1(Bad)\\
&(Weak, [3.49, 5.0), 4.27, r_2), r_2 \in I_2(Weak)\\
&(Medium, [5.0, 6.21), 5.79, r_3), r_3 \in I_2(Medium)\\
&(Fair, [6.21, 7.36), 6.77, r_4), r_4 \in I_2(Fair)\\
&(Good, [7.36, 8.43), 7.91, r_5), r_5 \in I_2(Good)\\
&(V_Good, [8.43, 9.51), 8.99, r_6), r_6 \in I_2(V_Good)\\
&(Excellent, [9.51, 10.0], 10.0, r_7), r_7 \in I_2(\mathbf{1})
\end{aligned}$$

Note that although the term *Bad* is defined to be the join of the terms $\mathbf{0}, V\_bad$ and *bad*, the 4-tuple-assessment associated with *Bad* still carries specific semantic feature of term *bad*. In fact, similarly as for the 2-tuple approach, utilizing the quantity $(r_1 - 2.75)$, where $2.75 = \nu(bad)$, the core meaning of *bad*, we observe that the larger the absolute value of this quantity, the lower its compatibility with the meaning of the term *bad*.

We observe that these above interval-semantics are approximately equal to the respective intervals of the given linguistic scale in practice exhibited in (14). This highlights the advantage of the proposed approach. It is worth stressing the flexibility of the method itself.

We show that the proposed semantic linguistic scales exhibit a desired computational structure. For simplicity, the indexes of interval-semantics of terms indicating its similarity degree will be ignored, if it is not necessary that they need to be declared explicitly.

4.3.1 Order relation on 4-tuple linguistic scales

Definition 4.5 Let two 4-tuples $(x, I(x), \nu(x), r)$ and $(x', I(x'), \nu(x'), r')$ be given. We shall write

$$(x, I(x), \nu(x), r) \leq (x', I(x), \nu(x'), r')$$

if and only if one of the following conditions holds:

(i) $x \leq x'$.
(ii) $x = x'$ and $r \leq r'$.

It can easily be seen that if

$(x, I(x), \nu(x), r) \leq (x', I(x'), \nu(x'), r')$ and $(x, I(x), \nu(x), r) \geq (x', I(x'), \nu(x'), r')$

then $(x, I(x), \nu(x), r) \equiv (x', I(x'), \nu(x'), r')$, i.e. all their components are respectively identical. Moreover, the linearity of the sets $\mathcal{S}$ and $[0, 1]$ implies the linearity of the 4-tuple semantic linguistic scales.

The following corollary is an immediate consequence of the above definition and Definition 4.4:

Corollary 4.2 *Every 4-tuple semantic linguistic scale associated with $\mathcal{S}$ is totally ordered.*

4.3.2 Aggregation on 4-tuple linguistic scales

One of the main aims of the study is to construct a semantic linguistic scale on which we can realize easily necessary operations to aggregate linguistic as well as numeric assessments of experts properly. Related to this there are two examination tasks. The first one that we have tried to solve up to now is to ensure that every 4-tuple of the semantic linguistic scale represents the order-based semantics of a term as much as possible. The second one we will show that the developed semantic linguistic scales have necessary computational structure.

To realize this, we introduce the following definition.

Definition 4.6 Let be given a 4-tuple semantic linguistic scale associated with $\mathcal{S}$ with a specificity of level l, $\mathcal{S}_{Q,l}$, and a numeric aggregation operator α in Yager's sense working on a normalized interval $[0, 1]$. The operator α will induce an aggregation operation Agg_α on $\mathcal{S}_{Q,l}$ defined as follows:

$$Agg_\alpha(\mathbf{a}) = Agg_\alpha(a_1, \ldots, a_n) = (t(r_\mathbf{a}), I(t(r_\mathbf{a}), \nu(t(r_\mathbf{a}), r_\mathbf{a})$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ is a vector of 4-tuple representations of terms, i.e., $a_i = (x_i, I(x_i), \nu(x_i), r_i)$, $i = 1, \ldots, n$, $r_\mathbf{a} = \alpha(r_1, \ldots, r_n)$ and $(t(r_\mathbf{a}), I(t(r_\mathbf{a}), \nu(t(r_\mathbf{a}), r_\mathbf{a})$ is the 4-tuple in the scale uniquely defined by Proposition 4.4.

This definition is very similar to aggregation operators used in practical decision making applications. For instance, in evaluation of students or of bids for project contract, experts may evaluate them with respect to different criteria using numeric scales and then aggregate their assessments. The evaluation objects will be classified into the classes labeled by certain linguistic terms, the semantic intervals associated with which contain their respective aggregation results. However, with 4-tuple semantic linguistic scales experts may express their assessments by either numeric values or linguistic values present in the scales.

Proposition 4.6 *The constructed 4-tuple semantic linguistic scale associated with a given superior-closed linguistic scale is closed with respect to the aggregation operator Agg_α. Moreover, Agg_α is increasingly monotonic and if the numeric aggregation operation α strictly increases, then so does Agg_α.*

Proof: The first statement of the proposition is obviously true in light of the correctness of the definition of Agg_α. By Proposition 4.4 and Definition 4.5, the monotonicity is derived from the monotonicity of Agg_α. Now, using the notations in Definition 4.6, assume that $\mathbf{a} = (a_1, \ldots, a_n) < \mathbf{b} = (b_1, \ldots, b_n)$, which implies that $a_i = (x_i, I(x_i), \nu(x_i), r_{\mathbf{a},i}) < b_i = (y_i, I(y_i), \nu(y_i), r_{\mathbf{b},i})$, for some i. From Definition 4.5, it follows that $r_{\mathbf{a},i} < r_{\mathbf{b},i}$. By the strict monotonicity of α, we have $r_\mathbf{a} = \alpha(r_{\mathbf{a},1}, \ldots, r_{\mathbf{a},n}) < r_\mathbf{b} = \alpha(r_{\mathbf{b},1}, \ldots, r_{\mathbf{b},n})$, which by Definition 4.5 leads to $(t(r_\mathbf{a}), I(t(r_\mathbf{a}), \nu(t(r_\mathbf{a}), r_\mathbf{a}) < (t(r_\mathbf{b}), I(t(r_\mathbf{b}), \nu(t(r_\mathbf{b}), r_\mathbf{b})$, as the terms $t(r_\mathbf{a})$ and $t(r_\mathbf{b})$ satisfying $t(r_\mathbf{a}) \leq t(r_\mathbf{b})$, by Proposition 4.4. ■

Now, we can show that the proposed 4-tuple semantic linguistic scales meet well all three requirements discussed in Section 2.

+ *For Requirement* 1: To meet this requirement is a main purpose of the study, in which the qualitative semantics of linguistic scales modeled by hedge algebra structure is proposed, and their quantitative semantics is declared explicitly by introducing 4-tuple representation of terms as examined above. There is a strict formalism to relate the quantitative semantic aspects of the

4-tuple semantic representations of the terms and their qualitative semantics, as presented in Section 3. The method of the construction of semantic linguistic scales proposed in this study is a unique approach up to now that bases on the order-based qualitative semantics of terms.

+ *For Requirement* 2: It has been shown that the components $I(s)$ and $\nu(s)$ are computed based on essential concepts of the quantitative semantics of terms, the SQMs, fuzziness measure, fuzziness intervals and similarity intervals of terms, that can be determined by formal procedures, providing fuzziness parameter values. Since once $I(s)$ and $\nu(s)$ have been defined, every value $r \in I(s)$ determines a 4-tuple semantic representation of s, the constructed semantic linguistic scale is completely determined by a formal procedure. Thus, these semantic linguistic scale has the properties (Pr1) and (Pr2) and, hence, this requirement is well satisfied.
+ *For Requirement* 3: The 4-tuple semantic scales have many significant advantages in the field of fuzzy decision making. The first, it can be considered as an immediate generalization of many practical linguistic scales of similar forms as that given in (1). The second, these semantic linguistic scales associated with a given superior-closed linguistic scale can be automatically constructed, providing its fuzziness parameter values. The third, these scales can be considered as a unification of the linguistic scales and their numeric reference domains that allows experts to express their assessments in terms of numeric values or linguistic values. The fourth, similarly as in the LOWA-based or 2-tuple representation approaches, the aggregation operators are defined in a very simple way, as they work on numeric values. It is important that the scales are closed with respect to many useful aggregation operations examined by Yager in [28] and it is easy to convert their numeric results into their respective linguistic terms. These show that these scales meet this requirement.

5 A comparative study using multi-criteria decision making

In this section, we consider a multi-criteria decision problem using two linguistic scales of different cardinalities for a comparative study of two approaches, the proposed 4-tuple based approach and the 2-tuple based approach.

A method for solving group multi-criteria decision making problems comprises two main tasks: (i) collecting assessments of experts with respect to criteria for the alternatives in question, utilizing the given linguistic scale, and (ii) aggregating the collected expert assessments. For simplicity, we will consider multi-criteria decision making problems with a unique expert, since in the case with a group of experts it is required to implement an additional aggregation scheme.

Let us consider a decision making problem with two alternatives A_1 and A_2 and three criteria $C_k, k = 1, 2, 3$. For simplicity, we assume that the expert use the same linguistic scale to express the assessments of her/his evaluation of all the alternatives under consideration with respect to these distinct criteria. In addition, to make a clearly visible difference of the proposed approach from the 2-tuple based approach, two linguistic scales are applied in turn, the one is a proper subset of the other, given as follows:

(1) The scale

$$\begin{aligned}\mathcal{S}_1 &= \{s_i : i = 1, \ldots, 9\}\\ &= \{E\_bad \triangleq \mathbf{0}, V\_bad, bad, R\_bad, medium,\\ &\quad R\_good, good, V\_good, Excellent \triangleq \mathbf{1}\}\end{aligned}$$

(2) The scale examined in Example 4.1 with

$$\begin{aligned}\mathcal{S}_2 &= \{s_{2,i} : i = 1, \ldots, 5\}\\ &= \{bad, R\_bad, medium, good, Excellent \triangleq \mathbf{1}\}\\ &= \mathcal{S}_1 \setminus \{E\_bad \triangleq \mathbf{0}, V\_bad, R\_good, V\_good\}\end{aligned}$$

The 4-tuple semantic linguistic scale associated with $\mathcal{S}_1 = X_{(2)} \cup C$ constructed by the method as described in Section 4, using the same fuzziness parameter values given in Example 4.1, is exhibited as follows:

$(E\_b., [0, 0.65), 0.31, r_1), r_1 \in I_2(\mathbf{0}); (V\_b., [0.65, 2.07), 1.33, r_2), r_2 \in I_2(V\_b.)$
$(b., [2.07, 3.49), 2.75, r_3), r_3 \in I_2(b.); (R\_b., [3.49, 0.5), 4.27, r_4), r_4 \in I_2(R_b.)$
$(\mathbf{W}, [0.5, 6.21), 5.69, r_5), r_5 \in I_2(\mathbf{W}); (R\_g., [6.21, 7.36), 6.77, r_6), r_6 \in I_2(R_g.)$
$(g., [7.36, 8.43), 7.91, r_7), r_7 \in I_2(g.); (V\_g., [8.43, 9.51), 8.99, r_8), r_8 \in I_2(V\_g.)$
$(Excellent, [9.51, 1.0), 10.0, r_9), r_9 \in I_2(\mathbf{1})$

The aim of the comparative study is to show that while the decision results of the proposed method depend strongly on the inherent qualitative semantics of terms in the context of their scale, themselves, but do not depend on the cardinality of the scales, it does not happen for the 2-tuple based approach. With this aim and since $\mathcal{S}_2$ is a proper subset of $\mathcal{S}_1$, we have to assume that the hedge algebras for representing the terms in the given linguistic scales and the fuzziness parameter values for constructing the 4-tuple semantic linguistic scales are the same. This ensures that the linguistic assessments of the expert using terms in $\mathcal{S}_2 \subseteq \mathcal{S}_1$ can be regarded as the same assessments of himself, when he uses the scale $\mathcal{S}_1$. With these assumptions, the 4-tuple semantic linguistic scale associated with $\mathcal{S}_2$ can be constructed and it consists of the following 4-tuples:

$(b., [0, 3.49), 2.75, r_1), r_1 \in I_1(b.); (R\_b., [3.49, 0.5), 4.27, r_2), r_2 \in I_2(R\_b.)$
$(\mathbf{W}, [5.0, 6.77), 5.69, r_3), r_3 \in I_1(\mathbf{W}); (Good, [6.77, 8.99), 7.91, r_4), r_4 \in I_1(G.)$
$(Excellent, [8.99, 10.0], 10.0, r_7), r_7 \in I_1(Excellent)$

5.1 *Case 1 – the use of scale $\mathcal{S}_1$*

Assume that the expert expresses his assessments of the two given alternatives with respect to the three criteria as represented in Table 2, in which are given also the weights of the criteria, supposing that the selected aggregation operation is the weighted average.

Now, we are ready to compute the results of aggregation for both approaches and their aggregation results are represented in Table 3. For the 2-tuple approach, the aggregation result of A_1 and A_2 are represented by $(R\_good, 0.48)$ and $(good, -0.45)$, respectively, and, therefore, A_2 is the best alternative. For the 4-tuple approach, the first alternative A_1 is preferable over A_2, as $(R\_g., [6.21, 7.36), 6.77, 7.30) > (R\_g., [6.21, 7.36), 6.77, 7.19)$, where the first 4-tuple is the aggregation result of A_1 and the second one is the aggregation result of A_2. Note that since his assessments are linguistic, the fourth components of the 4-tuples representing his assessments are identical with the third components, i.e. the SQM-values of his linguistic assessments.

Table 2
The evaluation provided by experts with respect to given criteria and their weights

Alternatives	Criteria and weights		
	$C_1, w_1 = 0.25$	$C_2, w_2 = 0.51$	$C_3, w_3 = 0.24$
A_1	$s_9 = Excellent$	$s_5 = medium$	$s_7 = good$
A_2	$s_4 = R\_bad$	$s_9 = Excellent$	$s_4 = R\_bad$

So, the two approaches produce different preferable alternatives. Because the 4-tuple approach exhibits a richer semantic representation of terms and more formalized fundamentals for constructing linguistic scales than the 2-tuple approach does, we are convinced that the result produced by the 4-tuple approach is more appropriate than the one produced by the 2-tuple approach.

5.2 *Case 2 – using the scale $\mathcal{S}_2$*

The purpose of this subsection is to show the following situation that the result of multi-criteria decision methods depend strongly on the number of terms in

Table 3
The weighted average results

	C_1	C_2	C_3	Aggregation
	$w_1 = 0.25$	$w_2 = 0.51$	$w_3 = 0.24$	
2-Tuple approach				
A_1	$(s_9, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_6, .48)$
A_2	$(s_4, 0)$	$(s_9, 0)$	$(s_4, 0)$	$(s_7, -.45)$
4-Tuple approach				
A_1	10.0	5.69	7.91	7.30
A_2	4.27	10.0	4.27	7.19

the given linguistic scales for the 2-tuple approach, while this situation does not occur for the proposed 4-tuple approach. So, we have assumed that $\mathcal{S}_2$ given above is a proper sub-scale of $\mathcal{S}_1$ and, hence, the number of the terms in $\mathcal{S}_1$ is reduced from 9 to 5, the number of the terms in $\mathcal{S}_2$.

We observe that the expert's linguistic assessments given in Table 2 are included in $\mathcal{S}_2$ and, the expert tries to focus on the inherent qualitative semantics of the terms in the given scale, when he expresses his linguistic assessments. This is a basis to permit us consider these linguistic assessments also as being his assessments using the linguistic scale $\mathcal{S}_2$, although their semantics may be changed a bit by the influence of a possible change in their left and right adjacent terms in $\mathcal{S}_2$. This constructed example aims to describe the intended situation that while the semantics of the expert's linguistic assessments in the context of $\mathcal{S}_1$ for the 4-tuple approach can be considered as almost unchanged in the context of $\mathcal{S}_2$, their semantics in the 2-tuple approach will be expected to be changed considerably so that the decision results may also be changed.

Table 4
The weighted average results

	C_1	C_2	C_3	Aggregation
	$w_1 = 0.25$	$w_2 = 0.51$	$w_3 = 0.24$	
2-Tuple approach				
A_1	$(s_5, 0)$	$(s_3, 0)$	$(s_4, 0)$	$(s_4, -.26)$
A_2	$(s_2, 0)$	$(s_5, 0)$	$(s_2, 0)$	$(s_4, -.47)$

In fact, since in the 4-tuple approach the SQM-values depend only on the linguistic variable $\mathcal{X}$ and its fuzziness parameter values provided, the SQM-values of the expert's linguistic assessments are identical with those that have been computed in Case 1. Therefore, the results of the expert's assessment aggregation produced by the numeric aggregation operation in question, which acts on the SQM-values appearing in the forth components, must be the same as in Case 1. Only the interval-semantics of the terms in $\mathcal{S}_2$ here are changed a bit, as discussed above, and they include the interval-semantics of the respective terms in $\mathcal{S}_1$, as some terms of $\mathcal{S}_1$ are absent in $\mathcal{S}_2$. Consequently, the interval-semantics of $\mathcal{S}_1$ is finer than those of $\mathcal{S}_2$. For instance, the interval-semantics of *good* in $\mathcal{S}_1$ is [7.36, 8.43), which is included in [6.77, 8.99), the interval-semantics of the same term *good* in $\mathcal{S}_2$. Hence, for the linguistic scale $\mathcal{S}_2$, the 4-tuple representing the aggregation results of alternatives A_1 and A_2 are given respectively as follows: $(good, [6.77, 8.99), 7.91, 7.30)$ and $(good, [6.77, 8.99), 7.91, 7.19)$, i.e. the same results as in Case 1, except their interval-semantics. So, A_1 is still the best alternative. However, in the 2-tuple approach the aggregation results of these alternatives are given in Table 4, which imply that A_1 is more preferable over A_2, giving a solution different from the one obtained in Case 1.

6 Conclusions

The semantics of vague linguistic terms is a complicated problem. Although in fuzzy sets we have witnessed a significant progress, there are still a number of open questions. For instance, we are interested in understanding the meaning of terms of a linguistic variable in the context of relationships between terms viewed as strings of symbols and the objects or phenomena present in the real world they point at. In the literature of fuzzy decision making, there is a lack of investigations on the semantics of vague linguistic terms expressed in the linguistic scales.

In decision making, the semantics of terms should be able to serve comparison tasks of alternatives in order to help decision makers to choose the best ones. Thus, the meaning of terms in a term-domain aims to express semantic order relationships between them. Note that to express expert opinions, the experts try to focus to choose appropriate terms of the scale based on their semantic order relationships, i.e. based on their qualitative semantics. This shows that the qualitative semantics of terms should play a pivotal role in the development of sound semantic linguistic scales.

In this study, we have proposed an interpretation of the inherent order-based semantics of terms of a linguistic variable as qualitative semantics of linguistic terms that are directly associated with the term string expressions regarded as their syntax. The quantitative semantics of terms serves as fundamental basis for computation on words, but it should represent the term qualitative semantics in a certain formal way. Therefore, it becomes necessary to establish a formal bridge to connect the quantitative semantics of terms with their qualitative semantics, i.e., their inherent order-based semantics. However, up to now, the examination of the semantics of linguistic terms has not been carefully taken into consideration and the qualitative semantics of terms still have not been declared and, hence, there exists a visible gap between the inherent qualitative semantics of terms and their quantitative semantics in the literature. The lack of determination of what is the actual qualitative semantics of linguistic terms and the lack of a formal discussions of the relation between the qualitative semantics of linguistic terms, which one can recognize in real world, and the quantitative semantics, which one wishes to assign to them, might be the reason that gives rise to some sophisticated problems in developing linguistic representation models and computation with words in the field of decision making.

Suggested by this complex situation, we articulated some requirements, of general nature, for construction of semantic linguistic scales for decision-making problems. These requirements are proposed based on the analysis of the actual relation between the qualitative semantics and the quantitative semantics of vague linguistic terms and of the relationship between linguistic scales the experts use to express their linguistic assessments and their associated computational semantic linguistic scales considered as domain of the desired operations for solving decision-making problems. These questions may be crucial for developing a sound and legitimate semantic linguistic scale associated with the given linguistic scale of a linguistic variable. We demonstrated that hedge algebras, which aim to model qualitative semantics of terms, can be applied to solve these questions. The 4-tuple (quantitatively) semantic linguistic scale has been developed, which is able to meet all the three proposed requirements for the construction of legitimate semantic linguistic scales. The important thing has been shown that there exists a strict mathematical formalism to construct a 4-tuple semantic linguistic scale, utilizing the quantitative semantic aspects of terms, based on the proposed formalized qualitative semantics of terms and the quantification of hedge algebras. It is natural and practical that the qualitative semantics of the terms in a linguistic scale is context-dependent, i.e. the semantics of a term in the scale depends on which are its adjacent terms in the given scale. It should be emphasized that the hedge-algebra-based qualitative semantics of terms provides a formalism to solve this problem. Hence, these 4-tuple semantic linguistic scales can be computationally determined by giving the fuzziness parameter values of the linguistic variable under consideration. This demonstrates that it is able to establish a formal bridge to link the quantitative semantics of terms in the linguistic scales to their qualitative semantics determined in the context of the respective linguistic scales, based on hedge algebra structures.

We showed also a number of their immediate advantages. In general, the proposed methodology for construction 4-tuple semantic linguistic scales can be considered as being characterized by several distinct aspects of the semantics of terms and is based on a clear mathematical formalism. These semantic linguistic scales can be viewed as a generalization of the respective numeric scales and, therefore, this concept becomes natural and easy understood. They are completely determined based on the fuzziness intervals of terms, when the user provides numeric values of the fuzziness parameters of linguistic variables and, hence, they can be produced computationally. Therefore, to construct an efficient semantic linguistic scale for a decision-making problem, the user can concentrate on the selection of linguistic hedges for the generation of suitable linguistics scales and the determination of appropriate values of the fuzziness parameters.

All of these observations point out that the semantic linguistic scales are very useful in addressing a variety of problems of decision making. For instance, aggregation operations can be induced by the corresponding numeric ones and work on the developed 4-tuple semantic linguistic scales, which are closed with respect to these aggregations. An additional linguistic approximation is not required. Since these scales can be considered as an immediate generalization of the respective numeric scales, in further applications one may allow the experts to express their assessments in terms of linguistic terms as well as in terms of numeric values in their respective scales. A comparative study examined by an example of a multi-criteria decision making problem in Section 5 illustrated these advantages.

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