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Description	

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THE GATHERING PROBLEM FOR TWO OBLIVIOUS ROBOTS WITH UNRELIABLE COMPASSES*

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Abstract. Anonymous mobile robots are often classified into synchronous, semi-synchronous, and asynchronous robots when discussing the pattern formation problem. For semi-synchronous robots, all patterns formable with memory are also formable *without* memory, with the single exception of forming a point (i.e., the gathering) by two robots. (All patterns formable with memory are formable without memory for synchronous robots, and little is known for asynchronous robots.) However, the gathering problem for two semi-synchronous robots without memory (called oblivious robots in this paper) is trivially solvable when their local coordinate systems are consistent, and the impossibility proof essentially uses the inconsistencies in their coordinate systems. Motivated by this, this paper investigates the magnitude of consistency between the local coordinate systems necessary and sufficient to solve the gathering problem for two oblivious robots under semi-synchronous and asynchronous models. To discuss the magnitude of consistency, we assume that each robot is equipped with an *unreliable* compass, the bearings of which may deviate from an absolute reference direction, and that the local coordinate system of each robot is determined by its compass. We consider two families of unreliable compasses, namely, *static compasses* with (possibly incorrect) constant bearings and *dynamic compasses* the bearings of which can change arbitrarily (immediately before a new look-compute-move cycle starts and after the last cycle ends). For each of the combinations of robot and compass models, we establish the condition on deviation ϕ that allows an algorithm to solve the gathering problem, where the deviation is measured by the largest angle formed between the x-axis of a compass and the reference direction of the global coordinate system: $\phi < \pi/2$ for semi-synchronous and asynchronous robots with static compasses, $\phi < \pi/4$ for semi-synchronous robots with dynamic compasses, and $\phi < \pi/6$ for asynchronous robots with dynamic compasses. Except for asynchronous robots with dynamic compasses, these sufficient conditions are also necessary.

Key words. mobile robot, distributed algorithm, gathering problem

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1. Introduction.

Geometric pattern formation by anonymous mobile robots has gained much attention [1, 2, 3, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16]. In the literature, a robot

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is represented by a point and repeatedly executes a "look-compute-move" cycle, during which it observes the positions of all robots (look phase), computes the next position using a given algorithm (compute phase), and moves to that position (move phase). A robot does not have access to a global coordinate system, and all its computations are done in terms of its local coordinate system. The robots do not have identifiers, are not equipped with communication devices, and execute the same algorithm.

The robots' behaviors are in general asynchronous. Their executions of the look, compute, and move phases may be interleaved in the sense that a robot may observe, for instance, another robot while it is moving.[1] The robots are said to be *semi-synchronous* when the execution of their cycles is assumed to be "instantaneous," which intuitively means that a robot is never observed while it is moving. Robots are said to be *synchronous* if all of them always execute the instantaneous cycles simultaneously. A robot is said to be *oblivious* if it has no memory to remember its execution history and its computations depend only on what it is observing in the current cycle. A robot is said to be *nonoblivious* if it has sufficient memory to remember the whole execution history and its action can depend also on what it has observed in the past.

The set of patterns formable by semi-synchronous oblivious robots is, by definition, a subset of the patterns formable by semi-synchronous nonoblivious robots. This inclusion relation is proper since the point formation (i.e., the gathering) problem for two robots is solvable for semi-synchronous nonoblivious robots, but it is unsolvable for semi-synchronous oblivious robots, which exhibits the impact of memory in forming a pattern [15]. Note that the gathering problem for more than two semi-synchronous oblivious robots is solvable provided that a robot can count the number of robots residing at the same point (i.e., detect multiple robots residing at the same point). Interestingly, with the sole exception of gathering of two robots mentioned above, any pattern formable by semi-synchronous nonoblivious robots is also formable by semi-synchronous oblivious robots [17]. Thus, memory helps only in the case of gathering two robots. All patterns formable by nonoblivious synchronous robots are formable by oblivious synchronous robots, and little is known for asynchronous robots. These facts motivate our study of the gathering problem for two oblivious semi-synchronous and asynchronous robots.

The impossibility proof of the gathering problem for two oblivious semi-synchronous robots relies on the "full" inconsistency of their local coordinate systems [15], while there is a simple gathering algorithm when they are "fully" consistent. A natural question then arises: what is the minimum magnitude of consistency between the local coordinate systems that is necessary and sufficient to solve the gathering problem for two oblivious robots? We answer this question in the paper for both semi-synchronous and asynchronous robots.

To discuss the magnitude of consistency, we consider that a robot is equipped with an *unreliable* compass, the bearings of which may deviate from the absolute ones (i.e., the bearings of a global coordinate system), and we assume that the compass determines the local coordinate system.

We consider two families of unreliable compasses with respect to the difference of timings that a compass can change the bearings. A *static compass* never changes its (possibly incorrect) bearings once an execution of algorithm starts. A *dynamic compass*, on the other hand, can change the bearings arbitrary times immediately before a new look-compute-move cycle starts, after the last cycle ends. We can consider

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[1] The robot, however, cannot determine its velocity, in particular, whether it is moving.

a more general family of compasses which can change the bearings even during the execution of a look-compute-move cycle. However, we do not investigate this case in this paper, since the impossibility of gathering in this case is trivial.

To measure the magnitude of deviation of a compass from the global coordinate system, we use the angle formed by the x-axis of the compass and the reference direction of the global coordinate system. In this paper, we investigate the maximum deviation that is necessary and sufficient for two oblivious robots to solve the gathering problem. We consider each of the four combinations of robot and compass models, and essentially show the following results:

Semi-synchronous robots with static compasses (SS). There is a gathering algorithm for two oblivious semi-synchronous robots that uses static compasses with maximum deviation ϕ if and only if $0 \leq \phi < \pi/2$.

Semi-synchronous robots with dynamic compasses (SD). There is a gathering algorithm for two oblivious semi-synchronous robots that uses dynamic compasses with maximum deviation ϕ if and only if $0 \leq \phi < \pi/4$.

Asynchronous robots with static compasses (AS). There is a gathering algorithm for two oblivious asynchronous robots that uses static compasses with maximum deviation ϕ if and only if $0 \leq \phi < \pi/2$.

Asynchronous robots with dynamic compasses (AD). There is a gathering algorithm for two oblivious asynchronous robots that uses dynamic compasses with maximum deviation ϕ if $0 \leq \phi < \pi/6$.

Note that whether $0 \leq \phi < \pi/6$ is necessary is left as an open problem for asynchronous robots with dynamic compasses.

The remainder of this paper is organized as follows. After a brief survey of related works in section 2, section 3 defines formal models of robots and compasses. We discuss the solvability of gathering by semi-synchronous and asynchronous robots with compasses in sections 4 and 5, respectively. Finally, section 6 concludes the paper.

2. Related works.

The set of geometric patterns formable/convergable[2] by a set of anonymous semi-synchronous robots was characterized by Suzuki and Yamashita for nonoblivious robots [15] and for oblivious robots [17]. From these two studies, it turns out that memory can help with the formation/convergence of geometric patterns only in very specific cases. Indeed, nonoblivious and oblivious semi-synchronous robots can solve formation/convergence for the same set of geometric patterns, except for the formation of a point with exactly two robots[3] (i.e., the gathering of two robots). As for asynchronous robots, little is known, except that the gathering problem for more than two robots is solvable [5, 6]. These positive results for the gathering problem rely on the ability of robots to detect multiplicity or, in other words, the ability to count the number of robots that share a given location. This assumption is indeed necessary. Otherwise, the gathering for more than two robots is reducible to the problem with two robots [13].

Essentially, the difficulty in forming (and even in converging to) a pattern by robots lies in the difficulty of breaking symmetry among the robots. In fact, any pattern is formable, given a symmetry-breaking tool, like a compass. The use of a compass was first introduced by Flocchini et al. [9]. They showed that asynchronous

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[2] *Formation* requires that all robots form the pattern within a finite number of steps, while *convergence* only requires the robots to approach the pattern asymptotically.

[3] Two oblivious robots can converge to a point with a naive algorithm that consists of always moving toward the midpoint of their positions.

robots with limited visibility can solve the gathering problem when every robot has access to a *correct* compass. Souissi, Défago, and Yamashita [14] extended the above result to the situation where compasses are *eventually consistent.* A compass is said to be eventually consistent if it is unstable and inaccurate for some arbitrary long period but eventually stabilizes to show the accurate direction.

In contrast, an *unreliable* compass may never correctly indicate the correct direction, although its maximum deviation is bounded. This type of compass was first introduced by Katayama et al. [12]. They showed that the gathering problem for two oblivious asynchronous robots is solvable if their compasses are either (1) static whose deviation is less than $\pi/6$ or (2) dynamic whose deviation is less than $\pi/8$.

Other work has focused on fault-tolerant formation/convergence for anonymous robots. Let F be the number of faulty robots. Cohen and Peleg [7] showed that convergence to a point is solvable for n asynchronous robots by simply converging to their center of gravity, even if some of the robots may crash (as long as there exists a nonfaulty robot). Bouzid, Potop-Butucaru, and Tixeuil [3, 4] proposed three Byzantine resilient convergence algorithms in one-dimensional space: (1) for synchronous robots provided $n > 2F$, (2) for semi-synchronous robots provided $n > 3F$, and (3) for asynchronous robots provided $n > 4F$. Agmon and Peleg [1] showed that (1) there is no Byzantine resilient gathering algorithm for semi-synchronous robots even if $F = 1$, and (2) there is a Byzantine resilient gathering algorithm for synchronous robots if and only if $n \geq 3F + 1$.

Finally, effects of sensor/control errors in convergence to a point were discussed by Cohen and Peleg [8] and Yamamoto et al. [16], assuming that the robots are aware of the global coordinate system. They measured sensor/control errors by a pair of the maximum angle and distance errors and obtained necessary and/or sufficient conditions for robots to have a convergence algorithm in terms of the pair.

3. System model and problem definition.

Robot with compass. In this paper, we investigate an autonomous mobile robot system $\mathcal{R}$ consisting of two oblivious robots r_0 and r_1 working in a two-dimensional Euclidean space $\mathbb{R}^2$, where $\mathbb{R}$ is the set of real numbers. The robots are *anonymous* and do not have identifiers; the subscript i of r_i is used only for the purpose of explanation. Let $\mathbf{r}_i(t)$ be the coordinates in the global x-y coordinate system Z of a robot r_i at time t. The *configuration* $C(t)$ of $\mathcal{R}$ at time t is defined by $(\mathbf{r}_0(t), \mathbf{r}_1(t))$.

The robots do not have access to Z, and each robot r_i at time t observes, computes, and moves in its local x-y coordinate system $Z_{(i,t)}$. The origin of $Z_{(i,t)}$ is always at the current position of r_i, and the direction of the x-axis corresponds to the bearings of its compass. Z, $Z_{(0,t)}$, and $Z_{(1,t)}$ are right-handed systems. Thus, for any point with coordinates $\mathbf{p}$ in Z, its coordinates $Z_{(i,t)}(\mathbf{p})$ in $Z_{(i,t)}$ are calculated by

$$Z_{(i,t)}(\mathbf{p})^T = sc_i(t) \begin{pmatrix} \cos\phi_i(t) & \sin\phi_i(t) \\ -\sin\phi_i(t) & \cos\phi_i(t) \end{pmatrix} (\mathbf{p} - \mathbf{r}_i(t))^T,$$

where the *scaling ratio* $sc_i(t)$ (with $0 < sc_i(t) < \infty$) is the ratio of the unit length in Z to that in $Z_{(i,t)}$, and the *deviation* $\phi_i(t)$ (with $-\pi < \phi_i(t) \leq \pi$) is the angle formed by the x-axes of Z and $Z_{(i,t)}$. The deviation abstracts the compass, and $Z_{(i,t)}(\mathbf{r}_i(t)) = \mathbf{0}$ always holds. Since the scaling ratio and the deviation (i.e., compass) may change as time goes, the local coordinate system $Z_{(i,t)}$ may change accordingly.

Look-compute-move cycle of robot. Each oblivious robot r_i repeatedly executes the look-compute-move cycle. The local coordinate system $Z_{(i,t)}$, and thus both the scaling ratio $sc_i(t)$ and the compass deviation $\phi_i(t)$ remain unchanged during a cycle (i.e., from look to move).

Suppose that a robot, say, r_0, starts executing the cycle at time t_0. In the look phase, r_0 observes the other robot r_1 and obtains the coordinates of r_1's position in the local coordinate system $Z_{(0,t_0)}$. We assume that this observation is an instantaneous action; r_0 obtains $Z_{(0,t_0)}(\mathbf{r}_1(t))$ as the result of observation, where t is a time instant in the look phase. If the gathering has already been achieved, then r_0 observes exactly one point at the origin.[4]

Next, robot r_0 computes, based on the coordinates $Z_{(0,t_0)}(\mathbf{r}_1(t))$, the coordinates in $Z_{(0,t_0)}$ of its next position. The algorithm is simply a total function ψ on $\mathbb{R}^2$. That is, when the compute phase finishes, r_0 obtains $\psi(Z_{(0,t_0)}(\mathbf{r}_1(t)))$ as the coordinates of its next position.

In the move phase, r_0 moves linearly toward coordinates $\psi(Z_{(0,t_0)}(\mathbf{r}_1(t)))$ in $Z_{(0,t_0)}$ at a (possibly variable) finite speed that r_0 cannot control. Since $Z_{(i,t)}$ does not change during the look-compute-move cycle, $Z_{(0,t_0)}$ is the current local coordinate system of r_0. The move phase may be too short for r_0 to reach the next position, and thus r_0 may finish the current execution of the look-compute-move cycle on the way to its next position. We assume, however, that the move phase is long enough to move over a small distance δ (in Z).[5]

We make three simplifying assumptions that incur no loss of generality. Let $\mathbb{N}$ denote the set of nonnegative integers.

1. Each execution of the look-compute-move cycle starts at the time at which the observation action is taken in the cycle.
2. The system is initialized at time 0, i.e., the first observation action is taken by a robot at 0.
3. The set of time instants at which the robots start executions of the look-compute-move cycle (or equivalently, the time instants at which they take observation actions) is $\mathbb{N}$. A robot is said to be *activated* at a time $t \in \mathbb{N}$ if it starts executing the cycle at t.

Execution. Given an algorithm and an initial configuration $C(0)$, let us observe the behavior of robot system $\mathcal{R}$. Let $C(t)$ be the configuration of $\mathcal{R}$ at $t \in \mathbb{N}$. An infinite sequence $\mathcal{E} : C(0), C(1), \ldots$ is called an *execution* of $\mathcal{R}$. Recall that $C(t)$ is the configuration at time t, in which at least one robot is activated. An execution must be *fair* in the sense that both robots are activated infinitely many times in any infinite execution.

Asynchronous and semi-synchronous robots. Robots are said to be *asynchronous* if we do not make any assumption on the execution of the look-compute-move cycle. Thus, a robot may be moving (move phase) while the other robot starts the look phase.

Robots are said to be *semi-synchronous* if every execution of the cycle is instantaneous. An execution of the cycle started at time $t \in \mathbb{N}$ is said to be *instantaneous* if the look and the compute phases immediately finish at t and the move phase finishes before $t + 1$.

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[4] The converse (one point at origin implies gathering) may not hold for asynchronous robots. See the definition of the gathering problem.

[5] Obviously, no gathering algorithm exists if a robot can finish the move phase at any position between the current and the next position.

Static and dynamic compasses. A compass is a ϕ-*compass* if $\phi \geq |\phi_i(t)|$ for every $i \in \{0,1\}$ and all $t \in \mathbb{N}$, i.e, a compass such that the absolute value of the deviation angle is bounded by ϕ. A compass is said to be *static* if $\phi_i(t)$ is constant in t. A compass is said to be *dynamic* if it can change its bearings at any time $t \in \mathbb{N}$ prior to the look phase. A ϕ-static compass is a ϕ-compass that is static and a ϕ-dynamic compass is a ϕ-compass that is dynamic.

Gathering problem. Let $\mathcal{L} = \{(\mathbf{p}, \mathbf{p}) : \mathbf{p} \in \mathbb{R}^2\}$ be the set of all configurations in which two robots are co-located. An execution $\mathcal{E} : C(0), C(1), \ldots$ is called a *gathering execution* if there are a configuration $C \in \mathcal{L}$ and a time instant $f \in \mathbb{N}$ such that $C(t) = C$ holds for all $t \geq f$. An algorithm is said to be a *gathering algorithm* if for any configuration $C(0)$, every execution $\mathcal{E} : C(0), C(1), \ldots$ with initial configuration $C(0)$ is a gathering execution.

An algorithm is given as a function and is deterministic. Nevertheless, execution $\mathcal{E}$ is not uniquely determined from a given initial configuration $C(0)$. Execution $\mathcal{E}$ varies depending on many factors, e.g., when each robot is activated, how and when each scale ratio and compass change, how far each robot moves, and so on. We consider that all these factors are controlled by an adversary playing against the gathering algorithm. This paper investigate the problem of designing a gathering algorithm.

Noetherian termination. The definition of a gathering algorithm is based on the Noetherian termination. It does not request a gathering algorithm to eventually terminate. All gathering algorithms that we present in this paper are of this type. A stronger (and perhaps more conventional) definition additionally imposes the termination condition upon a gathering algorithm. We will observe that the gathering algorithms for semi-synchronous robots in section 4 are transformable into gathering algorithms satisfying the termination condition.[6]

4. Semi-synchronous robots with compasses.

In this section, we investigate the gathering problem for two oblivious semi-synchronous anonymous mobile robots with static and then dynamic compasses. By definition, if the problem is solvable for the robots with ϕ-dynamic compasses, it is also solvable for the robots with ϕ-static compasses. We establish, for each of the static and the dynamic cases, the tight bound on ϕ for the problem to become solvable.

Consider, for an algorithm, a finite execution $\mathcal{E} : C(0), C(1), \ldots, C(f)$ with an initial configuration $C(0)$ and an execution $\mathcal{E}' : C'(0), C'(1), \ldots$ with an initial configuration $C'(0)$. If $C'(0) = C(f)$, then the concatenation $\mathcal{E}\mathcal{E}'$ of $\mathcal{E}$ and $\mathcal{E}'$, i.e., $C(0), C(1), \ldots,$ $C(f)(= C'(0)), C'(1), \ldots,$ is an execution with initial configuration $C(0)$, since the robots are semi-synchronous.[7] In this section, we implicitly rely on this property.

4.1. Semi-synchronous robots with static compasses.

We now investigate the static case, i.e., the gathering problem for two oblivious robots with static compasses under the semi-synchronous model. Since the compasses are static, let $\phi_i(t) = \phi_i$ for $i \in \{0,1\}$. The following theorem is a restatement of Theorem 3.1 of [15].

THEOREM 1 (see [15]). *There is no gathering algorithm for two oblivious anonymous robots with $\pi/2$-static compasses under the semi-synchronous model.*

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[6] This transformation is not applicable to the gathering algorithms for asynchronous robots in section 5.

[7] Asynchronous robots do not have this property, since a robot in $\mathcal{E}$ may still be engaged in its move phase at time f.

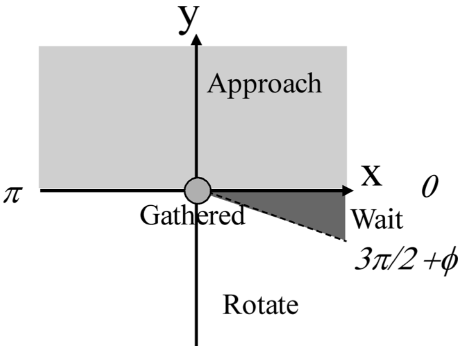

FIG. 1. *An illustration of Algorithm* A^{ϕ}_{SS}.

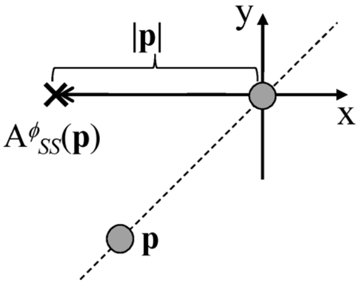

FIG. 2. *The move of a robot state Rotate who looks at the other robot at* $\mathbf{p}$ *in its local coordinate system.*

We present an algorithm A^{ϕ}_{SS}[8] and show that it is a correct gathering algorithm for two oblivious semi-synchronous robots with ϕ-static compasses, provided that $0 \le \phi < \pi/2$. Recall that an algorithm ψ is a total function on $\mathbb{R}^2$. For any $\mathbf{p} = (u, v) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, let $\arg(\mathbf{p}) = \omega$ be the argument (or phase) of $\mathbf{p}$, i.e., $0 \le \omega < 2\pi$ and $(u, v) = |\mathbf{p}|(\cos\omega, \sin\omega)$. Angles are calculated modulo 2π in the sequel.

ALGORITHM $A^{\phi}_{SS}(\mathbf{p})$.

G(athered): If $\mathbf{p} = \mathbf{0}$ then $A^{\phi}_{SS}(\mathbf{p}) = \mathbf{0}$.

A(pproach): If $0 < \arg(\mathbf{p}) \le \pi$ then $A^{\phi}_{SS}(\mathbf{p}) = \mathbf{p}$.

R(otate): If $\pi < \arg(\mathbf{p}) \le 3\pi/2 + \phi$ then $A^{\phi}_{SS}(\mathbf{p}) = (-|\mathbf{p}|, 0)$.

W(ait): If $3\pi/2 + \phi < \arg(\mathbf{p}) \le 2\pi$ then $A^{\phi}_{SS}(\mathbf{p}) = \mathbf{0}$.

Figure 1 illustrates Algorithm $A^{\phi}_{SS}(\mathbf{p})$. It divides the plane into four regions, Gathered, Approach, Rotate, and Wait, and asks a robot to take the move corresponding to the region to which the current position $\mathbf{p}$ of the other robot belongs. Each region is specified by the angle of its boundary (except Gathered, whose region is a singleton $\{\mathbf{0}\}$). For example, in the case of Approach, the corresponding region is specified by two boundary angles 0 and π. A robot moves toward the other robot in state A (Approach), it moves westward (i.e., negative direction) on its local x-axis in state R (Rotate), and stops in state W (Wait). The robots in state G (Gathered) are co-located, and no further actions are necessary. Although the actions at A, W, and G are intuitive, the action at R is less straightforward. Figure 2 illustrates the move of a robot in state Rotate who looks at the other robot at $\mathbf{p}$ in its local coordinate

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[8] SS in A^{ϕ}_{SS} stands for semi-synchronous robots with static compasses.

system. We shall observe in more details how it rotates, which is the core of A^{ϕ}_{SS}. In general, the state of a robot depends on both its current local coordinate system and the current position of the other robot.

Let $C(0)$ be a configuration and $\mathcal{E} : C(0), C(1), \ldots$ be an execution (of A^{ϕ}_{SS} on $\mathcal{R}$) with initial configuration $C(0)$, where $C(t) = (\mathbf{r}_0(t), \mathbf{r}_1(t))$ is the configuration at time instant t, i.e., $\mathbf{r}_i(t)$ ($i \in \{0,1\}$) is the location of r_i in Z at time t. The *state pair* $S(C(t))$ of configuration $C(t)$ at time t is a pair (s_0, s_1), where s_i ($i \in \{0,1\}$) is the state of robot r_i at time t. As mentioned, s_i may depend on both $C(t)$ and $Z_{(i,t)}$.

First, we confirm that state G corresponds to a "goal" configuration. Suppose that the state of a robot, say, r_0, is G at time t. Since $Z_{(0,t)}(\mathbf{r}_1(t)) = \mathbf{0}$, $\mathbf{r}_0(t) = \mathbf{r}_1(t)$ and the state of r_1 at t is also G. Since they do not move in the time interval $[t, t+1]$ regardless of whether they are activated at t, we obtain that $C(t') = C(t) \in \mathcal{L}$ for all $t' \geq t$. We can thus conclude that A^{ϕ}_{SS} is correct if there is a time instant $t \in \mathbb{N}$ such that $S(C(t)) = (G, G)$.

Since the state of a robot is G if and only if the state of the other robot is G, then $S(C(t)) \notin \{(G, s), (s, G) | s \in \{A, R, W\}\}$. By the definition of A^{ϕ}_{SS}, $S(C(t)) = (W, W)$ obviously never occurs; the execution never reaches a deadlock configuration $C(t)$ such that $S(C) = (W, W)$, in which neither robot moves.

We next examine the case in which $\mathcal{E}$ reaches a configuration $C(t)$ such that $S(C(t)) \in \{(W, A), (A, W)\}$. The robot in state A, say, r_0, moves toward r_1, while r_1, in state W, stays motionless in time interval $[t, t+1]$. By definition, if the distance between r_0 and r_1 is δ or less, then r_0 has reached the position of r_1 by $t+1$ and $S(C(t+1)) = (G, G)$ holds. If r_0 has not reached the position of r_1 at $t+1$, $S(C(t+1)) = S(C(t)) = (A, W)$ and the distance between r_0 and r_1 is now shorter, since the position of r_0 at $t+1$ lies on the line segment $\overline{\mathbf{r}_0(t)\mathbf{r}_1(t)}$. We thus conclude that A^{ϕ}_{SS} is correct if there is a time instant $t \in \mathbb{N}$ such that $S(C(t)) \in \{(W, A), (A, W)\}$.

We have already shown that A^{ϕ}_{SS} is correct if $S(C(0)) \in \{(G, G), (A, W), (W, A)\}$. If $S(C(0))$ is in none of (G, G), (A, W), (W, A), then the robots "rotate" the line segment $\overline{\mathbf{r}_0(t)\mathbf{r}_1(t)}$ counterclockwise until a state pair of either (A, W) or (W, A) occurs. This task is done by robots in the R (i.e., Rotate) state. The rest of this subsection is devoted to showing this. We now summarize some of the basic properties observed above.

PROPERTY 1.

1. *State pair (G, G) corresponds to a goal configuration.*
2. *For any configuration $C(t)$, $S(C(t)) \in \{(G, G), (A, A), (R, R), (A, R), (A, W), (R, A), (R, W), (W, A), (W, R)\}$.*
3. *If a configuration $C(t)$ such that $S(C) \in \{(A, W), (W, A)\}$ is reached, then a goal configuration (G, G) will be reached eventually.*

LEMMA 1. *Suppose that $\phi_0 = \phi_1$. Then A^{ϕ}_{SS} correctly solves the gathering problem for two oblivious robots under the semi-synchronous model.*

Proof. Recall that $C(t) = (\mathbf{r}_0(t), \mathbf{r}_1(t))$ is the configuration at time t. The coordinates of the position of robot r_i ($i \in \{0,1\}$) in Z at t are denoted by $\mathbf{r}_i(t) = (x_i(t), y_i(t))$. By Property 1, it suffices to show that $\mathcal{E}$ eventually reaches a configuration $C(f)$ such that $S(C(f)) \in \{(G, G), (A, W), (W, A)\}$ for some $f \in \mathbb{N}$.

The state pair $S(C(0))$ of initial configuration $C(0)$ must contain A as the state of a robot because $\phi_0 = \phi_1$, and there is nothing to show if $S(C(0)) \in \{(G, G), (A, W), (W, A)\}$. Hence we need to prove the lemma only for configurations $C(0)$ such that $S(C(0)) \in \{(A, R), (R, A)\}$. Without loss of generality, we assume the following:

1. $\phi_0 = \phi_1 = 0$,
2. $S(C(0)) = (R, A)$,

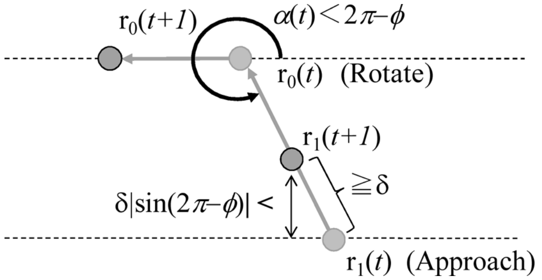

Fig. 3. *An illustration used in the proof of Lemma* 1.

3. $\mathbf{r}_0(0) = (x_0(0), y_0(0)) = \mathbf{0}$, i.e., the position of r_0 is at the origin in Z, and
4. $y_1(0) < 0$.[9]

We assume that $\mathcal{E}$ never reaches a configuration $C(f)$ such that $S(C(f)) \in \{(G,G),(A,W),(W,A)\}$ and derive a contradiction. Because $x_0(1) \le 0, y_0(1) = 0$ and $y_1(1) \le 0$, $S(C(1)) \in \{(G,G),(W,A),(R,A)\}$, which implies that $S(C(1)) = (R,A)$, since $S(C(1)) \notin \{(G,G),(W,A)\}$. Hence $S(C(t)) = (R,A)$ holds for all $t \in \mathbb{N}$.

Note that $x_0(t) \le 0, y_0(t) = 0$, and $y_1(t) \le 0$ hold. Let $\alpha(t) = \arg(Z_{(0,t)}(\mathbf{r}_1(t)))$. By definition, $\pi < \alpha(t) \le 2\pi - \phi$. Let $T = \{t_1, t_2, \dots\}$ be the time instants at which r_0 is activated. T is an infinite set, since the execution is fair. Obviously, by definition $\alpha(t_i) < \alpha(t_i + 1)$ for any $i \in \mathbb{N}$, which implies that $\alpha(t)$ converges to an angle $\alpha \le 2\pi - \phi$. By the definition of state R, $3\pi/2 \le \alpha$.

We now derive a contradiction. Let $U = \{u_1, u_2, \dots\}$ be the time instants at which r_1 is activated. U is also an infinite set, since the execution is fair. Robot r_1 is in state A at any time instant $u_i \in U$. If r_1 reaches the next position at $u_i + 1$, $y_1(u_i + 1) = 0$ and the state pair of $C(u_i + 1)$ is either (G,G) or (W,A), this is a contradiction. Thus, it moves by at least δ in time interval $[u_i, u_i + 1]$. Since $3\pi/2 \le \alpha(u_i) < 2\pi - \phi$, $y_1(u_i + 1) - y_1(u_i) > \delta|\sin(2\pi - \phi)|$ (see Figure 3 for an illustration). It is a contradiction, since $y_1(t) \le 0$ for any $t \in \mathbb{N}$. ☐

Lemma 2. *Suppose that $\phi_0 \ne \phi_1$. Then, A^{ϕ}_{SS} correctly solves the gathering problem for two oblivious robots under the semi-synchronous model.*

Proof. It suffices to show, by Property 1, that $\mathcal{E}$ eventually reaches a configuration $C(f)$ such that $S(C(f)) \in \{(G,G),(A,W),(W,A)\}$ for some $f \in \mathbb{N}$. Since $\phi_0 \ne \phi_1$, we assume $\phi_0 < \phi_1$ without loss of generality. Since $\phi < \pi/2$, the angle formed by their x-axes is less than π. Consider $C(t)$ for any $t \in \mathbb{N}$. Let $\mathbf{o}(t)$ be the intersection of the x-axes of $Z_{(0,t)}$ and $Z_{(1,t)}$. We may assume that a robot is not on the x-axis of the other at t, since $S(C(t)) \notin \{(G,G),(A,W),(W,A)\}$. Let $Z_{(i,t)}(\mathbf{p}) = (x_{(i,t)}(\mathbf{p}), y_{(i,t)}(\mathbf{p}))$ for $i \in \{0,1\}$ and $\mathbf{p} \in \mathbb{R}^2$. By definition, for $i \in \{0,1\}$, $x_{(i,t)}(\mathbf{o}(t))$ and $x_{(i,t)}(\mathbf{r}_i(t))$ are the x-coordinates, in $Z_{(i,t)}$ at time t, of the intersection $\mathbf{o}(t)$ and the position of r_i, respectively. According to their relative positions on the x-axis of $Z_{(i,t)}$, we partition the configurations that may occur in $\mathcal{E}$ into four classes:

P(ositive)P(ositive): $x_{(0,t)}(\mathbf{o}(t)) < x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) < x_{(1,t)}(\mathbf{r}_1(t))$,

P(ositive)N(egative): $x_{(0,t)}(\mathbf{o}(t)) < x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) > x_{(1,t)}(\mathbf{r}_1(t))$,

N(egative)P(ositive): $x_{(0,t)}(\mathbf{o}(t)) > x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) < x_{(1,t)}(\mathbf{r}_1(t))$, and

N(egative)N(egative): $x_{(0,t)}(\mathbf{o}(t)) > x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) > x_{(1,t)}(\mathbf{r}_1(t))$.

Figure 4 illustrates these four cases. In the following, we show that $\mathcal{E}$ eventually

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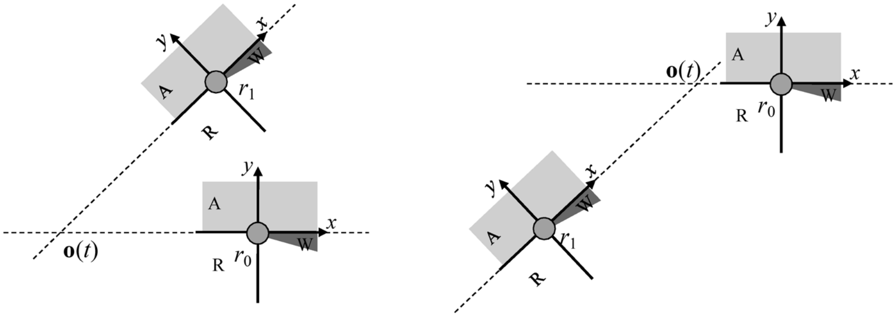
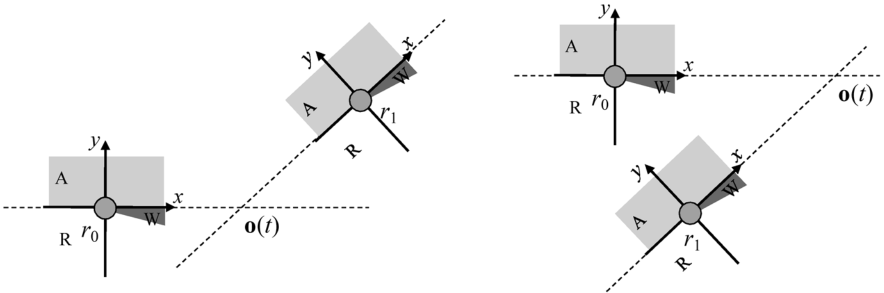
[9] Observe that the state pair of $C(0)$ is either (G,G), (A,W), or (W,A) if $y_1(0) = 0$.



(a) Class PP (b) Class PN



(c) Class NP (d) Class NN

FIG. 4. *Illustrations of the four cases PP, PN, NP, and NN used in the proof of Lemma* 2.

reaches a configuration $C(f)$ such that $S(C(f)) \in \{(G,G),(A,W),(W,A)\}$, for each of the four cases to which $C(0)$ may belong: PP, PN, NP, and NN.

Case NN. Suppose that $C(0)$ is in class NN, which implies that $S(C(0)) = (R,A)$. In this case, by using an argument similar to the one used in the proof of Lemma 1, we can show that $\mathcal{E}$ eventually reaches a configuration $C(f)$ such that $S(C(f)) \in \{(G,G),(W,A)\}$.

Case PN. Suppose that $C(0)$ is in class PN, which implies that $S(C(0)) \in \{(R,R),(R,W)\}$. Since r_0 goes west and r_1 goes west or stays motionless and thus r_0 decreases its x-coordinate and r_1 does not increase its x-coordinate (without changing their y-coordinates) in their local coordinate systems, $\mathcal{E}$ eventually reaches a configuration $C(f)$ in class NN. Thus, this case is reduced to Case NN.

Case NP. Suppose that $C(0)$ is in class NP, which implies that $S(C(0)) = (A,A)$. Since the robots move toward each other's position, $\mathcal{E}$ eventually reaches a configuration $C(f)$ in class PN, unless it reaches a configuration $C(f)$ such that $S(C(f)) = (G,G)$ directly. Thus, this case is reduced to Case PN.

Case PP. Suppose finally that $C(0)$ is in class PP, which implies that $S(C(0)) = (A,R)$. Since r_0 never reaches the x-axis of $Z_{(1,t)}$, and r_1 decreases its x-coordinate in $Z_{(1,t)}$, it follows that $\mathcal{E}$ eventually reaches a configuration $C(f)$ in class PN. Thus, this case is also reduced to Case PN. $\square$

By Lemmas 1 and 2, we derive the following theorem.

THEOREM 2. *For any* $0 \le \phi < \pi/2$, *Algorithm* A^{ϕ}_{SS} *for two oblivious robots that uses* ϕ*-static compasses solves the gathering problem under the semi-synchronous model.*

4.2. Semi-synchronous robots with dynamic compasses. We investigate the gathering problem for two semi-synchronous robots with ϕ-dynamic compasses for some ϕ. Unlike the previous subsection, $\phi_i(t)$ can now vary in time, as long as $|\phi_i(t)| \le \phi$ always holds.

The proof of Theorem 3.1 of [15] shows that an algorithm solves the gathering problem under the semi-synchronous model only if there is a configuration such that one robot, say, r_0, moves to the position of r_1 while r_1 stays motionless. We can restate this condition, using the notation introduced in subsection 4.1, as follows. A gathering algorithm is correct only if there is a configuration C such that the corresponding state pair $S(C)$ is either (A, W) or (W, A). We say that a configuration C is *stable* if $S(C)$ is determined uniquely, regardless of the current local coordinate systems $Z_{(i,t)}$. Following the proof of Theorem 3.1 in [15] for semi-synchronous robots, and additionally taking into account that they have dynamic compasses, we have the following property.

PROPERTY 2. *An algorithm solves the gathering problem for two oblivious semi-synchronous robots with dynamic compasses only if there is a stable configuration C such that $S(C)$ is either (A, W) or (W, A).*

THEOREM 3. *There is a gathering algorithm for two oblivious semi-synchronous robots with ϕ-dynamic compasses, only if $\phi < \pi/4$.*

Proof. It suffices to show that there is no gathering algorithm for $\phi = \pi/4$. We assume that such an algorithm exists, called *ALG*, in order to derive a contradiction. Then, by Property 2, there is a stable configuration C such that the corresponding state pair $S(C)$ is either (A, W) or (W, A). We assume, without loss of generality, that $C = ((0, 0), (1, 0))$ and $S(C) = (W, A)$. Since C is stable and $\phi_0(t)$ can be $\pm\pi/4$, we have that $S(C') = S(C'') = (W, A)$, where $C' = ((0, 0), (\sqrt{2}/2, \sqrt{2}/2))$, and $C'' = ((0, 0), (\sqrt{2}/2, -\sqrt{2}/2))$.

Consider an execution starting with initial configuration $C(0) = ((0, 0), (0, 1))$, and assume that $\phi_0(0) = \pi/4$ and $\phi_1(0) = -\pi/4$. Then $S(C(0)) = (W, W)$, a contradiction. $\square$

We next present Algorithm A^{ϕ}_{SD} and show that it is a correct gathering algorithm for two oblivious semi-synchronous robots with ϕ-dynamic compasses, provided that $0 \le \phi < \pi/4$.[10] For any $\mathbf{p} \in \mathbb{R}^2$ and angle ω, let $\rho_\omega(\mathbf{p}) = \mathbf{q}$, where

$$\mathbf{q}^T = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \mathbf{p}^T,$$

that is, $\rho_\omega(\mathbf{p})$ is the point obtained by rotating $\mathbf{p}$ by angle ω with respect to the rotation center $\mathbf{0}$.

ALGORITHM $A^{\phi}_{SD}(\mathbf{p})$.

G(athered): If $\mathbf{p} = \mathbf{0}$ then $A^{\phi}_{SD}(\mathbf{p}) = \mathbf{0}$.

A(pproach): If $\pi/2 + \phi < \arg(\mathbf{p}) \le 3\pi/2 - \phi$ then $A^{\phi}_{SD}(\mathbf{p}) = \mathbf{p}$.

W(ait): If $-\pi/2 + \phi < \arg(\mathbf{p}) \le \pi/2 - \phi$ then $A^{\phi}_{SD}(\mathbf{p}) = \mathbf{0}$.

R(otate): If $\pi/2 - \phi < \arg(\mathbf{p}) \le \pi/2 + \phi$ or $3\pi/2 - \phi < \arg(\mathbf{p}) \le 3\pi/2 + \phi$ $(= -\pi/2 + \phi)$, then $A^{\phi}_{SD}(\mathbf{p}) = \rho_{\frac{\pi}{2}+\phi}(\mathbf{p})$.

An illustration of this algorithm is shown in Figure 5. Like A^{ϕ}_{SS}, a robot moves toward the other robot in state A (i.e., Approach) and stays there motionless in state W (i.e., Wait). Although the action taken in state R (i.e., Rotate) seems slightly more complex than that for A^{ϕ}_{SS}, the idea behind the definition is similar to A^{ϕ}_{SS}. This additional complexity, illustrated in Figure 6, comes from the need to handle the

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[10] SD in A^{ϕ}_{SD} stands for semi-synchronous robots with dynamic compasses.

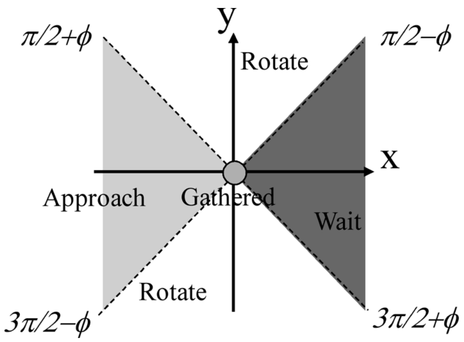


FIG. 5. *An illustration of Algorithm* A^{ϕ}_{SD}.

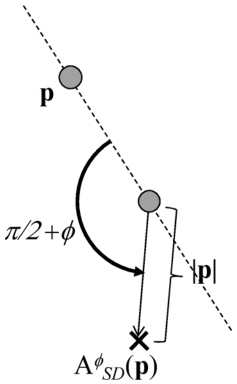


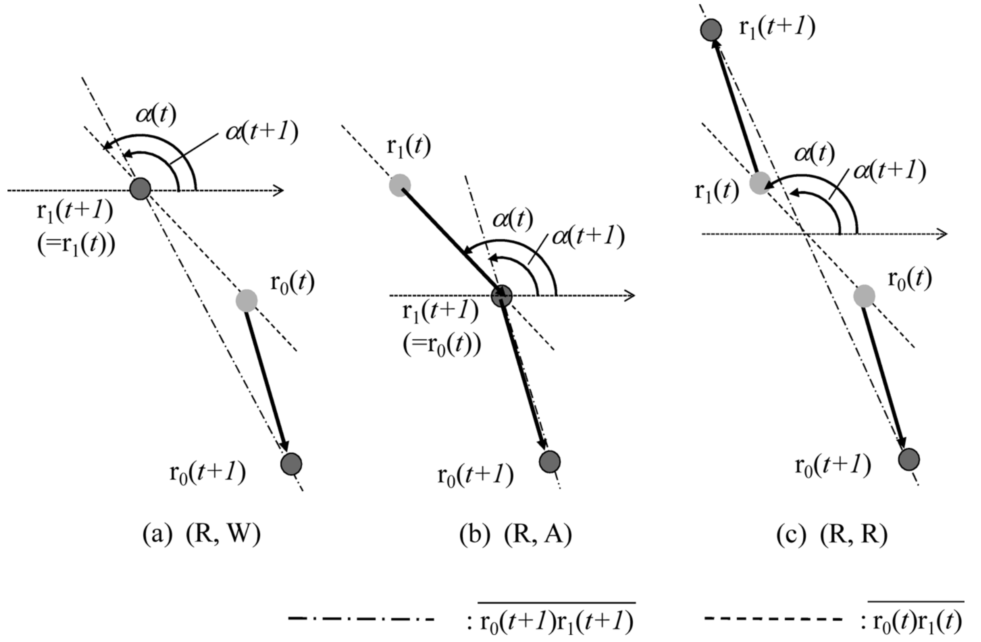
FIG. 6. *The move of a robot in state Rotate who looks at the other robot at* $\mathbf{p}$ *in its local coordinate system.*

dynamic compasses. Roughly, a robot at R rotates the line segment connecting the current robots' positions clockwise, until its deviation from the x-axis of Z becomes smaller than $\pi/2 - \phi (> 0)$. Since such a configuration C is stable and $S(C)$ is either (A, W) or (W, A), the gathering is eventually achieved.

Just like with A^{ϕ}_{SS}, the state pair (G, G) corresponds to a goal (i.e., gathered) configuration. Unlike with A^{ϕ}_{SS}, the state pair (A, A) never occurs, in addition to the state pairs in $\{(G, s), (s, G), (W, W) | s \in \{A, R, W\}\}$. Note that item 3 of Property 1 does not hold for A^{ϕ}_{SD}, i.e., not all configurations C, with $S(C) \in \{(A, W), (W, A)\}$, are stable.

We explain the intention of the definition $A^{\phi}_{SD}(\mathbf{p}) = \rho_{\pi/2+\phi}(\mathbf{p})$ of R. Suppose that an execution reaches at time t a configuration $C = (\mathbf{r}_0, \mathbf{r}_1)$, in which a robot, say, r_0, is in state R. For simplicity of explanation, assume that $\mathbf{r}_0 = \mathbf{0}$ and that $y_1 > 0$, where $\mathbf{r}_1 = (x_1, y_1)$. Since r_0 is at R, $\pi/2 - \phi < \arg(Z_{(0,t)}(\mathbf{r}_1)) \leq \pi/2 + \phi$ in $Z_{(0,t)}$. As $|\phi_0(t)| \leq \phi$, $\pi/2 - 2\phi < \arg(\mathbf{r}_1) \leq \pi/2 + 2\phi$ in Z. The direction θ of the next position hence satisfies $\pi < \theta < \arg(\mathbf{r}_1) + \pi (< 2\pi)$. Since $\theta < \arg(\mathbf{r}_1) + \pi$, a robot at R, once activated, rotates the line segment $\overline{\mathbf{r}_0\mathbf{r}_1}$ clockwise. Since $\pi < \theta$, the rotation of $\overline{\mathbf{r}_0\mathbf{r}_1}$ never exceeds the x-axis of Z.

THEOREM 4. *For any* $0 \leq \phi < \pi/4$*, Algorithm* A^{ϕ}_{SD} *for two oblivious robots that use* ϕ*-dynamic compasses solves the gathering problem under the semi-synchronous model.*



(a) (R, W) (b) (R, A) (c) (R, R)

FIG. 7. *Illustrations to explain why $\alpha(t)$ monotonically decreases, which are used in the proof of Theorem* 4.

Proof. It suffices to show that any execution $\mathcal{E} = C(0), C(1), \ldots$ eventually reaches a configuration $C(f)$ such that $S(C(f)) = (G, G)$. We assume that there is a configuration $C(0)$ such that there is an execution $\mathcal{E} = C(0), C(1), \ldots$ in which $S(C(t)) \neq (G, G)$ holds for any t. We then derive a contradiction. Let $C(t) = (\mathbf{r}_0(t), \mathbf{r}_1(t))$, where $\mathbf{r}_i(t) = (x_i(t), y_i(t))$ for $i \in \{0, 1\}$ and $\alpha(t) = \arg(\mathbf{r}_1(t) - \mathbf{r}_0(t))$. If $y_1(0) = 0$ holds, it is a contradiction since $C(0)$ is a stable configuration such that $S(C(0))$ is either (W, A) or (A, W). Without loss of generality, we thus assume that $\mathbf{r}_0(0) = \mathbf{0}$ and $y_1(0) > 0$. Hence $0 < \alpha(0) < \pi$.

By the respective definitions of states A and R, and by the above observation about R, we obtain that $0 < \alpha(t+1) \leq \alpha(t)$ for any $t \in \mathbb{N}$. If a robot at R is activated only a finite number of times, then there is an infinite subexecution $C(f), C(f+1), \ldots$ such that $S(C(t))$ is either (A, W) or (W, A) for any $t \geq f$ for some $f \in \mathbb{N}$—a contradiction. A robot at R is thus activated infinitely many times. Since a robot at R rotates segment $\overline{\mathbf{r}_0(t)\mathbf{r}_1(t)}$ clockwise whenever it is activated (Figure 7), and $0 < \alpha(t)$ for all $t \in \mathbb{N}$, then $\alpha(t)$ converges to an angle $\alpha > 0$.

We again derive a contradiction. Since $\alpha(t)$ converges to $\alpha > 0$ for any small $\epsilon > 0$, there is a time $f \in \mathbb{N}$ such that $\alpha(t) - \alpha < \epsilon$ for all $t \geq f$. Because an activation of r_1 does not increase $\alpha(t)$, we may assume without loss of generality that only r_0 is activated after f. Let $\tau = \pi/2 - \phi(= \pi - (\pi/2 + \phi))$.

For convenience, imagine that $\mathbf{r}_1(f) = \mathbf{0}$ and $\mathbf{r}_0(f) = (-1, 0)$. Since r_1 is not activated after f, r_1 stays at $\mathbf{0}$. Let ℓ (resp., ℓ') be a half line ended at $\mathbf{0}$ (resp., $(-1, 0)$) with $\pi - \epsilon$ (resp., $\pi - \tau$) being the angle it makes with the x-axis, and let $\mathbf{p}$ be the intersection of ℓ and ℓ' (see Figure 8 for an illustration).

Since $\tau \gg \epsilon$, $\mathbf{p}$ is in the second quadrant. Let $X = \mathbf{r}_0(f)(= (-1, 0)), \mathbf{r}_0(f+1), \ldots$ be the polygonal chain constructed from the positions of r_0 after f. It is easy to observe that X is entirely contained within the triangle formed by vertices $\mathbf{0}, \mathbf{p}$ and

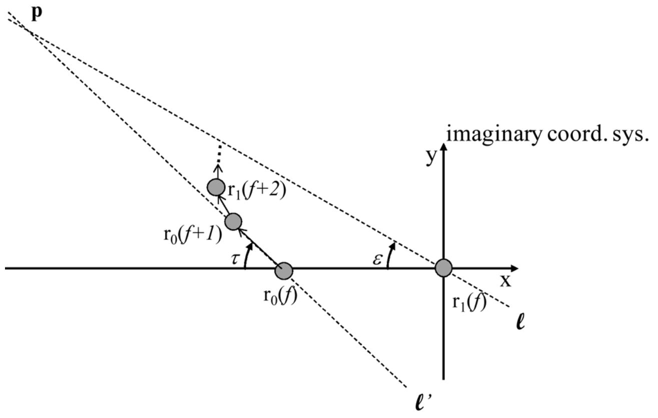


FIG. 8. *An illustration of a contradictive situation in the proof of Theorem* 4.

$(-1,0)$. This is a contradiction with the fact that $|\mathbf{r}_0(t+1)-\mathbf{r}_0(t)| \geq \delta$ and $\arg(\mathbf{r}_0(t+1)-\mathbf{r}_0(t)) \geq \pi-(\tau+\epsilon)$, for any $t \geq f$. ☐

Remark 1. Algorithms A^{ϕ}_{SS} and A^{ϕ}_{SD} run forever. Let us modify them by replacing Gathered into the following Terminate:

T(erminate): If $\mathbf{p}=\mathbf{0}$ then terminate.

The modified A^{ϕ}_{SS} and A^{ϕ}_{SD} then eventually terminate at a gathered configuration because the original A^{ϕ}_{SS} and A^{ϕ}_{SD} have the following property: for any execution $\mathcal{E}: C(0), C(1), \ldots,$ there is a time instant $f \in \mathbb{N}$ such that $S(C(f))=(G,G)$, and for any $f \in \mathbb{N}$ such that $S(C(f))=(G,G)$, $S(C(t))=(G,G)$ holds for any $t \geq f$.

5. Asynchronous robots with compasses.

We now address the case of asynchronous mobile robots. As emphasized earlier, a main difference between asynchronous and semi-synchronous robots is that in the former case, a concatenation $\mathcal{E}\mathcal{E}'$ of a finite execution $\mathcal{E}$ and an execution $\mathcal{E}'$ may not be a correct execution, even if the last configuration $C(f)$ of $\mathcal{E}$ is the initial configuration of $\mathcal{E}'$. This is because, due to the asynchrony of the three phases, one of the two robots may be caught in the middle of its move phase in $C(f)$ in $\mathcal{E}$. We say that a robot is *settled* at time t if (1) it is not activated at t or (2) it is activated at t and it will not change its position until it is next activated. Obviously $\mathcal{E}\mathcal{E}'$ is a correct execution if both robots are settled at time f.

5.1. Asynchronous robots with static compasses.

In subsection 4.1, we presented Algorithm A^{ϕ}_{SS} and showed per Theorem 2 that it correctly solves the gathering problem for two oblivious robots using ϕ-static compasses under the semi-synchronous model if $0 \leq \phi < \pi/2$. We now show that Theorem 2 can be extended to asynchronous robots; i.e., we show that A^{ϕ}_{SS} correctly solves the gathering problem for two oblivious robots using ϕ-static compasses under the asynchronous model if $0 \leq \phi < \pi/2$. We then conclude by Theorem 1 that there is a gathering algorithm for two oblivious asynchronous robots using static compasses with maximum deviation ϕ if and only if $0 \leq \phi < \pi/2$.

We keep the same notation defined in subsection 4.1 and follow the scenario that we adopted in the proof of Theorem 2. We show the correctness of A^{ϕ}_{SS} under the asynchronous model, taking into account the features that characterize an asynchronous execution. Like semi-synchronous robots, $S(C) \notin \{(G,s),(s,G),(W,W) | s \in \{A,R,W\}\}$ for any configuration C. Hence the execution never reaches a deadlock configuration C, in which neither robot can move, before reaching (G,G). However, unlike semi-synchronous robots, state pair (G,G) no longer characterizes a goal configuration, since an asynchronous robot may still be unsettled. In other words, it may be moving away without noticing that the gathering had just been completed.[11]

In order to handle such configurations $C(t)$ at which a robot is not settled, we also pay attention to times $a_i(t)$ and $b_i(t)$ with $i \in \{0,1\}$, where $a_i(t)$ (resp., $b_i(t)$) is the last time before (and including) t (resp., the first time after (and including) t) at which robot r_i is activated. If r_i is activated at t, then $a_i(t) = b_i(t) = t$. If r_i is not activated at t, then r_i is not activated in time interval $(a_i(t), b_i(t))$. Like subsection 4.1, let us start with the simple case of $\phi_0 = \phi_1$.

LEMMA 3. *Suppose that* $\phi_0 = \phi_1$. *Then* A^{ϕ}_{SS} *correctly solves the gathering problem for two oblivious robots under the asynchronous model.*

Proof. Our proof is very similar to that of Lemma 1. Let $C(0)$ be any initial configuration. Consider any execution $\mathcal{E} : C(0), C(1), \ldots$, where $\mathbf{r}_i(t) = (x_i(t), y_i(t))$ for any $i \in \{0,1\}$ and $t \in \mathbb{N}$. Except that both states of robots are G, $S(C(0))$ must contain A as the state of a robot. Thus, $S(C(0)) \in \{(G,G),(A,R),(A,W),(R,A),(W,A)\}$ holds.

If $S(C(0)) = (G,G)$, then by the definition of A^{ϕ}_{SS}, $S(C(t)) = (G,G)$ for any $t \in \mathbb{N}$, i.e., the gathering completes.

If $S(C(0)) \in \{(A,W),(W,A)\}$, then only the robot with state A, say, r_0, can move (toward r_1) at $C(0)$, and thus $S(C(1)) = (A,W)$. Hence, $\mathcal{E}$ eventually reaches a configuration $C(t)$ (possibly after taking a number of configurations C such that $S(C) = (A,W)$) such that $S(C(t)) = (G,G)$. Let t_0 be the earliest time instant t at which $S(C(t)) = (G,G)$ holds.

We observe that both robots are settled. The state of r_1 is W at $a_1(t_0)$ and hence r_1 is settled at t_0. Robot r_0 (whose state at $a_0(t_0)$ is A) is also settled at t_0 since, by the definition of A^{ϕ}_{SS}, the next position of r_0 at $a_0(t_0)$ is the position of r_1. It follows that gathering completes, like the case where $S(C(0)) = (G,G)$.

We continue with the cases where $S(C(0)) \in \{((A,R),(R,A)\}$. Like the proof of Lemma 1, assume without loss of generality that

1. $\phi_0 = \phi_1 = 0$;
2. $S(C(0)) = (R,A)$;
3. $\mathbf{r}_0(0) = (x_0(0), y_0(0)) = \mathbf{0}$, i.e., the position of r_0 is at the origin in Z; and
4. $y_1(0) < 0$.[12]

By the same argument used in the proof of Lemma 1, $y_1(t) \le 0$ holds for any $t \in \mathbb{N}$. If $y_1(t) = 0$ for some $t \in \mathbb{N}$, then let t_0 be the earliest time instant t at which $y_1(t) = 0$ holds. It follows that $S(C(t_0)) \in \{(W,A),(G,G)\}$, and r_1 is settled at t_0 (because $\mathbf{r}_1(t_0)$ is the next position of r_1 at $a_1(t_0)$). If r_0 is settled at t_0, the gathering eventually completes since it reduces to the case where $S(C(0)) \in \{(W,A),(G,G)\}$.

If r_0 is not settled at t_0, then r_0 is moving (or will move) in the negative direction on its x-axis, since $S(C(a_0(t_0))) = (R,A)$. Thus r_0 is settled at $b_0(t_0)$ and $S(C(b_0(t_0))) = (W,A)$. If r_1 is settled at $b_0(t_0)$, then the gathering eventually completes as discussed above. If r_1 is not settled at $b_0(t_0)$, then $S(C(t_1)) \in \{(W,A),(G,G)\}$ and both r_0 and r_1 are settled at t_1, where $t_1 = b_1(b_0(t_0))$. Therefore, the gathering eventually completes.

To derive a contradiction, we assume that $y_1(t) = 0$ does not hold for any $t \in \mathbb{N}$. By the same argument used in the proof of Lemma 1, $\mathcal{E}$ eventually reaches a

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[11] We call such a configuration a *pseudo-gathered* configuration in the next subsection.

[12] The state pair of $C(0)$ is any of (G,G), (A,W), or (W,A) if $y_1(0) \ge 0$.

configuration $C(t)$ such that $S(C(t)) = (W, A)$. Let t_0 be the first time instant t such that $S(C(t)) = (W, A)$ holds. If both r_0 and r_1 are settled at t_0, the gathering eventually completes, leading to a contradiction. If r_0 is not settled and is moving (or will move) in the negative direction on its x-axis at t_0, the gathering eventually completes by a similar argument as above. A contradiction is thus derived. ☐

LEMMA 4. *Suppose that* $\phi_0 \neq \phi_1$. *Then* A^{ϕ}_{SS} *correctly solves the gathering problem for two oblivious robots under the asynchronous model.*

Proof. Again, our proof is similar to that of Lemma 2. We continue to use the same concepts and notations but introduce them again for the convenience of the reader.

Consider any configuration $C(0)$ and any execution $\mathcal{E} = C(0), C(1), \ldots$ starting at $C(0)$, where $C(t) = (\mathbf{r}_0(t), \mathbf{r}_1(t))$ for any $t \in \mathbb{N}$. We assume $\phi_0 < \phi_1$ without loss of generality. Since $\phi < \pi/2$, we denote by $\mathbf{o}(t)$ the intersection of the x-axes of $Z_{(0,t)}$ and $Z_{(1,t)}$. Let $Z_{(i,t)}(\mathbf{p}) = (x_{(i,t)}(\mathbf{p}), y_{(i,t)}(\mathbf{p}))$ for any $i \in \{0, 1\}$ and $\mathbf{p} \in \mathbb{R}^2$. By definition, $x_{(i,t)}(\mathbf{o}(t))$ and $x_{(i,t)}(\mathbf{r}_i(t))$ are the x-coordinates, in $Z_{(i,t)}$ at time t, of the intersection $\mathbf{o}(t)$ and the position of r_i, respectively.

As explained in the proof of Lemma 2, under the semi-synchronous model we could assume without loss of generality that a robot is not on the x-axis of the other at t. Unfortunately, in the asynchronous model, we can no longer assume this. That is, a robot can possibly be located at $\mathbf{o}(t)$ at t. Taking this into account, we partition the configurations into four classes as follows (the partition is slightly different from the one defined in the proof of Lemma 2):

P(ositive)P(ositive): $x_{(0,t)}(\mathbf{o}(t)) < x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) < x_{(1,t)}(\mathbf{r}_1(t))$,
P(ositive)N(egative): $x_{(0,t)}(\mathbf{o}(t)) < x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) \geq x_{(1,t)}(\mathbf{r}_1(t))$,
N(egative)P(ositive): $x_{(0,t)}(\mathbf{o}(t)) \geq x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) < x_{(1,t)}(\mathbf{r}_1(t))$,
and
N(egative)N(egative): $x_{(0,t)}(\mathbf{o}(t)) \geq x_{(0,t)}(\mathbf{r}_0(t))$ and $x_{(1,t)}(\mathbf{o}(t)) \geq x_{(1,t)}(\mathbf{r}_1(t))$

Case NN. Suppose that $C(0)$ is in class NN, which implies that $S(C(0)) \in \{(W, A), (R, A), (G, G)\}$. We can show that gathering eventually completes in the first two cases by using arguments similar to those in the proof of Lemma 3. The last case obviously completes gathering.

Case PN. Suppose that $C(0)$ is in class PN, which implies that $S(C(0)) \in \{(R, R), (R, W)\}$. Since a robot r_i at R moves in the negative direction along its x-axis and thus decreases its x-coordinate (without changing its y-coordinate) in its local coordinate system, $\mathcal{E}$ eventually reaches a configuration $C(f)$ in class NN for the first time at f.

If both robots are settled at f, the case is reduced to Case NN. If r_1 is settled at f, the case is also reduced to Case NN, as follows: $C(0)$ is in class PN, $S(C(0)) = (R, R)$ or (R, W), only r_0 is activated at time 0, and r_1 is activated at time 1 while r_0 is still moving. Finally, if r_0 is settled at f, consider the time $b_1(f)$ at which r_1 is activated next time after f. It is easy to observe that $C(b_1(f))$ is in NN and $S(C(b_1(f))) \in \{(W, A), (R, A)\}$. Since r_1 is settled at $b_1(f)$, as above, the case is reduced to Case NN.

Case NP. Suppose that $C(0)$ is in class NP, which implies that $S(C(0)) \in \{(A, A), (W, A)\}$. If $S(C(0)) = (W, A)$, then obviously the gathering eventually completes. By arguments similar to those used to show Case PN and Lemma 2, the case is reduced to Case NN, or else gathering completes.

Case PP. Suppose that $C(0)$ is in class PP, which implies that $S(C(0)) = (A, R)$. Applying arguments similar to Case PN and the proof of Lemma 2, the case is reduced to Case PN, unless gathering completes. ☐

By Lemmas 3 and 4, we have the following theorem.

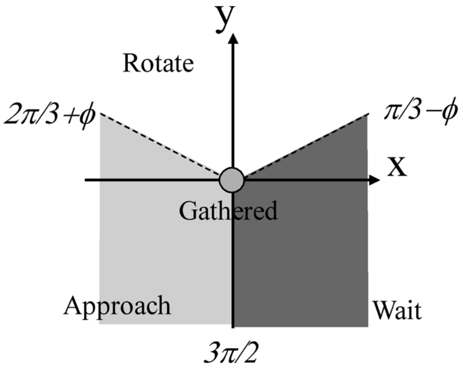

FIG. 9. *An illustration of Algorithm* A^{ϕ}_{AD}.

THEOREM 5. *For any* $0 \leq \phi < \pi/2$*, Algorithm* A^{ϕ}_{SS} *for two oblivious robots using* ϕ*-static compasses solves the gathering problem under the asynchronous model.*

Remark 2. At the end of section 4, we modified A^{ϕ}_{SS} by replacing the action Gathered into Terminate, and we showed that the modified A^{ϕ}_{SS} is a gathering algorithm for semi-synchronous robots with the termination condition. The modified A^{ϕ}_{SS}, however, is not a correct gathering algorithm for asynchronous robots, as the following counterexample shows. Let $C(0) = ((0,0),(0,-1))$ and suppose that the unit distances of Z and $Z_{(i,t)}$ are the same, i.e., $sc_i(t) = 1$ for all $i \in \{0,1\}$ and $t \in \mathbb{N}$ and that the compasses have no deviation, i.e., $\phi_0 = \phi_1 = 0$. Then $S(C(0)) = (R, A)$. Consider the following scenario:

Time 0: r_0 and r_1 are activated, where $S(C(0)) = (R, A)$.

Time Interval (0,1): r_1 moves and reaches $(0,0)$, but r_0 does not move.

Time 1: r_1 is activated, where $S(C(0)) = (G, G)$. Then r_1 halts.

Time Interval (1,2): r_0 moves and reaches $(-1,0)$.

Time 2: r_0 is activated, where $S(C(2)) = (W, A)$. Since r_1 has terminated, neither robot can move.

This shows that the modified A^{ϕ}_{SS} is not a correct gathering algorithm for asynchronous robots.

5.2. Asynchronous robots with dynamic compasses. We present a gathering algorithm A^{ϕ}_{AD} for two oblivious asynchronous robots using dynamic compasses and show its correctness, provided $0 \leq \phi < \pi/6$.[13]

ALGORITHM $A^{\phi}_{AD}(\mathbf{p})$.

G(athered): If $\mathbf{p} = \mathbf{0}$ then $A^{\phi}_{AD}(\mathbf{p}) = \mathbf{0}$.

A(pproach): If $2\pi/3 + \phi \leq \arg(\mathbf{p}) < 3\pi/2$ then $A^{\phi}_{AD}(\mathbf{p}) = \mathbf{p}$.

W(ait): If $-\pi/2 (= 3\pi/2) \leq \arg(\mathbf{p}) \leq \pi/3 - \phi$ then $A^{\phi}_{AD}(\mathbf{p}) = \mathbf{0}$.

R(otate): If $\pi/3 - \phi < \arg(\mathbf{p}) < 2\pi/3 + \phi$ then $A^{\phi}_{AD}(\mathbf{p}) = \rho_{\frac{2\pi}{3}+2\phi}(\mathbf{p})$.

Figures 9 and 10 illustrate Algorithm A^{ϕ}_{AD} and the move of a robot in state R (Rotate) who looks at the other robot at $\mathbf{p}$ in its local coordinate system under Algorithm A^{ϕ}_{AD}, respectively. We show the correctness of A^{ϕ}_{AD}.

THEOREM 6. *For any* $0 \leq \phi < \pi/6$*, Algorithm* A^{ϕ}_{AD} *for two oblivious robots using* ϕ*-dynamic compasses solves the gathering problem under the asynchronous model.*

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[13] AD in A^{ϕ}_{AD} stands for asynchronous robots with dynamic compasses.

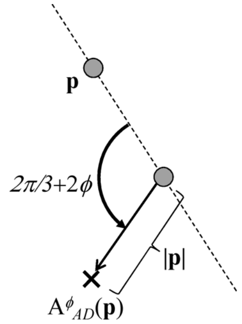

FIG. 10. *The move of a robot in state Rotate who looks at the other robot at* $\mathbf{p}$ *in its local coordinate system under Algorithm* A^{ϕ}_{AD}.

Proof. Consider any configuration $C(0)$ and any execution $\mathcal{E} = C(0), C(1), \ldots$ of A^{ϕ}_{AD} with initial configuration $C(0)$. For any $t \in \mathbb{N}$, let $C(t) = (\mathbf{r}_0(t), \mathbf{r}_1(t))$ and $\mathbf{r}_i(t) = (x_i(t), y_i(t))$. By the definition of A^{ϕ}_{AD}, we have $S(C(t)) \in \{(G,G), (A,W), (W,A), (A,R), (R,A), (W,R), (R,W)\}$ for any $t \in \mathbb{N}$. In the following, we show that $\mathcal{E}$ is a gathering execution.

A configuration $C(t)$ such that $S(C(t)) = (G,G)$ is said to be *pseudo-gathered* if $S(C(t')) \neq (G,G)$ for some $t' > t$ or, equivalently, if a robot is not settled at t. Unlike semi-synchronous robots' execution, $\mathcal{E}$ (of A^{ϕ}_{AD}) may reach a pseudo-gathered configuration.

Suppose that $C(t)$ is a pseudo-gathered configuration and a robot, say, r_0, is not settled. Since the state of r_1 is G (i.e., stay motionless), the execution can reach the same configuration $C(t+1)$ even if r_1 is not activated at t. Formally, if $C(t)$ is a pseudo-gathered configuration, then $\mathcal{E}' = C(0), C(1), \ldots, C(t-1), C(t+1), C(t+2), \ldots$ is also an execution of A^{ϕ}_{AD}.

The proof is by contradiction. We assume that $\mathcal{E}$ is not a gathering execution and derive a contradiction. If $\mathcal{E}$ is not a gathering execution, then there is an execution $\mathcal{E}'$ such that it is not a gathering execution and does not contain a pseudo-gathered configuration. Without loss of generality, we also assume that pseudo-gathering executions never appear in $\mathcal{E}$.

If $y_0(0) = y_1(0)$, since $0 \leq \phi < \pi/6$, $C(0)$ is stable and $S(C(0)) \in \{(A,W), (W,A), (G,G)\}$ by the definition of A^{ϕ}_{AD}. Since the case where $S(C(0)) = (G,G)$ is trivial, let us assume, without loss of generality, that $S(C(0)) = (A,W)$. Then obviously, r_0 always moves toward r_1 by the definition of A^{ϕ}_{AD} and the gathering eventually completes. We thus assume $y_0(0) < y_1(0)$ without loss of generality.

To show the correctness of A^{ϕ}_{SD} in subsection 4.2, we observed that a robot at R rotates the line segment connecting the current robots' positions clockwise until the state pair becomes either (W,A) or (A,W). The scenario of the correctness proof of A^{ϕ}_{AD} is similar. Define $\alpha(t) = \arg(\mathbf{r}_1(t) - \mathbf{r}_0(t))$, provided that $y_0(t) < y_1(t)$.

For the time being, we assume (1) $y_0(t) < y_1(t)$ for any $t \in \mathbb{N}$ (and hence $0 < \alpha(t) < \pi$), and (2) $0 < \alpha(t+1) \leq \alpha(t)$. The verification of their correctness is the core of the proof and will be given later.

Obviously, $\alpha(t)$ converges to an angle $\alpha \geq 0$ (under the above two assumptions). Indeed, $\alpha = 0$; that is, $\alpha(t)$ converges to 0. To observe this, let us assume that $\alpha > 0$. Then we can derive a contradiction by an argument identical to the last three paragraphs of the proof of Theorem 4.

When $\alpha(t) \approx 0$, by the definition of A^{ϕ}_{AD}, $C(t)$ is stable[14] and $S(C(t)) = (W, A)$. We now show that the gathering eventually completes from such $C(t)$, which contradicts the assumption that $\mathcal{E}$ is not a gathering execution.

Suppose that $\mathcal{E}$ eventually reaches a configuration $C(t)$ such that $\alpha(t) \approx 0$. Since $\alpha(t) \approx 0$, $C(t)$ is stable and $S(C(t)) = (W, A)$. Moreover, $C(t')$ is stable and $S(C(t')) = (W, A)$ for all $t' \geq t$. Let $f = \max\{b_0(t), b_1(t)\}$. Then, r_0 is settled after (and including) time f. By definition, $\mathcal{E}$ eventually reaches (G, G).

Now we return to the verification of the two assumptions mentioned above. That is, we prove (1) $y_0(t) < y_1(t)$ and (2) $0 < \alpha(t+1) \leq \alpha(t)$ for any $t \in \mathbb{N}$.

To this end, we still need a few more concepts. Let $s_i(t)$ be the state of robot r_i at time t. That is, letting $S(t) = S(C(t))$, $S(t) = (s_0(t), s_1(t))$. Since a robot, say, r_0, may not be settled at t, $s_0(t)$ may not coincide with the action $s^*_0(t)(= s_0(a_0(t)))$ that r_0 is engaging at t. (For consistency, we assume that $s^*_i(t) = W$ if robot r_i has never been activated.) Let $S^*(t) = (s^*_0(t), s^*_1(t)) = (s_0(a_0(t)), s_1(a_1(t)))$.

Suppose that $y_0(t) < y_1(t)$. We partition the working space $\mathbb{R}^2$ of the robots into two half planes delimited by the line L connecting their positions. Recall that $\alpha(t)$ is the angle that L forms with the x-axis of Z. We assume that both half planes contain L as a part and denote by $\Gamma_0(t)$ (resp., $\Gamma_1(t)$) the left-hand (resp., right-hand) half plane of L. Robot r_i may or may not be activated at time t. However, if r_i is activated, it calculates and moves toward the next position, the coordinates of which are expressed by $\mathbf{d}_i(t)$ in Z.[15]

As mentioned, we may assume $y_0(0) < y_1(0)$ without loss of generality. We then prove the following four statements. For any $t \geq 1$,

1. $y_0(t) < y_1(t)$,
2. $0 < \alpha(t) \leq \alpha(t-1)$,
3. $\mathbf{d}_i(t) \in \Gamma_i(t)$ for $i \in \{0, 1\}$, and
4. $S^*(t) \neq (A, A)$.

Recall that we assume that $\mathcal{E}$ is not a gathering execution and does not contain a pseudo-gathered configuration. The proof is by induction on t. Since the base case is obvious, let us concentrate on the induction step.

(A) First, we show $y_0(t) < y_1(t)$. In the proof, we implicitly use the fact that R always decreases the robot's y-coordinate. Assume that $y_0(t) \geq y_1(t)$ to derive a contradiction. Assume first that r_0 is activated at $t-1$ (r_1 may or may not be activated at $t-1$). Let $v = a_1(t-1) \leq t-1$. Since $y_0(v) < y_1(v)$, $s_1(v) \in \{A, W\}$. If $s_1(v) = W$, then $y_1(t) = y_1(t-1)$. Since r_0 is activated at $t-1$, $y_0(t) \leq y_1(t-1)$ and the equality holds only if $s_0(t-1) = A$. If $s_0(t-1) = A$, and $y_0(t) = y_1(t-1) = y_1(t)$, then $S(t) = (G, G)$ (since $y_0(t-1) < y_1(t-1)$), a contradiction. If $s_1(v) = A$, since $S^*(u) \neq (A, A)$ for all $v \leq u \leq t-1$, $s^*_0(u) \in \{R, W\}$ for all $v \leq u \leq t-1$, which implies that $y_0(t) \leq y_0(v) \leq y_1(t)$. If $y_0(t) = y_0(v) = y_1(t)$, then we can conclude $S(t) = (G, G)$, a contradiction. If $y_0(t) < y_0(v)$ or $y_0(v) < y_1(t)$, it directly implies $y_0(t) < y_1(t)$, a contradiction.

Assume next that r_1 is activated at $t-1$. Let $v = a_0(t-1) \leq t-1$. If $s_0(v) = R$, then $y_0(t) < y_0(t-1) \leq y_1(t)$, a contradiction. If $s_0(v) = W$, then $y_0(v) = y_0(t) = y_0(t-1) < y_1(t-1)$, and a contradiction is derived, since $s_1(t-1) \in \{A, W\}$ and thus $y_0(t) = y_1(t)$ implies $S(t) = (G, G)$. If $s_0(v) = A$, since $S^*(u) \neq (A, A)$ for all

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[14] Recall that a configuration C is said to be stable if $S(C)$ is determined uniquely, regardless of the current local coordinate systems $Z_{(i,t)}$.

[15] The coordinates of the next position in $Z_{(i,t)}$ are $A^{\phi}_{AD}(\mathbf{p})$, where $\mathbf{p}$ represents the coordinates of the other robot in $Z_{(i,t)}$.

$v \le u \le t-1$, $s_1(u) \neq A$, which implies that $s_1(u)$ is always W, a contradiction, since $y_0(t) \le y_1(v) = y_1(t)$ and $y_0(t) = y_1(t)$ implies $S(t) = (G, G)$.

(B) Second, we show $0 < \alpha(t) \le \alpha(t-1)$. If both robots are settled at $t-1$, then the claim is obvious, since $\mathbf{d}_i(t-1) \in \Gamma_i(t-1)$ for $i \in \{0,1\}$, $y_0(t) < y_1(t)$, $0 < \alpha(t) \le \alpha(t-1)$, and $S(t) \neq (G, G)$.

If robot r_0 is not settled at $t-1$, then $\mathbf{d}_0(a_0(t-1)) \in \Gamma_0(a_0(t-1))$ and $\alpha(t-1) \le \alpha(a_0(t-1))$, which implies that $\mathbf{d}_0(a_0(t-1) \in \Gamma_0(t-1)$. Then, by the same argument, $0 < \alpha(t) \le \alpha(t-1)$, since $y_0(t) < y_1(t)$. The case in which r_1 is not settled at $t-1$ is symmetrical.

(C) Third, we show $\mathbf{d}_i(t) \in \Gamma_i(t)$ for $i \in \{0,1\}$. Since we showed $y_0(t) < y_1(t)$ in (A) and $0 \le \alpha(t) < \alpha(t-1)$ in (B), the claim is obvious by the definition of A^{ϕ}_{AD}.

(D) Finally, we show $S^*(t) \neq (A, A)$. There are two cases to be considered. Assume first that r_0 is activated at t. Since $s_0(t) = A$, $2\pi/3 \le \alpha(t) < \pi$ (because $y_0(t) < y_1(t)$). Let $v = a_1(t) \le t-1$. Since $s_1(v) = A$, $0 < \alpha(v) \le \pi/2 + \phi$. Since $\pi/2 + \phi < 2\pi/3$ (because $\phi < \pi/6$), a contradiction is derived, since $\alpha(t) \le \alpha(v)$.

Next, assume that r_1 is activated at t. Let $v = a_0(t) \le t-1$. Since $S^*(u) \neq (A, A)$ for any $v \le u \le t-1$, $s_0(v) = A$, and $s_1(u) = W$ for any $v \le u \le t-1$ (since r_1 can take either A or W), a contradiction. ▯

6. Concluding remarks. This paper investigates the gathering problem for two oblivious anonymous mobile robots under disagreement of local coordinate systems. To discuss the magnitude of consistency between the local coordinate systems, we assumed that each robot is equipped with an unreliable compass, the bearings of which may deviate from an absolute reference direction, and that the local coordinate system of each robot is determined by its compass. We considered four classes of robot systems, which are specified by the combination of synchrony assumption (semi-synchronous/asynchronous robots) and compass models (static/dynamic), and established the maximum deviation ϕ allowing an algorithm to solve the gathering problem for each class: $\phi < \pi/2$ for semi-synchronous and asynchronous robots with static compasses, $\phi < \pi/4$ for semi-synchronous robots with dynamic compasses, and $\phi < \pi/6$ for asynchronous robots with dynamic compasses. Except for asynchronous robots with dynamic compasses, these sufficient conditions are also necessary. As for a necessary condition on ϕ for asynchronous robots with dynamic compasses we could show that $\phi < \pi/6$ is necessary for almost all cases and thus conjecture it and would like to leave it as a challenging future work. The results are summarized in Table 1.

CONJECTURE 1. *Condition* $\phi < \pi/6$ *is necessary for asynchronous oblivious robots with dynamic compasses to have a gathering algorithm.*

Remarks 1 and 2 emphasize that the modified A^{ϕ}_{SS} is a gathering algorithm for semi-synchronous robots with the termination property but is not for asynchronous robots. An interesting question thus is to ask if there is a gathering algorithm for asynchronous oblivious robots with the termination property. The gathering process with termination property could be viewed as a process of obtaining a point that the robots will gather as their common knowledge, and common knowledge is in general impossible to obtain under asynchronous settings. The plausible answer is thus *no*, and we would like to conjecture it. However, in order to complete a proof, we first need to deeply understand why gathering with termination is possible for semi-synchronous robots, although they share some asynchronous nature with asynchronous robots.

As a final note, in [17] the authors show that there is no gathering algorithm for oblivious robots under the semi-synchronous model even if the symmetricity of the initial configuration is 1 (i.e., even if the deviation of their local coordinate systems

Table 1

Summary of results about two oblivious robots gathering with unreliable compasses.

			Compass	
			Static	Dynamic
Timing model	*Semi-synch.*	Possible	$\phi < \pi/2$ (*Sec.* 4.1)	$\phi < \pi/4$ (*Sec.* 4.2)
		Impossible	$\phi \geq \pi/2$ ([13, 15])	$\phi \geq \pi/4$ (*Sec.* 4.2)
	Asynchronous	Possible	$\phi < \pi/2$ (*Sec.* 5.1)	$\phi < \pi/6$ (*Sec.* 5.2)
		Impossible	$\phi \geq \pi/2$ (*deduction*)	$\phi \geq \pi/4$ (*deduction*)

is less than π). We would like to note that this fact does not contradict Theorem 2 since Algorithm A_{SS}^{ϕ} relies on the existence of upper bound ϕ.

REFERENCES

[1] N. Agmon and D. Peleg, *Fault-tolerant gathering algorithms for autonomous mobile robots*, SIAM J. Comput., 36 (2006), pp. 56–82.

[2] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, *Distributed memoryless point convergence algorithm for mobile robots with limited visibility*, IEEE Trans. Robotics Automat., 15 (1999), pp. 818–828.

[3] Z. Bouzid, M. Potop-Butucaru, and S. Tixeuil, *Optimal Byzantine resilient convergence in asynchronous robot networks*, in Proceedings of the International Symposium on Stabilization, Safety, and Security of Distributed Systems, 2009, pp. 165–179.

[4] Z. Bouzid, M. Potop-Butucaru, and S. Tixeuil, *Byzantine-resilient convergence in oblivious robot networks: The price of asynchrony*, in Proceedings of the International Conference on Distributed Computing and Networks, 2009, pp. 275–280.

[5] M. Cieliebak, P. Flocchini, G. Prencipe, and N. Santoro, *Solving the robots gathering problem*, in Proceedings of the International Colloquium on Automata, Languages and Programming, 2003, pp. 1181–1196.

[6] M. Cieliebak and G. Prencipe, *Gathering autonomous mobile robots*, in Proceedings of the International Colloquium on Structural Information and Communication Complexity, 2002, pp. 57–72.

[7] R. Cohen and D. Peleg, *Convergence properties of the gravitational algorithm in asynchronous robot systems*, SIAM J. Comput., 34 (2005), pp. 1516–1528.

[8] R. Cohen and D. Peleg, *Convergence of autonomous mobile robots with inaccurate sensors and movements*, SIAM J. Comput., 38 (2008), pp. 276–302.

[9] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, *Gathering of asynchronous mobile robots with limited visibility*, Theoret. Comput. Sci., 337 (2005), pp. 147–168.

[10] N. Inuzuka, Y. Tomida, T. Izumi, Y. Katayama, and K. Wada, *Gathering problem of two asynchronous mobile robots with semi-dynamic compasses*, in Proceedings of the International Colloquium on Structural Information and Communication Complexity, 2008, pp. 5–19.

[11] T. Izumi, Y. Katayama, N. Inuzuka, and K. Wada, *Gathering autonomous mobile robots with dynamic compasses: An optimal result*, in Proceedings of the International Symposium on Distributed Computing, 2007, pp. 298–312.

[12] Y. Katayama, Y. Tomida, H. Imazu, N. Inuzuka, and K. Wada, *Dynamic compass models and gathering algorithms for autonomous mobile robots*, in Proceedings of the International Colloquium on Structural Information and Communication Complexity, 2007, pp. 274–288.

[13] G. Prencipe, *On the feasibility of gathering by autonomous mobile robots*, in Proceedings of the International Colloquium on Structural Information and Communication Complexity, 2005, pp. 246–261.

[14] S. Souissi, X. Défago, and M. Yamashita, *Using eventually consistent compasses to gather memory-less mobile robots with limited visibility*, ACM Trans. Autonomous Adaptive Systems, 4 (2009).

[15] I. Suzuki and M. Yamashita, *Distributed anonymous mobile robots: Formation of geometric patterns*, SIAM J. Comput., 28 (1999), pp. 1347–1363.

[16] K. Yamamoto, T. Izumi, Y. Katayama, N. Inuzuka, and K. Wada, *Convergence of mobile robots with uniformly-inaccurate sensors*, in Proceedings of the International Colloquium on Structural Information and Communication Complexity, 2009, pp. 320–333.

[17] M. Yamashita and I. Suzuki, *Characterizing geometric patterns formable by oblivious anonymous mobile robots,* Theoret. Comput. Sci., 411 (2010), pp. 2433–2453.