# A Logic with Implication Expressing Temporality

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# Chapter 1

# Introduction

In Kripke semantics which is well-known as semantics for modal logics, each formula is interpreted on triples consisting of a set of possible worlds, a binary relation (called an accessibility relation) on the set, and a valuation which assigns to each possible world the basic propositions that hold at the world. In Kripke semantics for non-modal propositional logics such as intuitionistic logic, intermediate logics, and weak logics with strict implication [1], a formula $A \to B$ is true at a possible world if and only if at every possible world accessible from there, whenever $A$ is true, also $B$ is true. In the case of intuitionistic logic an accessibility relation is a partial order, so regarding it as the order of time we may consider that $A \to B$ is true if and only if at any future world (including the present), whenever $A$ is true, also $B$ is true.

In this paper, we propose new Kripke semantics in which interpretation of implication is different from that in intuitionistic logic etc. In that semantics, an accessibility relation is a partial order as in intuitionistic logic, but the informal meaning of $A \to B$ is "$B$ holds as long as $A$ holds (in the future)." This interpretation of implication enables us to express temporality by comparing how long each formula remains true.

This paper consists of comparative studies between intuitionistic logic and the logic corresponding to the new Kripke semantics. In Chapter 3, we define each Kripke semantics and discuss their properties. The following is some informal motivation of Kripke semantics for intuitionistic logic. Think of an idealized mathematician (in this context traditionally called the creative subject), who extends his knowledge in the course of time. At each moment $x$ he has a stock of sentences, which he, by some means, has recognized as true. Since at every moment $x$ the idealized mathematician has various choices for his future activities (he may even stop altogether), the stages of his activity must be thought of as being partially ordered, and not necessarily linearly ordered. How will the idealized mathematician interpret the logical connectives? Evidently the interpretation of a composite statement must depend on the interpretation of its parts, e.g. the idealized mathematician has established $A$ or (and) $B$ at stage $x$ if he has established $A$ at stage $x$ or (and) $B$ at stage $x$. The implication is more cumbersome, since $A \to B$ may be known at stage $x$ without $A$ or $B$ being known. Clearly, the idealized mathematician knows $A \to B$ at stage $x$ if he knows that if at any future stage (including $x$) $A$ is established, also $B$ is established.

On the other hand, new Kripke semantics we propose deals with not only mathematical statements but also propositions whose truth and falsity may depend on time, so we make no assumption that if a propositional variable $p$ is true at stage $x$, then $p$ is true at any future stage (including $x$). Furthermore, we modify the interpretation of implication in intuitionistic logic, so that we can express temporality by comparing how long each formula remains true. Namely, the informal meaning of $A \to B$ is "$B$ holds as long as $A$ holds (in the future)." In this interpretation, there exist Kripke models with possible worlds at which formulas $A \to (B \to A)$ or $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$ is not true, while those are valid in any Kripke model of intuitionistic logic. Conversely, in this interpretation, $A \vee \neg A$ and $\neg\neg A \to A$ are valid in any Kripke model, while there exist Kripke models of intuitionistic logic such that those formula is not valid. Therefore, neither of intuitionistic logic and the new logic contains the other in Kripke validity.

In Chapter 4, we provide formal systems for each Kripke semantics, which are denoted by the generic name of Hilbert-type systems. The Hilbert-type system of intuitionistic logic contains nine axioms and one inference rule (modus ponens). Soundness Theorem which says that any formula derivable in this system is intuitionistically Kripke valid, is proved as usual by induction on derivation. Every axiom is intuitionistically Kripke valid and the inference rule (modus ponens) preserves the validity. Conversely, for the Completeness Theorem which says that any intuitionistically Kripke valid formula is derivable in the system, we need some notions and a few lemma's. To prove this theorem, we construct so-called canonical model whose possible worlds are the sets of all maximal consistent sets, whose accessibility relation is including relation on those sets, and whose valuation is defined such that a basic proposition is a member of a maximal consistent set if and only if the proposition is true at the maximal consistent set. Now if a formula is not derivable in the system above, then there exists a maximal consistent set such that the formula is not an element in it. Therefore the formula is not true at the possible world, so it is not valid in the canonical model. In this way, the Completeness Theorem is proved in the case of intuitionistic logic. On the other hand, the Hilbert-type formal system of the new logic is based on eleven axioms and four inference rules. We can prove that the system is also sound for the new Kripke semantics in the same way as that in intuitionistic logic. To prove the Completeness Theorem we try to construct the canonical model as in intuitionistic logic, but a problem rise as follow. A Kripke model in which $(p \to q) \wedge (q \to p) \to (p \vee q \to q)$ is not valid requires infinitely many elements at which $p$ is true and $q$ is not true as the example above. This implies that one set of formulas have to be recognized as different elements in the canonical model, so we cannot construct the canonical model in the same way as that in intuitionistic logic.

In Chapter 5, we discuss relationships between intuitionistic logic and the new logic. As we see in the examples of formulas above, neither intuitionistic logic nor the new logic contains the other in Kripke validity. We can, however, consider some connections between intuitionistic logic and the new logic. Namely, we can try to embed intuitionistic logic into the new logic in the sense that there is a translation $Tr$ such that for every formula $A$, $A$ is intuitionistically Kripke valid if and only if $Tr(A)$ is Kripke valid in the new logic. Such a relation holds between classical propositional logic and intuitionistic propositional logic, where the translation is a simple one from a formula $A$ of classical propositional logic to

$\neg\neg A$ of intuitionistic propositional logic. The translation presented in this paper is one of such translations, and makes it possible to interpret logical connectives in intuitionistic logic in terms of those in the new logic.

# Chapter 2

# Preliminaries

First we will define our propositional language and formulas, and give some notational conventions.

**Definition 2.1** The propositional language has an alphabet consisting of

1. the propositional variables: $p_0, p_1, p_2, \ldots$,
2. the propositional constant: $\bot$ (falsehood),
3. the logical connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication),
4. the auxiliary symbols: (, ).

**Definition 2.2** The formulas are defined inductively:

1. the propositional variables and the propositional constant are atomic formulas (or simply atoms),
2. if $A$ and $B$ are formulas, then so are $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$.

We will denote propositional variables by the small Roman letters $p$, $q$, $r$, possibly with subscripts or superscripts, the capital Roman letters $A$, $B$, $C$ and maybe some others are reserved for formulas, and capital Greek letters like $\Gamma$, $\Delta$, $\Sigma$ are used for denoting sets of formulas.

The sets of all propositional variables and all formulas are denoted by Vars and $\Phi$, respectively.

**Definition 2.3** The logical connectives $\neg$ (negation), $\leftrightarrow$ (equivalence) and the propositional constant $\top$ (truth) are defined as abbreviations:

$$(\neg A) := (A \rightarrow \bot),$$

$$(A \leftrightarrow B) := ((A \rightarrow B) \wedge (B \rightarrow A)),$$

$$\top := (\bot \rightarrow \bot).$$

In bracketing we adopt the usual convention that $\neg$ bind stronger than any of the binary connectives, and that $\wedge$, $\vee$ bind stronger than $\to$, $\leftrightarrow$.

We need the following notion to define Kripke semantics.

**Definition 2.4** A structure $\langle X, \leq \rangle$ is a partially ordered set if $\leq$ is a relation on $X$ ($\leq \subseteq X \times X$) such that

1. $x \leq x$,
2. $x \leq y$ and $y \leq x \Rightarrow x = y$,
3. $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

# Chapter 3

# Kripke semantics

In this chapter, we introduce Kripke semantics for intuitionistic logic and new Kripke semantics we propose in this paper, and discuss properties that hold on each Kripke semantics.

## 3.1 Kripke semantics for intuitionistic logic

We will first give some informal motivation of Kripke semantics for intuitionistic logic.

Think of an idealized mathematician (in this context traditionally called the creative subject), who extends his knowledge in the course of time. At each moment $x$ he has a stock of sentences, which he, by some means, has recognized as true. Since at every moment $x$ the idealized mathematician has various choices for his future activities (he may even stop altogether), the stages of his activity must be thought of as being partially ordered, and not necessarily linearly ordered. How will the idealized mathematician interpret the logical connectives? Evidently the interpretation of a composite statement must depend on the interpretation of its parts, e.g. the idealized mathematician has established $A$ or (and) $B$ at stage $x$ if he has established $A$ at stage $x$ or (and) $B$ at stage $x$. The implication is more cumbersome, since $A \to B$ may be known at stage $x$ without $A$ or $B$ being known. Clearly, the idealized mathematician knows $A \to B$ at stage $x$ if he knows that if at any future stage (including $x$) $A$ is established, also $B$ is established.

Now we will formalize the above sketched semantics.

**Definition 3.1** $\langle X, \leq, \models \rangle$ is an intuitionistic Kripke model if

1. $\langle X, \leq \rangle$ is a non-empty partially ordered set,
2. $\models$ is a binary relation on $X$ and Vars such that

$$x \models p \text{ and } x \leq y \Rightarrow y \models p.$$

$\models$ can be extended to a binary relation on $X$ and $\Phi$ as follow

not $x \models \bot$,

$$x \models A \wedge B \quad := \quad x \models A \text{ and } x \models B,$$

$$x \models A \vee B \quad := \quad x \models A \text{ or } x \models B,$$

$$x \models A \to B \quad := \quad \text{for all } y \geq x, \text{ if } y \models A \text{ then } y \models B.$$

If $x \models A$ does not hold then we write $x \not\models A$.

It follows from this definition that

$$x \models \neg A \Leftrightarrow \text{for all } y \geq x,\ y \not\models A.$$

The monotonicity of $\models$ for atoms is carried over to arbitrary formulas.

**Lemma 3.2** Let $\langle X, \leq, \models\rangle$ be an intuitionistic Kripke model, let $x, y \in X$, and let $A$ be a formula. Then

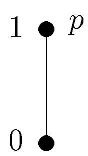
$$x \models A \text{ and } x \leq y \Rightarrow y \models A.$$

*Proof.* By induction on $A$.
atomic $A$: the lemma holds by Definition 3.1.
$A \equiv B \wedge C$: let $x \models B \wedge C$ and $x \leq y$, then $x \models B \wedge C \Leftrightarrow x \models B$ and $x \models C \Rightarrow$ (induction hypothesis) $y \models B$ and $y \models C \Leftrightarrow y \models B \wedge C$.
$A \equiv B \vee C$: mimic the conjunction case.
$A \equiv B \to C$: let $x \models B \to C$, $y \geq x$. Suppose $z \geq y$ and $z \models B$ then, since $z \geq x$, $z \models C$. Hence $y \models B \to C$.

**Definition 3.3** Let $\langle X, \leq, \models\rangle$ be an intuitionistic Kripke model, and let $A$ be a formula. Then we say

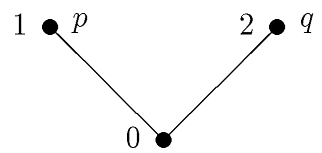
- $A$ is true at $x \in X$ if $x \models A$,
- $A$ is valid in $\langle X, \leq, \models\rangle$ if $x \models A$ for all $x \in X$,
- $A$ is intuitionistically Kripke valid (notation: $\models_i A$) if $A$ is valid in every intuitionistic Kripke model.

**Example 3.4** $\not\models_i \neg\neg p \to p$. To see this, we specify an intuitionistic Kripke model by indicating the partially ordered structure as a diagram, writing next to each element the propositional variables which are true at that element




Since $1 \models p$, we have $1 \not\models \neg p$, $0 \not\models \neg p$ and so $0 \models \neg\neg p$. We also have $0 \not\models p$, and hence $0 \not\models \neg\neg p \to p$.

**Example 3.5** $\not\models_i (p \to q) \vee (q \to p)$. We see this by the following intuitionistic Kripke model.




We have $1 \models p$ and $1 \not\models q$, so $0 \not\models p \to q$. Similarly, $0 \not\models q \to p$ since $2 \models q$ and $2 \not\models p$. Hence $0 \not\models (p \to q) \vee (q \to p)$.

**Lemma 3.6** Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model. Then

$$A \to B \text{ is valid in } \langle X, \leq, \models \rangle \Leftrightarrow \text{for all } x \in X, \text{ if } x \models A \text{ then } x \models B.$$

*Proof.* ($\Rightarrow$) By definition, for all $x \in X$, $x \models A \to B$ i.e. for all $x \in X$, for all $y \geq x$, if $y \models A$ then $y \models B$. So, for all $x \in X$, if $x \models A$ then $x \models B$.
($\Leftarrow$) For all $y \geq x$, if $y \models A$ then by assumption $y \models B$. This holds for all $x \in X$, hence $A \to B$ is valid in $\langle X, \leq, \models \rangle$.

## 3.2 Kripke semantics for the new logic

Now we will introduce new Kripke semantics we propose, which is different from that for intuitionistic logic in the following two respects. First, we make no assumption that if a propositional variable $p$ is true at stage $x$, then $p$ is true at any future stage (including $x$). Furthermore, we modify the interpretation of implication in intuitionistic logic, so that we can express temporality by comparing how long each formula remains true.

**Definition 3.7** $\langle X, \leq, \models \rangle$ is a Kripke model if

1. $\langle X, \leq \rangle$ is a non-empty partially ordered set,
2. $\models$ is a binary relation on $X$ and Vars.

$\models$ can be extended to a binary relation on $X$ and $\Phi$ as follow

$$x \not\models \bot,$$

$$x \models A \wedge B \quad := \quad x \models A \text{ and } x \models B,$$

$$x \models A \vee B \quad := \quad x \models A \text{ or } x \models B,$$

$$x \models A \to B \quad := \quad \text{for all } y \geq x, \text{ if for all } z (y \geq z \geq x)\ z \models A \text{ then } y \models B.$$

In the definition of $x \models A \to B$, we can replace $y \models B$ with for all $z'(y \geq z' \geq x)$ $z' \models B$. Indeed, let for all $z(y \geq z \geq x)$ $z \models A$ and $y \geq z' \geq x$. Then for all $z(z' \geq z \geq x)$ $z \models A$, so if $x \models A \to B$ already holds then $z' \models B$.

Hence the definition of $x \models A \to B$ is equivalent to

$$\text{for all } y \geq x, \text{ if for all } z(y \geq z \geq x)\ z \models A \text{ then for all } z(y \geq z \geq x)\ z \models B.$$

Thus the informal meaning of $A \to B$ is "$B$ holds as long as $A$ holds (in the future)."

**Lemma 3.8** Let $\langle X, \leq, \models \rangle$ be a Kripke model, let $x \in X$, and let $A$ be a formula. Then

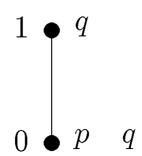
$$x \models \neg A \Leftrightarrow x \not\models A.$$

*Proof.* ($\Rightarrow$) By definition, for all $y \geq x$, if for all $z(y \geq z \geq x)$ $z \models A$ then $y \models \bot$. Take $x$ for $y$, then if for all $z(x \geq z \geq x)$ $z \models A$ then $x \models \bot$. Since $\langle X, \leq \rangle$ is a partially ordered set, this means that if $x \models A$ then $x \models \bot$. By definition, $x \not\models \bot$, so we have $x \not\models A$.
($\Leftarrow$) Let $y \geq x$ and for all $z(y \geq z \geq x)$ $z \models A$. Take $x$ for $z$, then $x \models A$ contrary to $x \not\models A$. Thus, for all $y \geq x$, if for all $z(y \geq z \geq x)$ $z \models A$ then $y \models \bot$. Hence $x \models \neg A$.

**Definition 3.9** Let $\langle X, \leq, \models \rangle$ be a Kripke model, and let $A$ be a formula. Then we say

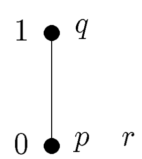
- $A$ is true at $x$ if $x \models A$,
- $A$ is valid in $\langle X, \leq, \models \rangle$ if $x \models A$ for all $x \in X$,
- $A$ is Kripke valid (notation: $\models A$) if $A$ is valid in every Kripke model.

**Example 3.10** $\not\models p \to (q \to p)$. We see this by the following Kripke model.




We have $0 \models q$, $1 \models q$, but $1 \not\models p$, and so $0 \not\models q \to p$. We also have $0 \models p$, and hence $0 \not\models p \to (q \to p)$.

**Example 3.11** $\not\models (p \to r) \wedge (q \to r) \to (p \vee q \to r)$. We see this by the following Kripke model.




Clearly, $0 \models p \to r$ and $0 \models q \to r$, so $0 \models (p \to r) \wedge (q \to r)$. We, however, have $0 \not\models p \vee q \to r$, since $0 \models p \vee q$, $1 \models p \vee q$, but $1 \not\models r$. Therefore $0 \not\models (p \to r) \wedge (q \to r) \to (p \vee q \to r)$.

**Lemma 3.12** Let $\langle X, \leq, \models \rangle$ be a Kripke model. Then

$$A \to B \text{ is valid in } \langle X, \leq, \models \rangle \Leftrightarrow \text{for all } x \in X, \text{ if } x \models A \text{ then } x \models B.$$

*Proof.* ($\Rightarrow$) By definition, for all $x \in X$, $x \models A \to B$ i.e. for all $y \geq x$, if for all $z(y \geq z \geq x)$ $z \models A$ then $y \models B$. Take $x$ for $y$, then if for all $z(x \geq z \geq x)$ $z \models A$ then $x \models B$. So, if $x \models A$ then $x \models B$.
($\Leftarrow$) Let $y \geq x$ and for all $z(y \geq z \geq x)$ $z \models A$. Take $y$ for $z$, then $y \models A$, and so by assumption $y \models B$. Thus, for all $y \geq x$, if for all $z(y \geq z \geq x)$ $z \models A$ then $y \models B$. This holds for all $x \in X$, and hence $A \to B$ is valid in $\langle X, \leq, \models \rangle$.

# Chapter 4

# Formal systems

In this chapter, we present Hilbert-type formal systems of intuitionistic logic and the new logic, and investigate their soundness and completeness for each Kripke semantics introduced in the previous chapter.

## 4.1 A formal system of intuitionistic logic

**Definition 4.1** The Hilbert-type system of intuitionistic logic contains the following axiom schemata:

I1. $A \to (B \to A)$,

I2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

I3. $A \land B \to A$,

I4. $A \land B \to B$,

I5. $A \to (B \to A \land B)$,

I6. $A \to A \lor B$,

I7. $B \to A \lor B$,

I8. $(A \to C) \to ((B \to C) \to (A \lor B \to C))$,

I9. $\bot \to A$,

and the following inference rule:

$$\frac{A \quad A \to B}{B} \text{ (modus ponens).}$$

A formula $A$ is said to be derivable in this system (notation: $\vdash_i A$) if there is a derivation of $A$ in this system, i.e., a sequence $A_1, \ldots, A_n$ of formulas such that $A_n = A$ and for every $k$, $1 \le k \le n$, $A_k$ is either an axiom schema or obtained from some of the preceding formulas in the sequence by the inference rule.

We will give a few examples needed for the Completeness Theorem.

**Example 4.2** The following sequence is a derivation of $A \to A$, for any formula $A$:

1. $(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$ (I2)
2. $A \to ((A \to A) \to A)$ (I1)
3. $(A \to (A \to A)) \to (A \to A)$ (by modus ponens from 1, 2)
4. $A \to (A \to A)$ (I1)
5. $A \to A$ (by modus ponens from 3, 4)

**Example 4.3** Below is a derivation of $A \wedge (A \to B) \to B$, for any formulas $A$ and $B$:

1. $A \wedge (A \to B) \to (A \to B)$ (I4)
2. $A \wedge (A \to B) \to A$ (I3)
3. $(A \wedge (A \to B) \to (A \to B)) \to ((A \wedge (A \to B) \to A) \to (A \wedge (A \to B) \to B))$ (I2)
4. $(A \wedge (A \to B) \to A) \to (A \wedge (A \to B) \to B)$ (by modus ponens from 1, 3)
5. $A \wedge (A \to B) \to B$ (by modus ponens from 2, 4)

**Example 4.4** The following sequence shows that $\vdash A \wedge B \to C$ implies $\vdash A \to (B \to C)$, for any formulas $A$, $B$ and $C$:

1. $A \wedge B \to C$
2. $(B \to (A \wedge B \to C)) \to ((B \to A \wedge B) \to (B \to C))$ (I2)
3. $(A \wedge B \to C) \to (B \to (A \wedge B \to C))$ (I1)
4. $B \to (A \wedge B \to C)$ (by modus ponens from 1, 3)
5. $((B \to (A \wedge B \to C)) \to ((B \to A \wedge B) \to (B \to C))) \to$
   $(A \to ((B \to (A \wedge B \to C)) \to ((B \to A \wedge B) \to (B \to C))))$ (I1)
6. $A \to ((B \to (A \wedge B \to C)) \to ((B \to A \wedge B) \to (B \to C)))$
   (by modus ponens from 2, 5)
7. $(B \to (A \wedge B \to C)) \to (A \to (B \to (A \wedge B \to C)))$ (I1)
8. $A \to (B \to (A \wedge B \to C))$ (by modus ponens from 4, 7)
9. $(A \to ((B \to (A \wedge B \to C)) \to ((B \to A \wedge B) \to (B \to C)))) \to$
   $((A \to (B \to (A \wedge B \to C))) \to (A \to ((B \to A \wedge B) \to (B \to C))))$ (I2)

10. $(A \to (B \to (A \wedge B \to C))) \to (A \to ((B \to A \wedge B) \to (B \to C)))$ (by modus ponens from 6, 9)

11. $A \to ((B \to A \wedge B) \to (B \to C))$ (by modus ponens from 8, 10)

12. $A \to (B \to A \wedge B)$ (I5)

13. $(A \to ((B \to A \wedge B) \to (B \to C))) \to$
    $((A \to (B \to A \wedge B)) \to (A \to (B \to C)))$ (I2)

14. $(A \to (B \to A \wedge B)) \to (A \to (B \to C))$ (by modus ponens from 11, 13)

15. $A \to (B \to C)$ (by modus ponens from 12, 14)

**Example 4.5** The following sequence shows that $\vdash A \to B \vee C$ and $\vdash A \wedge C \to B$ imply $\vdash A \to B$, for any formulas $A$, $B$ and $C$:

1. $A \to B \vee C$
2. $A \wedge C \to B$
3. $A \to (C \to B)$ (by Example 4.4 from 2)
4. $B \to B$ (by Example 4.2)
5. $(B \to B) \to ((C \to B) \to (B \vee C \to B))$ (I8)
6. $(C \to B) \to (B \vee C \to B)$ (by modus ponens from 4, 5)
7. $((C \to B) \to (B \vee C \to B)) \to (A \to ((C \to B) \to (B \vee C \to B)))$ (I1)
8. $A \to ((C \to B) \to (B \vee C \to B))$ (by modus ponens from 6, 7)
9. $(A \to ((C \to B) \to (B \vee C \to B))) \to$
   $((A \to (C \to B)) \to (A \to (B \vee C \to B)))$ (I2)
10. $(A \to (C \to B)) \to (A \to (B \vee C \to B))$ (by modus ponens from 8, 9)
11. $A \to (B \vee C \to B)$ (by modus ponens from 3, 10)
12. $(A \to (B \vee C \to B)) \to ((A \to B \vee C) \to (A \to B))$ (I2)
13. $(A \to B \vee C) \to (A \to B)$ (by modus ponens from 11, 12)
14. $A \to B$ (by modus ponens from 1, 13)

**Theorem 4.6 (Soundness)**

$$\vdash_i A \Rightarrow \models_i A.$$

*Proof.* By induction on $\vdash_i A$.
Base case: We verify that all axiom schemata are intuitionistically Kripke valid. By Lemma 3.6 if for every intuitionistic Kripke model $\langle X, \leq, \models \rangle$ and every $x \in X$, $x \models A$ implies $x \models B$, then $\models_i A \to B$.

I1. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Suppose $x \models A$, then by Lemma 3.2 for all $y \geq x$, $y \models A$, and so $x \models B \rightarrow A$. Therefore $\models_i A \rightarrow (B \rightarrow A)$.

I2. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Suppose $x \models A \rightarrow (B \rightarrow C)$, $y \geq x$ and $y \models A \rightarrow B$. Then if $z \geq y$ and $z \models A$ then $z \models C$ since $z \models B \rightarrow C$ and $z \models B$. So $y \models A \rightarrow C$, and hence $x \models (A \rightarrow B) \rightarrow (A \rightarrow C)$. Therefore $\models_i (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$.

I3. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Then $x \models A \wedge B \Rightarrow x \models A$ and $x \models B \Rightarrow x \models A$. Therefore $\models_i A \wedge B \rightarrow A$.

I4. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Then $x \models A \wedge B \Rightarrow x \models A$ and $x \models B \Rightarrow x \models B$. Therefore $\models_i A \wedge B \rightarrow B$.

I5. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Suppose $x \models A$, then by Lemma 3.2 for all $y \geq x$, if $y \models B$ then $y \models A \wedge B$, and so $x \models B \rightarrow A \wedge B$. Therefore $\models_i A \rightarrow (B \rightarrow A \wedge B)$.

I6. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Then $x \models A \Rightarrow x \models A$ or $x \models B \Rightarrow x \models A \vee B$. Therefore $\models_i A \rightarrow A \vee B$.

I7. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Then $x \models B \Rightarrow x \models A$ or $x \models B \Rightarrow x \models A \vee B$. Therefore $\models_i B \rightarrow A \vee B$.

I8. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Suppose $x \models A \rightarrow C$, $y \geq x$ and $y \models B \rightarrow C$. Then if $z \geq y$ and $z \models A \vee B$ i.e. $z \models A$ or $z \models B$ then $z \models C$ since $x \models A \rightarrow C$ and $y \models B \rightarrow C$. So $y \models A \vee B \rightarrow C$, and hence $x \models (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$. Therefore $\models_i (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.

I9. Let $\langle X, \leq, \models \rangle$ be an intuitionistic Kripke model, and let $x \in X$. Then $x \not\models \bot$, and so $x \models \bot \Rightarrow x \models A$. Therefore $\models_i \bot \rightarrow A$.

Induction step: We show that the inference rule preserves the validity. Suppose that $\models_i A$ and $\models_i A \rightarrow B$. Then for every intuitionistic Kripke model $\langle X, \leq, \models \rangle$ and every $x \in X$, $x \models A$ and $x \models A \rightarrow B$, and so $x \models B$. Therefore $\models_i B$.

For the Completeness Theorem we need some notions and a few lemmas.

**Definition 4.7** Let $\Gamma$, $\Delta$ be sets of formulas. The pair $(\Gamma, \Delta)$ is consistent if for any finite subset $\{A_1, \ldots, A_m\}$ of $\Gamma$ and $\{B_1, \ldots, B_n\}$ of $\Delta$,

$$\nvdash A_1 \wedge \ldots \wedge A_m \rightarrow B_1 \vee \ldots \vee B_n.$$

Here the conjunction of the empty set of formulas is $\top$ and the disjunction $\bot$. $(\Gamma, \Delta)$ is maximal consistent if $(\Gamma, \Delta)$ is consistent and $\Gamma \cup \Delta = \Phi$.

**Lemma 4.8** Let $\Gamma_0$, $\Delta_0$ be sets of formulas such that $(\Gamma_0, \Delta_0)$ is consistent. Then there exists a maximal consistent pair $(\Gamma, \Delta)$ such that $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$.

*Proof.* Let $C_1, C_2, \ldots$ be an enumeration of all formulas. Define a sequence of pairs $t_0 = (\Gamma_0, \Delta_0), t_1 = (\Gamma_1, \Delta_1), \ldots$ by taking

$$t_{i+1} := \begin{cases} (\Gamma_i, \Delta_i \cup \{C_{i+1}\}) \text{ if } (\Gamma_i, \Delta_i \cup \{C_{i+1}\}) \text{ is consistent} \\ (\Gamma_i \cup \{C_{i+1}\}, \Delta_i) \text{ otherwise.} \end{cases}$$

Let $\Gamma := \bigcup\{\Gamma_i \mid i < \omega\}$, $\Delta := \bigcup\{\Delta_i \mid i < \omega\}$. Notice that $\Gamma \cup \Delta = \Phi$. Let us show that $t_{i+1}$ is consistent whenever $t_i$ is consistent. Indeed, otherwise we could find formulas $A_1, \ldots, A_m \in \Gamma_i$ and $B_1, \ldots, B_n \in \Delta_i$ such that

$$\vdash A_1 \wedge \ldots \wedge A_m \to B_1 \vee \ldots \vee B_n \vee C_{i+1},$$

$$\vdash A_1 \wedge \ldots \wedge A_m \wedge C_{i+1} \to B_1 \vee \ldots \vee B_n.$$

But then, by Example 4.5,

$$\vdash A_1 \wedge \ldots \wedge A_m \to B_1 \vee \ldots \vee B_n,$$

contrary to the consistency of $t_i$. Therefore $(\Gamma, \Delta)$ is consistent.

**Lemma 4.9** Let $\Gamma$, $\Delta$ be sets of formulas such that $(\Gamma, \Phi - \Gamma)$ and $(\Delta, \Phi - \Delta)$ are both maximal consistent. Then

1. $A \wedge B \in \Gamma \Leftrightarrow A \in \Gamma$ and $B \in \Gamma$.
2. $A \vee B \in \Gamma \Leftrightarrow A \in \Gamma$ or $B \in \Gamma$.
3. $A \to B \in \Gamma \Leftrightarrow$ if $\Gamma \subseteq \Delta$ and $A \in \Delta$ then $B \in \Delta$.

*Proof.*

1. ($\Rightarrow$) Suppose that $A \wedge B \in \Gamma$ and $A \notin \Gamma$. However by I3, $\vdash_i A \wedge B \to A$, which is a contradiction, since $(\Gamma, \Phi - \Gamma)$ is consistent. Hence $A \wedge B \in \Gamma \Rightarrow A \in \Gamma$. $A \wedge B \in \Gamma \Rightarrow B \in \Gamma$ is checked analogously with the help of I4. Therefore $A \wedge B \in \Gamma \Rightarrow A \in \Gamma$ and $B \in \Gamma$.

   ($\Leftarrow$) Suppose that $A \in \Gamma$ and $B \in \Gamma$ and $A \wedge B \notin \Gamma$. However by Example 4.2, $\vdash_i A \wedge B \to A \wedge B$, which is a contradiction, since $(\Gamma, \Phi - \Gamma)$ is consistent. Hence $A \in \Gamma$ and $B \in \Gamma \Rightarrow A \wedge B \in \Gamma$.

2. ($\Rightarrow$) Suppose that $A \vee B \in \Gamma$ and $A \notin \Gamma$ and $B \notin \Gamma$. However by Example 4.2, $\vdash_i A \vee B \to A \vee B$, which is a contradiction, since $(\Gamma, \Phi - \Gamma)$ is consistent. Hence $A \vee B \in \Gamma \Rightarrow A \in \Gamma$ or $B \in \Gamma$.

   ($\Leftarrow$) Suppose that $A \in \Gamma$ and $A \vee B \notin \Gamma$. However by I6, $\vdash_i A \to A \vee B$, which is a contradiction, since $(\Gamma, \Phi - \Gamma)$ is consistent. Hence $A \in \Gamma \Rightarrow A \vee B \in \Gamma$. $B \in \Gamma \Rightarrow A \vee B \in \Gamma$ is checked analogously with the help of I7. Therefore $A \in \Gamma$ or $B \in \Gamma \Rightarrow A \vee B \in \Gamma$.

3. (⇒) Suppose that $A \to B \in \Gamma$, $\Gamma \subseteq \Delta$, $A \in \Delta$ and $B \notin \Delta$. Then $A \to B \in \Delta$, $A \in \Delta$ and $B \notin \Delta$. However by Example 4.3, $\vdash_i A \wedge (A \to B) \to B$, which is a contradiction, since $(\Delta, \Phi - \Delta)$ is consistent. Hence $A \to B \in \Gamma \Rightarrow$ if $\Gamma \subseteq \Delta$ and $A \in \Delta$ then $B \in \Delta$.

   (⇐) Suppose that $A \to B \notin \Gamma$. Then $(\Gamma \cup \{A\}, \{B\})$ is consistent, for otherwise we would have formulas $A_1, \ldots, A_n \in \Gamma$ such that $\vdash A_1 \wedge \ldots \wedge A_n \wedge A \to B$ and so, by Example 4.4, $\vdash A_1 \wedge \ldots \wedge A_n \to (A \to B)$, contrary to the consistency of $(\Gamma, \Phi - \Gamma)$. Therefore, by Lemma 4.8, $(\Gamma \cup \{A\}, \{B\})$ can be extended to a maximal consistent pair $(\Delta, \Phi - \Delta)$ such that $\Gamma \subseteq \Delta$, $A \in \Delta$ and $B \notin \Delta$.

**Definition 4.10** $\langle X^*, \subseteq, \models^* \rangle$ is the canonical model if

1. $X^* = \{\Gamma(\subseteq \Phi) \mid (\Gamma, \Phi - \Gamma) \text{ is maximal consistent}\}$,
2. $\models^*$ is the binary relation on $X^*$ and Vars such that $\Gamma \models^* p$ if and only if $p \in \Gamma$.

**Lemma 4.11** Let $\langle X^*, \subseteq, \models^* \rangle$ be the canonical model, let $\Gamma \in X^*$, and let $A$ be a formula. Then

$$\Gamma \models^* A \Leftrightarrow A \in \Gamma.$$

*Proof.* By induction on $A$, using Lemma 4.9.

**Theorem 4.12 (Completeness)**

$$\models_i A \Rightarrow \vdash_i A.$$

*Proof.* Suppose $\not\vdash_i A$. Then $(\emptyset, \{A\})$ is consistent, hence by Lemma 4.8 there is a maximal consistent pair $(\Gamma, \Phi - \Gamma)$ such that $A \notin \Gamma$. By Lemma 4.11 $A$ is not valid in the canonical model. Therefore $\not\models_i A$.

## 4.2 A formal system of the new logic

**Definition 4.13** The Hilbert-type system of the new logic contains the following axiom schemata:

A1. $A \to A$,

A2. $(A \to B) \wedge (B \to C) \to (A \to C)$,

A3. $A \wedge B \to A$,

A4. $A \wedge B \to B$,

A5. $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$,

A6. $A \to A \vee B$,

A7. $B \to A \vee B$,

A8. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$,

A9. $\bot \to A$,

A10. $A \wedge (A \to B) \to B$,

A11. $\neg\neg A \to A$,

and the following inference rules:

$$\frac{A \quad A \to B}{B} \text{ (modus ponens)},$$

$$\frac{A}{B \to A} \text{ (a fortiori)},$$

$$\frac{A \quad B}{A \wedge B} \text{ (adjunction)},$$

$$\frac{A \to C \quad B \to C}{A \vee B \to C} \text{ ($\vee$-elimination)}.$$

A formula $A$ is said to be derivable in this system (notation: $\vdash A$) if there is a derivation of $A$ in this system.

**Theorem 4.14 (Soundness)**

$$\vdash A \Rightarrow \models A.$$

*Proof.* By induction on $\vdash A$.
Base case: We verify that all axiom schemata are Kripke valid. By Lemma 3.12 if for every Kripke model $\langle X, \leq, \models \rangle$ and every $x \in X$, $x \models A$ implies $x \models B$, then $\models A \to B$.

A1. For every Kripke model $\langle X, \leq, \models \rangle$ and every $x \in X$, $x \models A$ implies $x \models A$. Therefore $\models A \to A$.

A2. Let $\langle X, \leq, \models \rangle$ be a Kripke model, and let $x \in X$. Suppose $x \models (A \to B) \wedge (B \to C)$ i.e. $x \models A \to B$ and $x \models B \to C$. Then

$$\text{for all } y \geq x, \text{ if for all } z(y \geq z \geq x)\, z \models A \text{ then for all } z(y \geq z \geq x)\, z \models B,$$

and

$$\text{for all } y \geq x, \text{ if for all } z(y \geq z \geq x)\, z \models B \text{ then for all } z(y \geq z \geq x)\, z \models C.$$

So,

$$\text{for all } y \geq x, \text{ if for all } z(y \geq z \geq x)\, z \models A \text{ then for all } z(y \geq z \geq x)\, z \models C,$$

and hence $x \models A \to C$. Therefore $\models (A \to B) \wedge (B \to C) \to (A \to C)$.

A3. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models A \wedge B \Rightarrow x \models A$ and $x \models B \Rightarrow x \models A$. Therefore $\models A \wedge B \rightarrow A$.

A4. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models A \wedge B \Rightarrow x \models A$ and $x \models B \Rightarrow x \models B$. Therefore $\models A \wedge B \rightarrow B$.

A5. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Suppose $x \models (A \rightarrow B) \wedge (A \rightarrow C)$ i.e. $x \models A \rightarrow B$ and $x \models A \rightarrow C$. Then for all $y \geq x$, if for all $z (y \geq z \geq x)$ $z \models A$ then $y \models B$ and $y \models C$. Hence $x \models A \rightarrow B \wedge C$. Therefore $\models (A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$.

A6. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models A \Rightarrow x \models A$ or $x \models B \Rightarrow x \models A \vee B$. Therefore $\models A \rightarrow A \vee B$.

A7. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models B \Rightarrow x \models A$ or $x \models B \Rightarrow x \models A \vee B$. Therefore $\models B \rightarrow A \vee B$.

A8. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models A \wedge (B \vee C) \Rightarrow x \models A$ and $(x \models B$ or $x \models C) \Rightarrow (x \models A$ and $x \models B)$ or $(x \models A$ and $x \models C) \Rightarrow x \models (A \wedge B) \vee (A \wedge C)$. Therefore $\models A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$.

A9. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \not\models \bot$, and so $x \models \bot \Rightarrow x \models A$. Therefore $\models \bot \rightarrow A$.

A10. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then $x \models A \wedge (A \rightarrow B) \Rightarrow x \models A$ and $x \models A \rightarrow B \Rightarrow x \models B$. Therefore $\models A \wedge (A \rightarrow B) \rightarrow B$.

A11. Let $\langle X, \leq, \models\rangle$ be a Kripke model, and let $x \in X$. Then by Lemma 3.8 $x \models \neg\neg A \Leftrightarrow x \not\models \neg A \Leftrightarrow x \models A$. Therefore $\models \neg\neg A \rightarrow A$.

Induction step: We show that the inference rules preserve the validity.

(modus ponens) Suppose that $\models A$ and $\models A \rightarrow B$. Then for every Kripke model $\langle X, \leq, \models\rangle$ and every $x \in X$, $x \models A$ and $x \models A \rightarrow B$, and so $x \models B$. Therefore $\models B$.

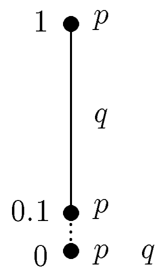
(a fortiori) Suppose that $\models A$. Then for every Kripke model $\langle X, \leq, \models\rangle$ and every $x \in X$, $x \models A$. By Lemma 3.12 we have $\models B \rightarrow A$.

(adjunction) Suppose that $\models A$ and $\models B$. Then for every Kripke model $\langle X, \leq, \models\rangle$ and every $x \in X$, $x \models A$ and $x \models B$, and so $x \models A \wedge B$. Therefore $\models A \wedge B$.

($\vee$-elimination) Suppose that $\models A \rightarrow C$ and $\models B \rightarrow C$. Then by Lemma 3.12 for every Kripke model $\langle X, \leq, \models\rangle$ and every $x \in X$, $x \models A$ implies $x \models C$ and $x \models B$ implies $x \models C$, and so $x \models A \vee B$ implies $x \models C$. Therefore $\models A \vee B \rightarrow C$.

To see a problem on completeness, we will consider the following example.

**Example 4.15** $\not\models (p \to q) \wedge (q \to p) \to (p \vee q \to q)$. Let $\langle Q, \leq, \models \rangle$ be a Kripke model, where $Q$ is the set of rational numbers, $\leq$ is the ordinary order on $Q$, and $p$, $q$ are true at 0, $p$ is true at $1/10^n (n = 0, 1, \ldots)$, and $q$ is true at all the other elements in $Q$.




Clearly, $x \models p \vee q$ for all $x \in Q$. We have $0 \models p \to q$, since for all $x > 0$, there exists a $y (x \geq y \geq 0)$ such that $y \not\models p$. Similarly, we also have $0 \models q \to p$, and so $0 \models (p \to q) \wedge (q \to p)$. We, however, have $0 \not\models p \vee q \to q$ since $1 \not\models q$. Therefore $0 \not\models (p \to q) \wedge (q \to p) \to (p \vee q \to q)$.

A Kripke model in which $(p \to q) \wedge (q \to p) \to (p \vee q \to q)$ is not valid requires infinitely many elements at which $p$ is true and $q$ is not true as the example above. This implies that one set of formulas have to be recognized as different elements in the canonical model, so we cannot construct the canonical model in the same way as that in intuitionistic logic.

# Chapter 5

# An embedding of intuitionistic logic

In this chapter, we will consider some connections between intuitionistic logic and the new logic. As we see in the preceding chapters, neither of them contains the other in Kripke validity. We can, however, try to embed intuitionistic logic into the new logic in the sense that there is a translation $Tr$ such that for every formula $A$, $A$ is intuitionistically Kripke valid if and only if $Tr(A)$ is Kripke valid in the new logic. The following is one of such translations, and makes it possible to interpret logical connectives in intuitionistic logic in terms of those in the new logic.

**Theorem 5.1** Let $Tr$ be a mapping from $\Phi$ to $\Phi$ defined as follow

$$Tr(p) := \top \to p, \quad \text{for all } p \in \text{Vars},$$

$$Tr(\bot) := \bot,$$

$$Tr(A \wedge B) := Tr(A) \wedge Tr(B),$$

$$Tr(A \vee B) := Tr(A) \vee Tr(B),$$

$$Tr(A \to B) := \top \to (\neg Tr(A) \vee Tr(B)).$$

Then for every formula $A$,

$$\models_i A \Leftrightarrow \models Tr(A).$$

*Proof.* ($\Rightarrow$) Suppose $\not\models Tr(A)$. Then there is a Kripke model $\langle X, \leq, \models\rangle$ in which $Tr(A)$ is not valid. Let $\models'$ be a relation on $X$ and Vars such that

$$x \models' p \quad \text{iff} \quad x \models \top \to p.$$

Then $\langle X, \leq, \models'\rangle$ is an intuitionistic Kripke model, since

$$\begin{aligned} x \models' p \text{ and } x \leq y \quad &\Rightarrow \quad x \models \top \to p \text{ and } x \leq y \\ &\Rightarrow \quad \text{for all } z \geq x,\ z \models p \text{ and } x \leq y \\ &\Rightarrow \quad \text{for all } z \geq y,\ z \models p \\ &\Rightarrow \quad y \models \top \to p \\ &\Rightarrow \quad y \models' p. \end{aligned}$$

We show that for every $x \in X$ and every formula $B$,

$$x \models' B \Leftrightarrow x \models Tr(B)$$

by induction on $B$.
If $B$ is an atomic formula then it is clear.
If $B \equiv C \wedge D$ then

$$\begin{aligned} x \models' B &\Leftrightarrow x \models' C \text{ and } x \models' D \\ &\Leftrightarrow x \models Tr(C) \text{ and } x \models Tr(D) \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow x \models Tr(B). \end{aligned}$$

The case $B \equiv C \vee D$ is considered in the same way.
If $B \equiv C \rightarrow D$ then

$$\begin{aligned} x \models' B &\Leftrightarrow \text{for all } y \geq x,\ y \not\models' C \text{ or } y \models' D \\ &\Leftrightarrow \text{for all } y \geq x,\ y \not\models Tr(C) \text{ or } y \models Tr(D) \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \text{for all } y \geq x,\ y \models \neg Tr(C) \text{ or } y \models Tr(D) \quad \text{(by Lemma 3.8)} \\ &\Leftrightarrow \text{for all } y \geq x,\ y \models \neg Tr(C) \vee Tr(D) \\ &\Leftrightarrow x \models Tr(B). \end{aligned}$$

Thus for every $x \in X$ and every formula $B$, $x \models' B \Leftrightarrow x \models Tr(B)$. Now we have $x \not\models Tr(A)$ for some $x \in X$ and so $x \not\models' A$ for some $x \in X$. Hence $A$ is not valid in $\langle X, \leq, \models' \rangle$ and therefore $\not\models_i A$.
($\Leftarrow$) Suppose $\not\models_i A$. Then there is an intuitionistic Kripke model $\langle X, \leq, \models \rangle$ in which $A$ is not valid. Let $\models''$ be a relation on $X$ and Vars such that

$$x \models'' p \quad \text{iff} \quad x \models p.$$

Then $\langle X, \leq, \models'' \rangle$ can be treated as a Kripke model. We show that for every $x \in X$ and every formula $B$,

$$x \models B \Leftrightarrow x \models'' Tr(B)$$

by induction on $B$.
For every propositional variable $p$, we have

$$\begin{aligned} x \models p &\Leftrightarrow \text{for all } y \geq x,\ y \models p \\ &\Leftrightarrow \text{for all } y \geq x,\ y \models'' p \\ &\Leftrightarrow x \models'' \top \rightarrow p \\ &\Leftrightarrow x \models'' Tr(p). \end{aligned}$$

The case $B \equiv \bot$ is clear.
If $B \equiv C \wedge D$ then

$$\begin{aligned} x \models B &\Leftrightarrow x \models C \text{ and } x \models D \\ &\Leftrightarrow x \models'' Tr(C) \text{ and } x \models'' Tr(D) \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow x \models'' Tr(B). \end{aligned}$$

The case $B \equiv C \vee D$ is considered in the same way.
If $B \equiv C \to D$ then

$$
\begin{aligned}
x \models B \quad &\Leftrightarrow \quad \text{for all } y \geq x,\ y \not\models C \text{ or } y \models D \\
&\Leftrightarrow \quad \text{for all } y \geq x,\ y \not\models'' Tr(C) \text{ or } y \models'' Tr(D) \quad \text{(by induction hypothesis)} \\
&\Leftrightarrow \quad \text{for all } y \geq x,\ y \models'' \neg Tr(C) \text{ or } y \models'' Tr(D) \quad \text{(by Lemma 3.8)} \\
&\Leftrightarrow \quad \text{for all } y \geq x,\ y \models'' \neg Tr(C) \vee Tr(D) \\
&\Leftrightarrow \quad x \models'' Tr(B).
\end{aligned}
$$

Thus for every $x \in X$ and every formula $B$, $x \models B \Leftrightarrow x \models'' Tr(B)$. Now we have $x \not\models A$ for some $x \in X$ and so $x \not\models'' Tr(A)$ for some $x \in X$. Hence $Tr(A)$ is not valid in $\langle X, \leq, \models'' \rangle$ and therefore $\not\models Tr(A)$.

# Chapter 6

# Concluding remarks

In this paper, we proposed new Kripke semantics in which implication is interpreted to express temporality, and provided a formal system which is sound for the semantics, while completeness remains a problem.

We also showed that intuitionistic logic can be embedded into the logic corresponding to the new semantics. Thus, in that logic it is certainly possible to express further details than in intuitionistic logic.

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