Title	二階述語論理の表現能力に関するゲーム論的アプローチ
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Citation	
Issue Date	1999-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/1245
Rights	
Description	Supervisor:小野 寛晰, 情報科学研究科, 修士

A Gametheoretic Approach to Second-Order Definability

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February 15, 1999

Keywords: definability, descriptive complexity, Ehrenfeucht game, second-order logic.

Our interest in this paper is to study the expressive power of second-order logic in finite structures, using the methodology of *finite model theory*.

1 Background

Model theory or the theory of models, first named by Tarski in 1954, is the part of the semantics of formalized languages that is concerned with the relationships between syntactic constructions of axiom systems and (mainly algebraic) properties of mathematical structures ("models"). In classical model theory, the expressive power of first-order language has been studied extensively already, for it obeys some fundamental principles such as the compactness theorem, which says that if each finite subset of a set Σ of sentences has a model then the whole set Σ has a model. Another typical classical result is Löwenheim's theorem: if a sentence has an infinite model, then it has a countable model. These results help us to observe some limitations of the expressive power of first-order logic; Löwenheim's theorem shows that no consistent set of sentences can imply that a model is uncountable, and the compactness theorem has been used to show that many mathematical properties cannot be expressed by a set of first-order sentences – for instance, there is no set of sentences whose models are precisely all the finite models. These two theorems we have stated are proved using classical methods of model constructions, which rest essentially upon the realm of infinite structures.

However, principal theorems of first-order logic fail and important methods become useless when we restrict ourselves to finite structures. The first landmark is Trakhtenbrot's Theorem (1950) which implies that first-order logic, when restricted to the finite, does not admit a complete proof calculus. Later, for the last twenty years we really come to ask questions of a modeltheoretic flavor with the restriction to finite structures, and it turned out that the questions are deeply connected to computational aspects, as the proof of Trakhtenbrot's theorem is based on the undecidability of the halting problem for Turing machines.

2 Descriptive complexity theory

The computational complexity of a query is the amount of resources, such as time or space, required by a Turing machine that answers the query. The *descriptive complexity* of a query is the complexity of describing the query in some logical formalism over finite structures. One of the fruitful results of finite model theory is the discovery of close relationships between computational and descriptive complexity, as shown below.

Theorem 1 [2] Let K be a class of ordered finite structures.

$$\begin{array}{lll} K \in \mathrm{LOGSPACE} & \text{iff} & K \text{ is axiomatizable in FO(DTC)} \\ K \in \mathrm{NLOGSPACE} & \text{iff} & K \text{ is axiomatizable in FO(TC)} \\ K \in \mathrm{PTIME} & \text{iff} & K \text{ is axiomatizable in FO(IFP)} \\ K \in \mathrm{NPTIME} & \text{iff} & K \text{ is axiomatizable in } \Sigma_1^1 \\ K \in \mathrm{PSPACE} & \text{iff} & K \text{ is axiomatizable in FO(PFP)} \end{array}$$

(Σ_1^1 denotes the fragment of second-order logic consisting of the sentences of the form $\exists X_1 \cdots \exists X_n \psi$, where ψ is first-order) □

The logics listed on the right sides of the equivalences except Σ_1^1 are called fixed-point logics, which have been introduced to strengthen the expressive power of first-order logic by adding the operations that represent recursive procedures. Theorem 1 provides the logical characterizations of complexity classes, therefore we are in a position to obtain logical analogies of major problems in complexity theory. For example, the P $\neq$ NP-problem now amounts to the question whether two logics FO(IFP) and Σ_1^1 have the same expressive power in finite structures or not.

3 Ehrenfeucht game

From the point of view indicated in Theorem 1, one will be convinced that it is of great use to inquire the expressive powers of the logics in finite structures, for it resolves itself to check up the relations among complexity classes. However, as already stated above, many important methods to serve this purpose in classical model theory become useless when restricting oneself to finite structures. Still, the gametheoretic method of Ehrenfeucht survives, which in fact is almost the only technique available in finite model theory. The notion of the Ehrenfeucht game ([3]) provides a simple characterization of the definability in first-order logic, as described in the following.

The *Ehrenfeucht game* $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$ is played by two players called *spoiler* and *duplicator*. In a play of the game, the players take turns at placing a pebble on structures $\mathcal{A}$ and

$\mathcal{B}$. First the spoiler choose one of the two structures $\mathcal{A}$ or $\mathcal{B}$, and places a pebble on one of its elements. The duplicator responds by placing a pebble on an element of the other structure. The spoiler chooses again one of the two structures and the game continues this way until m pebbles have been placed on each structure. The duplicator *wins* the game if the map caused by the pebbles constitutes a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Now the following are equivalent:

(i) The duplicator has a winning strategy in $\mathrm{G}_m(\mathcal{A}, \mathcal{B})$. (ii) $\mathcal{A} \equiv_m \mathcal{B}$.

Here, $\mathcal{A} \equiv_m \mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$. This result provides a method of examining the axiomatizability of a class K of structures in first-order logic, making use of the equivalence of the following statements:

(i) K is not axiomatizable in first-order logic.

(ii) For each $m \in N$ there are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$ and $\mathcal{A} \equiv_m \mathcal{B}$.

As an application of this method, we can show that the class EVEN, the class of finite structures of even cardinality, is not axiomatizable in first-order logic ([2]). In fact, for each $m \in N$ it is enough to choose structures $\mathcal{A}_0 \in$ EVEN and $\mathcal{B}_0 \notin$ EVEN such that either of them contains at least m elements, for one easily verifies that the duplicator has a winning strategy in $\mathrm{G}_m(\mathcal{A}_0, \mathcal{B}_0)$.

It has been one of the major issue in this area to find out the variant of the Ehrenfeucht method that characterizes a logic other than first-order, and to investigate the applications of it. For example, the straightforward generalization for MSO (monadic second-order logic), that is, second-order logic in which only unary relation variables are allowed, has been known already ([2]). Now in the MSO game, each player is allowed to make his choice of a *set of elements* in addition to his choice of an *element* in the structures.

4 Game for second-order logic

In the present paper we give a new application of the method for MSO. Here we show that, even in MSO, the class of finite structures of even cardinality is not expressible, demonstrating that the duplicator has a winning strategy in the corresponding MSO game.

Another result is that the generalization of the Ehrenfeucht method for second-order logic is also available, which is a simple generalization of the MSO game; this time, for any $k \in N$, the spoiler and the duplicator are allowed to make a choice of a *k-ary relation* over the structures.

Further, this generalization for second-order logic enables us to introduce a method for the fragment Σ^1_1 of second-order logic (cf. Theorem 1). We present it in two forms that appear slightly different from each other, for one of the formulations looks easier for the duplicator to win than the other. According to one of its formulations (easier for the duplicator), we can show that a class K of finite structures is not axiomatizable in Σ^1_1 iff the duplicator has a winning strategy for the following game:

- ▷ The spoiler selects $k \in N$.
- ▷ The spoiler selects $m \in N$.
- ▷ The duplicator selects $\mathcal{A}_0 \in K$.
- ▷ The spoiler sets a k-ary relation P_0 on $\mathcal{A}_0$.
- ▷ The duplicator selects $\mathcal{B}_0 \notin K$.
- ▷ The duplicator sets a k-ary relation Q_0 on $\mathcal{B}_0$.
- ▷ The spoiler and the duplicator play $\mathrm{G}_m((\mathcal{A}_0, P_0), (\mathcal{B}_0, Q_0))$.

In view of descriptive complexity theory, the axiomatizability in Σ^1_1 is the same as the computability in NPTIME (cf. Theorem 1). Therefore the question as to whether $\Sigma^1_1 = \Pi^1_1$ is equivalent to the famous problem of whether NPTIME = co-NPTIME. In other words, we have NPTIME $\neq$ co-NPTIME if we find a property which is in Π^1_1 and not in Σ^1_1. Now the statement "not in Σ^1_1" can be examined by the game above. In fact, it is enough to construct a winning strategy for the duplicator in the corresponding Σ^1_1 game. However, this may be difficult if we leave the game as it is, for it is complicated enough for the duplicator to find out correct responses to the spoiler's choices in this game. The difficulty is that the spoiler is allowed to set an *arbitrary* k-ary relation on a structure, so that it is possible for him to choose a random k-ary relation. Therefore we need to simplify the Σ^1_1 game by restricting his choice of a k-ary relation, such as the arity, the figure of a relation, and so on. This problem is left as a future work.

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