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Some results on bimodal logics

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Chapter 1

Introduction

Many reasonings which appear in daily thought are often influenced by situations, states, the passage of time and so on. The introduction of modal logics to take the situations, states and the passage of time into consideration is very useful. **K** which is the smallest normal modal logic, **KT** and **S4** which characterize temporal logic , **S5** which characterizes epistemic logic and so on are well-known monomodal logics . The various logical properties of them have been found out already. There are many syntactical results on them. Many monomodal logics have been investigated well since the early time of 20th century. On the other hand, in many multimodal logics with several modalities, however, even the most elementary questions concerning completeness, decidability and so on haven't been unsolved. From the point of view of application of modal logics, modal logics with one modality are sometimes not sufficient, and hence introduction of several modalities will be necessary in many situations. For example, epistemic logics with notion of tense will be able to express knowledge in the past and the future. Epistemic logics and temporal logics themselves can be also regarded as a kind of bimodal logics. Moreover, we assume in general that modalities in temporal logics have certain relations among them. On the other hand, for independently axiomatizable modal logics, the notion of the fusion of them were firstly introduced by S. Thomason 1980. Recently, independently axiomatizable bimodal logics as special bimodal logics are investigated semantically by M. Kracht and F. Wolter [3].

This paper presents a study of bimodal logics, that is, modal logics with two modalities, and will discuss these by both syntactic and semantical method. The cut-elimination properties of the fusions will be discussed in the syntactical studies. In the semantical approaches, dependently axiomatizable bimodal logics will be mainly discussed. We will consider the fusion of well-known logics (**K**, **KT**, **S4** and **S5**) and see several logical properties like Kripke completeness, finite model property and so on for bimodal logics. In Chapter 2, several basic monomodal logics and their sequent systems are introduced in the preliminaries, and several notations and definitions which will be used in later section are given. In Chapter 3, we will describe some sequent systems corresponding to fusions and derive cut-elimination properties, subformula properties and so on of them. The cut-elimination theorem for some systems, first, will be proved in usual way [4]. On the other hand, some systems for fusions are shown not to enjoy cut-elimination property. For systems which lack cut-elimination property, however, we will show that the systems have subformula property, by extending Takano's method [11] for $\mathbf{S5}^*$. Then as a corollary, the decidability for the systems will be derived in the same way as that by Gentzen [4]. The last topic in Chapter 3 is the Craig's interpolation theorem for these systems. To prove the theorem, we will use Maehara's method [13]. The method works well even for the systems lacking cut-elimination property, since they have the subformula property. As contrasted with the independently axiomatizable bimodal logics, for instance, a bimodal logic with the axiom $\Box p \supset \blacksquare p$ is a dependently axiomatizable bimodal logic. For the dependently axiomatizable bimodal logics, however, it is difficult for us to find out the systems with cut-elimination property. Next we will study semantics for bimodal logics. As Kripke completeness and finite model property of fusions have been extensively studied by M. Kracht and F. Wolter [3], here we will study these properties for dependently axiomatizable bimodal logics. It is quite hard to develop a general semantical study of dependently axiomatizable bimodal logics at this moment. So, as a steppingstone to future study of this topics, we will restrict ourselves to the study of Kripke completeness and the finite model property of bimodal logics which are obtained from a fusion of two monomodal logics by adding an axiom of the form $\sigma p \supset \tau p$, where σ and τ are sequences of two box operators.

Chapter 2

Preliminaries

In this chapter, several notations and definitions for some monomodal logics are given. The language $\mathcal{L}_\Box$ of propositional monomodal logic consists of

- propositional variables: p, q, r, $\cdots$
- logical symbols: $\wedge$, $\vee$, $\supset$, $\neg$, $\Box$.

Formulas, denoted by A, B, C, $\cdots$, are constructed in the usual way from propositional variables and logical symbols. In particular, $\Box A$ is a formula when A is a formula. We may append indexes to propositional variables and formulas. Greek capital letters Γ, Δ, Π, Σ, Θ and Ξ denote sequences of formulas. $\Box\Gamma$ denotes $\Box A_1, \Box A_2, \cdots, \Box A_n$, when Γ is $A_1, A_2, \cdots, A_n$.

Definition 2.1 (modal logic) *A set* $\mathbf{L}$ *of formulas in* $\mathcal{L}_\Box$ *is a* modal logic*, if the following conditions are satisfied:*

- *all tautologies belong to* $\mathbf{L}$*,*
- *if* $A, A \supset B \in \mathbf{L}$*, then* $B \in \mathbf{L}$*,*
- *if* $A \in \mathbf{L}$*, then* $\Box A \in \mathbf{L}$*,*
- *if* $A \in \mathbf{L}$*, then any substitution instance of* A *belongs to* $\mathbf{L}$*.*

Let $\mathbf{L}$ be a modal logic (of $\mathcal{L}_\Box$), and Q be a set of formulas (of $\mathcal{L}_\Box$).Then the least modal logic containing the set $\mathbf{L} \cup Q$ is denoted by $\mathbf{L} \bigoplus Q$. $\mathbf{K}$ denotes the least modal logic containing the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$. Any modal logic with the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is called a normal modal logic. The following modal logics are well-known :

$\mathbf{KT} = \mathbf{K} \bigoplus \{\Box p \supset p\}$

$\mathbf{K4} = \mathbf{K} \bigoplus \{\Box p \supset \Box\Box p\}$

$\mathbf{S4} = \mathbf{K} \bigoplus \{\Box p \supset p, \Box p \supset \Box\Box p\}$

$\mathbf{S5} = \mathbf{K} \bigoplus \{\Box p \supset p, \neg\Box\neg p \supset \Box\neg\Box\neg p\}$
We will discuss the combinations of these basic monomodal logics as fusions.

2.1 Sequent calculus LK

As a formalization for modal logics, we will adopt sequent calculus. It based on the system **LK** introduced by G.Gentzen. Any expression of the form $\Gamma \to \Delta$ is called a *sequent*. Here, Γ and Δ are called the *antecedent* and the *succedent* of the sequent, respectively.

An *inference* is expressed by the form

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_2 \quad S_3}{S} \quad ,$$

where S_1, S_2, S_3 and S are sequents. S_1, S_2 and S_3 are called the *upper sequents* and S is called the *lower sequent* of the inference. In particular, S_2 (S_3) is called the *left (right) upper sequent* of the inference.
The sequent system **LK** for the classical logic has the following initial sequents and inferences.

- Initial sequents:

the sequents of the form $A \to A$,

- Structural rules:

(weakening rule)

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} (w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A} (\to w)$$

(contraction rule)

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} (c \to) \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} (\to c)$$

(exchange rule)

$$\frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta} (e \to) \qquad \frac{\Gamma \to \Delta, A, B, \Sigma}{\Gamma \to \Delta, B, A, \Sigma} (\to e)$$

(cut rule)

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma} (cut)$$

- Logical rules:

$$\frac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge \to) \qquad \frac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge \to)$$

$$\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\ (\to \wedge) \qquad \frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B}\ (\to \vee) \qquad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B}\ (\to \vee)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Pi \to \Sigma}{A \supset B, \Gamma, \Pi \to \Delta, \Sigma}\ (\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ (\to\supset)$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}\ (\neg \to) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}\ (\to \neg)\ .$$

Weakening, contraction and exchange rules are called *weak inferences*; *w.i.* for short. The formula A in cut rule is called the *cut formula* of the cut. In the logical rules, $A \wedge B$, $A \vee B$, $A \supset B$ and $\neg A$ which appear in the lower sequent are called *principal* formulas of the rules.

Definition 2.2 (proof and end-sequent) *Proof and the end-sequent are defined inductively as follows:*

- *Initial sequent is proof, and end-sequent of the proof is itself.*
- *Let P_1 and P_2 be proofs with the end-sequents S_1 and S_2, respectively. If*

$$\frac{S_1}{S} \ \ or\ \frac{S_1 \quad S_2}{S}$$

is one of the inferences in **LK***, then*

$$\frac{P_1}{S} \ \ or\ \frac{P_1 \quad P_2}{S}$$

is proof, and the end-sequent is S.

Definition 2.3 (thread) *A sequence of sequents in a proof is called a* thread *of the proof if the following conditions are satisfied:*

- *the sequence begins with an initial sequent and ends with the end-sequent,*
- *every sequent in the sequence except the last is an upper sequent of an inference, and is followed immediately by the lower sequent of this inference.*

These notions and examples of the proofs are referred in [4], [13].

2.2 Mix rule

As an instrument to eliminate cut rules, we introduce the mix rule:

$$\frac{\Gamma \rightarrow \Delta \quad \Pi \rightarrow \Sigma}{\Gamma, \Pi_A \rightarrow \Delta_A, \Sigma}\ (A) \quad ,$$

where $A \in \Pi \cap \Delta$, and Π_A and Δ_A denote the sequences obtained from Π and Δ by deleting all occurrences of the formula A in them, respectively. The formula A in the above rule is called the *mix formula* of this mix . By means of mix, cut can be represented as follows:

$$\cfrac{\cfrac{\Gamma \rightarrow \Delta \quad \Pi \rightarrow \Sigma}{\Gamma, \Pi_A \rightarrow \Delta_A, \Sigma}\ (A)}{\Gamma, \Pi \rightarrow \Delta, \Sigma}\ (w.i.) \quad .$$

To the contrary, by means of cut, mix can be represented as follows:

$$\cfrac{\cfrac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta_A, A}\ (w.i.) \quad \cfrac{\Pi \rightarrow \Sigma}{A, \Pi_A \rightarrow \Sigma}\ (w.i.)}{\Gamma, \Pi_A \rightarrow \Delta_A, \Sigma}\ (cut) \quad .$$

In this sense, the mix rule and the cut rule are equivalent. So cut- elimination theorem can be proved by mix-elimination. The outline of the proof of mix-elimination is as follows:

1) concentrate to one of the uppermost mixes,

2) eliminate the mix by double induction on the degree and the rank which refer to the following definitions.

Definition 2.4 (degree) *The degree of a formula A, denoted by $deg(A)$, is the number of logical symbols which occur in A.*

Definition 2.5 (rank) *Let P be a proof which contains a mix rule only as the last inference. The* left*(*right*)* rank *of P, denoted by $\rho_l(\rho_r)$, is the max number of consecutive sequents which contain mix formula in the succedent(antecedent), counting upward from the left(right) upper sequent of the mix. Then $\rho = \rho_l + \rho_r$ is the* rank *of P. (Since $\rho_l \geq 1$ and $\rho_r \geq 1$, $\rho \geq 2$.)*

The cut-elimination theorem of **LK** is proved in detail by [4], [13].

2.3 Monomodal systems

In this section, we will describe the system $\mathbf{K}^*$, $\mathbf{KT}^*$, $\mathbf{S4}^*$ and $\mathbf{S5}^*$. First, $\mathbf{K}^*$ is obtained from the system **LK** by adding the following inference rule:

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box)\quad .$$

We can obtain various modal systems from **LK** by adding some inference rules. $\mathbf{KT}^*$ is the system obtained from **LK** by adding the following inference rules:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to) \qquad \frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\to \Box)\quad .$$

It is easy to see that, for any formula C, C is in **KT** if and only if $\to C$ is provable in $\mathbf{KT}^*$. In this sense, **KT** and $\mathbf{KT}^*$ are equivalent. $\mathbf{S4}^*$ is the system obtained from **LK** by adding the following inference rules:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\ (\to \Box)\quad .$$

Also, **S4** and $\mathbf{S4}^*$ are shown to be equivalent. $\mathbf{S5}^*$ is the system obtained from **LK** by adding the following inference rule:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to) \qquad \frac{\Box\Gamma \to \Box\Delta, A}{\Box\Gamma \to \Box\Delta, \Box A}\ (\to \Box)\quad .$$

S5 and $\mathbf{S5}^*$ are equivalent. See e.g. [8] [9], for the details. It is known that $\mathbf{K}^*$, $\mathbf{KT}^*$ and $\mathbf{S4}^*$ enjoy cut- elimination property, but $\mathbf{S5}^*$ lacks it. The cut-elimination theorem of the system $\mathbf{S4}^*$ are shown by M. Ohnishi and K. Matsumoto [6]. The cut-free systems for **S5** are given by G. E. Mints [5], M. Sato [10] and so on, but they are complicated. So, the decidability and Craig's interpolation theorem of $\mathbf{K}^*$, $\mathbf{KT}^*$ and $\mathbf{S4}^*$ are obtained from cut-elimination property by using the standard method. As for $\mathbf{S5}^*$, the subformula property has been shown by Takano [11]. Hence, the decidability and Craig's interpolation theorem follows from this.

Chapter 3

Syntactic results on bimodal logics

M. Kracht and F. Wolter developed a semantical study of fusions of independently axiomatizable modal logics [3]. In this chapter, we will introduce sequent systems of fusions of some basic monomodal logics and study logical properties like the decidability and Craig's interpolation theorem by using these systems. To show them, we first discuss cut-elimination property and subformula property for these systems. Since it is shown that the system for **S5*** lacks cut-elimination property, any system of fusions, one of whose component is **S5***, lack cut-elimination property. We will show, however, that Takano's method works well also for these systems, and hence we can get subformula property of them [11] [12].

3.1 Bimodal logics

The language $\mathcal{L}_{\Box\blacksquare}$ of propositional bimodal logic has

- propositional variables: p, q, r, $\cdots$
- logical symbols: $\wedge$, $\vee$, $\supset$, $\neg$, $\Box$, $\blacksquare$.

Formulas, denoted by A, B, C, $\cdots$, are constructed in the usual way from propositional variables and logical symbols. In particular, both $\Box A$ and $\blacksquare A$ are formulas when A is a formula. We may append indexes to the propositional variables and the formulas. Greek capital letter Γ, Δ, Π, Σ, Θ, Ξ and Υ denote sequences of formulas. $\Box\Gamma$ and $\blacksquare\Gamma$ denote $\Box A_1$, $\Box A_2$, $\cdots$, $\Box A_n$ and $\blacksquare A_1$, $\blacksquare A_2$, $\cdots$, $\blacksquare A_n$ respectively, when Γ is A_1, A_2, $\cdots$, A_n. We may also append indexes to the Greek letters. $Sub(A)$ denote the set of all subformulas of a formula A. A *bimodal logic* $\mathbf{L}$ of $\mathcal{L}_{\Box\blacksquare}$ is defined by adding the following condition to the definition of monomodal logic $\mathbf{L}$:

- *if* $A \in \mathbf{L}$, *then* $\blacksquare A \in \mathbf{L}$.

Definition 3.1 (fusion) *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be monomodal logics in* $\mathcal{L}_\Box$ *and* $\mathcal{L}_\blacksquare$*, respectively.The* fusion *of* $\mathbf{M}$ *and* $\mathbf{N}$*, denoted by* $\mathbf{M} \otimes \mathbf{N}$*, is the least bimodal logic in* $\mathcal{L}_{\Box\blacksquare}$ *containing both* $\mathbf{M}$ *and* $\mathbf{N}$*.*

When we want to specify the modality in a logic $\mathbf{M}$, we will attach the modality to $\mathbf{M}$. For instance, $\mathbf{M}_\Box$ denotes a modal logic in $\mathcal{L}_\Box$.

For $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*, \mathbf{S5}^*\}$, we will introduce sequent systems of the form $\mathbf{M}^* \otimes \mathbf{N}^*$. The systems $\mathbf{M}^* \otimes \mathbf{N}^*$ is obtained from $\mathbf{M}^*$ and $\mathbf{N}^*$, simply by combining their inferences. Of course, it is necessary to distinguish one modality from another. For example, $\mathbf{S4}^* \otimes \mathbf{S5}^*$ is defined as follows. Let $\blacksquare$ and $\Box$ be the modalities for $\mathbf{S4}^*$ and $\mathbf{S5}^*$, respectively. The system $\mathbf{S4}^* \otimes \mathbf{S5}^*$ is obtained from $\mathbf{LK}$ by adding the following inference rules:

$$\frac{A, \Gamma \to \Delta}{\blacksquare A, \Gamma \to \Delta}\,(\blacksquare \to) \qquad \frac{\blacksquare\Gamma \to A}{\blacksquare\Gamma \to \blacksquare A}\,(\to \blacksquare)$$

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,(\Box \to) \qquad \frac{\Box\Gamma \to \Box\Delta, A}{\Box\Gamma \to \Box\Delta, \Box A}\,(\to \Box)\ .$$

It is easy to see the following.

Lemma 3.2 *For any formula* C*,* $\to C$ *is provable in* $\mathbf{M}^* \otimes \mathbf{N}^*$ *if and only if* C *is in* $\mathbf{M} \otimes \mathbf{N}$*.*

Without any difficulty, we can show the following.

Theorem 3.3 (cut-elimination theorem)
Let $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*\}$*. Then every proof in* $\mathbf{M}^* \otimes \mathbf{N}^*$ *can be transformed, without changing the end-sequent, into cut-free one.*

Proof. We first replaced all cuts by mixes, and can show this theorem by double induction on the degree and the rank as usual. ∎

Corollary 3.4 (subformula property)
Let $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*\}$*. Then all formulas which construct cut-free proof in* $\mathbf{M}^* \otimes \mathbf{N}^*$ *consist of the subformulas of formulas which occur in the lowest sequent.*

Proof. For any inferences rule I except cut rule, the upper sequents of I consist of the subformulas of formulas in the lower sequent of I. ∎

3.2 $\mathbf{S5}^*$ lacks cut-elimination property

Next we will discuss the cut-elimination property of sequent systems for fusions, at least, one of whose component is $\mathbf{S5}^*$. We first note that $\mathbf{S5}^*$ lacks the cut-elimination property, while each of $\mathbf{K}^*$, $\mathbf{KT}^*$ and $\mathbf{S4}^*$ has it.

Lemma 3.5 $\mathbf{S5}^*$ *lacks cut-elimination property.*

Proof. This example was noticed first by M. Ohnishi and K. Matsumoto [7]. The sequent $p \to \Box\neg\Box\neg p$ is provable in $\mathbf{S5}^*$. In fact, the following proof is a proof of $p \to \Box\neg\Box\neg p$ in $\mathbf{S5}^*$:

$$\dfrac{\dfrac{\dfrac{\Box\neg p \to \Box\neg p}{\to \neg\Box\neg p, \Box\neg p}}{\to \Box\neg\Box\neg p, \Box\neg p} \qquad \dfrac{\dfrac{p \to p}{\neg p, p \to}}{\Box\neg p, p \to}}{p \to \Box\neg\Box\neg p}\ (\text{cut})$$

Next, we will show that $p \to \Box\neg\Box\neg p$ is not provable in $\mathbf{S5}^*$. Suppose otherwise. Then it is easy to see that the lowest inference of the proof must be either weakening rule or contraction rule.

Case 1: The lowest inference is weakening rule.

In this case, the upper sequent of the inference rule is $p \to$ or $\to \Box\neg\Box\neg p$. But clearly both sequents are not provable in $\mathbf{S5}^*$.

Case 2: The lowest inference is contraction rule.

In this case, the upper sequent of the inference rule is $p, p \to \Box\neg\Box\neg p$ or $p \to \Box\neg\Box\neg p, \Box\neg\Box\neg p$. The inference rule which infer one of these sequents is weakening rule or contraction rule. If the inference rule is weakening rule, any upper sequent of the inference is a sequent which is former one or not provable in $\mathbf{S5}^*$. If the inference rule is contraction rule, an upper sequent of the inference is $p, p, p \to \Box\neg\Box\neg p$; $p, p \to \Box\neg\Box\neg p, \Box\neg\Box\neg p$ or $p \to \Box\neg\Box\neg p, \Box\neg\Box\neg p, \Box\neg\Box\neg p$. Further, the inference rule which one of these sequents is also weakening rule or contraction rule. If the inference rule is weakening rule, the upper sequent of the inference is the sequent which is previous one or not provable in $\mathbf{S5}^*$. So, only possible sequents in the proof are sequents of the form $p, \cdots, p \to \Box\neg\Box\neg p, \cdots, \Box\neg\Box\neg p$. Thus, we can never get an initial sequent. ∎

3.3 Subformula property by Takano's method

We have seen that $\mathbf{S5}^*$ lacks cut-elimination property. On the other hand, Takano showed the following.

Theorem 3.6 *Every proof in* $\mathbf{S5}^*$ *can be transformed, without changing the end-sequent, into the proof which has subformula property.*

For example, all formulas which are occurred in the above proof of $p \to \Box\neg\Box\neg p$ are actually in $Sub(\Box\neg\Box\neg q)$. Since $\mathbf{S5}^*$ lacks cut-elimination property, any of $\mathbf{K}^* \otimes \mathbf{S5}^*$, $\mathbf{KT}^* \otimes \mathbf{S5}^*$, $\mathbf{S4}^* \otimes \mathbf{S5}^*$ and $\mathbf{S5}^* \otimes \mathbf{S5}^*$ lacks it. We will show the subformula property for fusions by extending Takano's method for $\mathbf{S5}^*$ into that for fusions. This section presents syntactical approach for the systems corresponding to fusions. In this section, we will show that the cuts in proof can be restrict the cuts with subformula property by extending Takano's method for $\mathbf{S5}^*$ into that for the systems corresponding to fusion (Theorem 3.9 and Corollary 3.20).

Definition 3.7 (acceptable cut) *A cut*

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma}$$

is acceptable, *if the cut formula forms the subformula of a formula which occurs in the lower sequent of the cut, namely* $A \in Sub(\Gamma, \Delta, \Pi, \Sigma)$. *The cut which is not acceptable is called an unacceptable cut.*

Definition 3.8 (suitable proof) *A proof is* suitable, *if every cut applied in it is acceptable.*

Theorem 3.9 *Every proof in* $\mathbf{K}^* \otimes \mathbf{S5}^*$, $\mathbf{KT}^* \otimes \mathbf{S5}^*$, $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *or* $\mathbf{S5}^* \otimes \mathbf{S5}^*$ *can be transformed, without changing the end-sequent, into suitable one.*

The system in which every proof can be transformed, without changing the end-sequent, into suitable one has subformula property, since in the inference rules which construct suitable proofs, the upper sequents of the inferences consist of the subformulas of formulas in lower sequents. Importance of this result will be shown in later section. We will concentrate mainly on $\mathbf{S4}^* \otimes \mathbf{S5}^*$ in the following, as other cases can be treated similarly.

Definition 3.10 (A-suitable proof) *For any formula A, a proof is A-*suitable*, if every cut applied in it is acceptable or has a subformula of A as its cut formula.*

Definition 3.11 (partition of sequence) *A pair $\langle \Upsilon_1; \Upsilon_2 \rangle$ of sequences is a* partition *of sequence Ξ, if $\Upsilon_1 \cap \Upsilon_2 = \phi$ and $\Upsilon_1 \cup \Upsilon_2 = \Xi$.*

Lemma 3.12 *Suppose $\Xi \subseteq Sub(\Gamma, \Theta, A)$. If $\Gamma, \Upsilon_1 \rightarrow \Upsilon_2, \Theta$ has an A-suitable proof in* $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *for every partition $\langle \Upsilon_1; \Upsilon_2 \rangle$ of Ξ, then so does $\Gamma \rightarrow \Theta$.*

Proof. We prove this by induction on the length of Ξ.
(i) If Ξ is the empty, $\langle ; \rangle$ is the partition of Ξ. So claim holds.
(ii) If Ξ denotes (Ξ', B), then $\Xi' \subseteq \Xi \subseteq Sub(\Gamma, \Theta)$. Let $\langle \Pi; \Sigma \rangle$ be any partition of Ξ'. Then $\langle \Pi; \Sigma, B \rangle$ and $\langle \Pi, B; \Sigma \rangle$ are partitions of Ξ. So $\Gamma, \Pi \rightarrow \Sigma, B, \Theta$ and $\Gamma, \Pi, B \rightarrow \Sigma, \Theta$ have A-suitable proofs. Hence A-suitable proof of $\Gamma, \Pi \rightarrow \Sigma, \Theta$ can be obtained from these sequents by means of a cut and weak inferences, since $B \in \Xi \subseteq Sub(\Gamma, \Theta, A)$. Therefore, $\Gamma \rightarrow \Theta$ has an A-suitable proof by induction hypothesis. ∎

Definition 3.13 (regular) *A proof is* regular, *if for any cut in the proof the cut formula doesn't occur in the lower sequent of the cut.*

If the lower sequent of a cut rule contains the cut formula, it is obtainable from one of the upper sequents by means of weak inferences. So any proof can be transformed into regular one. Note that $\Box$ under the discussion is the modality of $\mathbf{S5}^*$.

Lemma 3.14 *Let P be a regular, suitable proof of $\Gamma \rightarrow \Theta$. Suppose $\Box A \notin Sub(\Gamma, \Theta_{\Box A})$. Then $\Box A$ doesn't occur in the antecedent of any sequent of P.*

Proof. We prove this by induction on the number of sequents. Now assume that $\Box A$ occurs in the antecedent of a sequent.

Case 1: If $\Gamma \rightarrow \Theta$ is the initial sequent, $\Gamma \rightarrow \Theta$ is $\Box A \rightarrow \Box A$. Then $\Box A \in Sub(\Gamma)$. It is contradictory to $\Box A \notin Sub(\Gamma, \Theta_{\Box A})$.

Case 2: $\Box A$ occur in the antecedents of the upper sequents of structural rules except cut rule (weakening rule, contraction rule and exchange rule) and logical rules. In these cases, clearly $\Box A$ occur in the antecedents of the lower sequents of the rules.

Case 3: $\Box A$ occurs in at least one of the antecedents of the upper sequents of cut rule.

3.1. If $\Box A$ is not the cut formula, then clearly $\Box A$ occurs in the antecedent of the lower sequent.

3.2. If $\Box A$ is the cut formula, then $\Box A$ is the proper subformula of a formula which occurs in the antecedent of the lower sequent since P is regular and suitable.

Hence, if $\Box A$ occurs in the antecedent of a sequent in P, then there exists a formula B in Γ such that $\Box A$ is the proper subformula of B. So $\Box A \in Sub(B) \subseteq Sub(\Gamma)$. It is however contradictory to $\Box A \notin Sub(\Gamma, \Theta_{\Box A})$. ∎

Definition 3.15 (family) *The* family *of $\Box A$ in a suitable proof is the sequence of all formulas except $\Box A$ which occur in the lower sequents of $(\to \Box)$ in the proof with the principal formula $\Box A$.*

Since the lower sequents of $(\to \Box)$ consist of $\Box$-formulas, the family of $\Box A$ in a suitable proof of $\Gamma \to \Theta$ consists of $\Box$-formulas in $Sub(\Gamma, \Theta)$ except $\Box A$.

Definition 3.16 (covered) *Let $\langle \Box\Upsilon_1; \Box\Upsilon_2 \rangle$ be a partition of the family of $\Box A$ in a suitable proof. An application $(\to \Box)$*

$$\frac{\Box\Pi \to \Box\Sigma, B}{\Box\Pi \to \Box\Sigma, \Box B}\ (\to \Box)$$

in the proof is covered *by $\langle \Upsilon_1; \Upsilon_2; A \rangle$, if $\Pi \subseteq \Upsilon_1$, $\Sigma_A \subseteq \Upsilon_2$ and $B = A$.*

Lemma 3.17 *Let P be a regular, suitable proof of $\Gamma \to \Theta$, and $\langle \Box\Upsilon_1; \Box\Upsilon_2 \rangle$ a partition of the family of $\Box A$ in P. Suppose $\Box A \notin Sub(\Gamma, \Theta_{\Box A})$.*
1) If $\Delta \to \Lambda$ is a sequent in P such that no application of $(\to \Box)$ which is applied above $\Delta \to \Lambda$ is covered by $\langle \Upsilon_1; \Upsilon_2; A \rangle$, then $\Delta, \Box\Upsilon_1 \to \Box\Upsilon_2, \Lambda_{\Box A}$ has an A-suitable proof in $\mathbf{S4}_{\blacksquare}{}^{*} \otimes \mathbf{S5}_{\Box}{}^{*}$.
2) Either $\Box\Upsilon_1 \to \Box\Upsilon_2, A$ or $\Gamma, \Box\Upsilon_1 \to \Box\Upsilon_2, \Theta_{\Box A}$ has an A-suitable proof in $\mathbf{S4}_{\blacksquare}{}^{*} \otimes \mathbf{S5}_{\Box}{}^{*}$.

Proof. 1) We prove this by induction on the number of sequents which are above $\Delta \to \Lambda . X$ and $X^{\#}$ denote the sequents $\Delta \to \Lambda$ and $\Delta, \Box\Upsilon_1 \to \Box\Upsilon_2, \Lambda_{\Box A}$ respectively. If X is the lower sequent of weak inferences, $(\wedge \to)$, $(\to \wedge)$, $(\vee \to)$, $(\to \vee)$, $(\supset\to)$, $(\to\supset)$, $(\neg \to)$ or $(\to \neg)$, the conclusion follows from the induction hypothesis immediately. So we will mention the other cases.

Case 1: X is the initial sequent $B \to B$. Since $B \neq \Box A$ by Lemma 3.14, $X^{\#}$ (i.e. $B, \Box\Upsilon_1 \to \Box\Upsilon_2, B$) has a suitable proof.

Case 2: X is the lower sequent of an acceptable cut

$$\frac{\Pi_1 \rightarrow \Sigma_1, B \quad B, \Pi_2 \rightarrow \Sigma_2}{\Pi_1, \Pi_2 \rightarrow \Sigma_1, \Sigma_2} \ (acceptable\ cut)\ ,$$

where $B \in Sub(\Pi_1, \Pi_2, \Sigma_1, \Sigma_2)$. By the induction hypothesis, $\Pi_1, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Sigma_{1\,\Box A}, B$ and $B, \Pi_2, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Sigma_{2\,\Box A}$ have A-suitable proofs.

If $B \in Sub(\Box A)$ (i.e. $B \in Sub(A) \cup \{\Box A\}$), then $B \in Sub(A)$ since $B \neq \Box A$ by Lemma 3.14. In this case,

$$\cfrac{\cfrac{\Pi_1, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Sigma_{1\,\Box A}, B \quad B, \Pi_2, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Sigma_{2\,\Box A}}{\Pi_1, \Pi_2, \Box\Upsilon_1, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Box\Upsilon_2, \Sigma_{1\,\Box A}, \Sigma_{2\,\Box A}} \ (cut)}{\Pi_1, \Pi_2, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Sigma_{1\,\Box A}, \Sigma_{2\,\Box A}} \ (weak\ inferences) \quad \cdots\cdot (*)$$

is an A-suitable proof.

If $B \notin Sub(\Box A)$, then $B \in Sub(\Pi_1, \Pi_2, \Sigma_{1\,\Box A}, \Sigma_{2\,\Box A})$.So $(*)$ is a suitable proof.

Case 3: X is the lower sequent of an

$$\frac{\Box\Pi \rightarrow \Box\Sigma, B}{\Box\Pi \rightarrow \Box\Sigma, \Box B} \ (\rightarrow \Box)\ .$$

If $B = A$, then $X^{\#}$ is $\Box\Pi, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Box(\Sigma_A)$. By the assumption this inference is not covered by $\langle \Upsilon_1; \Upsilon_2; A \rangle$. So either $\Pi \cap \Upsilon_2 \neq \phi$ or $\Sigma_A \cap \Upsilon_1 \neq \phi$.In both cases, a suitable proof of $X^{\#}$ can be obtained by means of weakening rules.

If $B \neq A$, then $X^{\#}$ is $\Box\Pi, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Box(\Sigma_A), \Box B$. By induction hypothesis, $\Box\Pi, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \Box(\Sigma_A), B$ has an A-suitable proof, and in the case of $B = \Box A$, it has an A-suitable proof by means of weakening rule. Hence $X^{\#}$ has an A-suitable proof.

Case 4: X is the lower sequent of an

$$\frac{\blacksquare\Pi \rightarrow B}{\blacksquare\Pi \rightarrow \blacksquare B} \ (\rightarrow \blacksquare)\ .$$

Since $\blacksquare B$ is $\blacksquare$-formula, then $\blacksquare B_{\Box A}$ is $\blacksquare B$. So $X^{\#}$ is $\blacksquare\Pi, \Box\Upsilon_1 \rightarrow \Box\Upsilon_2, \blacksquare B$. Hence an A-suitable proof of $X^{\#}$ can be obtained by means of weak inferences.

2) Case 1 : Some application of $(\rightarrow \Box)$ is covered by $\langle \Upsilon_1; \Upsilon_2; A \rangle$. Take one of the uppermost such application

$$\frac{\Box\Pi \rightarrow \Box\Sigma, A}{\Box\Pi \rightarrow \Box\Sigma, \Box A} \ (\rightarrow \Box)\ ,$$

where $\Pi \subseteq \Upsilon_1$ and $\Sigma_A \subseteq \Upsilon_2$. By applying 1) to the upper sequent of this inference, we can obtain $\Box\Pi, \Box\Upsilon_1 \to \Box\Upsilon_2, \Box(\Sigma_A), A$. So $\Box\Upsilon_1 \to \Box\Upsilon_2, A$ has an A-suitable proof.

Case 2 : Otherwise. By applying 1) to the end-sequent, we can obtain an A-suitable proof of $\Gamma, \Box\Upsilon_1 \to \Box\Upsilon_2, \Theta_{\Box A}$. ∎

Corollary 3.18 *If* $\Gamma \to \Theta$ *has a suitable proof in* $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$, *then* $\Gamma \to \Theta_{\Box A}, A$ *has an* A*-suitable one.*

Proof. Let P be a regular, suitable proof of $\Gamma \to \Theta$.
(i) If $\Box A \in Sub(\Gamma, \Theta_{\Box A})$, $\Gamma \to \Theta_{\Box A}, A$ has the suitable proof :

$$\dfrac{\Gamma \to \Theta \quad \dfrac{A \to A}{\Box A \to A}}{\Gamma \to \Theta_{\Box A}, A}\ (\textit{acceptable cut})\ .$$

(ii) $\Box A \notin Sub(\Gamma, \Theta_{\Box A})$. Let Ξ be the family of $\Box A$ in P, and $\langle \Box\Upsilon_1; \Box\Upsilon_2 \rangle$ any partition of Ξ. Then either $\Box\Upsilon_1 \to \Box\Upsilon_2, A$ or $\Gamma, \Box\Upsilon_1 \to \Box\Upsilon_2, \Theta_{\Box A}$ has an A-suitable proof by Lemma 3.17 , and so too has $\Gamma, \Box\Upsilon_1 \to \Box\Upsilon_2, \Theta_{\Box A}, A$ by means of weak inferences. Thus $\Gamma \to \Theta_{\Box A}, A$ has an A-suitable one by Lemma 3.12. ∎

Proof of Theorem 3.9 for $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. We first replaced all unacceptable cuts by mixes, and show it by double induction on the degree and the rank that any proof with a mix for its lowest inference and not containing any other mix can be transformed into a suitable one without changing the end-sequent. By eliminating one of uppermost such a mixes in turn, all mixes can be eliminated. In particular we will mention the cases which the upper sequents of the mixes are the lower sequents of $(\Box \to), (\to \Box), (\blacksquare \to), (\to \blacksquare)$ and (*acceptable cut*), since the other cases can be proved in usual way by the induction hypothesis immediately.

Case 1 : $\rho = 2$.

1.1. The left and right upper sequents of the mix are the lower sequents of $(\to \Box)$ and $(\Box \to)$ respectively. Then the proof runs as follows :

$$\dfrac{\dfrac{\Box\Gamma \to \Box\Theta, A}{\Box\Gamma \to \Box\Theta, \Box A}\,(\to \Box) \quad \dfrac{A, \Delta \to \Sigma}{\Box A, \Delta \to \Sigma}\,(\Box \to)}{\Box\Gamma, \Delta \to \Box\Theta, \Sigma}\,(\Box A)\ ,$$

where $\Box A$ is the mix formula and $\Box A \notin \Box\Theta \cup \Delta$. We transform it into the proof :

$$\frac{\Box\Gamma \to \Box\Theta, A \quad A, \Delta \to \Sigma}{\Box\Gamma \to \Box\Theta, \Sigma}\,(A) \quad .$$

The degree of A is smaller than that of $\Box A$. Hence we can eliminate the mix by the hypothesis of induction on the degree.

1.2. The left and right upper sequents of the mix are the lower sequents of $(\to \blacksquare)$ and $(\blacksquare \to)$ respectively. Then the proof runs as follows :

$$\frac{\dfrac{\blacksquare\Gamma \to A}{\blacksquare\Gamma \to \blacksquare A}\,(\to \blacksquare) \quad \dfrac{A, \Delta \to \Sigma}{\blacksquare A, \Delta \to \Sigma}\,(\blacksquare \to)}{\blacksquare\Gamma \to \Sigma}\,(\blacksquare A) \quad ,$$

where $\blacksquare A$ is the mix formula and $\blacksquare A \notin \Delta$. We transform it into the proof :

$$\frac{\blacksquare\Gamma \to A \quad A, \Delta \to \Sigma}{\blacksquare\Gamma, \Delta \to \Sigma}\,(A) \quad .$$

The degree of A is smaller than that of $\blacksquare A$. Hence we can eliminate the mix by the hypothesis of induction on the degree.

Case 2 : $\rho > 2$.

Subcase 2.1: $\rho_r = 1$. In this case $\rho_l > 1$ since $\rho > 2$ and $\rho_r = 1$.

2.1.1. The right upper sequent of the mix is the lower sequent of $(\Box \to)$. Then the proof runs as follows :

$$\frac{\Gamma \to \Theta \quad \dfrac{A, \Delta \to \Sigma}{\Box A, \Delta \to \Sigma}\,(\Box \to)}{\Gamma, \Delta \to \Theta_{\Box A}, \Sigma}\,(\Box A) \quad ,$$

where $\Box A$ is the mix formula, $\Box A \in \Theta$ and $\Box A \notin \Delta$. Since $\Gamma \to \Theta_{\Box A}, A$ has an A-suitable proof by Corollary 3.18, we can construct the proof :

$$\dfrac{\dfrac{\Gamma \to \Theta_{\Box A}, A \quad A, \Delta \to \Sigma}{\Gamma, \Delta_A \to (\Theta_{\Box A})_A, \Sigma}\,(A)}{\Gamma, \Delta \to \Theta_{\Box A}, \Sigma}\,(w.i.) \quad .$$

Even if mixes appear in a proof of $\Gamma \to \Theta_{\Box A}, A$, all the mix formulas are subformulas of A. Since the degree of A is smaller than that of $\Box A$, the mixes can be eliminated by the hypothesis of induction on the degree. In the similar way, the mix which is lowest inference can be also eliminated by the hypothesis of induction on the degree. Hence a suitable proof can be obtained without exchanging the end-sequent.

2.1.2. The left and right upper sequents of the mix are the lower sequents of ($\Box \to$) and ($\blacksquare \to$) respectively. Then the proof runs as follows :

$$\dfrac{\dfrac{A, \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}(\Box \to) \quad \dfrac{B, \Delta \to \Sigma}{\blacksquare B, \Delta \to \Sigma}(\blacksquare \to)}{\Box A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\blacksquare B) \ ,$$

where $\blacksquare B$ is the mix formula, $\blacksquare B \in \Theta$ and $\blacksquare B \notin \Delta$. We transform it into the proof :

$$\dfrac{\dfrac{A, \Gamma \to \Theta \quad \dfrac{B, \Delta \to \Sigma}{\blacksquare B, \Delta \to \Sigma}(\blacksquare \to)}{A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\blacksquare B)}{\Box A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\Box \to) \ .$$

The left rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

2.1.3. The left and right upper sequents of the mix are the lower sequents of ($\blacksquare \to$) and ($\to \blacksquare$) respectively. Then the proof runs as follows :

$$\dfrac{\dfrac{A, \Gamma \to \Theta}{\blacksquare A, \Gamma \to \Theta}(\blacksquare \to) \quad \dfrac{B, \Delta \to \Sigma}{\blacksquare B, \Delta \to \Sigma}(\blacksquare \to)}{\blacksquare A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\blacksquare B) \ ,$$

where $\blacksquare B$ is the mix formula, $\blacksquare B \in \Theta$ and $\blacksquare B \notin \Delta$. We transform it into the proof :

$$\dfrac{\dfrac{A, \Gamma \to \Theta \quad \dfrac{B, \Delta \to \Sigma}{\blacksquare B, \Delta \to \Sigma}(\blacksquare \to)}{A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\blacksquare B)}{\blacksquare A, \Gamma, \Delta \to \Theta_{\blacksquare B}, \Sigma}(\blacksquare \to) \ .$$

The left rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

Subcase 2.2: $\rho_r > 1$.

2.2.1. The both upper sequents of the mix are the lower sequents of ($\to \Box$). Then the proof runs as follows :

$$\dfrac{\Box \Gamma \to \Box \Theta \quad \dfrac{\Box \Delta \to \Box \Lambda, B}{\Box \Delta \to \Box \Lambda, \Box B}(\to \Box)}{\Box \Gamma, \Box(\Delta_A) \to \Box(\Theta_A), \Box \Lambda, \Box B}(\Box A) \ ,$$

where $\Box A$ is the mix formula and $\Box A \in \Box\Theta \cap \Box\Delta$. We transform it into the proof :

$$\dfrac{\dfrac{\Box\Gamma \to \Box\Theta \quad \Box\Delta \to \Box\Lambda, B}{\Box\Gamma, \Box(\Delta_A) \to \Box(\Theta_A), \Box\Lambda, B}(\Box A)}{\Box\Gamma, \Box(\Delta_A) \to \Box(\Theta_A), \Box\Lambda, \Box B}(\to \Box) \ .$$

The right rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

2.2.2. The left and right upper sequents of the mix are the lower sequents of $(\Box \to)$ and $(\to \Box)$ respectively. Then the proof runs as follows :

$$\dfrac{\dfrac{A, \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}(\Box \to) \quad \dfrac{\Box\Delta \to \Box\Lambda, B}{\Box\Delta \to \Box\Lambda, \Box B}(\to \Box)}{\Box A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\Box C) \ ,$$

where $\Box C$ is the mix formula and $\Box C \in \Theta \cap \Box\Delta$. We transform it into the proof :

$$\dfrac{\dfrac{A, \Gamma \to \Theta \quad \dfrac{\Box\Delta \to \Box\Lambda, B}{\Box\Delta \to \Box\Lambda, \Box B}(\to \Box)}{A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\Box C)}{\Box A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\Box \to) \ .$$

The left rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

2.2.3. The left and right upper sequents of the mix are the lower sequents of $(\blacksquare \to)$ and $(\to \Box)$ respectively. Then the proof runs as follows :

$$\dfrac{\dfrac{A, \Gamma \to \Theta}{\blacksquare A, \Gamma \to \Theta}(\blacksquare \to) \quad \dfrac{\Box\Delta \to \Box\Lambda, B}{\Box\Delta \to \Box\Lambda, \Box B}(\to \Box)}{\blacksquare A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\Box C) \ ,$$

where $\Box C$ is the mix formula and $\Box C \in \Theta \cap \Box\Delta$. We transform it into the proof :

$$\dfrac{\dfrac{A, \Gamma \to \Theta \quad \dfrac{\Box\Delta \to \Box\Lambda, B}{\Box\Delta \to \Box\Lambda, \Box B}(\to \Box)}{A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\Box C)}{\blacksquare A, \Gamma, \Box(\Delta_C) \to \Theta_{\Box C}, \Box\Lambda, \Box B}(\blacksquare \to) \ .$$

The left rank of this proof is smaller than that of former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

2.2.4. The left and right upper sequents of the mix are the lower sequents of $(\Box \to)$ and $(\to \blacksquare)$ respectively.

$$\cfrac{\cfrac{A, \Gamma \to \Theta}{\Box A, \Gamma \to \Theta}\,(\Box \to) \qquad \cfrac{\blacksquare\Delta \to B}{\blacksquare\Delta \to \blacksquare B}\,(\to \blacksquare)}{\Box A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\blacksquare C)\ ,$$

where $\blacksquare C$ is the mix formula and $\blacksquare C \in \Theta \cap \Box\Delta$. We transform it into the proof :

$$\cfrac{\cfrac{A, \Gamma \to \Theta \qquad \cfrac{\blacksquare\Delta \to B}{\blacksquare\Delta \to \blacksquare B}\,(\to \blacksquare)}{A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\blacksquare C)}{\Box A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\Box \to)\ .$$

The left rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

2.2.5. The left and right upper sequents of the mix are the lower sequents of $(\blacksquare \to)$ and $(\to \blacksquare)$ respectively. Then the proof runs as follows :

$$\cfrac{\cfrac{A, \Gamma \to \Theta}{\blacksquare A, \Gamma \to \Theta}\,(\blacksquare \to) \qquad \cfrac{\blacksquare\Delta \to B}{\blacksquare\Delta \to \blacksquare B}\,(\to \blacksquare)}{\blacksquare A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\blacksquare C)\ ,$$

where $\blacksquare C$ is the mix formula and $\blacksquare C \in \Theta \cap \Box\Delta$. We transform it into the proof :

$$\cfrac{\cfrac{A, \Gamma \to \Theta \qquad \cfrac{\blacksquare\Delta \to B}{\blacksquare\Delta \to \blacksquare B}\,(\to \blacksquare)}{A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\blacksquare C)}{\blacksquare A, \Gamma, \blacksquare(\Delta_C) \to \Theta_{\blacksquare C}, \blacksquare B}\,(\blacksquare \to)\ .$$

The left rank of this proof is smaller than that of the former one. Hence we can eliminate the mix by the hypothesis of induction on the rank.

Case 3 : If the right upper sequent of the mix is the lower sequent of acceptable cut in particular, then the proof runs as follows :

$$\cfrac{\Pi \to \Sigma \qquad \cfrac{\Gamma \to \Theta, B \quad B, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\,(\mathit{acceptable\ cut})}{\Pi, \Gamma_A, \Delta_A \to \Sigma_A, \Theta, \Lambda}\,(A)\ ,$$

where A is the mix formula and $A \in (\Gamma, \Delta) \cap \Sigma$. Since B is the cut formula of the acceptable cut, $B \in Sub(\Gamma, \Delta, \Theta, \Lambda)$. Suppose $A \in \Delta$ but $A \notin \Gamma$, since other cases are shown likewise.

3.1. If $B \in Sub(A) \setminus \{A\}$, we transform the given proof into the proof :

$$\cfrac{\Gamma \to \Theta, B \quad \cfrac{\Pi \to \Sigma \quad B, \Delta \to \Lambda}{\Pi, B, \Delta_A \to \Sigma_A, \Lambda}\,(A)}{\Pi, \Gamma_A, \Delta_A \to \Theta, \Sigma_A, \Lambda}\,(B) \quad ,$$

Hence we can eliminate the upper mix by the hypothesis of induction on the rank and the lower mix by the hypothesis of induction on the degree since $deg(B) < deg(A)$.

3.2. If $B \equiv A$, we transform the given proof into the proof :

$$\cfrac{\cfrac{\Pi \to \Sigma \quad B, \Delta \to \Lambda}{\Pi, \Delta_A \to \Sigma_A, \Lambda}\,(A)}{\Pi, \Gamma_A, \Delta_A \to \Sigma_A, \Theta, \Lambda}\,(w.i.) \quad .$$

Hence we can eliminate the mix by the hypothesis of induction on the rank.

3.3. If $B \notin Sub(A)$, we transform the given proof into the proof :

$$\cfrac{\Gamma \to \Theta, B \quad \cfrac{\Pi \to \Sigma \quad B, \Delta \to \Lambda}{\Pi, B, \Delta_A \to \Sigma_A, \Lambda}\,(A)}{\Pi, \Gamma_A, \Delta_A \to \Theta, \Sigma_A, \Lambda}\,(acceptable\ cut) \quad ,$$

where $B \in Sub(\Gamma_A, \Delta_A, \Theta, \Lambda)$. Hence we can eliminate the mix by the hypothesis of induction on the rank, and the lowest inference is an acceptable cut since $B \in Sub(\Gamma_A, \Delta_A, \Theta, \Lambda)$. ∎

Corollary 3.18 must be modified as follows:

Corollary 3.19 *If* $\Gamma \to \Theta$ *has an suitable proof in* $\mathbf{S5}^* \otimes \mathbf{S5}^*$*, then* $\Gamma \to \Theta_{\Box A}, A$ *and* $\Gamma \to \Theta_{\blacksquare A}, A$ *respectively have* A*-suitable proofs in* $\mathbf{S5}^* \otimes \mathbf{S5}^*$*.*

Proof of Theorem 3.9 for $\mathbf{S5}^* \otimes \mathbf{S5}^*$. We first replaced all unacceptable cut by mixes, and show the theorem for $\mathbf{S5}^* \otimes \mathbf{S5}^*$ by double induction on the degree and rank that any proof with a mix for its lowermost inference and not containing any other mix can be transformed into a suitable one with the same end-sequent. By eliminating one of uppermost such a mixes in turn, all mixes can be eliminated. We will particularly mention only crucial cases.

Case 1: The right upper sequent of the mix is the lower sequent of $(\to \Box)$. Then the proof runs as follows:

$$\cfrac{\Gamma \to \Theta \quad \cfrac{A, \Delta \to \Sigma}{\Box A, \Delta \to \Sigma}\,(\Box \to)}{\Gamma, \Delta \to \Theta_{\Box A}, \Sigma}\,(\Box A) \quad ,$$

where $\Box A$ is the mix formula, $\Box A \in \Theta$ and $\Box A \notin \Delta$. Since $\Gamma \to \Theta_{\Box A}, A$ has an A-suitable proof by Corollary 3.19, we can construct the proof:

$$\dfrac{\dfrac{\Gamma \to \Theta_{\Box A}, A \quad A, \Delta \to \Sigma}{\Gamma, \Delta_A \to (\Theta_{\Box A})_A, \Sigma}\,(A)}{\Gamma, \Delta \to \Theta_{\Box A}, \Sigma}\,(w.i.) \quad .$$

Even if mixes appear in a proof of $\Gamma \to \Theta_{\Box A}, A$, all the mix formulas are subformulas of A. Since the degree of A is smaller than that of $\Box A$, the mixes can be eliminated by the hypothesis of induction on the degree. In the similar way, the mix which is lowest inference can be also eliminated by the hypothesis of induction on the degree. Hence a suitable proof can be obtained without changing the end-sequent.

Case 2: The right upper sequent of the mix is the lower sequent of $(\blacksquare \to)$. Then the proof runs as follows:

$$\dfrac{\Gamma \to \Theta \quad \dfrac{A, \Delta \to \Sigma}{\blacksquare A, \Delta \to \Sigma}\,(\blacksquare \to)}{\Gamma, \Delta \to \Theta_{\blacksquare A}, \Sigma}\,(\blacksquare \to) \quad ,$$

where $\blacksquare A$ is the mix formula, $\blacksquare A \in \Theta$ and $\blacksquare A \notin \Delta$. Since $\Gamma \to \Theta_{\blacksquare A}, A$ has an A-suitable proof by Corollary 3.19, we can construct the proof:

$$\dfrac{\dfrac{\Gamma \to \Theta_{\blacksquare A}, A \quad A, \Delta \to \Sigma}{\Gamma, \Delta_A \to (\Theta_{\blacksquare A})_A, \Sigma}\,(A)}{\Gamma, \Delta \to \Theta_{\blacksquare A}, \Sigma}\,(w.i.) \quad .$$

Even if mixes appear in a proof of $\Gamma \to \Theta_{\blacksquare A}, A$, all the mix formulas are subformulas of A. Since the degree of A is smaller than that of $\blacksquare A$, the mixes can be eliminated by the hypothesis of induction on the degree. In the similar way, the mix which is lowest inference can be also eliminated by the hypothesis of induction on the degree. Hence a suitable proof can be obtained without changing the end-sequent. ∎

As for $\mathbf{K}^* \otimes \mathbf{S5}^*$ and $\mathbf{KT}^* \otimes \mathbf{S5}^*$, subformula property can be seen in a similar way to $\mathbf{S4}^* \otimes \mathbf{S5}^*$. Thus, we have the following.

Corollary 3.20 (subformula property)
Then all formulas which construct suitable proof in $\mathbf{K}^* \otimes \mathbf{S5}^*$, $\mathbf{KT}^* \otimes \mathbf{S5}^*$, $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *or* $\mathbf{S5}^* \otimes \mathbf{S5}^*$ *consist of the subformulas of formulas which occur in the lowest sequent.*

Proof. In all inferences except cut rule, the upper sequents of the inferences consist of the subformulas of formulas in the lower sequents.In the acceptable cut rule, of course, the upper sequents of the inferences consist of the subformulas of formulas in the lower sequent. ∎

3.4 Decidability

As an application of subformula property which was seen in the previous section, we will see the decidability for the systems corresponding to fusions. A concrete finite procedure which decides to be provable or not for any formula in a system is called *a decision procedure.* If there exists a decision procedure, the system is side to be *decidable.*

Definition 3.21 (reduced) *A sequent* $\Gamma \to \Delta$ *is* reduced*, if each formula occurs at most three times in both* Γ *and* Δ*.*

Lemma 3.22 *Let* $\Gamma \to \Delta$ *be arbitrary sequent. Then there exists a suitable* $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *proof of* $\Gamma' \to \Delta'$ *which consists solely of reduced sequents such that* $\Gamma' \to \Delta'$ *is provable in* $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *if and only if* $\Gamma \to \Delta$ *is provable in* $\mathbf{S4}^* \otimes \mathbf{S5}^*$*.*

Proof. Suppose $\Gamma \to \Delta$ is not reduced. Then a reduced sequent $\Gamma' \to \Delta'$ can be obtained from $\Gamma \to \Delta$ by means of contraction and exchange rules. Conversely, $\Gamma \to \Delta$ can be obtained from $\Gamma' \to \Delta'$ by means of weakening and exchange rules. So, for any sequent $\Gamma \to \Delta$, there exists a reduced sequent $\Gamma' \to \Delta'$ such that $\Gamma' \to \Delta'$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$ if and only if $\Gamma \to \Delta$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$. Then we can obtained a suitable proof of reduced sequent $\Gamma' \to \Delta'$ by Theorem 3.9.

The sequences Γ, Δ, Π and Σ of formulas in the structural and logical rules of $\mathbf{S4}^* \otimes \mathbf{S5}^*$, respectively, are not able to contain two or more same formulas by deleting the overlapping formulas. Then by means of weak inferences and by deleting the nonessential sequents, we can obtain the suitable proof of $\Gamma' \to \Delta'$ which consists solely of reduced sequents. ∎

Theorem 3.23 $\mathbf{S4}^* \otimes \mathbf{S5}^*$ *is decidable.*

Proof. We will show this theorem by giving a decision procedure. Suppose that any sequent $\Gamma \to \Delta$ is given. By Lemma 3.22 it is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$ if and only if there exists a reduced sequent $\Gamma' \to \Delta'$ obtained from $\Gamma \to \Delta$ which is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$. Let $\mathcal{G}$ be the set of all the reduced sequents which consist of all formulas in $Sub(\Gamma', \Delta')$. Since the set $Sub(\Gamma', \Delta')$ is finite, the set $\mathcal{G}$ is finite. Now we define $\mathcal{G}_n$ as follows:

• $\mathcal{G}_0$ is the set of all the initial sequents in $\mathcal{G}$,

• $\mathcal{G}_{i+1}$ is the union of $\mathcal{G}_i$ and the set of all the sequents in $\mathcal{G} \setminus \mathcal{G}_i$ which can be lower sequents when upper sequents are in $\mathcal{G}_i$.

Then there exists j such that $\mathcal{G}_{j+1} = \mathcal{G}_j$ since the set $\mathcal{G}$ is finite. If the sequent $\Gamma' \to \Delta'$ is in $\mathcal{G}_j$, then $\Gamma' \to \Delta'$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$, viz. $\Gamma \to \Delta$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$. Otherwise, $\Gamma \to \Delta$ is not provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$. ∎

Similarly, we can show the following.

Theorem 3.24 *For any* $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*, \mathbf{S5}^*\}$, $\mathbf{M} \otimes \mathbf{N}$ *is decidable.*

3.5 Craig's interpolation theorem

In this section, Craig's interpolation theorem for various fusions is shown syntactically by using Maehara's method. Since, for any $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*\}$, $\mathbf{M}^* \otimes \mathbf{N}^*$ has cut-elimination property by Theorem 3.3, Craig's interpolation theorem for $\mathbf{M}^* \otimes \mathbf{N}^*$ can be shown by using the usual way. Also, even when at least one of $\mathbf{M}^*$ and $\mathbf{N}^*$ is $\mathbf{S5}^*$, we can use Maehara,s method and get Craig's interpolation theorem for $\mathbf{M}^* \otimes \mathbf{N}^*$. In the following, we will given a detailed proof of it for $\mathbf{S4}^* \otimes \mathbf{S5}^*$.

For technical reasons, we introduce the constant symbol $\top$, and admit $\to \top$ as an initial sequent.

Definition 3.25 (partition of sequence)
$\langle\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$ *is a partition of sequent* $\Gamma \to \Delta$, *if* $\Gamma_1 \cap \Gamma_2 = \phi$, $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Delta_1 \cup \Delta_2 = \phi$ *and* $\Delta_1 \cup \Delta_2 = \Delta$.

The set of all propositional variables which occur in A and constant symbol is denoted by $V(A)$.

Lemma 3.26 *Suppose that a sequent* $\Gamma \to \Delta$ *is provable in* $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\square}{}^*$, *and also that* $\langle\{\Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$ *is an arbitrary partition of* $\Gamma \to \Delta$. *Then there exists a formula* C, *called an interpolant, such that*

1) $\Gamma_1 \to \Delta_1, C$ *and* $C, \Gamma_2 \to \Delta_2$ *are both provable in* $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\square}{}^*$,

2) $V(C) \subseteq V(\Gamma_1 \cup \Delta_1) \cap V(\Gamma_2 \cup \Delta_2)$.

Proof. This lemma is proved by induction on the length a suitable proof of $\Gamma \to \Delta$. We will give a proof only the cases where $\Gamma \to \Delta$ is initial sequent or the lower sequent of one of $(\square \to)$, $(\to \square)$, $(\blacksquare \to)$, $(\to \blacksquare)$ and (*acceptable cut*).

Case 1. The sequent $\Gamma \to \Delta$ has the form $D \to D$. There are four cases $\langle\{D; D\}, \{;\}\rangle$, $\langle\{;\}, \{D; D\}\rangle$, $\langle\{D;\}, \{; D\}\rangle$ and $\langle\{; D\}, \{D;\}\rangle$ as the partitions. Then $\neg\top$, $\top$, D and $\neg D$ are serves as the interpolants, respectively.

Case 2. The last inference is

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta} \ (\Box \to) \quad .$$

2.1. The partition is $\langle\{\Box A, \Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $A, \Gamma_1 \to \Delta_1, C$ and $C, \Gamma_2 \to \Delta_2$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$\frac{\dfrac{\vdots}{A, \Gamma_1 \to \Delta_1, C}}{\Box A, \Gamma_1 \to \Delta_1, C} \qquad \frac{\vdots}{C, \Gamma_2 \to \Delta_2} \quad .$$

Hence C serves as an interpolant of the present partition.

2.2. The partition is $\langle\{\Gamma_1; \Delta_1\}, \{\Box A, \Gamma_2; \Delta_2\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\Gamma_1 \to \Delta_1, C$ and $C, A, \Gamma_2 \to \Delta_2$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$\frac{\vdots}{\Gamma_1 \to \Delta_1, C} \qquad \frac{\dfrac{\vdots}{C, A, \Gamma_2 \to \Delta_2}}{C, \Box A, \Gamma_2 \to \Delta_2} \quad .$$

Hence C serves as an interpolant of the present partition.

Case 3. The last inference is

$$\frac{\Box\Gamma \to \Box\Delta, A}{\Box\Gamma \to \Box\Delta, \Box A} \ (\to \Box) \quad .$$

3.1. The partition is $\langle\{\Box\Gamma_1; \Box\Delta_1, \Box A\}, \{\Box\Gamma_2; \Box\Delta_2\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\Box\Gamma_1 \to \Box\Delta_1, A, C$ and $C, \Box\Gamma_2 \to \Box\Delta_2$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Box\Gamma_1 \to \Box\Delta_1, A, C}}{\neg C, \Box\Gamma_1 \to \Box\Delta_1, A}}{\Box\neg C, \Box\Gamma_1 \to \Box\Delta_1, A}}{\dfrac{\Box\neg C, \Box\Gamma_1 \to \Box\Delta_1, \Box A}{\Box\Gamma_1 \to \Box\Delta_1, \Box A, \neg\Box\neg C}}
\qquad
\dfrac{\dfrac{\dfrac{\vdots}{C, \Box\Gamma_2 \to \Box\Delta_2}}{\Box\Gamma_2 \to \Box\Delta_2, \neg C}}{\dfrac{\Box\Gamma_2 \to \Box\Delta_2, \Box\neg C}{\neg\Box\neg C, \Box\Gamma_2 \to \Box\Delta_2}} \quad .
$$

Hence $\neg\Box\neg C$ serves as an interpolant of the present partition.

3.2. The partition is $\langle\{\Box\Gamma_1; \Box\Delta_1\}, \{\Box\Gamma_2; \Box\Delta_2, \Box A\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\Box\Gamma_1 \to \Box\Delta_1, C$ and $C, \Box\Gamma_2 \to \Box\Delta_2, \Box A$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$
\dfrac{\dfrac{\vdots}{\Box\Gamma_1 \to \Box\Delta_1, C}}{\Box\Gamma_1 \to \Box\Delta_1, \Box C}
\qquad
\dfrac{\dfrac{\dfrac{\vdots}{C, \Box\Gamma_2 \to \Box\Delta_2, A}}{\Box C, \Box\Gamma_2 \to \Box\Delta_2, A}}{\Box C, \Box\Gamma_2 \to \Box\Delta_2, \Box A} \quad .
$$

Hence $\Box C$ serves as an interpolant of the present partition.

Case 4. The last inference is

$$
\dfrac{A, \Gamma \to \Delta}{\blacksquare A, \Gamma \to \Delta} \; (\blacksquare \to) \quad .
$$

4.1. The partition is $\langle\{\blacksquare A, \Gamma_1; \Delta_1\}, \{\Gamma_2; \Delta_2\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $A, \Gamma_1 \to \Delta_1, C$ and $C, \Gamma_2 \to \Delta_2$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$
\dfrac{\dfrac{\vdots}{A, \Gamma_1 \to \Delta_1, C}}{\blacksquare A, \Gamma_1 \to \Delta_1, C}
\qquad
\dfrac{\vdots}{C, \Gamma_2 \to \Delta_2} \quad .
$$

Hence C serves as an interpolant of the present partition.

4.2. The partition is $\langle\{\Gamma_1; \Delta_1\}, \{\blacksquare A, \Gamma_2; \Delta_2\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\Gamma_1 \to \Delta_1, C$ and $C, A, \Gamma_2 \to \Delta_2$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$\dfrac{\vdots}{\Gamma_1 \to \Delta_1, C} \qquad \dfrac{\dfrac{\vdots}{C, A, \Gamma_2 \to \Delta_2}}{C, \blacksquare A, \Gamma_2 \to \Delta_2} \quad .$$

Hence C serves as an interpolant of the present partition.

Case 5. The last inference is

$$\dfrac{\blacksquare\Gamma \to A}{\blacksquare\Gamma \to \blacksquare A}\ (\to \blacksquare) \quad .$$

5.1. The partition is $\langle\{\blacksquare\Gamma_1; \blacksquare A\}, \{\blacksquare\Gamma_2; \}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\blacksquare\Gamma_1 \to A, C$ and $C, \blacksquare\Gamma_2 \to$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\blacksquare\Gamma_1 \to A, C}}{\neg C, \blacksquare\Gamma_1 \to A}}{\blacksquare\neg C, \blacksquare\Gamma_1 \to A}}{\blacksquare\neg C, \blacksquare\Gamma_1 \to \blacksquare A}}{\blacksquare\Gamma_1 \to \blacksquare A, \neg\blacksquare\neg C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{C, \blacksquare\Gamma_2 \to}}{\blacksquare\Gamma_2 \to \neg C}}{\blacksquare\Gamma_2 \to \blacksquare\neg C}}{\neg\blacksquare\neg C, \blacksquare\Gamma_2 \to} \quad .$$

Hence $\neg\blacksquare\neg C$ serves as an interpolant of the present partition.

5.2. The partition is $\langle\{\blacksquare\Gamma_1; \}, \{\blacksquare\Gamma_2; A\}\rangle$. By applying the induction hypothesis to the proof of the upper sequent, there exists an interpolant C such that $\blacksquare\Gamma_1 \to C$ and $C, \blacksquare\Gamma_2 \to A$ are both provable in $\mathbf{S4}_{\blacksquare}{}^* \otimes \mathbf{S5}_{\Box}{}^*$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\vdots}{\blacksquare\Gamma_1 \to C}}{\blacksquare\Gamma_1 \to \blacksquare C} \qquad \dfrac{\dfrac{\dfrac{\vdots}{C, \blacksquare\Gamma_2 \to A}}{\blacksquare C, \blacksquare\Gamma_2 \to A}}{\blacksquare C, \blacksquare\Gamma_2 \to \blacksquare A} \quad .$$

Hence $\blacksquare C$ serves as an interpolant of the present partition.

Case 6. The last inference is

$$\dfrac{\Gamma \to \Delta, A \quad A, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma}\ (\textit{acceptable cut}) \quad ,$$

where $A \in Sub(\Gamma, \Pi, \Delta, \Sigma)$. Then the partition is $\langle\{\Gamma_1, \Pi_1; \Delta_1, \Sigma_1\}, \{\Gamma_2, \Pi_2; \Delta_2, \Sigma_2\}\rangle$. This is only the case which never happens when the cut-elimination theorem holds.

6.1. If $A \in Sub(\Gamma_1, \Pi_1, \Delta_1, \Sigma_1)$, by applying the induction hypothesis to the proof of the upper sequent, there exist interpolants C_1 and C_2 such that $\Gamma_1 \to \Delta_1, A, C_1$; $C_1, \Gamma_2 \to \Delta_2$; $A, \Pi_1 \to \Sigma_1, C_2$ and $C_2, \Pi_2 \to \Sigma_2$ are all provable in $\mathbf{S4}_{\blacksquare}{}^{*} \otimes \mathbf{S5}_{\Box}{}^{*}$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \to \Delta_1, A, C_1} \quad \dfrac{\vdots}{A, \Pi_1 \to \Sigma_1, C_2}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_1, C_2}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_1 \vee C_2, C_1 \vee C_2}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_1 \vee C_2} \qquad \dfrac{\dfrac{\dfrac{\vdots}{C_1, \Gamma_2 \to \Delta_2}}{C_1, \Gamma_2, \Pi_2 \to \Delta_2, \Sigma_2} \quad \dfrac{\dfrac{\vdots}{C_2, \Pi_2 \to \Sigma_2}}{C_2, \Gamma_2, \Pi_2 \to \Delta_2, \Sigma_2}}{C_1 \vee C_2, \Gamma_2, \Pi_2 \to \Delta_2, \Sigma_2} .$$

Hence $C_1 \vee C_2$ serves as interpolants of the present partition.

6.2. If $A \in Sub(\Gamma_2, \Pi_2, \Delta_2, \Sigma_2)$, by applying the induction hypothesis to the proof of the upper sequent, there exist interpolants C_1 and C_2 such that $\Gamma_1 \to \Delta_1, C_1$; $C_1, \Gamma_2 \to \Delta_2, A$; $\Pi_1 \to \Sigma_1, C_2$ and $C_2, A, \Pi_2 \to \Sigma_2$ are all provable in $\mathbf{S4}_{\blacksquare}{}^{*} \otimes \mathbf{S5}_{\Box}{}^{*}$. Then we can obtain following two proofs:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \to \Delta_1, C_1}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_1} \quad \dfrac{\dfrac{\vdots}{\Pi_1 \to \Sigma_1, C_2}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_2}}{\Gamma_1, \Pi_1 \to \Delta_1, \Sigma_1, C_1 \wedge C_2} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{C_1, \Gamma_2 \to \Delta_2, A} \quad \dfrac{\vdots}{C_2, A, \Pi_2 \to \Sigma_2}}{C_1, \Gamma_2, C_2, \Pi_2 \to \Delta_2, \Sigma_2}}{C_1 \wedge C_2, \Gamma_2, C_1 \wedge C_2, \Pi_2 \to \Delta_2, \Sigma_2}}{C_1 \wedge C_2, \Gamma_2, \Pi_2 \to \Delta_2, \Sigma_2} .$$

Hence $C_1 \wedge C_2$ serves as interpolants of the present partition. ∎

Theorem 3.27 (Craig's interpolation theorem)
If $A \supset B$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$, then there exists a formula C such that

1) $A \supset C$ and $C \supset B$ are both provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$,

2) $V(C) \subseteq V(A) \cap V(B)$.

Proof. Assume that $A \supset B$ is provable in $\mathbf{S4}^* \otimes \mathbf{S5}^*$. Clearly, the sequent $A \to B$ is provable in it. Then by Lemma 3.26, taking A as Γ_1 and B as Δ_2, there exists a formula C satisfying *1)* and *2)* of Theorem 3.27. ∎

Similarly, we can show the following.

Theorem 3.28 *Let $\mathbf{M}^*, \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S5}^*\}$. If $A \supset B$ is provable in $\mathbf{M}^* \otimes \mathbf{S5}^*$, then there exists a formula C such that*

1) $A \supset C$ and $C \supset B$ are both provable in $\mathbf{M}^* \otimes \mathbf{S5}^*$,

2) $V(C) \subseteq V(A) \cap V(B)$.

3.6 Remarks

As for fusions, M. Kracht and F. Wolter [3] proved the followings semantically:

- both $\mathbf{M}$ and $\mathbf{N}$ are decidable $\Rightarrow$ $\mathbf{M} \otimes \mathbf{N}$ is decidable,
- both $\mathbf{M}$ and $\mathbf{N}$ hold the Craig's interpolation theorem $\Rightarrow$ $\mathbf{M} \otimes \mathbf{N}$ holds it.

In this chapter, we obtained the syntactical results for these property. For any $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*, \mathbf{S5}^*\}$, we could see that $\mathbf{M}^* \otimes \mathbf{N}^*$ has subformula property either by deriving the cut-elimination property or by showing that every proof can be transformed into suitable one with same end-sequent. In either case, important logical properties like the decidability and the Craig's interpolation theorem can be derived.

As for dependently axiomatizable bimodal logics, however, it is difficult to find sequent systems in which the cut-elimination property holds. Therefore, it would be necessary to develop semantical methods for them. Some attempts to this direction are made in the next section.

Chapter 4

Kripke type semantics

There are several semantical researches on fusion of independently axiomatizable bimodal logics by M. Kracht and F. Wolter [3] and so on, and some syntactical approaches have been seen in previous section. In this chapter, we will examine several dependently axiomatizable bimodal logics, using semantical method. Using Kripke type semantics, logical properties like the completeness and the finite model property for these logics will be discussed. Though our study in the present chapter remains still an initial stage, the attempt made here will contribute to future extensive, semantical study in future.

4.1 Kripke frames and models

First we will extend Kripke type semantics to bimodal logics.

Definition 4.1 (Kripke frame)
Let M be a nonempty set, and $R_\square$ and $R_\blacksquare$ be binary relations on M; viz. $R_\square \subseteq M \times M$ and $R_\blacksquare \subseteq M \times M$. Then a frame *is a triple $(M, R_\square, R_\blacksquare)$, where M is called the set of* possible worlds*, and both $R_\square$ and $R_\blacksquare$ are called* accessibility relations.

Definition 4.2 (Kripke model)
Let $\mathcal{F} = (M, R_\square, R_\blacksquare)$ be a frame, and V be a mapping such that $V(p) \subseteq M$ for each propositional variable p. Then a Kripke model *is a pair $(\mathcal{F}, V)$, i.e. $(M, R_\square, R_\blacksquare, V)$, where V is called a* valuation *on $\mathcal{F}$. For a given Kripke model $(M, R_\square, R_\blacksquare, V)$, a binary relation $\models$ between $a \in M$ and formulas is defined inductively on the length of formulas as follows:*

- $a \models p \iff a \in V(p)$

- $a \models A \wedge B \iff a \models A$ *and* $a \models B$
- $a \models A \vee B \iff a \models A$ *or* $a \models B$
- $a \models A \supset B \iff a \models A$ *implies* $a \models B$
- $a \models \neg A \iff$ *not* $a \models A$
- $a \models \Box A \iff$ *for any* $b \in M$, $aR_{\Box}b$ *implies* $b \models A$
- $a \models \blacksquare A \iff$ *for any* $b \in M$, $aR_{\blacksquare}b$ *implies* $b \models A$.

The relation $\models$ is defined by the valuation V uniquely. So $\models$ and $(M, R_{\Box}, R_{\blacksquare}, \models)$ is also called a valuation, and a Kripke model, respectively, when no confusions will occur. A formula A is *true in model* $\mathcal{M} = (M, R_{\Box}, R_{\blacksquare}, \models)$, denoted by $\mathcal{M} \models A$, if $a \models A$ for any $a \in M$. A formula A is *valid in frame* $\mathcal{F} = (M, R_{\Box}, R_{\blacksquare})$, denoted by $\mathcal{F} \models A$, if $\mathcal{M} \models A$ for any model $\mathcal{M} = (M, R_{\Box}, R_{\blacksquare}, \models)$.

In the following, we will consider particular interdependences between $\Box$ and $\blacksquare$, which can be expressed by a formula of the form $\alpha_1 \cdots \alpha_m p \supset \beta_1 \cdots \beta_n p$, where $\alpha_i, \beta_j \in \{\Box, \blacksquare\}$. An example is $\Box p \supset \blacksquare p$, which can be interpreted in epistemic logic as " $H_{\blacksquare}$ knows everything what $H_{\Box}$ knows " when $\Box p$ ($\blacksquare p$) is interpreted as " $H_{\Box}$ ($H_{\blacksquare}$) knows p ".

Lemma 4.3 *Let* $\alpha_i, \beta_j \in \{\Box, \blacksquare\}$. *Suppose* $m, n \geq 1$. *For any frame* $\mathcal{F} = (M, R_{\Box}, R_{\blacksquare})$,
$\mathcal{F} \models \alpha_1 \cdots \alpha_m A \supset \beta_1 \cdots \beta_n A \iff$
$(**)$ $\forall c_k\ (0 \leq k \leq n-1)\ (c_k R_{\beta_{k+1}} c_{k+1} \Rightarrow \exists d_l\ (0 \leq l \leq m-1)\ d_l R_{\alpha_{l+1}} d_{l+1})$,
where $c_0 = d_0 = a$ *and* $c_n = d_m = b$.

Proof. [$\Leftarrow$] Let $(M, R_{\Box}, R_{\blacksquare}, \models)$ be a model, and a be in M.
$a \not\models \beta_1\beta_2 \cdots \beta_{n-1}\beta_n A$
$\Rightarrow \exists e_1(aR_{\beta_1}e_1$ and $e_1 \not\models \beta_2 \cdots \beta_n A)$
$\vdots$
$\Rightarrow \exists e_1 \cdots \exists e_{n-1}(aR_{\beta_1}e_1, \cdots, e_{n-2}R_{\beta_{n-1}}e_{n-1}$ and $e_{n-1} \not\models \beta_n A)$
$\Rightarrow \exists e_1 \cdots \exists e_{n-1}\exists e_n(aR_{\beta_1}e_1, \cdots, e_{n-2}R_{\beta_{n-1}}e_{n-1}, e_{n-1}R_{\beta_n}e_n$ and $e_n \not\models A)$
$\Rightarrow \exists d_1 \cdots \exists d_{m-1}\exists d_m(aR_{\alpha_1}d_1, \cdots, d_{m-2}R_{\alpha_{m-1}}d_{m-1}, d_{m-1}R_{\alpha_m}d_m$ and $d_m \not\models A)$ $(e_n = d_m)$
$\Rightarrow \exists d_1 \cdots \exists d_{m-1}(aR_{\alpha_1}d_1, \cdots, d_{m-2}R_{\alpha_{m-1}}d_{m-1}$ and $d_{m-1} \not\models \alpha_m A)$
$\vdots$
$\Rightarrow \exists d_1(aR_{\alpha_1}d_1$ and $d_1 \not\models \alpha_2 \cdots \alpha_m A)$
$\Rightarrow a \not\models \alpha_1\alpha_2 \cdots \alpha_{m-1}\alpha_m A$

[$\Rightarrow$] Suppose $c_k R_{\beta_{k+1}} c_{k+1}$ for any c_{k+1} $(0 \leq k \leq n-1)$, where $c_0 = a$ and $c_n = b$. Let $(M, R_{\Box}, R_{\blacksquare}, \models)$ be a model in which

$$x \models p \Leftrightarrow \exists d_l \ (0 \leq l \leq m-1), d_l R_{\alpha_{l+1}} d_{l+1}$$

where $d_0 = a$ and $d_m = x$. Then $a \models \alpha_1 \cdots \alpha_m p$, and so $a \models \beta_1 \cdots \beta_n p$ by the assumption. Since, for any c_k $(0 \leq k \leq n-1)$, $c_k R_{\beta_{k+1}} c_{k+1}$ and $a \models \beta_1 \cdots \beta_n p$, $c_n \models p$ (*i.e.* $b \models p$). Hence $\exists d_l (0 \leq l \leq m-1), d_l R_{\alpha_{l+1}} d_{l+1}$ where $d_0 = a$ and $d_m = b$. ∎

The condition in Lemma 4.3 corresponding to the schema $\alpha_1 \cdots \alpha_m A \supset \beta_1 \cdots \beta_n A$ is displayed by the following Figure 4.1.

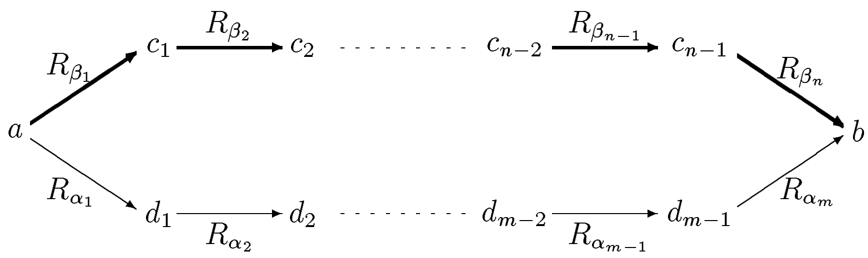

Figure 4.1:

4.2 Completeness

We will see Kripke completeness for the bimodal logics with the axiom $\alpha_1 \cdots \alpha_m p \supset \beta_1 \cdots \beta_n p$ where $\alpha_i, \beta_j \in \{\Box, \blacksquare\}$ by using the canonical models.

In the following, Φ denotes the set of all formulas of bimodal logics.

Definition 4.4 (L-consistent set) *For a bimodal logic* $\mathbf{L}$*, a set* $U \in \Phi$ *is* $\mathbf{L}$- consistent *if* $\neg(B_0 \wedge B_1 \wedge \cdots \wedge B_{n-1}) \notin \mathbf{L}$ *for any* $B_0, \cdots, B_{n-1} \in U$.

Definition 4.5 (L-maximal set) *For a bimodal logic* $\mathbf{L}$*, a set* $U \in \Phi$ *is* $\mathbf{L}$- maximal *is the following conditions are satisfied:*

- *U is $\mathbf{L}$-consistent,*
- *for any $A \in \Phi$, either $A \in U$ or $\neg A \in U$.*

Lemma 4.6 *Let* $\mathbf{L}$ *be a normal logic.*
(1) $(A_0 \wedge \cdots \wedge A_{n-1}) \supset A \in \mathbf{L}$
$\Rightarrow (\Box A_0 \wedge \cdots \wedge \Box A_{n-1}) \supset \Box A \in \mathbf{L}$ *and* $(\blacksquare A_0 \wedge \cdots \wedge \blacksquare A_{n-1}) \supset \blacksquare A \in \mathbf{L}$.
(2) • $(\Box A_0 \vee \cdots \vee \Box A_{n-1}) \supset \Box(A_0 \vee \cdots \vee A_{n-1}) \in \mathbf{L}$,
• $(\blacksquare A_0 \vee \cdots \vee \blacksquare A_{n-1}) \supset \blacksquare(A_0 \vee \cdots \vee A_{n-1}) \in \mathbf{L}$.

Proof. (1) By induction on n. We first prove the case $n = 0$. Suppose $A_0 \supset A \in \mathbf{L}$, and then $\Box(A_0 \supset A) \in \mathbf{L}$. Since $\mathbf{L}$ is normal, $\Box A_0 \supset \Box A \in \mathbf{L}$. If $(A_0 \wedge \cdots \wedge A_k) \supset A \in \mathbf{L}$, then $(A_0 \wedge \cdots \wedge A_{k-1}) \supset (A_k \supset A) \in \mathbf{L}$. By induction hypothesis, $(\Box A_0 \wedge \cdots \wedge \Box A_{k-1}) \supset \Box(A_k \supset A) \in \mathbf{L}$. Since $\mathbf{L}$ is normal, $(\Box A_0 \wedge \cdots \wedge \Box A_{k-1}) \supset (\Box A_k \supset \Box A) \in \mathbf{L}$. Hence $(\Box A_0 \wedge \cdots \wedge \Box A_k) \supset \Box A \in \mathbf{L}$. As for $\blacksquare$, we can see in the similar way to $\Box$.

(2) By induction on n. The case $n = 0$ is clearly. Suppose the case $n = k - 1$, and then $(\Box A_0 \vee \cdots \vee \Box A_{k-1}) \supset \Box(A_0 \vee \cdots \vee A_{k-1}) \in \mathbf{L}$. $(\Box A_0 \vee \cdots \vee \Box A_{k-1}) \supset \Box(A_0 \vee \cdots \vee A_{k-1} \vee A_k) \in \mathbf{L}$ since $(A_0 \vee \cdots \vee A_{k-1}) \supset (A_0 \vee \cdots \vee A_{k-1} \vee A_k) \in \mathbf{L}$. Further, $A_k \supset (A_0 \vee \cdots \vee A_{k-1} \vee A_k) \in \mathbf{L}$, and so $\Box A_k \supset \Box(A_0 \vee \cdots \vee A_{k-1} \vee A_k) \in \mathbf{L}$. Thus $(\Box A_0 \vee \cdots \vee \Box A_{k-1} \vee \Box A_k) \supset \Box(A_0 \vee \cdots \vee A_{k-1} \vee A_k) \in \mathbf{L}$. As for $\blacksquare$, we can see in the similar way to $\Box$. ∎

The following lemma is essential in proving the completeness.

Lemma 4.7 (Lindenbaum's lemma)
Every $\mathbf{L}$*-consistent set of formulas is contained in a* $\mathbf{L}$*-maximal set.*

Proof. Let $A_0, \cdots, A_i, \cdots$ be an enumeration of the set Φ, and U be any $\mathbf{L}$-consistent set. Now define Ω as follows:

$$\Omega_0 = U$$

$$\Omega_{n+1} = \begin{cases} \Omega_n \cup \{A_n\}, & \text{if } \Omega_n \cup \{A_n\} \text{ is } \mathbf{L}\text{-consistent;} \\ \Omega_n \cup \{\neg A_n\}, & \text{otherwise.} \end{cases}$$

$$\Omega = \bigcup_{n \geq 0} \Omega_n.$$

By using induction, we will show that Ω_n is $\mathbf{L}$ -consistent for any n. Clearly, Ω_0 is $\mathbf{L}$-consistent. Next assume that Ω_n is $\mathbf{L}$-consistent. If $\Omega_n \cup \{A_n\}$ is $\mathbf{L}$-consistent, then Ω_{n+1} is $\mathbf{L}$-consistent by the definition of Ω_n. Now consider the case that $\Omega_n \cup \{A_n\}$ is not $\mathbf{L}$-consistent, i.e. there exist $B_1, \cdots, B_k \in \Omega_n$ such that $\neg(B_1 \wedge \cdots \wedge B_k \wedge A_n) \in \mathbf{L}$. Suppose moreover that Ω_{n+1} $(= \Omega_n \cup \{\neg A_n\})$ is not $\mathbf{L}$-consistent. Then there exist $C_1, \cdots, C_l \in \Omega_n$ such that $\neg(C_1 \wedge \cdots \wedge C_l \wedge (\neg A_n)) \in \mathbf{L}$. Hence $C_1 \wedge \cdots \wedge C_l \supset \neg(B_1 \wedge \cdots \wedge B_k) \in \mathbf{L}$, i.e. $\neg(C_1 \wedge \cdots \wedge C_l \wedge B_1 \wedge \cdots \wedge B_k) \in \mathbf{L}$. But this contradicts the $\mathbf{L}$-consistency of Ω_n. Next, we will show that Ω is $\mathbf{L}$-consistent. Suppose otherwise. Then, $\neg(D_1 \wedge \cdots \wedge D_s) \in \mathbf{L}$ for some $D_1, \cdots, D_s \in \Omega$. Since $\Omega = \bigcup_n \Omega_n$, $D_i \in \Omega_n$ for each i. Let N be the maximum number in $\{n_1, \cdots, n_s\}$. Then, $D_i \in \Omega_N$ for all i. This means that Ω_N is inconsistent. But this is a contradiction.

It remains to show that either $A \in \Omega$ or $\neg A \in \Omega$ for each $A \in \Phi$. Suppose $A \in \Omega$ and $\neg A \in \Omega$ for some formula A. Since $\neg(A \wedge \neg A) \in \mathbf{L}$, this contradicts the $\mathbf{L}$-consistency of Ω. So exactly one of A and $\neg A$ must be in Ω for any $A \in \Phi$.

Hence Ω is a $\mathbf{L}$-maximal set containing U. ∎

Now we consider the method of canonical models.

Definition 4.8 (canonical model) *The* canonical model $(M^L, R^L_\Box, R^L_\blacksquare, \models^L)$ *of a normal modal logic* $\mathbf{L}$ *is defined as follows:*

- $M^L = \{U \subseteq \Phi | U \mathit{is} \mathbf{L}\text{-}\mathit{maximal}\}$,
- $U_1 R^L_\Box U_2 \iff \{A \in \Phi | \Box A \in U_1\} \subseteq U_2$,
- $U_1 R^L_\blacksquare U_2 \iff \{A \in \Phi | \blacksquare A \in U_1\} \subseteq U_2$,
- $U \models^L p \iff p \in U$.

The *canonical frame* for $\mathbf{L}$ is $\mathcal{F}^L = (M^L, R^L_\Box, R^L_\blacksquare)$.

Lemma 4.9 *Let* $\mathbf{L}$ *be a normal logic. Then, for any* $\mathbf{L}$*-maximal set,* U*, the following holds.*

(1) $\mathbf{L} \subseteq U$.
(2) $A \wedge B \in U \iff A \in U$ *and* $B \in U$.
(3) $A \vee B \in U \iff A \in U$ *or* $B \in U$.
(4) $A \supset B \in U \iff A \in U$ *implies* $B \in U$.
(5) $\neg A \in U \iff$ *not* $A \in U$.
(6) $\Box A \in U \iff$ *for any* $U' \in M^L$, $U R^L_\Box U'$ *implies* $A \in U'$.
(7) $\blacksquare A \in U \iff$ *for any* $U' \in M^L$, $U R^L_\blacksquare U'$ *implies* $A \in U'$.

Proof. (1) Take any $A \in \mathbf{L}$, and then $\neg(\neg)A \in \mathbf{L}$. Thus $\neg A \in U$ contradicts the $\mathbf{L}$-consistency of U. Hence $A \in U$.

(2) [$\Rightarrow$] Suppose thet $A \wedge B \in U$ and moreover that $A \notin U$ or $B \notin U$. Then $\neg A \in U$ or $\neg B \in U$. Since

$$\neg(A \wedge B \wedge (\neg A)) \in \mathbf{L} \text{ and } \neg(A \wedge B \wedge (\neg B)) \in \mathbf{L},$$

we have a contradiction in the either case.

[$\Leftarrow$] Suppose that $A, B \in U$ and moreover that $A \wedge B \notin U$. Then $\neg(A \wedge B) \in U$. Since

$$\neg(\neg(A \wedge B) \wedge A \wedge B) \in \mathbf{L},$$

we have a contradiction.

(3) [$\Rightarrow$] Suppose that $A \vee B \in U$ and moreover that $A \notin U$ and $B \notin U$. Then $\neg A \in U$ and $\neg B \in U$. Since

$$\neg((A \vee B) \wedge (\neg A) \wedge (\neg B)) \in \mathbf{L},$$

we have a contradiction.

[$\Leftarrow$] Suppose that $A \in U$ or $B \in U$, and moreover that $A \vee B \notin U$. Then $\neg(A \vee B) \in U$. Since

$$\neg(\neg(A \vee B) \wedge A) \in L \text{ and } \neg(\neg(A \vee B) \wedge B) \in L,$$

we have a conradiction in either case.

(4) [$\Rightarrow$] Suppose that $A \supset B \in U$ and moreover that $A \in U$ and $B \notin U$. Then $\neg B \in U$. Since

$$\neg((A \supset B) \wedge A \wedge (\neg B)) \in \mathbf{L},$$

we have a contradiction.

[$\Leftarrow$] Suppose that $\neg A \in U$ or $B \in U$, and moreover that $A \supset B \notin U$. Then $\neg(A \supset B) \in U$. Since

$$\neg(\neg(A \supset B) \wedge (\neg A)) \in \mathbf{L} \text{ and } \neg(\neg(A \supset B) \wedge B) \in \mathbf{L},$$

we have a contradiction in either case.

(5) It is clear by the definition of **L**-maximal.

(6) [$\Rightarrow$] If $\Box A \in U$ and $U R^L_\Box U'$, then $A \in U'$ by the definition of $R^L_\Box$.

[$\Leftarrow$] Suppose $\Box A \notin U$. We will show that $\{B \in \Phi | \Box B \in U\} \cup \{\neg A\}$ is **L**-consistent. Suppose otherwise. Then

$$\neg(B_1 \wedge \cdots \wedge B_k \wedge \neg A) \in \mathbf{L}, i.e. (B_1 \wedge \cdots \wedge B_k) \supset A \in \mathbf{L}$$

for some $B_1, \cdots, B_k$ such that $\Box B_i \in U$ for each $i \leq k$. By Lemma 4.6,

$$(\Box B_1 \wedge \cdots \wedge \Box B_k) \supset \Box A \in \mathbf{L}.$$

Since $\Box B_i \in U$ for each i, $\Box A \in U$. But this is a contradiction. Hence $\{B \in \Phi | \Box B \in U\} \cup \{\neg A\}$ is **L**-consistent. Then, by Lemma 4.7 there exists $U' \in M^L$ such that $\{B \in \Phi | \Box B \in U\} \cup \{\neg A\} \subseteq U'$. Clearly $U R^L_\Box U'$ and $A \notin U$.

(7) This can be seen in the similar way to (6). ∎

Lemma 4.10 *For any $A \in \Phi$ and any $U \in M^L$, $U \models^L A$ iff $A \in U$.*

Proof. We will prove this inductively by using Lemma 4.9 (2), (3), (4), (5), (6), (7) and the definition of $\models^L$. ∎

To show that a logic **L** with the axiom $\alpha_1 \cdots \alpha_m p \supset \beta_1 \cdots \beta_n p$ is complete with respect to some class of frames defined by certain conditions, we need to show the following.

Lemma 4.11 *Let $\alpha_{\hat{i}}, \beta_{\hat{j}} \in \{\Box, \blacksquare\}$. Suppose $m, n \geq 1$. If a normal logic **L** contains $\alpha_1 \cdots \alpha_m p \supset \beta_1 \cdots \beta_n p$, then $\forall U_{\hat{k}} \in M^L$ $(0 \geq \hat{k} \geq n)$ $(U_{\hat{k}} R^L_{\hat{k}+1} U_{\hat{k}+1} \Rightarrow \exists V_{\hat{l}} \in M^L$ $(0 \geq \hat{l} \geq m)$ $V_{\hat{l}} R^L_{\hat{l}+1} V_{\hat{l}+1})$, where $U_0 = V_0$ and $U_n = V_m$.*

Proof. Suppose that U_0 $(= V_0)$ and U_n $(= V_m)$ are given. Let

$$W_h = \{A | \alpha_h A \in V_{h-1}\} \cup \{\neg \alpha_{h+1} \cdots \alpha_m B | B \notin V_m\}.$$

By induction on h $(1 \leq h \leq m)$, we will show that W_h is **L**-consistent. Then there exists the **L**-maximal V_h containing W_h by Lemma 4.7.

Step 1: If $h = 1$,

$$W_1 = \{A | \alpha_1 A \in V_0\} \cup \{\neg \alpha_2 \cdots \alpha_m B | B \notin V_m\}.$$

Suppose that W_1 is not **L**-consistent. Then

$$\neg(A_1 \wedge \cdots \wedge A_k \wedge \neg\alpha_2 \cdots \alpha_m B_1 \wedge \cdots \wedge \neg\alpha_2 \cdots \alpha_m B_l) \in \mathbf{L}, \textit{i.e.}$$

$$(A_1 \wedge \cdots \wedge A_k) \supset (\alpha_2 \cdots \alpha_m B_1 \vee \cdots \vee \alpha_2 \cdots \alpha_m B_l) \in \mathbf{L}$$

for some $k, l \geq 0$, where $\alpha_1 A_i \in V_0$ for each i and $B_{i'} \notin V_n$ for each i'. Let $B = (B_1 \vee \cdots \vee B_l)$. Then since

$$(\alpha_2 \cdots \alpha_m B_1 \vee \cdots \vee \alpha_2 \cdots \alpha_m B_l) \supset \alpha_2 \cdots \alpha_m B \in \mathbf{L}$$

by Lemma 4.6 (2), it follows that

$$(A_1 \wedge \cdots \wedge A_k) \supset \alpha_2 \cdots \alpha_n B \in \mathbf{L}.$$

Hence, by Lemma 4.6 (1),

$$(\alpha_1 A_1 \wedge \cdots \wedge \alpha_1 A_k) \supset \alpha_1 \alpha_2 \cdots \alpha_m B \in \mathbf{L}.$$

Since $\alpha_1 A_i \in V_0$ for each i, $\alpha_1\alpha_2\cdots\alpha_m B \in \mathbf{L}$. As $\alpha_1\cdots\alpha_m B \supset \beta_1\cdots\beta_n B \in \mathbf{L}$ by hypothesis, $\beta_1\cdots\beta_n B \in V_0$ $(= U_0)$. Hence $B \in U_n$ $(= V_m)$, since $U_0 R^L_{\beta_1} U_1, \cdots, U_{n-1} R^L_{\beta_n} U_n$ hold. But this contradicts $B_{i'} \notin V_m$ for all i'.

Step 2: Assume that $W_j = \{A|\alpha_j A \in V_{j-1}\} \cup \{\neg\alpha_{j+1}\cdots\alpha_m B|B \notin V_m\}$ is $\mathbf{L}$- consistent. Then there exists a $\mathbf{L}$-maximal V_j containing W_j, which satisfies that

$$\alpha_{j+1}\cdots\alpha_m B \in V_j \text{ implies } B \in V_m.$$

Now suppose that $W_{j+1} = \{A|\alpha_{j+1} A \in V_j\} \cup \{\neg\alpha_{j+2}\cdots\alpha_m B|B \notin V_m\}$ is not $\mathbf{L}$-consistent. Then

$$\neg(A_1 \wedge \cdots \wedge A_k \wedge \neg\alpha_{j+2}\cdots\alpha_m B_1 \wedge \cdots \wedge \neg\alpha_{j+2}\cdots\alpha_m B_l) \in \mathbf{L},$$

for some $k, l \geq 0$ where $\alpha_{j+1} A_i \in V_j$ for each i and $B_{i'} \notin V_m$ for each i'. Let $B = (B_1 \vee \cdots \vee B_l)$. Then since

$$(\alpha_{j+2}\cdots\alpha_m B_1 \vee \cdots \vee \alpha_{j+2}\cdots\alpha_m B_l) \supset \alpha_{j+2}\cdots\alpha_m B \in \mathbf{L}$$

by Lemma 4.6 (2), it follows that

$$(A_1 \wedge \cdots \wedge A_k) \supset \alpha_{j+1}\cdots\alpha_m B \in \mathbf{L},$$

and so, by Lemma 4.6 (1),

$$(\alpha_{j+1} A_1 \wedge \cdots \wedge \alpha_{j+1} A_k) \supset \alpha_{j+1}\alpha_{j+2}\cdots\alpha_m B \in \mathbf{L}.$$

Since $\alpha_{j+1} A_i \in V_j$ for each i, $\alpha_{j+1}\alpha_{j+2}\cdots\alpha_m B \in V_j$. By hypothesis, $B \in V_m$. Hence $B_{\tilde{i}} \in V_m$ for some $\tilde{i}$. But this contradicts $B_{i'} \notin V_m$ for all i'. Thus W_{j+1} is $\mathbf{L}$-consistent. ∎

Theorem 4.12 (Kripke completeness)
Let $\mathbf{L} = \mathbf{M} \bigotimes \mathbf{N} \bigoplus \{\tau p_i \supset \sigma_i p \mid i \in I\}$ *for* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$, *where both* τ_i *and* σ_i *are sequences of* $\Box$ *and* $\blacksquare$. *Then for any* $A \in \Phi$, $A \in \mathbf{L}$ *iff* $\mathcal{F} \models A$ *for any* $\mathcal{F} = (M, R_\Box, R_\blacksquare)$ *in the class of frames in which* $R_\Box$ *and* $R_\blacksquare$ *correspond to* $\mathbf{L}$.

Proof. We will mention the case $\tilde{\mathbf{L}} = \mathbf{S4}_\Box \bigotimes \mathbf{S5}_\blacksquare \bigoplus \{\tau_i p \supset \sigma_i p \mid i \in I\}$, where both τ_i and σ_i are sequences of $\Box$ and $\blacksquare$. The other cases can be seen in the same way.

[$\Rightarrow$] If $R_\Box$ is reflexive and transitive, $R_\blacksquare$ is equivalence relation and $R_\Box$ and $R_\blacksquare$ satisfy $(**)$ in Lemma 4.3, then it is clear that $A \in \tilde{\mathbf{L}}$ implies $\mathcal{F} \models A$.

[$\Leftarrow$] If A is valid in all frames in which $R_\Box$ is reflexive and transitive, $R_\blacksquare$ is equivalence relation and $R_\Box$ and $R_\blacksquare$ satisfy $(**)$ in Lemma 4.3, then $\mathcal{F}^{\tilde{L}} \models A$. Now suppose $A \notin \tilde{\mathbf{L}}$, and then $\{\neg A\}$ is $\tilde{\mathbf{L}}$-consistent. By Lemma 4.7, there exists a $\tilde{\mathbf{L}}$-maximal U such that $\neg A \in U$ (i.e. $A \notin U$). So $U \not\models^{\tilde{L}} A$ by Lemma 4.10, but this contradicts $\mathcal{F}^{\tilde{L}} \models A$. Hence $A \in \tilde{\mathbf{L}}$. ∎

This theorem can be extended in the following way. Recall that a modal logic $\mathbf{L}$ is *canonical* if $\mathcal{F}^L \models \mathbf{L}$. Monomodal logics **K**, **KT**, **K4**, **S4** and **S5** are canonical. Similarly to Theorem 4.12, we can prove the following.

Theorem 4.13 (Kripke completeness) *If both* $\mathbf{M}$ *and* $\mathbf{N}$ *are canonical monomodal logic, then* $\mathbf{M} \otimes \mathbf{N} \bigoplus \{\tau_i p \supset \sigma_i p \mid i \in I\}$, *where both* τ_i *and* σ_i *are sequences of* $\Box$ *and* $\blacksquare$ *is Kripke complete.*

After we proved this theorem, we found a stronger result in [3], which says that formulas of the form $\tau_i p \supset \sigma_i p$ in our theorem can be replaced by Sahlqvist formulas.

4.3 Finite model property

In the canonical model $(M^L, R^L_\Box, R^L_\blacksquare, \models^L)$ for a logic $\mathbf{L}$, M^L can not be restricted to a finite set. But it would be quite useful if we could get a finite model in which a given unprovable formula is false. A logic $\mathbf{L}$ has the finite model property if the following condition is satisfied:

$$\text{if } A \notin \mathbf{L}, \text{ then there is a finite } \mathbf{L}\text{-model } \mathcal{M} \text{ such that } \mathcal{M} \not\models A.$$

In this section, the finite model property of several dependently axiomatizable bimodal logics are shown. These results may give us a hint of general results on the finite model property.

Suppose we have a model $(M, R_\Box, R_\blacksquare, \models)$, and $\Psi(A)$ is finite set which contains $Sub(A)$. Now we introduce the filtration method. By means of the filtrations, we will show finite model property of some bimodal logics. We define a binary relation $\sim$ on M as follows:

$a \sim b \iff$ for any $C \in \Psi(A)$, $a \models C$ iff $b \models C$.

Clearly $\sim$ is equivalence relation. Let $[a]$ denote the equivalence class of a, i.e. $[a] = \{x \in M | a \sim x\}$. Let $\alpha \in \{\Box, \blacksquare\}$. Then a binary relation S on $M/\sim$ is filtration if the following conditions are satisfied:

- $aR_\alpha b \implies [a]S_\alpha[b]$,
- $[a]S_\alpha[b] \implies$ for any $\alpha B \in \Psi(A)$, $a \models \alpha B$ implies $b \models B$.

In this section, we will prove the finite model property by means of the following three kinds of filtrations.

- Coarsest filtration:

$$[a]S_\alpha[b] \iff \text{for any } \alpha B \in \Psi(A),\ a \models \alpha B \text{ implies } b \models B.$$

- Finest filtration:

$$[a]S_\alpha[b] \iff \text{there exist } a', b' \text{ such that } a \sim a',\ b \sim b' \text{ and } a'R_\alpha b'.$$

- Filtration for **S5**:

$$[a]S_\alpha[b] \iff \text{for any } \alpha B \in \Psi(A),\ a \models \alpha B \text{ implies } b \models \alpha B.$$

Theorem 4.14 *Let* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$. *Then* $\mathbf{M} \otimes \mathbf{N} \bigoplus \{\Box p \supset \blacksquare p\}$ *has the finite model property.*

Here, we will give a detail proof of the finite model property of $\mathbf{S4} \otimes \mathbf{S4} \bigoplus \{\Box p \supset \blacksquare p\}$. Other cases can be treated similarly. Let $\mathbf{L_1} = \mathbf{S4} \otimes \mathbf{S4} \bigoplus \{\Box p \supset \blacksquare p\}$. We note that $\mathbf{L_1}$ is complete in any frame $(M, R_\Box, R_\blacksquare)$ where both $R_\Box$ and $R_\blacksquare$ are reflexive and transitive and $R_\blacksquare \subseteq R_\Box$. Suppose $A \notin \mathbf{L_1}$. Then by the completeness theorem (Theorem 4.12), there exists a $\mathbf{L_1}$-model $(M, R_\Box, R_\blacksquare, \models)$ such that for some $a_0 \in M$, $a_0 \not\models A$. Now we define $\Psi(A)$ as follows:

$$\begin{aligned}
&\Psi_1 = Sub(A),\\
&\Psi_2 = \{\blacksquare B | \Box B \in Sub(A)\},\\
&\Psi_3 = \{\blacksquare \Box B | \Box B \in Sub(A)\},\\
&\Psi_4 = \{\blacksquare \blacksquare \Box B | \Box B \in Sub(A)\},\\
&\Psi_5 = \{\Box \Box B | \Box B \in Sub(A)\},\\
&\Psi_6 = \{\blacksquare \blacksquare B | \blacksquare B \in Sub(A)\},\\
&\Psi_7 = \{\blacksquare \blacksquare B | \Box B \in Sub(A)\},
\end{aligned}$$

$$\Psi(A) = \bigcup_{i=1}^{7} \Psi_i.$$

It is easily seen that $\Psi(A)$ is finite.

$[a]S_{\Box}[b] \Longleftrightarrow$ for any $\Box B \in \Psi(A)$, $a \models \Box B$ implies $b \models B$,

$[a]S_{\blacksquare}[b] \Longleftrightarrow$ for any $\blacksquare B \in \Psi(A)$, $a \models \blacksquare B$ implies $b \models B$,

$[a] \models^* p \Longleftrightarrow a \models p$.

Further, consider the following models:

$(M/\sim, S_{\Box}, S_{\blacksquare}, \models^*)$.

If the number of formulas in $\Psi(A)$ is m, then the number of elements of $M/\sim$ is at most 2^m. So $M/\sim$ is a finite set.

Lemma 4.15

(1) For any $a, b \in M$, $aR_{\Box}b$ implies $[a]S_{\Box}[b]$.

(2) For any $a, b \in M$, $aR_{\blacksquare}b$ implies $[a]S_{\blacksquare}[b]$.

(3) For any $a \in M$, $[a]S_{\Box}[a]$

(4) For any $a, b, c \in M$, $[a]S_{\Box}[b]$ and $[b]S_{\Box}[c]$ implies $[a]S_{\Box}[c]$.

(5) For any $a \in M$, $[a]S_{\blacksquare}[a]$

(6) For any $a, b, c \in M$, $[a]S_{\blacksquare}[b]$ and $[b]S_{\blacksquare}[c]$ implies $[a]S_{\blacksquare}[c]$.

(7) For any $a, b \in M$, $[a]S_{\blacksquare}[b]$ implies $[a]S_{\Box}[b]$.

Proof. First, we note that both $R_{\Box}$ and $R_{\blacksquare}$ are reflexive and transitive.

(1) Suppose $aR_{\Box}b$. Then it is clear that for any $\Box B \in \Psi(A)$ if $a \models \Box B$ then $b \models B$.

(2) Similarly to (1).

(3) Since the relation $R_{\Box}$ is reflexive, $a \models \Box B$ implies $a \models B$.

(4) Suppose that $a \models \Box B$ for any $\Box B \in \Psi(A)$ and that $[a]S_{\Box}[b]$ and $[b]S_{\blacksquare}[c]$.

(i) The case of $\Box B \in \Psi_1$. If $a \models \Box B$, then $a \models \Box\Box B$ since $R_{\Box}$ is transitive, and thus $b \models \Box B$ by $\Box\Box B \in \Psi_5$ and $[a]S_{\Box}[b]$. Hence $c \models B$ by $\Box B \in \Psi_1$ and $[b]S_{\Box}[c]$.

(ii) The case of $\Box B \in \Psi_5$. Then the $\Box B$ is the form $\Box\Box B'$. If $a \models \Box\Box B'$, then $b \models \Box B'$ since $\Box\Box B' \in \Psi_5$ and $[a]S_{\Box}[b]$, and therefore $b \models \Box\Box B'$ since $R_{\Box}$ is transitive. Hence $c \models \Box B'$ by $\Box\Box B' \in \Psi_5$ and $[b]S_{\Box}[c]$.

(5) Similarly to (3).

(6) Suppose that $a \models \blacksquare B$ for any $\blacksquare B \in \Psi(A)$ and that $[a]S_{\blacksquare}[b]$ and $[b]S_{\blacksquare}[c]$.

(i) Similarly to (4) (i).

(ii) The case of $\blacksquare B \in \Psi_2$. In this case, $\blacksquare\blacksquare B \in \Psi_7$ since $\Box B$ is in Ψ_1. If $a \models \blacksquare B$, then $a \models \blacksquare\blacksquare B$ since $R_{\blacksquare}$ is transitive, and so $b \models \blacksquare B$ by $\blacksquare\blacksquare B \in \Psi_7$ and $[a]S_{\blacksquare}[b]$. Hence $c \models B$ by $\blacksquare B \in \Psi_2$ and $[b]S_{\blacksquare}[c]$.

(iii) The case of $\blacksquare B \in \Psi_3$. In this case, $\blacksquare B$ is the form $\blacksquare\Box B'$, and $\blacksquare\blacksquare\Box B' \in \Psi_4$ and since $\Box B'$ is in Ψ_1. If $a \models \blacksquare\Box B'$, then $a \models \blacksquare\blacksquare\Box B'$ since $R_\blacksquare$ is transitive, and so $b \models \blacksquare\Box B'$ by $\blacksquare\blacksquare\Box B' \in \Psi_4$ and $[a]S_\blacksquare[b]$. Hence $c \models \Box B'$ by $\blacksquare\Box B' \in \Psi_3$ and $[b]S_\blacksquare[c]$.

(iv) The case of $\blacksquare B \in \Psi_4$. Then $\blacksquare B$ is the form $\blacksquare\blacksquare\Box B'$. If $a \models \blacksquare\blacksquare\Box B'$, then $b \models \blacksquare\Box B'$ since $\blacksquare\blacksquare\Box B' \in \Psi_4$ and $[a]S_\blacksquare[b]$. So $b \models \blacksquare\blacksquare\Box B'$ since $R_\blacksquare$ is transitive. Hence $c \models \blacksquare\Box B'$ by $\blacksquare\blacksquare\Box B' \in \Psi_4$ and $[b]S_\blacksquare[c]$.

(v) The case of $\blacksquare B \in \Psi_6$. Then $\blacksquare B$ is the form $\blacksquare\blacksquare B'$. If $a \models \blacksquare\blacksquare B'$, then $b \models \blacksquare B'$ by $\blacksquare\blacksquare B' \in \Psi_6$ and $[a]S_\blacksquare[b]$. Hence $b \models \blacksquare\blacksquare B'$ since $R_\blacksquare$ is transitive. Hence $c \models \blacksquare B'$ by $\blacksquare\blacksquare B' \in \Psi_6$ and $[b]S_\blacksquare[c]$.

(vi) The case of $\blacksquare B \in \Psi_7$. Then $\blacksquare B$ is the form $\blacksquare\blacksquare B$. If $a \models \blacksquare\blacksquare B'$, then $b \models \blacksquare B'$ by $\blacksquare\blacksquare B \in \Psi_7$ and $[a]S_\blacksquare[b]$. Hence $b \models \blacksquare\blacksquare B'$ since $R_\blacksquare$ is transitive. Hence $c \models \blacksquare B'$ by $\blacksquare\blacksquare B' \in \Psi_7$ and $[b]S_\blacksquare[c]$.

(7) Suppose $[a]S_\blacksquare[b]$ and $a \models \Box B$ for any $\Box B \in \Psi(A)$.

(i) The case of $\Box B \in \Psi_1$. In this case, $\blacksquare B \in \Psi_2$. If $a \models \Box B$, then $a \models \blacksquare B$ since $R_\blacksquare \subseteq R_\Box$. Hence $b \models B$ by $\blacksquare B \in \Psi_2$ and $[a]S_\blacksquare[b]$.

(ii) The case of $\Box B \in \Psi_5$. Then $\Box B$ is the form $\Box\Box B'$, and $\blacksquare\Box B' \in \Psi_1$ since $\Box B'$ is in Ψ_1. If $a \models \Box\Box B'$, then $a \models \blacksquare\Box B'$ since $R_\blacksquare \subseteq R_\Box$. Hence $b \models \Box B'$ by $\blacksquare\Box B' \in \Psi_3$ and $[a]S_\blacksquare[b]$. ∎

We can see that the model $(M/\sim, S_\Box, S_\blacksquare, \models^*)$ is $\mathbf{L_1}$-model by Lemma 4.15 (3), (4), (5), (6) and (7). Now, the finite model property of $\mathbf{S4} \otimes \mathbf{S4} \oplus \{\Box p \supset \blacksquare p\}$ is derived by combining the following lemma.

Lemma 4.16 *If $B \in \Psi(A)$, then for any $a \in M$, $a \models B \Longleftrightarrow [a] \models^* B$.*

Proof. We will prove this by induction on the formation of B.

- The case where B is a propositional variables is given by the definition of $\models^*$.
- The case where B is of the form $C \wedge D$, $C \vee D$ or $C \supset D$ is straightforward.
- The case where $B = \Box C$. [$\Rightarrow$] Suppose that $a \models \Box C$. If $[a]S_\Box[b]$ then $b \models C$ since $\Box C \in \Psi(A)$. By the induction hypothesis $b \models^* C$. Hence $[a] \models^* \Box C$.

[$\Leftarrow$] Suppose $[a] \models^* \Box C$. If $aR_\Box b$ then $[a]S_\Box[b]$ by Lemma 4.15 (1). Since $[a]S_\Box[b]$ and $[a] \models^* \Box C$, and so $[b] \models^* C$. By induction hypothesis $b \models C$. Thus $a \models \Box C$.

The case where $B = \blacksquare C$ can be shown in the similar way. This time we use Lemma 4.15 (2). ∎

We can proved the other cases in the similar way. Let $\mathbf{M}, \mathbf{M}' \in \{\mathbf{K}, \mathbf{KT}\}$ and $\mathbf{N}, \mathbf{N}' \in \{\mathbf{K4}, \mathbf{S4}\}$, and then the finite model properties of $\mathbf{M} \otimes \mathbf{M}' \oplus \{\Box p \supset \blacksquare p\}$, $\mathbf{M}_{\Box} \otimes \mathbf{N}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$, $\mathbf{N}_{\Box} \otimes \mathbf{M}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$, and $\mathbf{N} \otimes \mathbf{N}' \oplus \{\Box p \supset \blacksquare p\}$ can be proved by taking $\Psi_1 \cup \Psi_2$, $\Psi_1 \cup \Psi_2 \cup \Psi_6 \cup \Psi_7$, $\Psi_1 \cup \Psi_2 \cup \Psi_3 \cup \Psi_5$ and $\bigcup_{i=1}^{7} \Psi_i$ as $\Psi(A)$, respectively. Next we will discuss logics with **S5** as one of the fusions.

Theorem 4.17 *Let* $\mathbf{M} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$. *Then* $\mathbf{M}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$ *has the finite model property.*

Here, the finite model property of $\mathbf{S4}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$ will be mainly discussed. Let $\mathbf{L_2} = \mathbf{S4}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$. Suppose $A \notin \mathbf{L_2}$. Then by the completeness theorem (Theorem 4.12), there exists a model $(M, R_{\Box}, R_{\blacksquare}, \models)$ such that for some $a_0 \in M$, $a_0 \not\models A$. Now we define $\Psi(A)$ as follows:

$$\begin{aligned}
\Psi_1 &= Sub(A),\\
\Psi_2 &= \{\blacksquare B | \Box B \in Sub(A)\},\\
\Psi_3 &= \{\blacksquare \Box B | \Box B \in Sub(A)\},\\
\Psi_4 &= \{\Box \Box B | \Box B \in Sub(A)\},
\end{aligned}$$

$$\Psi(A) = \bigcup_{i=1}^{4} \Psi_i.$$

It is easily seen that $\Psi(A)$ is finite.

$[a] S_{\Box} [b] \iff$ for any $\Box B \in \Psi(A)$, $a \models \Box B$ implies $b \models B$,
$[a] S_{\blacksquare} [b] \iff$ for any $\blacksquare B \in \Psi(A)$, $a \models \blacksquare B$ iff $b \models \blacksquare B$,
$[a] \models^* p \iff a \models p$.

Further, consider the following models:
$(M/\sim, S_{\Box}, S_{\blacksquare}, \models^*)$.
If the number of formulas in $\Psi(A)$ is m, then the number of elements of $M/\sim$ is at most 2^m. So $M/\sim$ is a finite set.

Lemma 4.18

(1) For any $a, b \in M$, $a R_{\Box} b$ *implies* $[a] S_{\Box} [b]$.
(2) For any $a, b \in M$, $a R_{\blacksquare} b$ *implies* $[a] S_{\blacksquare} [b]$.
(3) For any $a \in M$, $[a] S_{\Box} [a]$
(4) For any $a, b, c \in M$, $[a] S_{\Box} [b]$ *and* $[b] S_{\Box} [c]$ *implies* $[a] S_{\Box} [c]$.

(5) For any $a \in M$, $[a]S_{\blacksquare}[a]$

(6) For any $a, b, c \in M$, $[a]S_{\blacksquare}[b]$ and $[a]S_{\blacksquare}[c]$ implies $[b]S_{\blacksquare}[c]$.

(7) For any $a, b \in M$, $[a]S_{\blacksquare}[b]$ implies $[a]S_{\Box}[b]$.

Proof. (1), (2), (3), (4) and (5) can be proved in the similar way to Lemma 4.15 (1), (2), (3), (4) and (5). Then we will show the others.

(6) It is clear that $S_{\blacksquare}$ is transitive and symmetric. Then suppose $[a]S_{\blacksquare}[b]$ and $[a]S_{\blacksquare}[c]$, for any $a, b \in M$. Since $S_{\blacksquare}$ is symmetric, $[b]S_{\blacksquare}[a]$ and $[a]S_{\blacksquare}[c]$, and so $[b]S_{\blacksquare}[c]$ by the transitivity for $S_{\blacksquare}$.

(7) Suppose $[a]S_{\blacksquare}[b]$ and $a \models \Box B$ for $\Box B \in \Psi(A)$.

(i) The case of $\Box B \in \Psi_1$. In this case, $\blacksquare B \in \Psi_2$. Then $a \models \blacksquare B$ since $R_{\blacksquare} \subseteq R_{\Box}$. So $b \models \blacksquare B$ by $\blacksquare B \in \Psi_2$ and $[a]S_{\blacksquare}[b]$. Since $R_{\blacksquare}$ is reflexive, $b \models B$.

(ii) The case of $\Box B \in \Psi_3$. In this case, $\Box B$ is the form $\Box\Box B'$, and $\blacksquare\Box B' \in \Psi_4$ since $\Box B'$ is in Ψ_1. Then $a \models \blacksquare\Box B'$ since $R_{\blacksquare} \subseteq R_{\Box}$. So $b \models \blacksquare\Box B'$ by $\blacksquare\Box B' \in \Psi_4$ and $[a]S_{\blacksquare}[b]$. Since $R_{\blacksquare}$ is reflexive, $b \models \Box B'$.

In either case, we have shown that $b \models B$. Thus $[a]S_{\Box}[b]$. ∎

For the bimodal logic discussed above, we can also show that Lemma 4.16 holds by Lemma 4.18 (1)(2).

We can proved the other cases in the similar way. The finite model property of $\mathbf{K}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$ and $\mathbf{KT}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$ can be shown by taking $\Psi_1 \cup \Psi_2$ as $\Psi(A)$, and that of $\mathbf{K4}_{\Box} \otimes \mathbf{S5}_{\blacksquare} \oplus \{\Box p \supset \blacksquare p\}$ can be proved by taking $\Psi_1 \cup \Psi_2 \cup \Psi_3 \cup \Psi_4$ as $\Psi(A)$. But we don't know at this moment whether $\mathbf{S5}_{\Box} \otimes \mathbf{N}_{\blacksquare}$ has the finite model property, for $\mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$.

In the similar way, it is proved that some other logics have finite model property.

Theorem 4.19 *The following logics enjoy finite model property.*

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \blacksquare\Box p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M}_{\Box} \otimes \mathbf{N}_{\blacksquare} \oplus \{\Box p \supset \Box\blacksquare p\}$, *where* $\mathbf{M} \in \{\mathbf{K}, \mathbf{KT}\}$ *and* $\mathbf{N} \in \{\mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{S4}_{\Box} \otimes \mathbf{S4}_{\blacksquare} \oplus \{\Box p \supset \Box\blacksquare p\}$,

$\mathbf{S4}_{\Box} \otimes \mathbf{KT}_{\blacksquare} \oplus \{\Box p \supset \Box\blacksquare p\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \blacksquare\blacksquare p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \Box\blacksquare\Box p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{S4}\}$.

Proof. We can prove in the similar way to Theorem 4.14. ∎

Theorem 4.20
Let $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}\}$. *Then* $\mathbf{M} \otimes \mathbf{N} \bigoplus \{\alpha_1 \cdots \alpha_m p \supset \beta p\}$, *where* $\alpha_i, \beta \in \{\Box, \blacksquare\}$, *enjoy the finite model property.*

Here, the finite model property of $\mathbf{K} \otimes \mathbf{K} \bigoplus \{\alpha_1 \cdots \alpha_m p \supset \beta p\}$ will be mainly discussed. The other cases can be proved in the same way. Let $\mathbf{L_3} = \mathbf{K} \otimes \mathbf{K} \bigoplus \{\alpha_1 \cdots \alpha_m p \supset \beta p\}$. Suppose $A \notin \mathbf{L_3}$. Then by the completeness theorem (Theorem 4.12), there exists a model $(M, R_\Box, R_\blacksquare, \models)$ such that for some $a_0 \in M$, $a_0 \not\models A$.

$[a] S_\Box [b] \iff$ there exist a', b' such that $a \sim a'$, $b \sim b'$ and $a' R_\Box b'$
$[a] S_\blacksquare [b] \iff$ there exist a', b' such that $a \sim a'$, $b \sim b'$ and $a' R_\blacksquare b'$
$[a] \models^* p \iff a \models p$

Then, consider the following model:
$(M/\sim, S_\Box, S_\blacksquare, \models^*)$.

Lemma 4.21 *For any* $a, b \in M$, *if* $[a] S_\beta [b]$ *then there exist* c_i $(0 \leq i \leq m-1)$ *such that* $[c_i] S_{\alpha_i} [c_{i+1}]$, *where* $c_0 = a$ *and* $c_m = b$.

Proof. Suppose $[a] S_\beta [b]$. Then there exist a', b' such that $a \sim a'$, $b \sim b'$ and $a' R_\beta b'$. Since $R_\beta \subseteq R_{\alpha_1} \circ \cdots \circ R_{\alpha_m}$, there exist c_i $(0 \leq i \leq m-1)$ such that $c_i R_{\alpha_i} c_{i+1}$, where $c_0 = a$ and $c_m = b$. Hence $[c_i] S_{\alpha_i} [c_{i+1}]$. ∎

We can also show that Lemma 4.16 for the above logic holds.

It is hard to develop a general semantical study of dependently axiomatizable bimodal logics at this moment. The finite model property of general bimodal logics have been left unanswered as future work.

4.4 decidability

As an application of finite model property which was seen in the previous section, we will see the decidability for several bimodal logics. A bimodal logic $\mathbf{L}$ is finitely axiomatizable, if the logic $\mathbf{L}$ is obtained from the fusion of $\mathbf{K}_\Box$ and $\mathbf{K}_\blacksquare$ by adding finite axioms. In general, the following is known.

Theorem 4.22 *If finitely axiomatizable modal logic* $\mathbf{L}$ *has the finite model property, then* $\mathbf{L}$ *is decidable.*

The proof of the above can be referred in detail by [7] [2]. Then by Theorem 4.22, we can show the following.

Theorem 4.23 *The following bimodal logics are decidable.*

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \blacksquare p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M}_\Box \otimes \mathbf{S5}_\blacksquare \oplus \{\Box p \supset \blacksquare p\}$, *where* $\mathbf{M} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \blacksquare \Box p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M}_\Box \otimes \mathbf{N}_\blacksquare \oplus \{\Box p \supset \Box \blacksquare p\}$, *where* $\mathbf{M} \in \{\mathbf{K}, \mathbf{KT}\}$ *and* $\mathbf{N} \in \{\mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{S4}_\Box \otimes \mathbf{S4}_\blacksquare \oplus \{\Box p \supset \Box \blacksquare p\}$,

$\mathbf{S4}_\Box \otimes \mathbf{KT}_\blacksquare \oplus \{\Box p \supset \Box \blacksquare p\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \blacksquare \blacksquare p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{S4}\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\Box p \supset \Box \blacksquare \Box p\}$, *where* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}, \mathbf{S4}\}$,

$\mathbf{M} \otimes \mathbf{N} \oplus \{\alpha_1 \cdots \alpha_m p \supset \beta p\}$, *where* $\alpha_i, \beta \in \{\Box, \blacksquare\}$ *and* $\mathbf{M}, \mathbf{N} \in \{\mathbf{K}, \mathbf{KT}\}$.

Proof. Since these logics finitely axiomatizable and enjoy the finite model property, they are decidable by Theorem 4.22. ∎

Since the finite model property of general bimodal logics have been left unanswered, and so the decidability of them have been also left unanswered as future work.

Conclusions and Remarks

In the syntactic study, we could derive the cut-elimination property of $\mathbf{M}^* \otimes \mathbf{N}^*$ for any $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*\}$. Further, by extending Takano's method for $\mathbf{S5}^*$, we proved that every proof in $\mathbf{M'}^* \otimes \mathbf{S5}^*$ can be transformed into suitable one with same end-sequent for $\mathbf{M'}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*, \mathbf{S5}^*\}$. So these bimodal logics have subformula property. For these logics, as results, several logical properties like the decidability and the Craig's interpolation theorem have been shown. As for dependently axiomatizable bimodal logics, however, it is difficult to find sequent systems in which the cut-elimination property holds. Therefore, it would be necessary to develop semantical method for them.

In the semantical approach, we could prove the Kripke completeness of bimodal logics which are obtained from fusions of basic monomodal logics by adding axioms of the form $\alpha_1 \cdots \alpha_m p \supset \beta_1 \cdots \beta_n p$, where $\alpha_i, \beta_j \in \{\Box, \blacksquare\}$, by constructing the canonical model. But the finite model property of the logics hasn't been unsolved in general for its difficulty of the study of the dependently axiomatizable bimodal logics. So as a steppingstone to future work of the logics, the finite model property of some bimodal logics with special interdependent axioms could be obtained. As an application, the decidability of the bimodal logics which enjoy the finite model property could be proved.

As the future work, some logical properties of more general dependently axiomatizable bimodal logics are expected. Further, general study of multimodal logics with more modalities is interesting one.

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