Title	二重様相論理に関する幾つかの結果
Author(s)	丸山，晃生
Citation	
Issue Date	1999-03
Type	Thesis or Dissertation
Text version	author
URL	http://hdl.handle.net/10119/1247
Rights	
Description	Supervisor:小野 寛晰，情報科学研究科，修士

Some results on bimodal logics

Akio Maruyama

School of Information Science,
Japan Advanced Institute of Science and Technology

February 15, 1999

Keywords: bimodal logics, subformula property, finite model property, decidability, Kripke completeness .

1 Introduction

Many reasonings which appear in daily thought are often influenced by situations, states, the passage of time and so on. The introduction of modal logics to take the situations, states and the passage of time into consideration is very useful. The various logical properties of many monomodal logics have been found out already. Many monomodal logics have been investigated well since the early time of 20th century. On the other hand, for multimodal logics, that is, modal logics with several modalities, however, even the most elementary questions concerning completeness, decidability and so on haven't been unsolved. From the point of view of application of modal logics, modal logics with one modality are sometimes not sufficient, and hence introduction of several modalities will be necessary in many situations. For independently axiomatizable modal logics, the notion of the fusion of them were firstly introduced by S. Thomason 1980. Recently, M. Kracht and F. Wolter [1] obtained general results on independently axiomatizable bimodal logics by using Kripke type semantics.

This paper presents a study of bimodal logics, and discuss these logics by both syntactic and semantical method. 1) In syntactic study, for some sequent systems corresponding to fusions of basic monomodal logics, we obtain the several logical properties like the cut-elimination property, the decidability and the Craig's interpolation theorem. 2) In semantical approach, we discuss bimodal logics obtained from fusion by adding the interdependent axioms. For some of these logics, the logical properties like the Kripke completeness, finite model property and decidability.

2 Bimodal logics

In this paper, two modalities $\Box$ and $\blacksquare$ are introduced. Both $\Box A$ and $\blacksquare A$ are formulas when A is a formula. In particular, if A is in a logic $\mathbf{L}$, then both $\Box A$ and $\blacksquare A$ are in $\mathbf{L}$. Let $\mathbf{M}$ and $\mathbf{N}$ be monomodal logics with the modalities $\Box$ and $\blacksquare$, respectively. The fusion of $\mathbf{M}$ and $\mathbf{N}$, denoted by $\mathbf{M} \otimes \mathbf{N}$, is the least bimodal logic containing both $\mathbf{M}$ and $\mathbf{N}$.

$\mathbf{K}$ denotes the least modal logic containing the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$. Any modal logic with the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is called a normal modal logic. The least modal logic containing the set $\mathbf{L} \cup Q$ is denoted by $\mathbf{L} \bigoplus Q$. Then the well-known normal modal logics are obtained as follows:

$\mathbf{KT} = \mathbf{K} \bigoplus \{\Box p \supset p\}$,
$\mathbf{K4} = \mathbf{K} \bigoplus \{\Box p \supset \Box\Box p\}$,
$\mathbf{S4} = \mathbf{K} \bigoplus \{\Box p \supset p, \Box p \supset \Box\Box p\}$,
$\mathbf{S5} = \mathbf{K} \bigoplus \{\Box p \supset p, \neg\Box\neg p \supset \Box\neg\Box\neg p\}$.

We discuss the combinations of these basic monomodal logics as fusions.

3 Bimodal systems

We first describe the system $\mathbf{K}^*$, $\mathbf{KT}^*$, $\mathbf{S4}^*$ and $\mathbf{S5}^*$ for the logics $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{S4}$ and $\mathbf{S5}$, respectively. The system $\mathbf{K}^*$ is obtained from the system $\mathbf{LK}$ by adding the following inference rule:

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box)\ .$$

Let $\mathbf{LK}^+$ be the system obtained form $\mathbf{LK}$ by adding the following inference rules:

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to)\ .$$

Then the systems $\mathbf{KT}^*$, $\mathbf{S4}^*$ and $\mathbf{S5}^*$ are obtained from $\mathbf{LK}^+$ by adding the following inference rules, respectively:

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\to \Box)\ ,\quad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\ (\to \Box)\ \text{ and }\ \frac{\Box\Gamma \to \Box\Delta, A}{\Box\Gamma \to \Box\Delta, \Box A}\ (\to \Box)\ .$$

The sequent systems of the form $\mathbf{M}^* \otimes \mathbf{N}^*$ for $\mathbf{M}^*, \mathbf{N}^* \in \{\mathbf{K}^*, \mathbf{KT}^*, \mathbf{S4}^*, \mathbf{S5}^*\}$ is obtained from $\mathbf{M}^*$ and $\mathbf{N}^*$, simply by combining their inferences. Of course, it is necessary to distinguish one modality from another. It is easy to see that, for any formula C, $\to C$ is provable in $\mathbf{M}^* \otimes \mathbf{N}^*$ if and only if C is in $\mathbf{M} \otimes \mathbf{N}$.

When both $\mathbf{M}^*$ and $\mathbf{N}^*$ are not $\mathbf{S5}^*$, it is trivial to see that $\mathbf{M}^* \otimes \mathbf{N}^*$ enjoy cut-elimination property. If as least one of $\mathbf{M}^*$ and $\mathbf{N}^*$ is $\mathbf{S5}^*$, then $\mathbf{M}^* \otimes \mathbf{N}^*$ lacks it. For these systems, however, the subformula property can be proved by Takano's method [3]. As results, we can show some logical properties like the decidability and the Craig's interpolation theorem.

4 Semantical approach

Next, we discuss extensions of fusions by using Kripke type semantics. This paper presents bimodal logics obtained from fusions by adding axioms which are the form $\tau_i p \supset \sigma_i p$, where τ_i and σ_i are sequences of $\Box$ and $\blacksquare$.

If both **M** and **N** are canonical monomodal logics, then Kripke completeness of $\mathbf{M} \otimes \mathbf{N} \bigoplus \{\tau_i p \supset \sigma_i p \mid i \in I\}$, where τ_i and σ_i are sequences of $\Box$ and $\blacksquare$, can be proved by constructing a model, that is, the canonical model. The finite model property of the general logics,however, hasn't been left unanswered for its difficulty of the study of the dependently axiomatizable bimodal logics. So as a steppingstone to future work of the logics, for some of these logics, we can show the finite model property. As an application, then the decidability of the logics with the finite model property can be shown.

5 Conclusions

• the results of the syntactic study:

We proved the cut-elimination theorem of bimodal systems corresponding to fusions of several basic monomodal logics. Further, by extending Takano's method for $\mathbf{S5}^*$, we could proved that bimodal systems where at least one of the systems is $\mathbf{S5}^*$ enjoy the subformula property. As results, several logical properties like the decidability and the Craig's interpolation property have been shown.

• the results of the semantical approach:

We could see the Kripke completeness of bimodal logics which are obtained from fusions of two basic monomodal logics by adding certain interdependent axioms by constructing the canonical model. The finite model property of several dependently axiomatizable bimodal logics was obtained. As an application, the decidability of the logics with the finite model property could be shown.

As the future work, some logical properties of more general dependently axiomatizable bimodal logics are expected. Further, general study of multimodal logics with more modalities is interesting one.

References

[1] M.Kracht and F.Wolter, Properties of independently axiomatizable bimodal logics, Journal of Symbolic Logic 56, (1991) pp.1469-1485.

[2] M.Ohnishi and K.Matsumoto, Gentzen method in modal calculi, Osaka Math. J. 9, (1957) pp.113-130

[3] M.Takano, Subformula property as a substitute for cut-elimination in modal propositional logic, Mathematica Japonica 37, (1992) pp.1129-1145.