A relationship between classical linear logic and intuitionistic linear logic

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February 12, 2015

Keywords: classical linear logic, intuitionistic linear logic, double negation elimination.

Major logics can be decided into two kinds: classical ones and intuitionistic ones. Usually, intuitionistic one is a sublogic of its classical counterpart.

Classical logic (CL) is a logic that is more familiar to us. For example, it has been used in the class of high school mathmatics and truth table, and discussion of the general mathematics. In CL, the excluded middle formula ($\neg p \vee p$) and the double negation elimination formula ($\neg\neg p \to p$) are provable.

Intuitionistic logic (IL) is a sublogic of CL, and that has been formalized the Brouwer's intuitionism by Heyting. In IL, it is known that the excluded middle formula and the double negation elimination formula are not provable. The widely-known CurryHoward correspondence shows the correspondence between proofs in IL and computer programs. So, the research of IL is important from the viewpoint of information science.

By adding some set of excluded middle formulas or double negation elimination formulas to the assumption, IL has the same proof power as CL. i.e., IL proves same formulas as CL. It is formalized as the following proposition where $L \vdash \Gamma \Rightarrow A$ means $\Gamma \Rightarrow A$ is provable in logic L.

Proposition 1 ([2])**.** *For propositional logic, the following two conditions are equivalent;*

- $CL \vdash \Gamma \Rightarrow A$,
- $IL \vdash \Gamma, \Pi \Rightarrow A$

where Π *is a set of excluded middle formulas for variables in* $\Gamma \Rightarrow A$.

Let us consider about $CL \vdash p \Rightarrow p$. By the above proposition, we get $IL \vdash p, p \vee \neg p \Rightarrow p$. But, we can also get $IL \vdash p \Rightarrow p$. So, we may not need excluded middle formulas for all variables in $\Gamma \Rightarrow A$. Then, it is natured to ask, for which propositional variables, we really need the excluded middle formula. For this problem, Ishihara [4] gave a smaller set of Π.

Linear logic is a substructural logic introduced by Girard [3]. In Linear logic, we do not admit weakening and contraction rules. Therefore, assumptions are used exactly once in a proof, and treated as resources.

This thesis describes a relationship between the classical and intuitionistic multiplicative implicational linear logics (MILL). This is a sublogic of linear logic consisting of a implicational connective. Since most tricky in propositional connectives, we chose a multiplicative implicational connective. Intuitionistic MILL (MIILL) has the same proof power as classical MILL (MICLL). Next, we describe for multiplicative implicational linear logic with zero (MIZLL). 0 is treated as the bottom symbol. The double negation elimination for linear logic is $((p \multimap 0) \multimap 0) \multimap p$. Since we do not have the disjunction, we can not describe the excluded middle formula. So, we add a set of double negation elimination formulas.

Our goal is to find a smaller multiset of *DNE* such that Φ in the above proposition. In previous research by [4, 5], the set added to assumption of intuitionistic is decided only by the final formula. It seems that lacking the weakening and contraction rules in linear logic makes the situation more complicated. At the current moment we could not decide by the final formula, but we need to analyze the derivation process of final formula to find a appropriate set.

We use the inductive proof structure (IPS). This is a graph theoretical formal system which is used in linear logic. IPS for the multiplicative fragment is given by Girard [3]. We propose IPSc for the implication and 0 fragments of linear logic, and IPSi for MIILL. We show that the following two conditions are equivalent;

- There exists an IPSc α such that $TN(\alpha) = \Gamma_{\mathrm{L}} \cup \{x^{A_{\mathrm{R}}}\}$,

- There exists an IPSi α_i such that $TN(\alpha_i) = \Gamma_{\mathrm{L}} \cup DNE \cup \{x^{A_{\mathrm{R}}}\}$

where $\Delta = \Delta'_{\mathrm{R}} \cup \{x^{A_{\mathrm{R}}}\}$, and DNE is defined as follows;

$$DNE = \{z^{(\sim\sim p \multimap p)_{\mathrm{L}}} \mid (y, p_{\mathrm{R}}) \in L_\alpha, y \notin \Delta'_{\mathrm{R}}\}.$$

We mention $z^{(\sim\sim p \multimap p)_{\mathrm{L}}}$ for $y^{p_{\mathrm{R}}}$ exists in Γ_{L}. As the find result of this thesis, we show that the set DNE can be replaced a smaller set S, where

$$S = \left\{ z^{(\sim\sim p \multimap p)_{\mathrm{L}}} \;\middle|\; \begin{array}{l} (y, p_{\mathrm{R}}) \in L_\alpha, \\ y \notin \Delta'_{\mathrm{R}}, \\ \neg \left(\begin{array}{l} LE(y) \neq EN(y) \wedge \\ \forall {y_0}^{0_{\mathrm{R}}} \in V_\alpha.(LE(y) \not\smile y_0) \wedge \\ EN(y) \in G_\alpha \end{array} \right) \end{array} \right\}$$

and where $G_\alpha = \{x \in V_\alpha \mid \forall x_1, x_2 \in V_\alpha.(x \smile x_1 \wedge x \smile x_2 \rightarrow x_1 \smile x_2)\}$.

The present thesis is organized as follows. In Chapter 2, we define linear logic, and the sequent calculus for linear logic. Using it, we describe various results such as the cut elimination in linear logic. We also mention formal systems other than the sequent calculus. In Chapter 3, we describe IPS. At the beginning, we show that IPS for the multiplicative fragment and proofnets are equivalent. For IPS which we defined, we redefine some concepts for formula such as positivity, which we need later. Chapter 4 is our main results. Because it seems our final result is not the optimal one, we conclude this with the discussion to improve it in Chapter 5.

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