Using conditional copula to estimate Value at Risk in Vietnam's foreign exchange market

Vu-Linh Nguyen and Van-Nam Huynh

Japan Advanced Institute of Science and Technology, JAIST
{vulinh, huynh}@jaist.ac.jp

Abstract. In this paper, we briefly review the basics of copula theory and the problem of estimating Value at Risk (VaR) of portfolio composed by several assets. We present two VaR estimation models in which each return series is assumed to follow AR(1)-GARCH(1, 1) model and the innovations are simultaneously generated using Gaussian copula and Student t copula. The presented models are applied to estimate VaR of a portfolio consisting of 6 currencies to VND. The results are compared with results from two VaR estimation models using AR(1)-GARCH(1, 1) model and the innovations are separately generated using univariate standard normal and Student t distribution.

Keywords: copula, value at risk (VaR), AR(1)-GARCH(1, 1), foreign exchange market

1 Introduction

The theory of copula is a very powerful tool for modeling joint distributions because it does not require the assumption of joint normality which is rarely adequate in application [3, 15]. Applications based on copula theory center around Sklar theorem which allows to decompose any N-dimensional joint distribution into its N marginal distributions and a copula function which describes the dependence structure between the variables [3, 11, 15, 17]. Furthermore, the converse of Sklar theorem can be used to learn the dependence structure given prior information about distribution and copula.

During the last years, copula based models have been increasingly applied in finance and economics. Those models have shown advantages comparing with the traditional models, specially where dependency is non-linear and the involved random variables follow different univariate distributions. The book of Nelsen (2005) provided a very good introduction about copula including the basics of copula theory and the advantages of using copulas to construct the joint distribution and learn the dependence [11]. Also, Cherubini *et al.* (2004) provided a comprehensive guide for applying copula in financial problems. For example, asset pricing, risk management and credit risk analysis [3]. Bouye *et al.* (2000) provided a readable statistical inference framework of copulas in the estimation problem [1]. Embrechts *et al.* (2002) highlighted the pitfalls when finding the multivariate models and suggested simulation algorithms to avoid those problem [6]. Georges (2001) used the normal copula to model options time of exercise and for derivative pricing[8]. Meneguzzo and Vecchiato (2002) used copula to model the risk of credit derivatives [10]. Cherubini and Luciano (2001) proposed a VaR estimation model using the Archimedean copula family and the historical empirical marginal distribution [2]. Fortin and Kuzmics (2002) used convex linear combinations of copula to estimate the VaR of a portfolio consisting of the FSTE and DAX stock indices [7]. Embrechts *et al.* (2003) used copula to learn the optimal bounds for risk measures of functions of dependent risks [5]. Rockinger and Jondeau (2001) used the Plackett copula with GARCH process with innovations modeled by the Student-t asymmetrical distribution to learn the change of dependence through time of daily return of stock market indices [16]. Patton and Andrew (2006) used conditional copula based models to explore the dependence structure of exchange rates [14]. Palaro and Hotta (2006) used the SJC, Plackett and Student-t conditional copula to modeled the innovations of GARCH process and used simulation methods to estimate VaR of portfolio composed by Nasdaq and S&P500 stock indices [15].

Vietnams foreign exchange (forex) market has remained relatively poorly developed despite more than two decades of general reform throughout the economy. Furthermore, existing works on this particular topic is rather limited. The papers of Nguyen *et al.* (2010), and Nguyen *et al.* (2009) point out that Vietnam's foreign exchange (forex) market has remained far less active and sophisticated than forex markets in many other countries [12, 13].

In this paper, we apply conditional copula based models to estimate VaR of a portfolio composed by 6 currencies to VND namely VND/AUD, VND/EUR, VND/GBP, VND/JPY, VND/USD and VND/CNY. The paper is organized as follow. Section 2 summarizes the markets risk problem and define VaR measure. The basic of copula is presented in Section 3. Section 4 presents two conditional copula based model for estimating VaR in which each return series is modeled using AR(1)-GARCH$(1, 1)$ model and innovations are simultaneously modeled using the Gaussian copula and Student t copula. In section 5, the presented models are applied to estimate VaR of a portfolio composed by 6 daily rate return and log return series of currencies. The results are compared with those obtained using simulation AR(1)-GARCH$(1, 1)$ models where innovations are separately modeled using univariate standard normal and student t distributions. The conclusion is given in section 6.

2 Market risk problem

Let us consider the problem of measuring the risk of holding an portfolio consists of N assets with returns at T-th day, denoted as $x_{n,T}$, given the historical data $\{x_{n,t}|t = 1, 2, \dots, T-1\}$, for $n = 1, 2, \dots, N$ [15]. The portfolio return at t-th day, denoted as x_t, is approximately equal to

$$x_t = \omega_1 x_{1,t} + \omega_2 x_{2,t} + \ldots + \omega_N x_{N,t}, \tag{1}$$

where ω_n is the portfolio weighs of asset n and $\sum_{n=1}^{N} \omega_{n,t} = 1$, for $t = 1, 2, \dots, T$, $n = 1, 2, \dots, N$.

In 1994, the American bank JP Morgan published a risk control method knows as Riskmetrics, based mainly on a parameter named *Value at Risk* (VaR). For a given time horizon T and confidence level p, the VaR is defined as the loss in market value over the time horizon T that is exceeded with probability $1-p$. More precisely, VaR of a portfolio can be defined as follows.

Definition 1. *Let $H_T(x_T \mid \Im)$ be the conditional distribution function of the returns of portfolio consists of x_1, x_2, ..., x_N at time T with conditional set $\Im$.*

$$\Im = \{X_{n,t} | n = 1, 2, \ldots, N, t = 1, 2, \ldots, T-1\} \tag{2}$$

$\Im$ represents the past information from day 1 to day $T-1$. Then the VaR of the portfolio at time T, with confidence level p, where $p \in (0,1)$ is defined by

$$VaR_T(p) = \inf\{s =: H_T(s \mid \Im) \geq 1-p\}. \tag{3}$$

Figure 1 illustrates VaR and p.

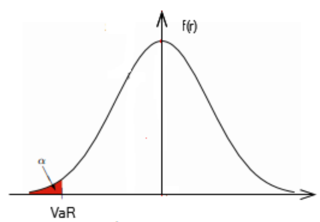

Fig. 1. The value at risk VaR and level $\alpha = 1-p$

In this paper, VaR is approximated using simulation models. The exchange rate series are assumed to fit the AR(1)-GARCH(1, 1) models with standard normal and student t innovations. The historical data of innovations is used to fit the multivariate copula which then used to generated values of innovations simultaneously. The generated values of portfolio distribution obtained by substituting the values of innovation into AR(1)-GARCH(1, 1) models. Finally, VaR is approximated as the corresponding element of simulation series after increasingly ordering the simulated values of portfolio distribution.

3 Copula

The concept of copulas was introduced by Sklar (1959), and has been recognized as a powerful tool for modeling dependence between random variables. Almost applications based on copula theory centralize around the Sklar theorem which ensures the relation between a N-dimensional distribution and a corresponding copula [3, 11].

A copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform.

Definition 2. *A N-dimensional copula (N-copula) is a function C, whose domain is $[0,1]^N$ and whose range is $[0,1]$ with the following properties:*

1, *For every* $u \in [0,1]^N$, $C(u) = 0$ *if at least one coordinate of* u *is* 0 *and if all coordinates of* u *are* 1 *except* u_n, *then* $C(u) = u_n$, $n = 1, 2, \ldots, N$.
2, *For every* $a, b \in [0,1]^n$ *such that* $a \leq b$, $V_C([a,b]) \geq 0$.

Sklar theorem is perhaps the most important result regarding copulas [17]. It ensures the relation between a N-dimensional distribution function and a corresponding copula and is used in essentially all applications of copula.

Theorem 1. *Let* H *be a* N*-dimensional distribution function with* 1 *dimensional margins* $F_1, F_2, \ldots, F_N$. *Then there exists a* N*-copulas* C *such that for all* x *in* $\mathbb{R}^N$,

$$H(x_1, x_2, \ldots, x_N) = C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)). \tag{4}$$

If $F_1, F_2, \ldots, F_N$ *are all continuous, then* C *is unique; Otherwise* C *is uniquely determined on* $RanF_1 \times RanF_2 \times \ldots \times RanF_N$.

Conversely, if C *is a* N*-copula and* $F_1, F_2, \ldots, F_N$ *are distribution functions, then the function* H *defined by (4) is a* N*-distribution function with margins.*

The following corollary is often known as the converse of Sklar theorem. We can use this corollary to find copula when the margins and joint distributions are given.

Corollary 1. *Let* $H, C, F_1, \ldots, F_N$ *be as in Theorem 1 and* $F_1^{(-1)}, \ldots, F_N^{(-1)}$ *be quasi-inverses of* $F_1, \ldots, F_N$, *respectively. Then, for any* u *in* $[0,1]^N$

$$C(u_1, u_2, \ldots, u_N) = H(F_1^{(-1)}(u_1), F_2^{(-1)}(u_2), \ldots, F_N^{(-1)}(u_N)). \tag{5}$$

By applying Sklar theorem and exploiting the relation between the distribution and the density function, we can easily derive the multivariate copula density

$$c(F_1(x_1), F_2(x_2), \ldots, F_N(x_N))$$

associated with a copula function $C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N))$:

$$\begin{aligned} h(x_1, x_2, \ldots, x_N) &= \frac{\partial^N [C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N))]}{\partial F_1(x_1) \ldots \partial F_N(x_N)} \prod_{n=1}^{N} f_n(x_n) \\ &= c(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) \prod_{n=1}^{N} f_n(x_n), \end{aligned} \tag{6}$$

where we define

$$c(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = \frac{f(x_1, x_2, \ldots, x_N)}{\prod_{n=1}^{N} f_n(x_n)}. \tag{7}$$

The results of Sklar theorem and its corollary can be extended in conditional case as follows

$$H(x_1, x_2, \ldots, x_N \mid \Im) = C(F_1(x_1 \mid \Im), F_2(x_2 \mid \Im), \ldots F_N(x_N \mid \Im) \mid \Im), \tag{8}$$

and

$$C(u_1, u_2, \ldots, u_N \mid \Im) = H(F_1^{-1}(u_1 \mid \Im), F_2^{-1}(u_2 \mid \Im), \ldots, F_N^{-1}(u_N \mid \Im) \mid \Im)), \tag{9}$$

where $\Im$ is given the conditional set.

The Gaussian copula is a distribution over the unit cube $[0,1]^N$. It is constructed from a multivariate normal distribution over $\mathbb{R}^N$ by using the probability integral transform. Formally, Gaussian copula is defined as follows

Definition 3. *Let R be a symmetric, positive definite matrix with $diag(R) = 1$ and let Φ_R the standardized multivariate normal distribution with correlation matrix R. Then the multivariate Gaussian copula is defined by*

$$C^{Gauss}(u_1, u_2, \ldots, u_N; R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_N)), \tag{10}$$

where Φ_R^{-1} denotes the inverse of the standard univariate normal distribution function Φ_R.

The associated multinormal copula density is

$$\begin{aligned} c^{Gauss}(\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_N); R) &= \frac{f^{Gauss}(x_1, x_2, \ldots, x_N)}{\prod_{n=1}^N f_n^{Gauss}(x_n)} \\ &= \frac{\frac{1}{(2\pi)^{\frac{N}{2}} |R|^{\frac{1}{2}}} exp(-\frac{1}{2} x^{'} R^{-1} x)}{\prod_{n=1}^N \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2} x_n^2)}, \end{aligned} \tag{11}$$

and hence, fixing $u_n = \Phi(x_n)$, and denote

$$\zeta = (\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_N)^{'})$$

the vector of the Gaussian univariate distribution functions, we have

$$c(u_1, u_2, \ldots, u_N; R) = \frac{1}{|R|^{\frac{1}{2}}} exp[-\frac{1}{2} \zeta^{'} (R^{-1} - I) \zeta]. \tag{12}$$

The student t copula is defined as follows

Definition 4. *Let R be a symmetric, positive definite matrix with $diag(R) = 1$ and let $T_{R,\nu}$ the standardized multivariate Student t distribution with correlation matrix R and ν degree of freedom. Then the multivariate Student t copula is defined as follows*

$$C(u_1, u_2, \ldots, u_N; R, \nu) = T_{R,\nu}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_N)), \tag{13}$$

where $t_\nu^{-1}(u_n)$ denotes the inverse of the Student t cumulative distribution function.

The associated Student t copula density is:

$$\begin{aligned} c(u_1, u_2, \ldots, u_N; R, \nu) &= \frac{f^{Student}(x_1, x_2, \ldots, x_N)}{\prod_{n=1}^N f_n^{Student}(x_n)} \\ &= |R|^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu+N}{2})}{\Gamma(\frac{\nu}{2})} [\frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}]^N \frac{(1 + \frac{\zeta^{'} R^{-1} \zeta}{\nu})^{-\frac{\nu+N}{2}}}{\prod_{n=1}^N (1 + \frac{\zeta_n^2}{\nu})^{-\frac{\nu+1}{2}}}, \end{aligned} \tag{14}$$

where $\zeta = (t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_N))^{'}$.

4 Using conditional copula to estimate VaR

This section presents two simulation models using conditional copulas to estimate VaR of a portfolio consists of several assets, namely AR(1)-GARCH$(1,1)$+ Gaussian copula and AR(1)-GARCH$(1,1)$+Student t copula. In those models, each return series is assumed to follow AR(1)-GARCH$(1,1)$ models and the innovations are simultaneously generated using copulas.

4.1 Modeling the marginal distributions

Returns series has been successfully modeled by ARMA-GARCH models [4,15]. In this paper, AR(1)-GARCH$(1,1)$ the models are used to modeled the margins as follows

$$\begin{aligned} x_{n,t} &= \mu_n + \phi_n x_{n,t-1} + \epsilon_{n,t}; \\ \epsilon_{n,t} &= \sigma_{n,t}\eta_{n,t}; \\ \sigma^2_{n,t} &= \alpha_n + \beta_n \epsilon^2_{n,t-1} + \gamma_n \sigma^2_{n,t-1}; \end{aligned} \tag{15}$$

where $\{\eta_{n,t}\}$ is white noise process, α_n, β_n, γ_n satisfy the condition of GARCH model: $\beta_n + \gamma_n < 1$, for $n = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, T$. The conditional distribution of the standardized innovations

$$\eta_{n,t} = \frac{\epsilon_{n,t}}{\sigma_{n,t}} | \Im_{n,T}, n = 1, 2, \ldots, N,$$

was modeled by white noises and denoted by $F_{n,t}$ in general case (the marginal distributions). We consider the case that $\eta_{n,t}$ are standard normal distributions and student t distributions with the same degree of freedom, $n = 1, 2, \ldots, N$.

The joint distribution of innovation vector $\eta_t = (\eta_{1,t}, \eta_{2,t}, \ldots, \eta_{N,t})$ is model by conditional copula.

Let $u_{n,t} = F_{n,t}(\eta_{n,t}|\Im)$, $F_{1,t}$, $F_{2,t}$, $\ldots$ and $F_{N,t}$ are marginal distributions conditioned to $\Im$, the information available up to time $T-1$. If the models were correctly specified then series $\{u_{n,t}|t = 1, 2, \ldots, T-1\}$ will be standard uniform series.

4.2 Modeling the copula

We assume that $(\eta_{1,T}, \eta_{2,T}, \ldots, \eta_{N,T})$ has multivariate distribution function

$$H_T(\eta_{1,T}, \ldots, \eta_{n,T}; \theta_{1,T}, \theta_{2,T} \mid \Im) \tag{16}$$

and continuous univariate marginal distribution functions $F_{n,T}(\eta_{n,T}; \theta_{n,T} \mid \Im)$ where $\Im = \{\eta_{n,t}|n = 1, 2, \ldots, N, t = 1, 2, \ldots, T-1\}$.

Since the marginal distributions are continuous, the conditional copula C_T is uniquely defined according to Sklar theory. Furthermore, we have

$$\begin{aligned} &C_T(F_{1,T}(\eta_{1,T}; \theta_{1,T} \mid \Im), \ldots, F_{N,T}(\eta_{N,T}; \theta_{N,T} \mid \Im); \theta_{2,T} \mid \Im) \\ &= H_T(\eta_{1,T}, \ldots, \eta_{N,T}; \theta_{1,T}, \theta_{2,T} \mid \Im), \end{aligned} \tag{17}$$

where $\theta_{1,T}$ is the margins' parameters and $\theta_{2,T}$ is copula's parameters of copula function C_T.

The parameters $\theta_{1,T}, \theta_{2,T}$ are estimated by using IFM (inference for the margins) method as follows

1. Firstly, we estimate the margin's parameters $\widehat{\theta}_{1,T}$ by performing the estimation of the univariate marginal distributions
$$\widehat{\theta}_{1,T} = \underset{\theta_{1,T}}{\operatorname{argmax}} \sum_{t=1}^{T-1} \sum_{n=1}^{N} \ln f_{n,T}(\eta_{n,t}; \theta_{1,T}). \tag{18}$$
2. Secondly, given $\widehat{\theta}_{1,T}$, we perform the estimation of the copula parameter $\widehat{\theta}_{2,T}$ as follows
$$\widehat{\theta}_{2,T} = \underset{\theta_{2,T}}{\operatorname{argmax}} \sum_{t=1}^{T-1} \ln c_T(F_{1,T}(\eta_{1,t}; \widehat{\theta}_{1,T}), \dots, F_{n,T}(\eta_{n,t}; \widehat{\theta}_{1,T}); \theta_{2,T}). \tag{19}$$

If the marginal distributions $F_{n,T}$ are standard normal distributions then C_T is a multivariate Gaussian copula with correlation matrix $\theta_{2,T} = R_T$. And if the marginal distributions $F_{n,T}$ are Student's t distributions with same degree of freedom $\theta_{1,T} = \nu_T$, then C_T is a Student's t copula with parameter $\theta_{2,tT} = R_T$. In this case, N marginal distributions are assumed to have the same degree of freedom.

4.3 Monte Carlo simulation

We use Gaussian and student t copula to simulate K vector
$$\eta_{T,k} = (\eta_{1,T,k}, \eta_{2,T,k}, \dots, \eta_{N,T,k}), \tag{20}$$
for $k = 1, 2, \dots, K$.

In case of multivariate Gaussian copula, the Monte Carlo simulation can be process as follows:

1. Find the Cholesky decomposition A of the linear correlation matrix R.
2. Simulate N i.i.d. $z = (z_1, z_2, \dots, z_N)^{'}$ from $N(0,1)$
3. Set $\eta^{'}_{T,k} = Az$

Similarly, we have the Monte Carlo simulation for multivariate student t copula

1. Find the Cholesky decomposition A of the linear correlation matrix R.
2. Simulate N i.i.d. $z = (z_1, z_2, \dots, z_N)^{'}$ from $N(0,1)$
3. Simulate a random variate s from χ^2_ν independent of z
4. Set $y = Az$
5. Set $\eta^{'}_{T,k} = \sqrt{(\nu/s)}y$

Then, we can simulate K vectors $(x_{1,T,k}, x_{2,T,k}, \dots, x_{N,T,K})$ and K values of $x_{T,k}$ by using model (15), for $k = 1, 2, \dots, K$. We order series $\{x_{T,k}\}$ in increasing order. Then we have the VaR of portfolio by $VaR_T(\alpha) = x_{T,K_p}$, equivalently, its exactly the Kp-th element of simulation series after ordering by increasing order.

5 Application

In this section, the presented copula based models are applied to estimate VaR of a portfolio composed by 6 currencies to VND namely VND/AUD, VND/EUR, VND/GBP, VND/JPY, VND/USD and VND/CNY. The results are compared with results of with AR(1)-GARCH(1, 1)+ N and AR(1)-GARCH(1, 1)+t in which that each return series is assume to follow AR(1)-GARCH(1, 1) models with the innovations are separately modeled by univariate standard normal and student t distribution.

5.1 Data description

The database contains 1328 daily closing prices, from January 2^{nd} 2007 to March 30^{th} 2012. We denote the log-returns, of 6 exchange rates by variable $x_1, x_2, \ldots, x_6$, respectively. Note that for each exchange rate n, the log-return at day t is defined by $x_{n,t} = \ln(p_{n,t}) - \ln(p_{n,t-1})$, where $p_{n,t}$ is the closing price of currency n at day t, $n = 1, 2, \ldots, 6$ and $t = 1, 2, \ldots, 1328$. Figure 2 presents the plots of 6 series and Table 1 contains descriptive statistics.

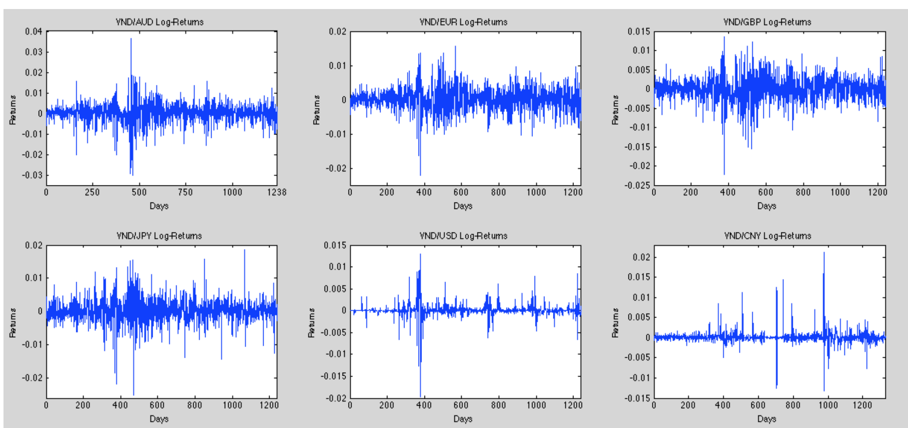

Fig. 2. Daily log returns of 6 currencies to VND.

Table 1. Descriptive statistics of daily log-returns of 6 currencies to VND.

Statistics	VND/AUD	VND/EUR	VND/GBP	VND/JPY	VND/USD	VND/CNY
Mean	17.16E-5	8.68E-05	1.72E-05	20.32E-5	8.53E-05	15.56E-5
Std	5.08E-3	3.50E-3	3.35E-3	3.66E-3	1.55E-3	1.56E-3
Minimum	-0.0303	-0.0222	-0.0221	-0.0250	-0.0198	-0.0133
Median	55.37E-5	19.14E-5	9.46E-05	10.47E-5	2.08E-05	5.94E-05
Maximum	0.0364	0.0157	0.0136	0.0186	0.0130	0.0212
Kurtosis	9.0186	5.5229	5.9954	7.8918	42.4722	59.0849
Asymmetry	-0.4711	-0.1548	-0.4275	-0.2536	-1.7482	3.1359

In Figure 2, we can see the evidence of volatility clustering which can be processed using GARCH models. Table 1 shows that all of 6 return series distributions have large kurtosis, especially VND/USD and VND/CNY have very large kurtosis which make difficulty to capture its perturbation. The asymmetry of VND/EUR and VND/JPY are small implies that its distribution is nearly symmetric while other series have larger asymmetry.

5.2 Results and evaluation

Let us consider a portfolio with equal weights for 6 indices or in other word, the log return of portfolio at day t-th is $x_t = \frac{1}{6}\sum_{n=1}^{6} x_{n,t}$.

In order to assess the accuracy of the estimated VaR we backtest the models at 95%, 97.5%, 99% and 99.5% confidence level by the following procedure. For each day $T = 751, 752, \ldots, 1327$, data in the 750 previous days are used to estimated VaR using AR(1)-GARCH(1,1)+Gaussian copula and AR(1)-GARCH(1,1)+Student t copula models. Since the dataset contains 1327 observations, we have a total of 577 tests for VaR at each level α. We also do backtesting with AR(1)-GARCH(1,1)+ N and AR(1)-GARCH(1,1)+t models in which each return series is assumed to follow AR(1)-GARCH(1,1) model with the innovations are separately modeled using univariate standard normal and student t distribution. For each model, we repeat the test 10 time to access the robustness. To compare the performance of VaR estimation models, we compare the maximum, minimum and average of proportion of observations and number of proportion where the portfolio loss exceeded the estimated VaR among 10 testing times. The average number is average of proportion of observation (number of observations) in 10 testing times. The results are presented in Table 2.

Table 2. Proportion of observations (number of observations in brackets), for $t = 751$ to 1327, where the portfolio loss exceeded the estimated VaR for $\alpha = 0.005, 0.01, 0.025$ and 0.05.

Alpha (α)	Proportion	GARCH+Gaussian	GARCH+Student	GARCH+N	GARCH+t
$\alpha = 0.5\%$	Average	**0.0279(16.1)**	0.0307(17.7)	0.0808(46.6)	0.0858(49.5)
	Minimum	**0.0260(15)**	0.0277(16)	0.0780(45)	0.0832(48)
	Maximum	**0.0312(18)**	0.0329(19)	0.0832(48)	0.0858(50)
$\alpha = 1\%$	Average	**0.0387(22.3)**	0.0392(22.6)	0.0984(56.8)	0.1083(62.5)
	Minimum	**0.0364(21)**	0.0381(22)	0.0936(54)	0.1057(61)
	Maximum	**0.0416(24)**	**0.0416(24)**	0.1040(60)	0.1083(64)
$\alpha = 2.5\%$	Average	**0.0655(37.8)**	0.0747(43.1)	0.1322(76.3)	0.1336(77.1)
	Minimum	**0.0624(36)**	0.0728(42)	0.1300(75)	0.1300(75)
	Maximum	**0.0693(40)**	0.0780(45)	0.1352(78)	0.1369(79)
$\alpha = 5\%$	Average	**0.0924(53.3)**	0.0978(56.4)	0.1537(88.7)	0.1535(88.6)
	Minimum	**0.0936(52)**	0.0953(55)	0.1525(88)	0.1525(88)
	Maximum	**0.0936(54)**	0.0988(57)	0.1560(90)	0.1560(90)

In Table 2, the first two columns are corresponding to VaR estimation models using Gaussian and Student t copulas. Similarly, the last two columns are corresponding to VaR estimation models using AR(1)-GARCH(1, 1) and innovations are generated using standard normal and Student t distribution. The results show that the AR(1)-GARCH(1, 1)+Gaussian copula model provided the best results for VaR estimation for all 4 levels of α. Two conditional copula based models provided better results comparing with 2 other models. Furthermore, the small difference between the minimum, maximum and average numbers of observations (proportion) among 10 repeated times shows that all 4 models are stable.

We also repeated the experiment for daily rate returns of 6 exchange rate with the rate return at day t is defined by $x_{n,t} = \frac{p_{n,t} - p_{n,t-1}}{p_{n,t-1}}$, where $t = 1, 2, \ldots, 1328$ and $n = 1, 2, \ldots, 6$. The results are presented in Table 3.

Table 3. Proportion of observations (number of observations in brackets), for $t = 751$ to 1327, where the portfolio loss exceeded the estimated VaR for $\alpha = 0.005, 0.01, 0.025$ and 0.05.

Alpha (α)	Proportion	GARCH+Gaussian	GARCH+Student	GARCH+N	GARCH+t
$\alpha = 0.5\%$	Average	**0.0289(16.7)**	0.0302(17.4)	0.0801(46.2)	0.0854(49.3)
	Minimum	**0.0243(14)**	0.0260(15)	0.0780(45)	0.0832(48)
	Maximum	**0.0312(18)**	0.0329(19)	0.0832(48)	0.0884(51)
$\alpha = 1\%$	Average	0.0397(22.9)	**0.0395(22.8)**	0.1010(58.3)	0.1092(63.0)
	Minimum	**0.0364(21)**	**0.0364(21)**	0.0953(55)	0.1057(61)
	Maximum	**0.0416(24)**	**0.0416(24)**	0.1057(61)	0.1127(65)
$\alpha = 2.5\%$	Average	**0.0660(38.1)**	0.0768(44.3)	0.1314(75.8)	0.1335(77.0)
	Minimum	**0.0641(37)**	0.0745(43)	0.1300(75)	0.1317(76)
	Maximum	**0.0693(40)**	0.0797(46)	0.1335(77)	0.1352(78)
$\alpha = 5\%$	Average	**0.0946(54.6)**	0.1003(57.9)	0.1323(75.8)	0.1539(88.8)
	Minimum	**0.0919(53)**	0.0988(57)	0.1525(88)	0.1525(88)
	Maximum	**0.0971(56)**	0.1040(60)	0.1560(90)	0.1560(90)

Similar to the case of log return, all the experiment results of copula models are better than other models. The reason is that the copula capture the dependence between series and use to estimate portfolio distribution while other models process without considering the dependency.

6 Conclusion

In this paper, we briefly review the basics of copula theory and two VaR estimation models namely AR(1)-GARCH(1, 1)+ Gaussian copula and AR(1)-GARCH(1, 1)+ Student t copula. Those models are applied to capture the dependency and estimate VaR of portfolio consists of 6 foreign exchange rate in Vietnam's market. The results of conditional copula based models are better than AR(1)-GARCH(1, 1)+ N and AR(1)-GARCH(1, 1)+ Student t models in which each return series is assumed to follow AR(1)-GARCH(1, 1) model and innovations are separately generated using standard normal and student t distribution. We repeat the estimation process 10 time and analyze the results to assess the stability of 4 models and make the conclusion that all considered models are stable.

References

1. Bouye, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T., *Copulas for finance, a reading guide and some applications*, Working paper, Financial Econometrics Research Center, City University, London, 2000.
2. Cherubini, U. and Luciano, E., *Value at risk trade-off and capital allocation with copulas*, Economic Notes, 30:235-256, 2001.
3. Cherubini, U., Luciano, E. and Vecchiato, W., *Copula Method in Finance*, John Wiley and Son, Ltd, 2004.
4. Dias, A., and Embrechts, P., *Dynamic copula models for multivariate high-frequency data in finance*, Working Paper, ETH Zurich: Department of Mathematics, 2003.
5. Embrechts, P. and Hoing, A., Juri, A., *Using copulae to bound the value-at-risk for functions of dependent risks*, Finance and Stochastics, 7:145-167, 2003.
6. Embrechts, P., McNeil, A., and Straumann, D., *Correlation and dependence in risk management: properties and pitfalls*, Risk management: value at risk and beyond, 176-223, 2002.
7. Fortin, I. and Kuzmics, C., *Tail dependence in stock return pairs*, International Journal of Intelligent Systems in Accounting, Finance & Management, 11: 89-107, 2002.
8. Georges, P., Lamy, A.G., Nicolas, E., Quibel, G. and Roncalli, T., *Multivariate survival modelling: a unified approach with copulas*, Working paper, Credit Lyonnais, Paris, 2001.
9. Hansen, B., *Autoregressive conditional density estimation*, International Economic Review, 35:705-730, 1994.
10. Meneguzzo, D. and Vecchiato, W., *Copulas sensitivity in collaterized debt obligations and basket defaults swaps pricing and risk monitoring*, Working paper, Veneto Banca, 2002.
11. Nelsen, R. B., *An introduction to Copulas*, Springer, 2005.
12. Nguyen, P. T. and Nguyen, T., *A Market Microstructure Approach to the Foreign Exchange Market in Vietnam*, 22nd Australasian Finance and Banking Conference 2009, 2009.
13. Nguyen, T. P. and Nguyen, D. T., *Vietnam's exchange rate policy and implication for its foreign exchange market, 1986-2009*, 2010.
14. Patton, Andrew J., *Modelling asymmetric exchange rate dependence*, International economic review 47(2):527-556, 2006.
15. Palaro, H. P. and Hotta, L. K., State University of Campinas, *Using Conditional Copula to Estimate Value at Risk*, Journal of Data Science, 2006.
16. Rockinger, M. and Jondeau, E., *Conditional dependency of financial series: an application of copulas*, Working paper NER # 82, Banque de France. Paris, 2001.
17. Sklar, A., *Random variables, Distribution functions, and Copulas, A personal look backward and forward*, Illinois Institute of Technology, 1996