Intersection Dimension of Bipartite Graphs

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ABSTRACT
We introduce a concept of intersection dimension of a graph with respect to a graph class. This generalizes Ferrers dimension, boxicity, and poset dimension, and leads to interesting new problems. We focus in particular on bipartite graph classes defined as intersection graphs of two kinds of geometric objects. We relate well-known graph classes such as interval bigraphs, two-directional orthogonal ray graphs, chain graphs, and (unit) grid intersection graphs with respect to these dimensions. As an application of these graph-theoretic results, we show that the recognition problems for certain graph classes are NP-complete.

Keywords
Ferrers dimension, Boxicity, Unit grid intersection graph, Segment-ray graphs, Orthogonal ray graph, NP-hardness.

1. INTRODUCTION
Given a family \( F \) of sets, the intersection graph of \( F \) is the graph in which each set in \( F \) is a vertex, and two vertices are adjacent if and only if the corresponding sets intersect. A typical example, when \( F \) is a family of intervals on a line, yields the well-known class of interval graphs. Interval graphs have linear time recognition algorithms [3, 11], and nice forbidden structure characterizations. (For instance, the theorem of Lekkerkerker and Boland [24] characterizes interval graphs by the absence of induced cycles of length four and five, and the absence of asteroidal triples.)

It is natural to study a bipartite version of intersection graphs: given two families \( F \) and \( F' \) of sets, the intersection bigraph of \( F, F' \) is the bipartite graph in which each set in \( F \) is a red vertex, each set in \( F' \) is a blue vertex, and a red vertex is adjacent to a blue vertex if and only if the corresponding sets intersect. When both \( F \) and \( F' \) are families of intervals on a line, we obtain interval bigraphs studied in [25, 31]. We denote the class of interval bigraphs by IBG. While the recognition of interval bigraphs is polynomial (in time \( O(n^{16}) \) [25]), there is no efficient algorithm known, and no characterization in terms of forbidden substructures.\(^1\) It turns out that there are better bipartite analogues of interval graphs. A two-directional orthogonal ray graph, or 2DOR graph, is an intersection bigraph of a family \( F \) of upward rays, and a family \( F' \) of rightward rays, in the plane [33]. These graphs were introduced in connection with defect tolerance schemes for nano-programmable logic arrays [29, 37]. There are several reasons these 2DOR graphs might be considered better bipartite analogues of interval graphs, including an ordering characterization [33, 21], and a Lekkerkerker-Boland type characterization [12], both analogous to the characterizations for interval graphs. Other forbidden structure characterizations of the class of 2DOR graphs can be found in [19, 20, 12].

Several other graph classes can be defined as intersection bigraphs of two families \( F, F' \). When both \( F \) and \( F' \) are inclusion-free families of intervals on a line, we obtain the class of proper interval bigraphs which turns out to be the same as the better known class BPG of bipartite permutation graphs [20], see below. When \( F \) is a family of points, and \( F' \) a family of rightward rays, in a line, we obtain the class CHAIN of chain graphs (cf. below). When \( F \) is a family of vertical segments, and \( F' \) a family of horizontal segments, in the plane, we obtain the class GIG of grid intersection graphs. Several other examples are included in the paper.

We note that the following inclusions are well known or

\(^1\)Recently, Rafiey [28] and Takaoka, Tayu, and Ueno [38] have independently reported faster algorithms for recognizing interval bigraphs.

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easy to derive

\[ \text{CHAIN} \subseteq \text{BPG} \subseteq \text{IBG} \subseteq \text{2DOR} \subseteq \text{GIG}. \]

We now introduce our concept of intersection dimension. Let \( G = (V, E) \) and \( G' = (V', E') \) be two graphs. The intersection \( G \cap G' \) of \( G \) and \( G' \) is the graph \( (V \cap V', E \cap E') \). For two graph classes \( C \) and \( C' \), we define the pairwise intersection of \( C \) and \( C' \) as \( C \cap C' = \{ G \cap G' : G \in C, G' \in C' \} \). We also write \( C^k = \{ G_1 \cap \ldots \cap G_k : G_i \in C \text{ for } 1 \leq i \leq k \} \). If both \( C \) and \( C' \) are closed under taking induced subgraphs, it is easy to check that \( C \cap C' = \{ G \cap G' : G \in C, G' \in C' \} \). Since every graph class in this paper is closed under taking induced subgraphs, we shall from now on use the latter equality, and assume that the vertex sets of the two graphs are the same, when defining the pairwise intersection of graph classes.

The dimension of a graph \( G \) with respect to the graph class \( C \) is the minimum \( k \) such that \( G \in C^k \). In the discussion below we shall point out how this definition generalizes Ferrers dimension, boxicity, cubicity, and poset dimension.

We are particularly interested in expressing one graph class \( C \) class \( C' \), respectively.

\[ G \cap C \cap \!
\]

Since chain graphs are exactly the \( 2K_2 \)-free bigraphs, it is easy to see that a matrix has the Ferrers property if and only if it has none of the following 2 \( \times \) 2 matrices as a submatrix:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}.
\]

Since chain graphs are exactly the \( 2K_2 \)-free bigraphs, it is easy to see that chain graphs are exactly the bigraphs whose biadjacency matrices have the Ferrers property.

### 2.1.2 Bipartite permutation graphs, convex graphs, biconvex graphs, interval bigraphs, and chordal bipartite graphs

A graph \( G = (V, E) \) with \( V = \{1, 2, \ldots, n\} \) is a permutation graph if there is a permutation \( \pi \) over \( V \) such that \( \{i, j\} \in E(G) \) if and only if \( (i - j)(\pi(i) - \pi(j)) < 0 \). A graph is a bipartite permutation graph if it is bipartite and a permutation graph. The class of bipartite permutation graphs is denoted by \( \text{BPG} \). Several equivalent definitions of the class \( \text{BPG} \) are collected in [20].

An ordering \( < \) of \( X \) in a bipartite graph \( B = (X, Y; E) \) has the adjacency property if for every vertex \( y \) in \( Y \), \( N(y) \) consists of vertices that are consecutive in the ordering \( < \) of \( X \). A bipartite graph \( (X, Y; E) \) is convex if there is an ordering of \( X \) or \( Y \) that fulfills the adjacency property. A bipartite graph \( (X, Y; E) \) is biconvex if there are orderings of \( X \) and \( Y \) that fulfill the adjacency property. We denote the classes of convex bipartite graphs and biconvex bipartite graphs by \( \text{Convex} \) and \( \text{Biconvex} \), respectively.

A biinterval representation of a bigraph \( B = (U, V; E) \) is a pair \( (\mathcal{I}_U, \mathcal{I}_V) \) of sets of closed intervals such that \( \mathcal{I}_U = \{ u_{[u_l, u_r]} : u \in U \} \) and \( \mathcal{I}_V = \{ v_{[v_l, v_r]} : v \in V \} \), and \( \{ u, v \} \in E \) for \( u \in U \) and \( v \in V \) if and only if \( u_{[u_l, u_r]} \cap v_{[v_l, v_r]} \neq \emptyset \). A biinterval representation \( (\mathcal{I}_U, \mathcal{I}_V) \) is unit if for each interval \( [\ell, r] \in \mathcal{I}_U \cup \mathcal{I}_V \), \( r - \ell = 1 \).

A bigraph is a chordal bipartite graph if every induced cycle is of length four. The class of chordal bipartite graphs is denoted by \( \text{CBG} \).

### 2.1.3 Orthogonal ray graphs

Here we define the graph classes we deal with in this paper. We also introduce some important properties of them. For their inclusion relations and other known results for them, the readers can refer to the standard textbooks in this field [4, 15, 36].

For a graph class \( C \), the recognition problem of \( C \) is the problem deciding whether a given graph belongs to \( C \).

#### 2.1.1 Chain graphs and Ferrers diagrams

A bipartite graph \( B = (X, Y; E) \) is a chain graph if there is an ordering \( (x_1, x_2, \ldots, x_p) \) on \( X \) such that \( N_B(x_1) \supseteq N_B(x_2) \supseteq \cdots \supseteq N_B(x_p) \). It is easy to see that if there exists such an ordering on \( X \), then there exists an ordering \( (y_1, y_2, \ldots, y_q) \) on \( Y \) such that \( N_B(y_1) \supseteq N_B(y_2) \supseteq \cdots \supseteq N_B(y_q) \). Chain graphs are also known as difference graphs and Ferrers bigraphs. It is known that chain graphs are exactly \( 2K_2 \)-free bigraphs [16]. The class of chain graphs is denoted by \( \text{CHAIN} \).

A \( 0 \times 1 \) matrix has the Ferrers property if its rows and columns can be reordered so that 1s in each row and column appear consecutively with the rows left-justified and the columns top-justified. The reordered matrix is called a Ferrers diagram. It is easy to see that a matrix has the Ferrers property if and only if it has none of the following \( 2 \times 2 \) matrices as a submatrix:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}.
\]
A bipartite graph $B = (X, Y; E)$ is an orthogonal ray graph if there is a pair $(R_X, R_Y)$ of families of rays (or half-lines) such that $R_X = \{R_x : x \in X\}$ is a family of pairwise non-intersecting horizontal rays, $R_Y = \{R_y : y \in Y\}$ is a family of pairwise non-intersecting vertical rays, and $(x, y) \in E$ if and only if $R_x$ and $R_y$ intersect. We call such a pair $(R_X, R_Y)$ an orthogonal ray representation of $B$. We denote the class of orthogonal ray graphs by OR.

Note that in a representation of an orthogonal ray graph horizontal rays can go rightward and leftward and vertical rays can go upward and downward. If we restrict horizontal rays to be only rightwards, then we have 3-directional orthogonal ray graphs. Furthermore, if we restrict horizontal rays to be only rightwards and vertical rays to be only upwards, then we have 2-directional orthogonal ray graphs. We denote the classes of 3-directional orthogonal ray graphs and 2-directional orthogonal ray graphs by $3DOR$ and $2DOR$, respectively.

For the class $2DOR$, several nice characterizations are known (see e.g. [21, 12, 30, 31, 19, 33]). Among those characterizations, the followings are useful for our purpose. In this language they appear in [32, 33], in an equivalent graph theoretic form they are given in [21, 19].

**Theorem 2.1.** For a bipartite graph $B$, the following conditions are equivalent:

1. $B$ is a 2-directional orthogonal ray graph;
2. $B$ is $\gamma$-freeable; that is, the rows and columns of a biadjacency matrix of $B$ can be independently permuted so that no 0 has a 1 both below and to its right;
3. $B$ is of Ferrers dimension at most 2. (The Ferrers dimension of a bipartite graph is defined in Section 2.1.10.)

There are other equivalent characterizations of the class $2DOR$, as suggested in the introduction, in terms of absence of induced cycles and bipartite versions of asteroids, in terms of invertible pairs, etc. [21, 12, 19].

It is known that the recognition of $2DOR$ can be done in polynomial time [12, 33], while it is open for $3DOR$ and OR. Recently, Felsner, Mertzios, and Musta\-t\-a \[14]\ have shown that if the direction (right, left, up, or down) for each vertex is given, then it can be decided in polynomial time whether a given graph has an orthogonal ray representation in which each vertex has the given direction.

**2.1.4 Grid intersection graphs**

A bipartite graph $B = (X, Y; E)$ is a grid intersection graph if there is a pair $(S_X, S_Y)$ of families of segments such that $S_X = \{S_x : x \in X\}$ is a family of pairwise non-intersecting horizontal segments, $S_Y = \{S_y : y \in Y\}$ is a family of pairwise non-intersecting vertical segments, and $(x, y) \in E$ if and only if $S_x$ and $S_y$ intersect. We call such a pair $(S_X, S_Y)$ a grid intersection representation of $B$. A bipartite graph is a unit grid intersection graph if it has a grid intersection representation in which each segment of length 1. We denote the classes of grid intersection graphs and unit grid intersection graphs by $GIG$ and $UGIG$, respectively.

**2.1.5 Segment-ray graphs**

A bipartite graph $B = (X, Y; E)$ is a segment-ray graph if there is a pair $(S_X, R_Y)$ of families of segments and rays such that $S_X = \{S_x : x \in X\}$ is a family of pairwise non-intersecting horizontal segments, $R_Y = \{R_y : y \in Y\}$ is a family of pairwise non-intersecting vertical upward rays, and $(x, y) \in E$ if and only if $S_x$ and $R_y$ intersect. We call such a pair $(S_X, R_Y)$ a segment-ray representation of $B$. We denote the class of segment-ray graphs by $SR$.

**2.1.6 Recognition problems and inclusion relations**

For the graph classes introduced above, the following relations are known [4, 27, 33]:

- CHAIN $\subseteq$ BPG $\subseteq$ Biconvex $\subseteq$ Convex $\subseteq$ IBG $\subseteq$ 2DOR $\subseteq$ OR $\subseteq$ UGIG $\subseteq$ GIG.

Also it is known that 2DOR $\not\subseteq$ CBG [33], and that CBG is incomparable to 3DOR and GIG [27].

It is known that the recognition problems of CHAIN [18], BPG [34], Biconvex [36], Convex [36], IBG [25], 2DOR [33], and CBG [35] can be solved in polynomial time. On the other hand, it is known that the recognition problems of GIG [23] and UGIG [26, 39] are NP-complete. The complexity of the recognition problems of 3DOR, OR, and SR is not known.

Note that even if three graph classes $A$, $B$, and $C$ satisfy $A \subseteq B \subseteq C$ and the recognition problems of $A$ and $C$ are both polynomial-time solvable (NP-hard), it does not mean the recognition problem of $B$ is polynomial-time solvable (NP-hard, resp.).

**2.1.7 Other graphs**

The $d$-dimensional hypercube $H_d$ is the graph with $2^d$ vertices in which the vertices corresponds to the subsets of $\{1, \ldots, d\}$ and two vertices are adjacent if and only if the symmetric difference of the corresponding sets is of size 1.

Let $K_{a,b}$ denote the complete bipartite graph having a vertices in one side and $b$ vertices in the other side. We denote by $K_{a,n} = nK_2$ the graph obtained by removing a perfect matching from the complete bipartite graph $K_{a,n}$.

**2.1.8 Boxicity and cubicity**

An interval graph is the intersection graph of closed intervals on the real line. A unit interval graph is the intersection graph of closed unit intervals on the real line. We denote the classes of interval graphs and unit interval graphs by INT and UINT, respectively.

The *boxicity* of a graph $G$ is the minimum integer $k$ such that $G \in INT^k$, and the *cubicity* of $G$ is the minimum integer $k$ such that $G \in UINT^k$. It is known that given a graph, deciding whether its boxicity (or cubicity) is at most 2 is NP-complete [23, 5].

**2.1.9 Bigraph intersection dimension**

For bipartite graph classes, if one of them is additionally closed under disjoint union, we may assume that the bipartitions of $G$ and $G'$ are the same when taking their intersection. More precisely, we have the following lemma.

**Lemma 2.2.** Let $B$ and $B'$ be bipartite graph classes. If at least one of them is closed under disjoint union and taking induced subgraphs, then $B \otimes B' = \{(X, Y; E) \cap (X', Y; E') : (X, Y; E) \in B, (X, Y; E') \in B'\}$.

**Proof.** Let $C = \{(X, Y; E) \cap (X', Y; E') : (X, Y; E) \in B, (X', Y; E') \in B'\}$. Clearly, $C \subseteq B \otimes B'$. In the following, we show that $B \otimes B' \subseteq C$. By symmetry, we may assume that $B'$ is closed under disjoint union and taking induced subgraphs.
Let \( H = (X; Y; E) \in \mathcal{B} \) and \( H' = (X', Y'; E') \in \mathcal{B}' \). Now let \( H'' = (X; Y; E'' = \{(x, y) : x \in X, y \in Y\}) \). It is easy to see that \( H \cap H'' = H \cap H'' \). Observe that \( H'' \) is the disjoint union of two induced subgraphs of \( H \), where one is induced by \( (X \times X', Y \times Y') \) and the other by \( (X \times Y'; X \times Y') \). Since \( \mathcal{B}' \) is closed under disjoint union and taking induced subgraphs, it follows that \( H'' \in \mathcal{B}' \). Since \( H \cap H'' = H \cap H'' \), we have \( H \cap H'' \in \mathcal{C} \). \( \square \)

Unfortunately, \( \text{CHAIN} \) is not closed under disjoint union. For example, \( K_2 \) is a chain graph but \( 2K_2 \) is not. It is the only exception in this paper. Fortunately, we have the following lemma for chain graphs.

**Lemma 2.3.** \( \text{CHAIN}^2 = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E), (X, Y; E') \in \text{CHAIN}\} \).

**Proof.** Let \( \mathcal{C} = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E), (X, Y; E') \in \text{CHAIN}\} \). Clearly, \( \mathcal{C} \subseteq \text{CHAIN}^2 \). In the following, we show that \( \text{CHAIN}^2 \subseteq \mathcal{C} \).

Let \( H_1 = (X_1, Y_1; E_1) \in \text{CHAIN} \) and \( H_2 = (X_2, Y_2; E_2) \in \text{CHAIN} \). Now let \( H'_1 = (X_1, Y_1; E'_1) \) and \( H'_2 = (X_2, Y_2; E'_2) \), where

\[
E'_1 = E_1 \cup \{(x, y) : x \in X_1 \times X_2, y \in Y_1 \times Y_2\},
\]

\[
E'_2 = E_2 \cup \{(x, y) : x \in X_1 \times Y_2, y \in Y_1 \times Y_2\},
\]

\[
\{ (x, y) : x \in X_1 \times Y_1, y \in Y_1 \times Y_2 \},
\]

\[
\{ (x, y) : x \in X_1 \times Y_2, y \in Y_1 \times Y_2 \}.
\]

See Fig. 2. It is not difficult to see that \( H_1 \cap H_2 = H'_1 \cap H'_2 \). Observe that both \( H'_1 \) and \( H'_2 \) are chain graphs. Therefore, \( H_1 \cap H_2 \subseteq \mathcal{C} \).

**2.1.11 Poset dimension**

The poset dimension \( \text{pd}(P) \) of a poset \( P \) is the minimum integer \( k \) such that there exist \( k \) linear extensions of \( P \) such that for any two elements \( x, y \) of \( P \), \( x < y \) in \( P \) if and only if \( x < y \) in all the linear extensions.

**Definition.** The Ferrers dimension \( \text{fd}(P) \) of a poset \( P \) is the Ferrers dimension of the digraph defined in such a way that the vertices are the elements of \( P \) and there is an arc \((u, v)\) if and only if \( u < v \). Cogis [10] showed that for any poset \( P \), \( \text{fd}(P) = \text{pd}(P) \).

A poset is of height 2 if every element is either a minimal element or a maximal element. The underlying graph of a height-2 poset is the bigraph \( B = (X; Y; E) \) such that \( X \) is the set of minimal elements, \( Y \) is the set of maximal elements, and \( \{x, y\} \in E \) if and only if \( x < y \). It is easy to see that any bigraph is the underlying graph of some poset of height 2.

**Example.**

3. \((P, Q, D)\)-bigraphs

We introduce the notion of \((P, Q, D)\)-bigraphs, where a bigraph \( B = (U, V; E) \) is said to be an \((P, Q, D)\)-bigraph if and only if for some domain \( D \) (e.g., the real number line \( \mathbb{R} \)) each vertex in \( u \in U \) can be represented as a type \( P \) subset \( P_u \) of \( D \) and each vertex \( v \in V \) can be represented as a type \( Q \) subset \( Q_v \) of \( D \) such that for every \( u \in U, v \in V, E \in E \) if and only if \( P_u \cap Q_v \neq \emptyset \). For example, in this setting, interval bigraphs are (interval, interval, \( \mathbb{R} \))-bigraphs. We will use \((P, Q, D)\) to denote the class of \((P, Q, D)\)-bigraphs.

Our discussion will focus on the cases when \( P, Q \) are the following subsets of \( \mathbb{R} \): points, rays, unit-intervals, and intervals; and the following axis-aligned subsets of \( \mathbb{R}^2 \): points, rays, unit-segments, segments, squares, and rectangles. Note: for rays, we will use \( \rightarrow, \downarrow, \leftarrow \), and \( \uparrow \) to denote the rightward, downward, leftward, and upward rays respectively.

Moreover, when we refer to a ray \( r \) (rather than using a specific arrow), \( r \) can be any axis-aligned ray from the domain.

3.1 \((P, Q; \mathbb{R})\)-Bigraphs

We begin with some easy observations characterizing \( \text{CHAIN}, \text{Convex}, \) and \( \text{Biconvex} \) bigraphs as \((P, Q; D)\)-bigraphs (see Proposition 3.1). This is followed by a couple essential lemmas that we will use to relate \((P, Q, \mathbb{R})\)-bigraphs to \((P', Q', \mathbb{R}^2)\)-bigraphs.

**Proposition 3.1.** For a bigraph \( B = (X; Y; E) \):

1. \( B \) is \( \text{CHAIN} \) if and only if \( B \) is \((\text{point}, \rightarrow; \mathbb{R}) \).
2. \( B \) is \( \text{Convex} \) if and only if \( B \) is \((\text{point}, \text{interval}; \mathbb{R}) \).
3. \( B \) is \( \text{Biconvex} \) if and only if \( B \) is \((\text{point}, \text{interval}; \mathbb{R}) \) and \((\text{interval}, \text{point}; \mathbb{R}) \).

**Proof.** These follow easily by definition. \( \square \)

It is also known that a bigraph is a bipartite permutation graph (BPG) if and only if it is a unit-interval bigraph [20]; i.e., \( \text{BPG} = (\text{unit-interval}, \text{unit-interval}; \mathbb{R}) \). Interestingly, we observe that (unit-interval, unit-interval; \( \mathbb{R} \))-bigraphs actually have a simpler representation. Specifically, (unit-interval, unit-interval; \( \mathbb{R} \)) = (point, unit-interval; \( \mathbb{R} \)) and we prove this via the following more general lemma.

**Lemma 3.2.** For a bigraph \( B = (U, V; E) \) and any \( Q \in \{\rightarrow, \text{ray}, \text{unit-interval}, \text{interval}\} \), \( B \in (\text{unit-interval}, Q; \mathbb{R}) \) if and only if \( B \in (\text{point}, Q; \mathbb{R}) \).

**Proof.**
 obviously $(x,x') \in \mathcal{R}$. Notice that, for any choice of $X,Y \in \mathcal{Q}$, the following classes of bigraphs are the same: $(\text{point}, \text{ray}; \mathcal{R})$ and $(\text{ray}, \text{point}; \mathcal{R})$. These are given in the following two corollaries.

**Corollary 3.3.** For each $Q \in \{\rightarrow, \text{ray}, \text{unit-interval}, \text{interval}\}$, the following classes of bigraphs are the same: $(\text{point}, Q; \mathcal{R}), (\text{ray}, Q; \mathcal{R}), (\text{unit-interval}, Q; \mathcal{R})$.

**Corollary 3.4.** For each $P,Q \in \{\text{point}, \rightarrow, \text{unit-interval}\}$, a bigraph $B$ is $(P,Q; \mathcal{R})$ if and only if $B$ is $(Q,P; \mathcal{R})$.

Notice that the statement of Corollary 3.4 does not allow either of $P$ or $Q$ to be ray-type sets. This is because Lemma 3.2 cannot be used to give us the desired biconvexity-like when rays are allowed for a given set. However, by Lemma 3.2, we can transform any $(\text{ray}, \text{ray}; \mathcal{R})$ representation into a $(\text{point}, \text{ray}; \mathcal{R})$ and $(\text{ray}, \text{point}; \mathcal{R})$. Thus, $(\text{ray}, \text{ray}; \mathcal{R})$ is a subset of the bigraph classes which are both $(\text{point}, \text{ray}; \mathcal{R})$ and $(\text{ray}, \text{point}; \mathcal{R})$.

Moreover, the graph $(P_5)$ given in Figure 3 is $(\text{point}, \text{ray}; \mathcal{R})$ but not both $(\text{point}, \text{ray}; \mathcal{R})$ and $(\text{ray}, \text{point}; \mathcal{R})$. This is easy to see since no three vertices in the same partition (say, $X$) can have pairwise incomparable neighborhoods; i.e., two of the three must be represented by rays in the same direction and thus must have nested neighborhoods. Moreover, the graph in Figure 3 has a, b, c in $X$ such that their neighborhoods are pairwise comparable. This is formalized in the following proposition.

**Proposition 3.5.** If a bigraph $B = (X,Y;E)$ is $(\text{ray},\text{ray};\mathcal{R})$ where each $x \in X$ is a ray then for every $(x,x',x'') \subseteq X$ and every $y \in Y$, there exists $x^* \in \{x,x',x''\}$ and $x^{**} \in \{x,x',x''\} \setminus \{x^*\}$ such that $N(x^*) \subseteq N(x^{**})$ or $N(x) \subseteq N(x^*)$.

### 3.2 $(P,Q;\mathbb{R}^2)$-Bigraphs

In this subsection we consider the domain $\mathbb{R}^2$ and describe several classes of bigraphs as the intersection of one dimensional bigraph classes (i.e., as $(P,Q; \mathcal{R}) \cap (P',Q'; \mathcal{R})$). Notice that, for $P,Q \in \{\text{point}, \text{unit-interval}, \text{interval}\}$, $(P,Q; \mathcal{R})$ is hereditary and closed under disjoint union. Thus, by Lemma 2.2, for $P,Q \in \{\text{point}, \text{unit-interval}, \text{interval}\}$ and any choices of $P'$ and $Q'$, $B = (X,Y;E)$ is $(P',Q'; \mathcal{R})$ if and only if $B = (X,Y;E')$ for $(X,Y;E') \in (P,Q; \mathcal{R})$.

**Theorem 3.6.** UGI $= \mathcal{B} \cap \mathcal{P}^2 = \{\text{point}, \text{unit-interval}; \mathcal{R}^2\}$.

**Proof.** First we show that $\text{UGI} \subseteq \mathcal{B} \cap \mathcal{P}^2$. Let $G = (U,V;E) \in \text{UGI}$ and $R = (U,V)$ be a unit grid representation of $G$, where the horizontal segments $U$ represent the vertices in $U$ and the vertical segments $V$ represent the vertices in $V$. That is, $U = \{(y_u) \times [x_u,x_u+1] : u \in U\}$, $V = \{(y_v,y_v+1) \times \{x_v\} : v \in V\}$, and $E = \{(u,v) : u \in U, v \in V, y_u \in [y_v,y_v+1], x_u \in [x_v,x_v+1]\}$. From $U$, we construct two point-unit bi-interval representations $R'$ and $R''$ as follows:

$$R' = \{(y_u) : u \in U\}, \quad R'' = \{(x_v) : v \in V\}.$$

By Lemma 3.2, $R'$ and $R''$ represent the bipartite permutation graphs $G' = (U,V;E')$ and $G'' = (U,V;E'')$, respectively, where

$$E' = \{(u,v) : u \in U, v \in V, y_u \in [y_v,y_v+1]\}, \quad E'' = \{(x_v) : v \in V, x_v \in [x_u,x_u+1]\}.$$

Since $(u,v) \in E' \cap E''$ for $u \in U$ and $v \in V$ if and only if $y_u \in [y_v,y_v+1]$ and $x_v \in [x_u,x_u+1]$, we have $E' = E' \cap E''$. Therefore, $G = G' \cap G''$.

Next we show that $\mathcal{B} \cap \mathcal{P}^2 \subseteq \text{UGI}$. Let $G' = (U,V,E')$ and $G'' = (U,V,E'')$ be bipartite permutation graphs. Let $R'$ and $R''$ be point-unit bi-interval representations of $G'$ and $G''$, respectively, such that $U$ is the point set of $R'$ and the unit interval set of $R''$. Such representations exist by Corollary 3.3. Let $u \in U$, and let $p_u$ and $[\ell_u,\ell_u+1]$ be the point in $R'$ and the unit interval in $R''$ representing the vertex $u$. We assign the unit horizontal segment $[p_u] \times ([\ell_u,\ell_u+1])$ to $u$. Similarly, for a vertex $v \in V$ with the unit interval $[\ell_v,\ell_v+1]$ in $R'$ and the point $p_v$ in $R''$, we assign the unit vertical segment $[\ell_v,\ell_v+1] \times [p_v]$. The obtained unit grid representation represents $G = G' \cap G''$, since $[p_u] \times ([\ell_u,\ell_u+1])$ and $[\ell_v,\ell_v+1] \times [p_v]$ intersect if and only if $p_u \in ([\ell_u,\ell_u+1])$ and $p_v \in ([\ell_v,\ell_v+1])$.

Using Theorem 3.6 and Corollary 3.4 the following is immediate.

**Corollary 3.7.** $(\text{unit-square}, \text{unit-square}; \mathbb{R}^2) = (\text{point}, \text{unit-interval}; \mathbb{R}^2)^2 = \text{UGI}$.

The corollary above implies that a bipartite graph of cubicity-2 is UGI. It is easy to see that the star $K_{1,5}$ is UGI, but its cubicity is more than 2. Therefore, we have the following corollary, which is a nice complement to the fact Biconvexity-2 \& Bipartite = GIG [2].

**Corollary 3.8.** Cubicity-2 \& Bipartite $\subseteq \text{UGI}$.

The proof of the following theorem is an easy modification of the proof of Theorem 3.6. The relation $GIG \neq \text{Convex}^2$ is shown by Fig. 5.
Therefore, the recognition problems are NP-hard for Biconvex polynomial time.

Chain dimension at most 4.

Figure 5: A (point, interval)² representation of the full subdivision $H$ of $K_{3,3}$; i.e., $H \in \text{Convex}^2$. On the other hand, $H \notin \text{GIG}$, since it is the full subdivision of a non-planar graph, and thus not a string graph.

**Theorem 3.9.** Biconvex² $\subseteq$ (Biconvex $\otimes$ Convex) $\subseteq$ GIG $\subseteq$ Convex².

Since Convex $\subseteq$ 2DOR, it holds that GIG $\subseteq$ 2DOR² = CHAIN⁴. Therefore, every grid intersection graph has Ferrers dimension at most 4.

**Corollary 3.10.** The recognition problems of BPG², Biconvex², and Biconvex $\otimes$ Convex are NP-complete.

**Proof.** The problems are in NP since the recognition problems of BPG and Biconvex are polynomial-time solvable and the intersection of two graphs can be computed in polynomial time.

Mustaţă and Pergel [26] showed that the recognition problem is NP-hard for any graph class $\mathcal{C}$ satisfying UGIG $\subseteq$ $\mathcal{C}$ $\subseteq$ GIG. By Theorems 3.6 and 3.9 and the fact that BPG $\subseteq$ Biconvex, it follows that UGIG $\subseteq$ Biconvex² $\subseteq$ GIG. Therefore, the recognition problems are NP-hard for BPG² and Biconvex².

4. SEGMENT-RAY GRAPHS

Let $F$ be a matrix with entries 0, 1, *, where * means “don’t care.” A matrix $M$ is $F$-free if $M$ does not have $F$ as a submatrix ignoring *-entries. A bipartite graph is $F$-free if it has a $F$-free biadjacency matrix.

It is known that a bipartite graph is a chordal bipartite graph if and only if it is $F$-freeable [22], a 2-directional orthogonal ray graph if and only if it is $\gamma$-freeable [33], and a grid intersection graph if and only if it is cross-freeable [17], where the forbidden matrices are defined as follows:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \quad \text{cross} = \begin{pmatrix} * & 1 & * \\ 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

In this section, using the following matrix $V$, we characterize segment-ray graphs:

$$V = \begin{pmatrix} 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

Obviously, a matrix is cross-free if it is $V$-free, and $V$-free if it is $\gamma$-free.

The proof of the following theorem is similar to the proofs of the cross-free characterization of GIG [17] and the $\gamma$-free characterization of 2DOR [33].

**Theorem 4.1.** A bipartite graph is a segment-ray graph if and only if it is $V$-free.

**Proof.** For the only-if part, let $B = (U, V; E)$ be a segment-ray graph and $R$ be its segment-ray representation such that each vertex in $U$ corresponds to a horizontal segment in $R$, and each vertex in $V$ corresponds to a vertical upward ray in $R$. Let $M$ be the bipartite adjacency matrix of $B$ with the rows indexed by $U$ and the columns indexed by $V$. Let $S_u$ be the segment corresponding to $u \in U$ with $y$-coordinate $b$, and $R_v$ be the ray corresponding to $v \in V$ with $x$-coordinate $a$. If $S_u$ intersects with rays on both sides of $x = a$ and $R_v$ intersects with a segment below $y = b$, then $S_u$ and $R_v$ must intersect at $(a, b)$. Thus we can make $M$ $V$-free by permuting the columns in nondecreasing order of the $x$-coordinates of the corresponding rays and the rows in nonincreasing order of the $y$-coordinates of the corresponding segments.

For the if part, let $B = (U, V; E)$ be a bipartite graph and $M$ be its $V$-free bipartite adjacency matrix with the rows indexed by $U$ and the columns indexed by $V$. For each $u \in U$, we put the horizontal segment with end points $(i, j_1)$ and $(i, j_2)$, where $i$ is the row index of $u$ and $j_1, j_2$ are the smallest and largest indices such that $M_{i,j} = 1$. For any two vertices $u \in U$ and $v \in V$, it is clear that the corresponding segment and ray intersect if the vertices are adjacent. Conversely, if $u$ and $v$ are not adjacent, then the corresponding segment and ray cannot intersect since $M$ is $V$-free.

Now we show that every segment-ray graph has Ferrers dimension at most 3. To this end, we need the following simple fact.

**Lemma 4.2.** An $m \times n$ 0-1 matrix $M$ is $V$-free if and only if for each entry $(i, j)$ with $M_{i,j} = 0$ at least one of the following holds:

1. $M_{i,k} = 0$ for all $1 \leq k \leq j$;
2. $M_{i,k} = 0$ for all $j \leq k \leq n$;
3. $M_{k,j} = 0$ for all $i \leq k \leq m$.
Theorem 4.3. Every segment-ray graph has Ferrers dimension at most 3.

Proof. Let $B$ be a segment-ray graph and $M$ be its V-free bipartite adjacency matrix. Let $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ be the following 0-1 matrices of the same size with $M$:

- $M_{i,j}^{(1)} = 0$ if and only if $M_{i,k} = 0$ for all $1 \leq k \leq j$;
- $M_{i,j}^{(2)} = 0$ if and only if $M_{i,k} = 0$ for all $j \leq k \leq n$;
- $M_{i,j}^{(3)} = 0$ if and only if $M_{h,j} = 0$ for all $i \leq k \leq m$.

It is easy to see that $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ have the Ferrers property. By Lemma 4.2, it holds that $M^{(1)} \cap M^{(2)} \cap M^{(3)} = M$. This completes the proof. $\square$

Note that the upper bounds of the Ferrers dimension for $\text{GIG} \leq 4$ and $\text{2DOR} \leq 2$ can be shown in similar ways by using the forbidden submatrix characterizations.

Corollary 4.4. OR is incomparable to both $\text{CHAIN}^3$ and SR.

Proof. By Theorem 4.3, it holds that $\text{SR} \subseteq \text{CHAIN}^3$. Hence it suffices to show that $\text{OR} \not\subseteq \text{CHAIN}^3$ and $\text{SR} \not\subseteq \text{OR}$.

Fig. 6(a) shows that $H_3 \in \text{OR}$. From the definitions, it holds that $H_3 = K_{4,4} - 4K_2$. It is known that $\text{fd}(K_{n,n} - nK_2) = n$ [40, 41], and thus $\text{fd}(H_3) = 4$. Thus $\text{OR} \not\subseteq \text{CHAIN}^3$. It is known that $C_{2n} \not\in \text{OR}$ if $n > 6$ [33]. On the other hand, it is easy to see that $C_{2n} \in \text{SR}$ for any $n$ (see Fig. 6(b)). Thus $\text{SR} \not\subseteq \text{OR}$. $\square$

Corollary 4.5. SR is a proper subset of $\text{GIG}$.

Proof. From the definition, SR is a subset of GIG. Since $H_3 \in \text{OR} \subset \text{GIG}$ and $H_3 \not\in \text{CHAIN}^3 \supseteq \text{SR}$, it holds that $\text{SR} \neq \text{GIG}$. $\square$

5. Boxicity and Ferrers Dimension

Chatterjee and Ghosh [9] presented some relations between the boxicity of undirected graphs and the Ferrers dimension of the directed graphs obtained somehow from the undirected graphs. Here we present a similar but more direct relation between the boxicity and the Ferrers dimension of bigraphs.

If $\text{fd}(B) = 1$, then $\text{box}(B) \leq 2$. This is because, $\text{fd}(B) = 1$ implies that $B$ is a chain graph, and thus $B$ is a grid intersection graph [27]. This bound is tight since $\text{fd}(K_{n,n}) = 1$ and $\text{box}(K_{n,n}) = 2$ for every $n \geq 2$.

Theorem 5.1. Let $B$ be a bigraph with $\text{fd}(B) \geq 2$. It holds that

$$\text{box}(B) \leq \text{fd}(B) \leq 2\text{box}(B).$$

Proof. Adiga, Bhowmick, and Chandran [1] showed that for a poset $Q$ of height 2 and its underlying graph $H$ it holds that $\text{box}(H) \leq \text{pd}(Q) \leq 2\text{box}(H)$ if $\text{pd}(Q) \geq 2$. (Recently Felsner [13] has shown a more general result.) Since $\text{fd}(Q) = \text{pd}(Q)$ [10], it holds that $\text{box}(H) \leq \text{fd}(Q) \leq 2\text{box}(H)$ if $\text{fd}(Q) \geq 2$.

Let $P$ be a poset that has $B$ as the underlying graph. From the argument above, it follows that $\text{box}(B) \leq \text{fd}(P) \leq 2\text{box}(B)$ if $\text{pd}(P) \geq 2$. Hence it suffices to show that $\text{fd}(P) = \text{fd}(B)$.

Let $M_B$ be a bipartite adjacency matrix of $B$. Then, an adjacency matrix $M_P$ of the digraph corresponding to $P$ can be represented by the following form:

$$M_P = \begin{pmatrix} M_B & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus it is easy to see that $\text{fd}(P) \geq \text{fd}(B)$ as $M_B$ is a submatrix of $M_P$. On the other hand, let $B_1, \ldots, B_{\text{fd}(B)}$ be Ferrers bigraphs that satisfy $B = \bigcap_{1 \leq i \leq \text{fd}(B)} B_i$. Let $M_{B_i}$ be the bipartite adjacency matrix of $B_i$ in which the rows and columns are ordered as in $M_B$. Now we define $M_P$ as follows:

$$M_P = \begin{pmatrix} M_{B_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly $M_P = \bigcap_{1 \leq i \leq \text{fd}(B)} M_{P_i}$, and each $M_{P_i}$ has the Ferrers property. This implies that $\text{fd}(P) \leq \text{fd}(B)$. $\square$

The upper bound in Theorem 5.1 is tight. It is known that $\text{box}(K_{n,n} - nK_2) = [n/2]$ [6] and $\text{fd}(K_{n,n} - nK_2) = n$ [40, 41].

Bellatoni, Hartman, Przytycka, and Whitesides [2] showed that the grid intersection graphs are exactly the bigraphs of boxicity at most 2. This implies that the Ferrers dimension of a grid intersection graph is at most 4. We show that the converse is not true.

Theorem 5.2. $\text{GIG} \not\subseteq \text{CHAIN}^4$.

Proof. We show that $H_4 \in \text{CHAIN}^4 \setminus \text{GIG}$. Chang and West [8] showed that $H_4$ cannot be represented as the intersection graph of axis-parallel rectangles in the plane. This implies that $H_4 \not\in \text{GIG}$. Let $M$ and $M'$ be the following

```
(a) $H_3 \in \text{OR}$.

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(b) $C_{2n} \in \text{SR}$.
```

Figure 6: Examples showing incomparabilities.
matrices:
\[
M = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
\[
M' = \begin{pmatrix}
a & 1 & 1 & 1 & a & a & a \\
1 & b & 1 & 1 & b & 1 & b \\
1 & c & 1 & c & c & 1 & c \\
1 & 1 & d & d & d & d & 1 \\
a & 1 & c & 1 & 1 & 1 & 1 \\
a & b & c & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

The matrix \(M\) is a biadjacency matrix of \(H_4\), and \(M'\) has the same 1-entries as \(M\) but has one of \(a, b, c, d\) for each 0-entry of \(M\). For \(x \in \{a, b, c, d\}\), let \(M_x\) be the 0-1 matrix obtained from \(M'\) by replacing all \(x\) with 0 and replacing all other non-numeric entries with 1. It is easy to see that \(M_x\), for all \(x \in \{a, b, c, d\}\), has none of the forbidden \(2 \times 2\) matrices in (1) as a submatrix, and thus has the Ferrers property. Since \(M = M_a \cap M_b \cap M_c \cap M_d\), it holds that \(H_4 \in \text{CHAIN}^4\).

Chandran, Francis, and Mathew [7] showed that boxicity is unbounded for chordal bipartite graphs. Thus we have the following.

**Corollary 5.3.** Ferrers dimension is unbounded for chordal bipartite graphs.

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