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Sankaku-Tori: An Old Western-Japanese Game Played on a Point Set

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Abstract. We study a combinatorial game named "sankaku-tori" in Japanese, which means "triangle-taking" in English. It is an old pencil-and-paper game for two players played in Western Japan. The game is played on points on the plane in general position. In each turn, a player adds a line segment to join two points, and the game ends when a triangulation of the point set is completed. The player who completes more triangles than the other wins. In this paper, we consider two restricted variants of this game. In the first variant, the first player always wins in a nontrivial way, and the second variant is NP-complete in general.

1 Introduction

"Sankaku-tori" is a classic pencil-and-paper game for two players, traditionally played in Western Japan. Sankaku-tori literally means "triangle taking" in English. The rule is as follows. First, two players put a number of points on a sheet of paper. Then, they join the points alternately by a line segment. Line segments cannot cross each other. When an empty triangle is completed by a move, it scores +1 to the player who draws the line segment (if two empty triangles are completed, it scores +2). When no more line segments can be drawn, the game ends, and the player who scores more wins (see Fig. 1; in the figure, solid lines and dotted lines are played by the first player $\mathcal{R}$ and the second player $\mathcal{B}$, respectively. Finally, $\mathcal{R}$ wins since she obtains four triangles, while $\mathcal{B}$ obtains two triangles).

We study the algorithmic aspects of the sankaku-tori game. First, we prove that if the points are in convex position, then the first player always has a

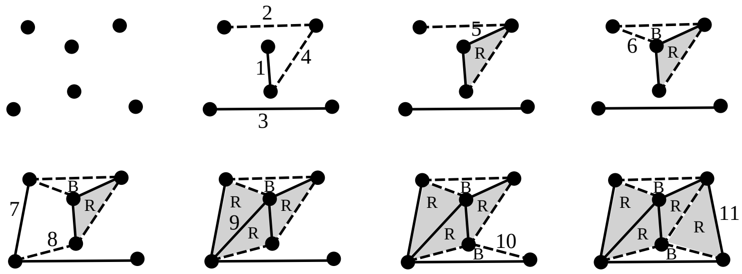


Fig. 1. Sample play.

winning strategy. Second, we consider a solitaire version of the sankaku-tori game. Namely, we are given a point set and some line segments connecting pairs of those points, and we want to maximize the number of triangles that can be constructed by drawing k more line segments. We prove that this problem is NP-complete.

The game has a similar flavor to those studied by Aichholzer et al. [1] under the name of "Games on Triangulations." Among variations they studied, the most significant resemblance can be seen in the *monochromatic complete triangulation game*. The only difference between the sankaku-tori game and the monochromatic complete triangulation game is the following. In the monochromatic complete triangulation game, if a player completes a triangle, then she can continue to draw a line segment. This rule is similarly seen in *Dots and Boxes*, where two players construct a grid instead of a triangulation. Dots and Boxes has been investigated in the literature (see [4, 2]), and especially, one book is devoted to the game [3], revealing a rich mathematical structure. Aichholzer et al. [1] proved that the monochromatic complete triangulation game is a first-player win if the number of points is odd, and a tie if it is even. We note that a few problems left by Aichholzer et al. [1] have recently been resolved by Manić et al. [5].

On the other hand, in the sankaku-tori game, even though a player completes a triangle, she should leave the token to the next player. Hence, we cannot directly use the previously known results, and we need to develop new techniques for our game.

2 Preliminaries

In this paper, a finite planar point set S is always assumed to be in general position, i.e., no three points in S are collinear. A *triangulation* of a finite planar point set S is a decomposition of its convex hull by triangles in such a way that their vertices are precisely the points in S. Two players $\mathcal{R}$(ed) and $\mathcal{B}$(lue) play in turns, and we assume that $\mathcal{R}$ is the first player. That is, the players construct

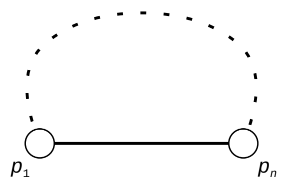


Fig. 2. (Case 1)

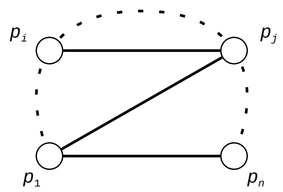


Fig. 3. (Case 2)

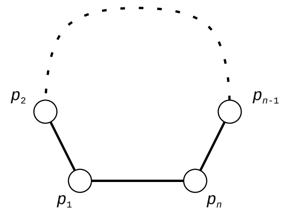


Fig. 4. (Case 3)

a triangulation on a given point set S.[8] Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turn by drawing one edge in each move. We note that each player draws precisely one edge. This is the difference from the dots-and-boxes-like games. The game ends when a triangulation is completed. Each triangle belongs to the player who draws the last edge of the triangle.[9] The player who has more triangles than the other wins.

We first note that, for any point set, the number of edges of a triangulation is determined by the position of the points. That is, the number of turns of the sankaku-tori game is determined when the position of the points are given.

3 The first player wins on convex position

In this section, the main theorem is the following.

Theorem 1. *When S is a point set in convex position, the first player $\mathcal{R}$ always wins.*

To prove the theorem, we describe a winning strategy for $\mathcal{R}$ in Lemma 1. Once the first player $\mathcal{R}$ draws a line p_ip_j in the first move, we have two intervals $I_1 = [p_i, p_{i+1}, \ldots, p_{j-1}, p_j]$ and $I_2 = [p_j, p_{j+1}, \ldots, p_{i-i}, p_i]$. Then any point p in I_1 can be joined to the other point q if and only if q is in I_1 when the points are in convex position. That is, each line segment separates an interval of the points into two independent intervals. The winning strategy is an inductive one that consists of three substrategies. We note that the strategy in Lemma 1 is applied simultaneously in each interval. For example, suppose that $\mathcal{R}$ has two strategies S_1 and S_2 on intervals $I_1 = [p_i, p_{i+1}, \ldots, p_{j-1}, p_j]$ and $I_2 = [p_j, p_{j+1}, \ldots, p_{i-i}, p_i]$, respectively. If $\mathcal{B}$ joins two points in I_1, $\mathcal{R}$ uses S_1 on the interval I_1, and then, if $\mathcal{B}$ joins two points in I_2, $\mathcal{R}$ now uses S_2 on the interval I_2, and so on. Since the points are in convex position, they can apply their strategies independently in each interval.

[8] In a real game, two players arbitrarily draw the point set by themselves simultaneously until both agree with.

[9] In a real game, when a player draws the last edge, she writes her initials in the triangle.

Lemma 1. *Suppose that, at a certain point of the game, $\mathcal{B}$ has to move and there are some intervals resembling Cases 1, 2, and 3 in Fig.s 2, 3, and 4, respectively. Then, after two moves, $\mathcal{R}$ can replicate the same configuration without losing points. Moreover, if the number of vertices in an interval is odd, at the end it is possible for $\mathcal{R}$ to get one more points.*

Proof. We show an induction for the number of turns of the game. As mentioned in the last paragraph in Preliminaries, if we have n points in convex position, the number of turns is exactly $2n-3$. In the figures, dotted lines illustrate the isolated points. In base cases, dotted lines mean that no points are there. We can check the claims in Lemma 1 in base cases by simple case analysis. Now we turn to general cases.

(Case 1) Player $\mathcal{B}$ has two choices. If $\mathcal{B}$ joins p_i and p_j with $1 < i < j < n$, $\mathcal{R}$ joins p_1 and p_j and obtain (Case 2). Therefore, without loss of generality, we assume that $\mathcal{B}$ joins p_1 and p_i with $1 < i < n$. In this case, $\mathcal{R}$ can join p_i and p_n, and obtain the triangle $p_1 p_i p_n$. Moreover, (Case 1) applies to both intervals $p_1, \ldots, p_i$ and $p_i, \ldots, p_n$. Therefore, by induction, $\mathcal{R}$ wins in this case because $\mathcal{R}$ already obtains $+1$ by the triangle $p_1 p_i p_n$.

(Case 2) The same analysis of (Case 1) can be applied in the interval $[p_i..p_j]$. Therefore, by inductive hypothesis, $\mathcal{B}$ cannot take an advantage in this interval. Without loss of generality, we can assume that $\mathcal{B}$ plays in interval $[p_1..p_i]$. Essentially, $\mathcal{B}$ has four choices.

(Subcase 2-1) If $\mathcal{B}$ joins p_1 and p_i, $\mathcal{R}$ joins p_j and p_n, and they have three intervals in (Case 1). Then it is easy to check that the claim holds.

(Subcase 2-2) If $\mathcal{B}$ picks $p_{i'}$ with $1 < i' < i$ and joins it to either p_1 or p_i, $\mathcal{R}$ again joins p_j and p_n. Then we have two intervals $[p_i..p_j]$ and $[p_j..p_n]$ in (Case 1). If $\mathcal{B}$ joins p_1 and $p_{i'}$, we have an interval $[p_1..p_{i'}]$ in (Case 1), and the other interval $[p_{i'}..p_i]$ in (Case 3). The other case ($\mathcal{B}$ joins $p_{i'}$ and p_i) is symmetric. In any case, by inductive hypothesis, the claim holds.

(Subcase 2-3) If $\mathcal{B}$ joins p_j and $p_{i'}$ for some $1 < i' < i$, $\mathcal{R}$ joins $p_{i'}$ to p_i. Then $\mathcal{R}$ obtains the triangle $p_i p_j p_{i'}$, and two intervals $[p_{i'}..p_i]$ and $[p_i..p_j]$ are in (Case 1), and two intervals $[p_1..p_{i'}]$ and $[p_j..p_n]$ together essentially in the same case as (Case 2). Therefore, $\mathcal{R}$ wins in this case.

(Subcase 2-4) The last case is that $\mathcal{B}$ picks up two points $p_{i'}$ and $p_{i''}$ with $1 < i' < i'' < i$ and join them by an edge. Then $\mathcal{R}$ joins $p_{i'}$ to p_j, and obtain two intervals $[p_1..p_{i'}]$ and $[p_j..p_n]$ together in (Case 2), an interval $[p_{i''}..p_i]$ with an edge $(p_{i''}, p_{i'})$ in (Case 3), and two intervals $[p_{i'}..p_{i''}]$ and $[p_i..p_j]$ in (Case 1). Therefore, we have the claim in this case again.

(Case 3) Now we have three subcases.

(Subcase 3-1) $\mathcal{B}$ joins two points in $\{p_1, p_2, p_{n-1}, p_n\}$. If $\mathcal{B}$ joins p_2 and p_{n-1}, $\mathcal{R}$ joins p_1 and p_{n-1}, and obtain two triangles ($p_1 p_2 p_{n-1}$ and $p_1 p_{n-1} p_n$), and they end up in Case 1. On the other hand, if $\mathcal{B}$ joins p_1 and p_{n-1}, $\mathcal{R}$ joins p_2 and p_{n-1} and obtains (Case 1). The other cases are symmetric. Thus we have the claim.

(Subcase 3-2) $\mathcal{B}$ joins one point in $\{p_1, p_2, p_{n-1}, p_n\}$ and another one p_i with $2 < i < n-1$. If $\mathcal{B}$ joins p_1 and p_i, $\mathcal{R}$ joins p_i and p_2 and obtain the triangle

$p_1p_2p_i$. Then they also have an interval $[p_2..p_i]$ in (Case 1) and $[p_i..p_n]$ with p_1 in (Case 3) again. Thus we have the claim. If $\mathcal{B}$ joins p_2 and p_i, $\mathcal{R}$ now joins p_i and p_1 and get the same situation. The other two cases are symmetric.

(Subcase 3-3) $\mathcal{B}$ joins two points p_i and p_j with $2 < i < j < n-1$. In the case, $\mathcal{R}$ joins p_i and p_n. Then both of the interval $[p_1..p_i]$ with p_n and the interval $[p_i..p_n]$ are independently in (Case 3). Therefore, we again use the induction.

By the induction for the number of points, we have the lemma. □

Now we prove Theorem 1:

Proof (of Theorem 1). When $n = 2k+1$ for some $k > 1$, $\mathcal{R}$ joins p_1 and p_k. Then two intervals $[p_1..p_k]$ and $[p_k..p_n]$ are both in (Case 1) in Lemma 1. Moreover, one of two intervals consists of odd number of points. Thus $\mathcal{R}$ obtains at least one more triangle than $\mathcal{B}$.

When $n = 2k$ for some $k > 1$, $\mathcal{R}$ joins p_1 and p_3. Then two intervals $[p_1..p_3]$ and $[p_3..p_n]$ are both in (Case 1), and they are of odd length. Thus $\mathcal{R}$ obtains at least two more triangles than $\mathcal{B}$.

In any case, $\mathcal{R}$ always wins. □

4 NP-completeness

In this section, we consider the solitaire variant by modifying the rule of the game. We start halfway through the game. That is, we are given a set of n points and $O(n)$ lines joining them. We are also given two integers $k = O(n)$ and t. The decision problem asks whether we can obtain t triangles after k moves for the set of points and lines.

Theorem 2. *The solitaire variant of Sankaku-Tori is NP-complete.*

The problem is in NP since we can guess k new edges and easily check whether we can obtain t triangles. Later in this section, we reduce the POSITIVE PLANAR 1-IN-3-SAT problem [6] to our problem. In POSITIVE PLANAR 1-IN-3-SAT, we are given a 3-CNF formula φ with n variables and m clauses, together with a planar embedding of its incidence graph $G(\varphi)$. Each clause of φ consists of three positive literals (i.e., variable itself). The incidence graph $G(\varphi)$ of φ consists of n vertices v_{x_i} corresponding to the variables x_i and m vertices v_{C_j} corresponding to the clauses C_j. There is an edge (v_{x_i}, v_{C_j}) if and only if x_i appears in C_j. The problem is to decide whether there exists a satisfying assignment to the variables of φ such that each clause in φ has exactly one literal assigned true. POSITIVE PLANAR 1-IN-3-SAT is NP-complete [6].

Basic idea. Suppose that we are given a set of points and lines illustrated in Fig. 5(a), which consists of two components called *crescents* and eight lines, calles *barriers*, that surround the crescents. A crescent consists of c points and $2c-3$ lines (we have $c = 5$ in the figure), and is triangulated by the lines. The barriers define a barrier region that prevent us from drawing a line between the

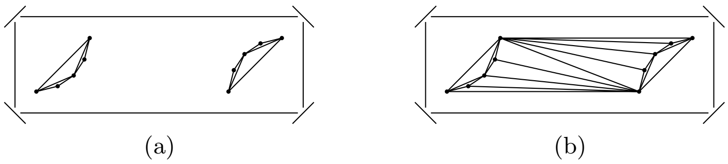


Fig. 5. Basic idea: two crescents and their enclosing line segments.

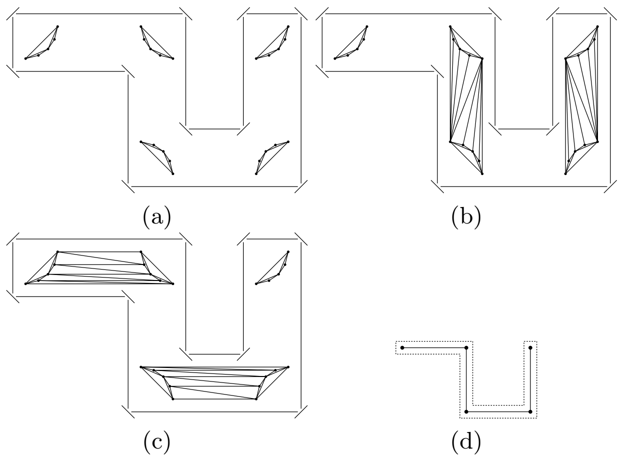


Fig. 6. Line gadget.

points inside and outside of the region. (Although there exist no points in the outside of the barrier in Fig. 5(a), such points will appear later.)

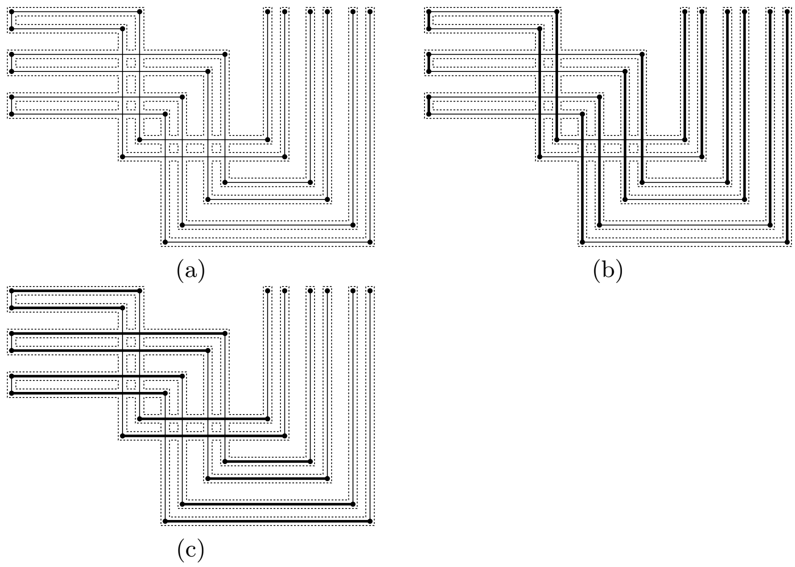
If we are given two integers $k = 2c - 1$ and $t = 2c - 2$, a unique solution for obtaining t triangles is illustrated in Fig. 5(b). We can deduce this observation by introducing a *loss* which is defined as $i - j$ if we obtain j triangles by drawing i lines. In Fig. 5(a), we can obtain triangles by drawing lines between two connected components (crescents and/or barriers), which requires at least one loss. (We are required to draw at least two lines for obtaining one triangle.) Here, $k = 2c - 1$ and $t = 2c - 2$ means the loss should be at most one. Thus, we are required to draw k lines so as not to connect three or more connected components. Here, we cannot obtain t or more triangles if we connect two barriers, or if we connect a crescent and a barrier. A unique solution is to connect two crescents by drawing k lines, which results in Fig. 5(b).

Line gadgets. A line gadget of length ℓ, illustrated in Fig. 6(a), consists of $\ell + 1$ crescents and $4(\ell + 1)$ barriers surrounding them. By drawing $2c - 1$ lines between two adjacent crescents, we can obtain $2c - 2$ triangles. Note that the barriers prevent us from drawing a line between nonadjacent crescents.

Suppose that we are required to obtain $i(2c-2)$ triangles by drawing $i(2c-1)$ lines ($0 < i \leq \lceil \ell/2 \rceil$). By an argument similar to one in the basic idea, loss should be at most i. We cannot obtain $i(2c-2)$ triangles, if we connect two barriers, if we connect a crescent and a barrier, or if we connect three or more crescents. A unique solution for obtaining $i(2c-2)$ triangles is to connect i pairs of crescents. For example, Fig. 6(a) is a line gadget of length 4, and we can obtain Fig. 6(b) and (c) by connecting two pairs of crescents by $2(2c-1)$ lines.

For convenience, we abbreviate Fig. 6(a) as in Fig. 6(d). The points in the figure denote crescents, and the (solid) edges denote the adjacency among the crescents. The dotted rectilinear polygon in Fig. 6(d) denotes the barriers. Each line segment of the dotted polygon has one barrier. At each corner of the dotted polygon, we have an additional barrier. Since any crescent can be connected with at most one adjacent crescent, we can associate the connecting pairs of crescents in Fig. 6(a) with a matching of the graph in Fig. 6(d).

Line gadgets have flexibility on their shapes: We can extend or shorten the distance between any two adjacent crescents. We can select a direction at each bend of the gadget. We can also set any angles at the bends within 90 degrees.



(a) (b) (c)

Fig. 7. Variable gadget.

Variable gadgets. As illustrated in Fig. 7(a), we arrange c_i line gadgets of length $2\ell+1$ ($c_i = 3$ and $2\ell+1 = 9$ in the figure). As in Fig. 6(d), the dotted polygons denote barriers. The uppermost line gadget crosses the remaining $c_i - 1$ line gadgets. We have $8(c_i - 1)$ crossing points, each of which requires eight additional lines as barriers. Since a non-crossing line gadget requires $4(2\ell+2)$

barriers, a variable gadget with c_i line gadgets requires $4(2\ell+2)c_i+8\cdot 8(c_i-1) = O(\ell c_i)$ barriers.

A unique non-crossing maximum matching of Fig. 7(a) is, as illustrated in the bold lines in Fig. 7(b), achieved by taking $\ell+1$ matching edges in each line gadget. In case we are not allowed to use crescents at both ends of the line gadgets, a unique non-crossing maximum matching for the remaining crescents is, as illustrated in the bold lines in Fig. 7(c), achieved by taking ℓ matching edges in each line gadget. As the rule of the Sankaku-tori prohibits crossing lines, matching edges for Fig. 7(a) cannot cross each other. Thanks to this property, as in Fig. 7(b) and (c), we can synchronize the selection of vertical or horizontal matching edges among all line gadgets in a variable gadget.

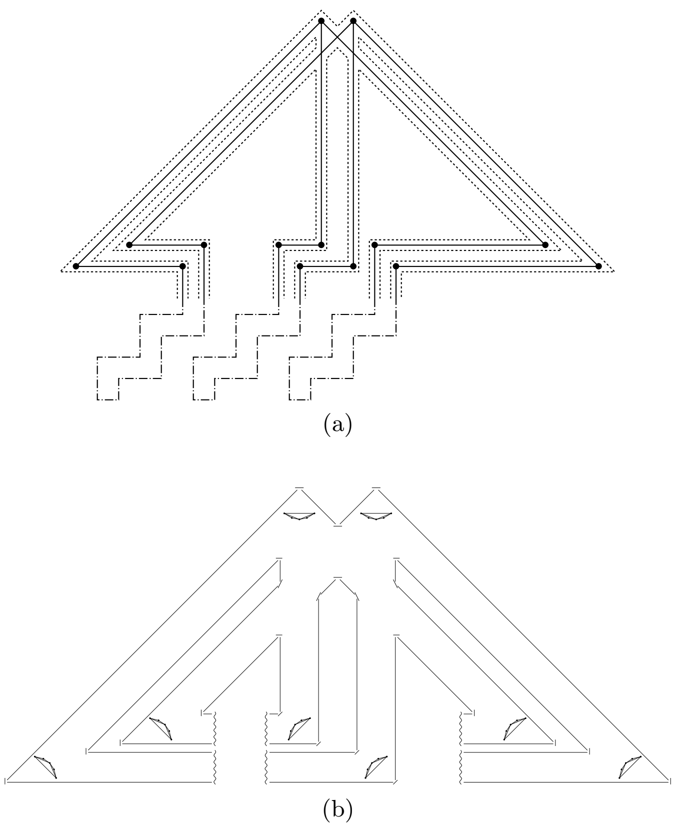


Fig. 8. Clause gadget.

Clause gadgets. As illustrated in Fig. 8(a), we share the crescents at the ends of three line gadgets of length $2\ell+1$. Since each clause has three literals, we use the line gadgets corresponding to them. A clause gadget has $6\ell+2$ crescents.

The crucial part is on the top of the clause gadget, whose details are illustrated in Fig. 8(b). We note here that the number of barriers in a clause gadget is the same as that of (non-sharing) three line gadgets.

In case one of the three line gadgets takes both of the top two crescents to realize the matching in Fig. 7(b), the other two line gadgets cannot use the top crescents. This means that the maximum matching for the other two is the matching in Fig. 7(c). In this case, the number of matching edges is $3\ell+1$, which is a perfect matching on $6\ell+2$ crescents.

If no line gadgets take either of the top two crescents, we cannot obtain a perfect matching. If two line gadgets take one of the top two respectively, each of them has unmatched crescents, which means we cannot obtain a perfect matching. Thus, we can obtain a perfect matching if and only if one of the three line gadgets takes the top two crescents, and the other two do not take any.

Reduction. Let c_i denote the number of occurrences of literal x_i in φ ($i = 1, 2, \ldots, n$). As illustrated in Fig. 7(a), a variable gadget for x_i has c_i line gadgets of length $2\ell+1$. Since we have $3m$ literals in φ, n variable gadgets have $3m$ line gadgets in total. Since line gadgets have flexibility on their shape, line gadgets are arranged along the planar embedding of $G(\varphi)$.

Each clause has variable gadgets as illustrated in Fig. 8(a), which consists of three line gadgets corresponding to the three literals in the clause. Since φ has m clauses, we have m clause gadgets with $(6\ell+2)m$ crescents and $O(\ell m)$ barriers. Two magic numbers are set to $k = (3\ell+1)m(2c-1)$ and $t = (3\ell+1)m(2c-2)$.

If φ is 1-in-3 satisfiable, by the following strategy, we can obtain t triangles. For literals assigned true, the corresponding line gadgets take the two crescents at both ends of the gadgets and achieve the maximum matching as illustrated in Fig. 7(b). For literals assigned false, as illustrated in Fig. 7(c), the corresponding line gadgets do not take the two crescents at both ends of the gadgets and achieve the maximum matching for other crescents. The formula φ has m literals assigned true, and they satisfy all clauses with 1-in-3 property. This means that, for every clause gadget, exactly one line gadget takes the top two crescents, and no two line gadgets take the same crescents at the same time. From the argument above, each clause gadget has one line gadget with $\ell+1$ matching edges and two line gadgets with ℓ matching edges. Thus, we have $(3\ell+1)m$ matching edges in total, which means we can obtain $t = (3\ell+1)m(2c-2)$ triangles by drawing $k = (3\ell+1)m(2c-1)$ lines. The opposite direction is clear from the above discussion.

As mentioned before, line gadgets have flexibility on their shapes. Using this fact, it is easy to see that all gadgets can be joined appropriately by polynomial number of line gadgets. Thus this is a polynomial time reduction.

Therefore, we complete the proof of Theorem 2.

5 Conclusion

In this paper, we formalized a combinatorial game that is an old pencil-and-paper game for two players played in Western Japan. This game has a similar flavor to "Games on Triangulations" investigated by Aichholzer et al. [1]. We have only showed the computational complexity in a few restricted cases of the game. Along the line in [1], we have a lot of unsolved variants in our game. For example, the hardness of a two-player variant of this game in general position, the strategies for the points in convex position with (fixed) k points inside of the points in convex position, and so on. At a glance at Theorem 1, the first player seems to have an advantage. It is interesting to investigate the positions of points where this intuition is false. For example, if the four points are placed as three points in convex position and one center point, the second player can win by simple case analysis.

Acknowledgments

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