

Title	Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid
Author(s)	Araki, Yoshiaki; Horiyama, Takashi; Uehara, Ryuhei
Citation	Lecture Notes in Computer Science, 8973: 294-305
Issue Date	2015-02-26
Type	Journal Article
Text version	author
URL	http://hdl.handle.net/10119/13803
Rights	This is the author-created version of Springer, Yoshiaki Araki, Takashi Horiyama and Ryuhei Uehara, Lecture Notes in Computer Science, 8973, 2015, 294-305. The original publication is available at www.springerlink.com , http://dx.doi.org/10.1007/978-3-319-15612-5_26
Description	WALCOM: Algorithms and Computation, 9th International Workshop, WALCOM 2015, Dhaka, Bangladesh, February 26-28, 2015. Proceedings

Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid

Yoshiaki Araki¹, Takashi Horiyama², and Ryuhei Uehara³

¹ Japan Tessellation Design Association, Japan
yoshiaki.araki@gmail.com

² Information Technology Center,
Saitama University, Japan

horiyama@al.ics.saitama-u.ac.jp

³ School of Information Science,
Japan Advanced Institute of Science and Technology, Japan
uehara@jaist.ac.jp

Abstract. Common unfolding of a regular tetrahedron and a Johnson-Zalgaller solid is investigated. More precisely, we investigate the sets of all edge unfoldings of Johnson-Zalgaller solids. Among 92 Johnson-Zalgaller solids, some of edge unfolding of J17 and J84 admit to fold into a regular tetrahedron. On the other hand, there are no edge unfolding of the other Johnson-Zalgaller solids that admit to fold into a regular tetrahedron.

1 Introduction

In 1525 the painter and printmaker Albrecht Dürer published a book, translated as “The Painter’s Manual,” in which he explained the methods of perspective [13]. In the book, he includes a description of many polyhedra, which he presented as surface unfoldings, are now called “nets.” An *edge unfolding* is defined by a development of the surface of a polyhedron to a plane, such that the surface becomes a flat polygon bounded by segments that derive from edges of the polyhedron. We would like an unfolding to possess three characteristics. (1) The unfolding is a single, simply connected piece. (2) The boundary of the unfolding is composed of (whole) edges of the polyhedron, that is, the unfolding is a union of polyhedron faces. (3) The unfolding does not self-overlap, that is, it is a simple polygon. We call a simple polygon that satisfies these conditions a *net* for the polyhedron.

Since then, nets for polyhedra have been widely investigated (rich background can be found in [5], and recent results can be found in [10]). For example, Alexandrov’s theorem states that every metric with the global topology and local geometry required of a convex polyhedron is in fact the intrinsic metric of some convex polyhedron. Thus, if P is a net of a convex polyhedron Q , then the shape (as a convex polyhedron) is uniquely determined. Alexandrov’s theorem was stated in 1942, and a constructive proof was given by Bobenko and Izmistiev in 2008 [4]. A pseudo-polynomial algorithm for Alexandrov’s theorem, given by Kane et al. in 2009, runs in $O(n^{456.5}r^{1891}/\epsilon^{121})$ time, where r is the ratio of the

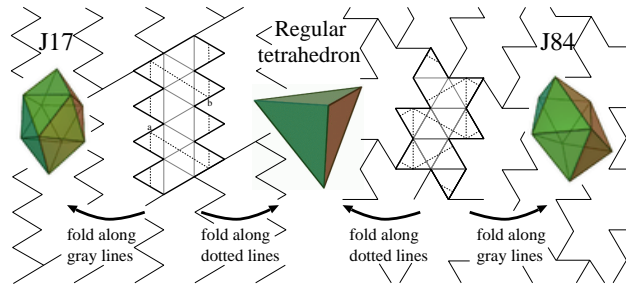


Fig. 1. (Left) an edge unfolding of the JZ solid J17, and (right) an edge unfolding of the JZ solid J84, which are also nets of a regular tetrahedron, respectively. These polygons are also p2 tilings.

largest and smallest distances between vertices, and ϵ is the coordinate relative accuracy [9]. The exponents in the time bound of the result are remarkably huge.

Therefore, we have to restrict ourselves to smaller classes of polyhedra to investigate from the viewpoint of efficient algorithms. In this paper, we consider some classes of polyhedra that have common nets. In general, a polygon can be a net of two or more convex polyhedra. Such a polygon is called a *common net* of the polyhedra¹. Recently, several polygons folding into two different polyhedra have been investigated (see [12] for comprehensive list). In this context, it is natural to ask whether there is a common net of two (or more) different Platonic solids. This question has arisen several times independently, and it is still open (see [5, Section 25.8.3]). In general nets, there is a polygon that can fold into a cube and an almost regular tetrahedron with small error $\epsilon < 2.89200 \times 10^{-1796}$ [12]. On the other hand, when we restrict ourselves to deal with only edge unfoldings, there are no edge unfolding of the Platonic solids except a regular tetrahedron that can fold into a regular tetrahedron [8]. This result is not trivial since a regular icosahedron and a regular dodecahedron have 43,380 edge unfoldings. In fact, it is confirmed that all the edge unfolding are nets (i.e., without self-overlapping) recently [6].

In this paper, we broaden the target of research from the set of five Platonic solids to the set of 92 Johnson-Zalgaller solids (JZ solids for short). A JZ solid is a strictly convex polyhedron, each face of which is a regular polygon, but which is not uniform, i.e., not a Platonic solid, Archimedean solid, prism, or antiprism (see, e.g., <http://mathworld.wolfram.com/JohnsonSolid.html>). Recently, the number of edge unfoldings of the JZ solids are counted [7], however, it has not been investigated how many nets (without self-overlapping) are there. On the other hand, the tilings of edge unfoldings of JZ solids are classified [2]. That is, they classified the class of the JZ solids whose edge unfoldings form tilings. Some tilings are well investigated in the context of nets; a polygon is a net of a regular tetrahedron if and only if it belongs to a special class of tilings [1].

¹ Note that an edge of an unfolding can pass through a flat face of the polyhedra.

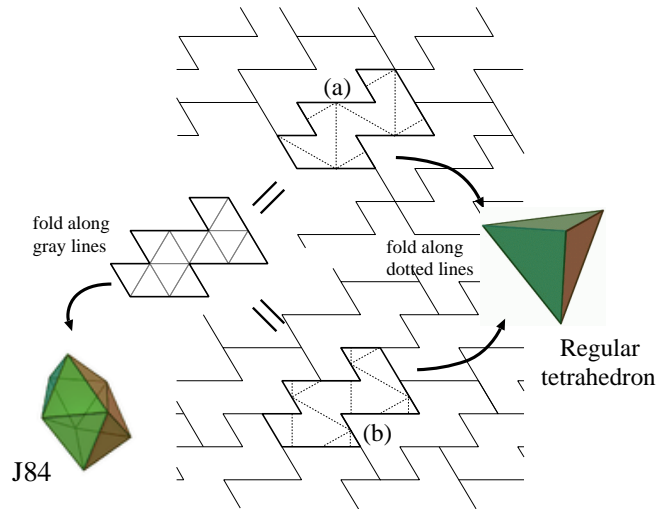


Fig. 2. An edge unfolding of the JZ solid J84. It has two different types of p2 tilings, and hence there are two different ways to fold into a regular tetrahedron.

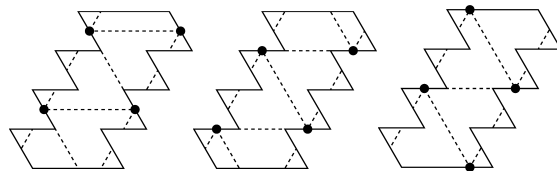


Fig. 3. An edge unfolding of the JZ solid J17 that can be folded into a regular tetrahedron in three different ways.

In this paper, we concentrate on common nets of a regular tetrahedron and the JZ solids. More precisely, we classify the set of edge unfoldings of the JZ solids such that each of them is also folded into a regular tetrahedron. We first show that there exists edge unfoldings of some JZ solids that are also nets of a regular tetrahedra:



















Theorem 1. *An edge unfolding of the JZ solid J17 and an edge unfolding of the JZ solid J84 fold into a regular tetrahedron.*

We will show that Fig. 1 certainly proves Theorem 1. Next we also compute all common nets that fold into both of a JZ solid and a regular tetrahedra:²

Theorem 2. *(1) Among 13,014 edge unfoldings of the JZ solid J17 [7], there are 87 nets that fold into a regular tetrahedron, which consist of 78 nets that have one way of folding into a regular tetrahedron, 8 nets that have two ways of*

² These numbers are counted on the “unlabeled” solids, and congruent unfoldings are not reduced. See [7] for further details.

Table 1. The JZ solids whose some edge unfoldings are nets of tetramonohedra.

Name	J1	J8	J10	J12	J13	J14
Image						
# of □s	1	5	1	0	0	3
# of △s	4	4	12	6	10	6
L_{J_i}	$\sqrt{\frac{\sqrt{3}}{3} + 1}$ = 1.255...	$\sqrt{\frac{5\sqrt{3}}{3} + 1}$ = 1.971...	$\sqrt{\frac{\sqrt{3}}{3} + 3}$ = 1.891...	$\sqrt{1.5}$ = 1.224...	$\sqrt{2.5}$ = 1.581...	$\sqrt{\sqrt{3} + \frac{3}{2}}$ = 1.797...
Name	J15	J16	J17	J49	J50	J51
Image						
# of □s	4	5	0	2	1	0
# of △s	8	10	16	6	10	14
L_{J_i}	$\sqrt{\frac{4\sqrt{3}}{3} + 2}$ = 2.075...	$\sqrt{\frac{5\sqrt{3}}{3} + \frac{5}{2}}$ = 2.320...	2 = 2	$\sqrt{\frac{2\sqrt{3}}{3} + \frac{3}{2}}$ = 1.629...	$\sqrt{\frac{\sqrt{3}}{3} + \frac{5}{2}}$ = 1.754...	$\sqrt{3.5}$ = 1.870...
Name	J84	J86	J87	J88	J89	J90
Image						
# of □s	0	2	1	2	3	4
# of △s	12	12	16	16	18	20
L_{J_i}	$\sqrt{3}$ = 1.732...	$\sqrt{\frac{2\sqrt{3}}{3} + 3}$ = 2.038...	$\sqrt{\frac{\sqrt{3}}{3} + 4}$ = 2.139...	$\sqrt{\frac{2\sqrt{3}}{3} + 4}$ = 2.270...	$\sqrt{\sqrt{3} + \frac{9}{2}}$ = 2.496...	$\sqrt{\frac{4\sqrt{3}}{3} + 5}$ = 2.703...

foldings into a regular tetrahedron, and 1 net that has three ways of folding into a regular tetrahedron. (2) Among 1,109 edge unfoldings of the JZ solid J84 [7], there are 37 nets that fold into a regular tetrahedron, which consist of 32 nets that have one way of folding into a regular tetrahedron, and 5 nets that have two ways of folding into a regular tetrahedron.

We note that some nets allow to fold into a regular tetrahedron in two or three different ways of folding. A typical example that has two ways of folding is shown in Fig. 2. We can tile the net of the JZ solid J84 in two different ways, hence we can fold a regular tetrahedron in two different ways according to the tilings. The unique net that has three ways of folding is shown in Fig. 3.

Among 92 JZ solids, Akiyama et al. found that 18 JZ solids have edge unfoldings that are also tilings [2]. We will show that all of them are also p2 tiling, which imply that they can be folded into tetramonohedra. As shown in Theorem 1, two of them can be folded into regular tetrahedra. On the other hand, the other 16 JZ solids do not have such edge unfoldings:

Theorem 3. *Except J17 and J84, there is no other JZ solid such that its edge unfolding is a net of a regular tetrahedron.*

Therefore, we classify the set of edge unfoldings of the JZ solids by the foldability of a regular tetrahedron.

2 Preliminaries

We first show some basic results about unfolding of a polyhedron.

Lemma 1 ([5, Sec. 22.1.3]). *All vertices of a polyhedron X are on the boundary of any unfolding of X .*

Let P be a polygon on the plane, and R be a set of four points (called *rotation centers*) on the boundary of P . Then P has a *tiling* called symmetry group $p2$ ($p2$ tiling, for short) if P fills the plane by the repetition of 2-fold rotations around the points in R . The filling should contain no gaps nor overlaps. The rotation defines an equivalence relation on the points in the plane. Two points p_1 and p_2 are mutually equivalent if p_1 can be moved to p_2 by the 2-fold rotations. More details of $p2$ tiling can be found, e.g., in [11]. Based on the notion of $p2$ tiling, any unfolding of a tetramonohedron³ can be characterized as follows:

Theorem 4 ([1, 3]). *P is an unfolding of a tetramonohedron if and only if (1) P has a $p2$ tiling, (2) four of the rotation centers consist in the triangular lattice formed by the triangular faces of the tetramonohedron, (3) the four rotation centers are the lattice points, and (4) no two of the four rotation centers belong to the same equivalent class on the tiling.*

We can obtain the characterization of the unfolding of a regular tetrahedron if each triangular face in Theorem 4 is a regular triangle. By Theorem 4, Theorem 1 is directly proved by Fig. 1. (Of course it is not difficult to check these nets in Fig. 1 by cutting and folding directly.)

In the classification in [2], they show only $p1$ tilings for the JZ solids J84, J86 and J89. However, they also have edge unfoldings that form $p2$ tilings as shown in Fig. 1 (J84) and Fig. 4 (J86 and J89), and hence they can fold into tetramonohedra.

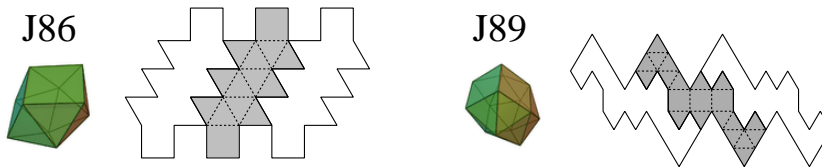


Fig. 4. $p2$ tilings by (left) an edge unfolding of JZ solid J86, and (right) an edge unfolding of JZ solid J89.

Let L_{J_i} be the length of an edge of a regular tetrahedron T_{J_i} that has the same surface area of the JZ solid J_i . We assume that each face of J_i is a regular polygon that consists of edges of unit length. Thus it is easy to compute L_{J_i} from its surface area of J_i as shown in Table 1. If an edge unfolding P_{J_i} of the JZ

³ A *tetramonohedron* is a tetrahedron that consists of four congruent triangular faces.

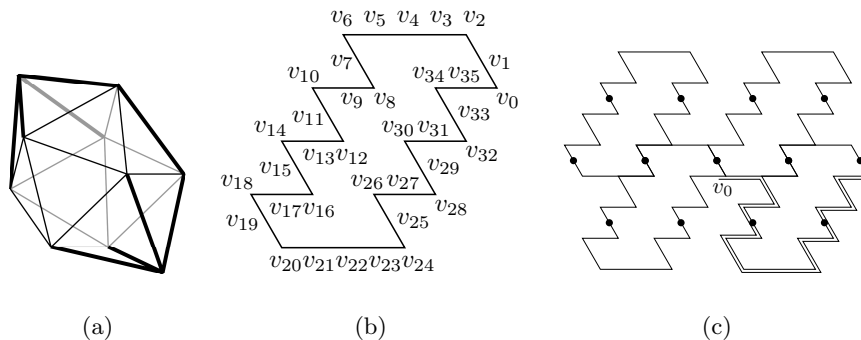


Fig. 5. (a) a spanning tree of the JZ solid J17, (b) its corresponding unfolding, and (c) a p2 tiling around the rotation centers.

solid J_i can be folded into a regular tetrahedron, the tetrahedron is congruent to T_{J_i} since they have the same surface area. Moreover, by Theorem 4, P_{J_i} is a p2 tiling, and its four of the rotation centers form the regular triangular lattice filled by regular triangles of edge length L_{J_i} . Let c_1 and c_2 be any pair of the rotation centers of distance L_{J_i} . Then, by Lemma 1, c_1 and c_2 are on boundary of P_{J_i} and P'_{J_i} for some polygons P_{J_i} and P'_{J_i} , respectively. By the same extension of Theorem 25.3.1 in [5] used in [8, Lemma 8], we can assume that c_1 and c_2 are on the corners or the middlepoints on some edges of regular faces of JZ solids J_i without loss of generality. Summarizing them, we obtain the following lemma:

Lemma 2. *Assume that a polygon P_{J_i} is obtained by an edge unfolding of a JZ solid J_i . If P_{J_i} can be folded into a regular tetrahedron T_{J_i} , P_{J_i} forms a p2 tiling \mathcal{T} . Let c_1 and c_2 be any two rotation centers on \mathcal{T} such that the distance between c_1 and c_2 is L_{J_i} , equal to the length of an edge of T_{J_i} . Then, the vertices c_1 and c_2 are on the corners or the middlepoints on edges of unit length in \mathcal{T} .*

3 The JZ solids J17 and J84

In this section, we describe an algorithm to obtain Theorem 2. By applying the technique in [6], we can enumerate a set of spanning trees of any polyhedron, where a spanning tree is obtained as a set of edges. By traversing each spanning tree, we can obtain its corresponding unfolding P_{J_i} . Since all edges of a JZ solid have the same length, P_{J_i} can be represented by a cyclic list C_{J_i} of its interior angles a_j , where vertices v_j of P_{J_i} correspond to the corners or the middlepoints on some edges of the original JZ solid. Since a spanning tree has $n - 1$ edges, each edge appears twice as the boundary of P_{J_i} , and each edge is broken into two halves, P_{J_i} has $4(n - 1)$ vertices. Fig. 5 illustrates (a) a spanning tree of the JZ solid J17, and (b) its corresponding unfolding, which can be represented by $C_{J_i} = \{60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180\}$.

Now, we use Theorem 4 and check if each edge unfolding is a p2 tiling or not. We can use the similar idea with the algorithm for gluing borders of a polyhedron (see [5, Chap. 25.2]): around each rotation center, check if the corresponding points make together 360° . If not, we dismiss this case, and otherwise, we obtain a gluing to form a regular tetrahedron.

We first consider the JZ solid J17. In this case, we can determine the length of each edge of the triangular lattice equals to 2, since each face of the (potential) regular tetrahedron consists of four unit tiles. We can check if each unfolding of the JZ solid J17 can be folded into a regular tetrahedron as follows:

1. For each pair of v_{j_1} and v_{j_2} , suppose they are rotation centers, and check if the distance between them is 2.
2. Obtain a path $v_{j'_1}-v_{j_1}-v_{j''_1}$ which is glued to $v_{j''_1}-v_{j_1}-v_{j'_1}$ by a 2-fold rotation around v_{j_1} . So do a path $v_{j'_2}-v_{j_2}-v_{j''_2}$ for v_{j_2} .
3. Replace interior angles of $a_{j'_1}, \dots, a_{j_1}, \dots, a_{j''_1}$ in C_{J_i} with angle a'_{j_1} , where $a'_{j_1} = a_{j'_1} + a_{j''_1}$ if $j'_1 \neq j''_1$ and $a'_{j_1} = a_{j_1} + 180$ if $j'_1 = j''_1 = j_1$. So do $a_{j'_2}, \dots, a_{j''_2}$. Let C'_{J_i} be the resulting cyclic list.
4. For each pair of v_{j_3} and v_{j_4} , suppose they are rotation centers, and check if a path $v_{j_3}-v_{j_4}$ in C'_{J_i} is glued to the remaining path $v_{j_4}-v_{j_3}$.
5. Check if v_{j_3} and v_{j_4} are the lattice points of the regular triangular lattice defined by v_{j_1}, v_{j_2} , and check if no two of $v_{j_1}, v_{j_2}, v_{j_3}$ and v_{j_4} belong to the same equivalent class.

In Step 1, since every face of the JZ solid J17 is a triangle, a_j is always a multiple of 60. The relative position of v_j from v_0 can be represented as a linear combination of two unit vectors \mathbf{u} and \mathbf{v} that make a 60° angle. Thus, we check if vector $\overrightarrow{v_{j_1}v_{j_2}}$ is one of $\pm 2\mathbf{u}, \pm 2\mathbf{v}, \pm 2(\mathbf{u} - \mathbf{v})$ in this step.

In Step 2, two vertices $v_{j'_1}$ and $v_{j''_1}$ are obtained as v_{j_1-k} and v_{j_1+k} with an integer k satisfying $a_{j_1-k} + a_{j_1+k} < 360$ and $a_{j_1-k'} + a_{j_1+k'} = 360$ for all $0 \leq k' < k$. In Fig. 5, v_1 and v_7 are supposed to be rotation centers, and paths $v_0-v_1-v_2$ and $v_4-v_7-v_{10}$ are glued to $v_2-v_1-v_0$ and $v_{10}-v_7-v_4$, respectively. By rotating P_{J_i} around v_{j_1} and v_{j_2} repeatedly, we obtain a horizontally infinite sequence of P_{J_i} as shown in Fig. 5(c), whose upper and lower borders are the repetition of the path denoted in double line. The list of the interior angles along the double line is obtained as C'_{J_i} in Step 3. In Fig. 5(c), C'_{J_i} is $\{180, 180, 240, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180\}$.

In Step 4, we check if $a_{j_3} = a_{j_4} = 180$ holds and $a_{j_3-k} + a_{j_3+k} = 360$ for other gluing of vertices v_{j_3-k} and v_{j_3+k} in C'_{J_i} . If P_{J_i} passes all checks in Steps 1–4, P_{J_i} has a p2 tiling with rotation centers $v_{j_1}, v_{j_2}, v_{j_3}$ and v_{j_4} . In Step 5, we check if the four points meet Theorem 4(2)–(4) and if each triangular face is a regular triangle. As in Step 1, this check can be done from the positions of vertices represented as a linear combination of \mathbf{u} and \mathbf{v} .

For the JZ solid J84, we can check in the same way by letting the length of the triangular lattice equal to $\sqrt{3}$, and thus, in Step 1, we check if vector $\overrightarrow{v_{j_1}v_{j_2}}$ is one of $\pm(\mathbf{u} + \mathbf{v}), \pm(2\mathbf{u} - \mathbf{v}), \pm(2\mathbf{v} - \mathbf{u})$. The complete catalogue of 87 and 37 nets of the JZ solids J17 and J84, respectively, that fold into a regular

tetrahedron is given in <http://www.al.ics.saitama-u.ac.jp/horiyama/research/unfolding/common/>.

4 The other JZ solids

In this section, we prove Theorem 3. Combining the results in [2] and the tilings in Fig. 1 and Fig. 4, the set \mathcal{J} of JZ solids whose edge unfoldings can be p2 tiling is $\mathcal{J} = \{J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90\}$. In other words, some edge unfoldings of the JZ solids in \mathcal{J} can be folded into tetramonohedra. Among them, J17 and J84 allow to fold into regular tetrahedra from their edge unfoldings as shown in Fig. 1. We will show that the other JZ solids do not. Hereafter, we only consider the JZ solids in \mathcal{J} . Then each face is either a unit square or a unit triangle. We call each of them a *unit tile* to simplify. We consider the rotation centers form the regular triangular lattice of size L_{J_i} . Let c_1 and c_2 be any pair of the rotation centers of distance L_{J_i} . We use the fact that the distance between c_1 and c_2 is equal to L_{J_i} , and show that any combination of unit tiles cannot achieve the length.

Intuitively, two points c_1 and c_2 are joined by a sequence of edges of unit length that are supported by unit tiles in \mathcal{T} . Thus, by Lemma 2, we can observe that there exists a linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \dots, q_k, p_k)$ such that (1) c_1 is on either p_0 or q_1 , (2) c_2 is on either q_k or p_k , (3) the length of $p_i p_{i+1}$ is 1, (4) the length of $p_{i-1} q_i$ and $q_i p_i$ is $1/2$ (in other words, q_i is the center point of $p_{i-1} p_i$), (5) each angle at q_i ($1 \leq i \leq k$) is 180° , (6) each angle at p_i ($1 \leq i \leq k-1$) is in $\{60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ\}$, and (7) the linkage is not self-crossing. (See [5] for the definition of the notion of *linkage*.) Without loss of generality, we suppose that L_{J_i} has the minimum length among the linkages satisfying the conditions from (1) to (7). By the minimality, we also assume that (8) $p_i \neq p_j$ for each $i \neq j$, and (9) if $|i - j| > 1$, the distance between p_i and p_j is not 1 (otherwise, we obtain a shorter linkage).

Therefore, by Theorem 4, for sufficiently large k , if all possible pairs c_1 and c_2 on the linkages satisfying the conditions from (1) to (9) do not achieve the required distance L_{J_i} , any edge unfolding of the JZ solid J_i cannot be folded into a corresponding regular tetrahedron T_{J_i} . We show an upper bound of k :

Theorem 5. *Let \mathcal{J} be the set $\{J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90\}$ of the JZ solids that have some edge unfoldings which are also p2 tilings. For some $J_i \in \mathcal{J}$, suppose that the linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \dots, q_k, p_k)$ defined above exist. Then $k \leq 10$.*

Proof. By simple calculation, L_{J_i} takes the maximum value $\sqrt{4\sqrt{3}/3 + 5} = 2.703\dots$ for $J90$ in \mathcal{J} . Thus the length of the line segment $c_1 c_2$ is at most $2.703\dots$.

Now we assume that the line segment $c_1 c_2$ passes through a sequence $C_1 C_2 \dots C_h$ of unit tiles in this order. That is, the line segment $c_1 c_2$ has nonempty intersection with each of C_i in this order. If $c_1 c_2$ passes an edge shared by two unit tiles,

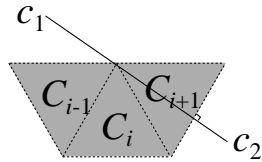


Fig. 6. The shortest way to pierce three consecutive unit tiles.

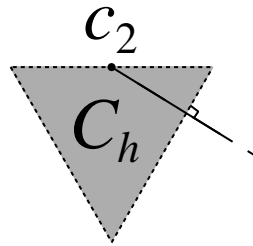


Fig. 7. The shortest intersection with the last unit tile C_h .

we take arbitrary one of two in the sequence. We consider the minimum length of the part of c_1c_2 that intersects three consecutive unit tiles $C_{i-1}C_iC_{i+1}$ in the sequence. Since they are unit triangles and/or squares, three unit tiles make greater than or equal to 180° at a vertex. Therefore the minimum length is achieved by the three consecutive triangles arranged in Fig. 6, and in this case, the length is greater than or equal to $\sqrt{3}/2 = 0.866\dots$. Thus, if c_1c_2 passes through nine unit tiles, the intersection has length at least $3\sqrt{3}/2 = 2.598\dots$. On the other hand, the last point c_2 is on the vertex or a midpoint of an edge of the last unit tile C_h . Then the intersection of c_1c_2 and C_h has at least $\sqrt{3}/4 = 0.433\dots$ (Fig. 7). Since $3\sqrt{3}/2 + \sqrt{3}/4 = 3.03\dots > 2.703$, c_1c_2 passes through at most 9 unit tiles.

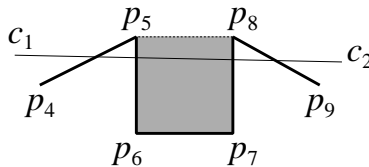


Fig. 8. Unit square can contribute three edges to the linkage.

Now we turn to the linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \dots, q_k, p_k)$ supported by the unit tiles $C_1C_2\dots C_h$ with $h \leq 9$. We consider the number of edges of a unit tile that contributes to L_{J_i} . Locally, the worst case is that a unit square that contributes three edges to L_{J_i} (Fig. 8). However, in this case, the length of the intersection of the square and c_1c_2 has length at least 1. Therefore, further analysis for the remaining length at most $2.703\dots - 1 = 1.703\dots$, and we can confirm that this case does not give the worst value of k . In the same reason, if c_1c_2 passes through an entire edge of length 1, it does not give the worst value of k . Next considerable case is that a unit tile C_i contributes two edges to L_{J_i} independent from C_{i-1} and C_{i+1} . That is, c_1c_2 passes through two vertices of

C_i . Then C_i is not a unit triangle since we can replace two edges by the third edge and obtain a shorter linkage. Thus C_i is a unit square, and c_1c_2 passes through the diagonal of C_i since two edges are not shared by C_{i-1} and C_{i+1} . Then the intersection of c_1c_2 and C_i has length $\sqrt{2} = 1.414 \dots$, and hence this case does not give the worst value of k again. Therefore, in the worst case, each unit tile contributes exactly two edges to L_{J_i} , and each edge is shared by two consecutive unit tiles in the sequence $C_1C_2 \dots C_h$, where $h \leq 9$. Therefore, the linkage consists of at most 10 unit length edges, that is, $k \leq 10$. \square

Now, for $k \leq 10$, if all possible pairs c_1 and c_2 on the linkages satisfying the conditions from (1) to (9) do not realize any distance L_{J_i} in Table 1, any edge unfolding of the JZ solid J_i cannot be folded into a corresponding regular tetrahedron T_{J_i} . However, the number of possible configurations of the linkage is still huge. To reduce the number, we use the following theorem:

Theorem 6. *Let \mathcal{J} be the set $\{J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90\}$ of the JZ solids that have some edge unfoldings which are also $p2$ tilings. For some $J_i \in \mathcal{J}$, suppose that the linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \dots, q_k, p_k)$ defined above exist. Let I be the set of integers and $I_{+1/2}$ be the set defined by $I \cup \{i + 1/2 \mid i \in I\}$. Let $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (\sqrt{3}/2, 1/2)$, $\mathbf{u}_3 = (1/2, \sqrt{3}/2)$, $\mathbf{u}_4 = (0, 1)$, $\mathbf{u}_5 = (-1/2, \sqrt{3}/2)$, and $\mathbf{u}_6 = (-\sqrt{3}/2, 1/2)$ be six unit length vectors (Fig. 9). Then there are four integers k_2, \dots, k_5 in I and two numbers k_1, k_6 in $I_{+1/2}$ such that $\sum_{i=1}^6 |k_i| \leq 10$ and $c_2 = c_1 + \sum_{i=1}^6 k_i \mathbf{u}_i$.*

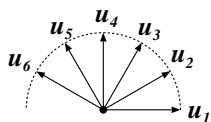


Fig. 9. Six unit vectors.

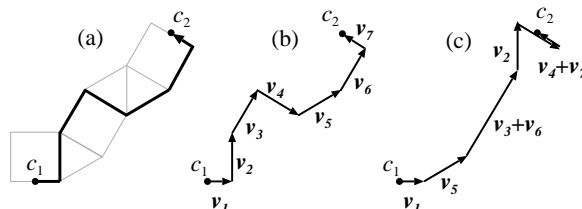


Fig. 10. Linkage as the set of unit vectors: (a) given tiling and linkage, (b) corresponding vectors, and (c) reorganized vectors.

Proof. When we regard each edge in the linkage as a unit vector, since vectors are commutative, we can swap two edges without changing the coordinate of c_2 (see Fig. 10; one can find the same idea in, e.g., [5, Section 5.1.1]). Thus we have the theorem. \square

Corollary 1. *For the two points c_1 and c_2 with $c_2 = c_1 + \sum_{i=1}^6 k_i \mathbf{u}_i$ in Theorem 6, There are four numbers h_1, h_2, h_3, h_4 in $I_{+1/2}$ such that $c_2 = c_1 + \sum_{i=1}^4 h_i \mathbf{u}_i$.*

Proof. Since $\mathbf{u}_6 = \mathbf{u}_4 - \mathbf{u}_2$ and $\mathbf{u}_5 = \mathbf{u}_3 - \mathbf{u}_1$, we can remove two vectors from the equation. Precisely, we have $\sum_{i=1}^6 k_i \mathbf{u}_i = (k_1 - k_5)\mathbf{u}_1 + (k_2 - k_6)\mathbf{u}_2 + (k_3 + k_5)\mathbf{u}_3 + (k_4 + k_4)\mathbf{u}_4$. \square

Now we prove the main theorem in this section:

Proof. (of Theorem 3) First we consider two points c_1 and c_2 given in Corollary 1: $c_2 = c_1 + \sum_{i=1}^4 h_i \mathbf{u}_i$ for some four numbers h_1, h_2, h_3, h_4 in $I_{+1/2}$. Then $|c_1 c_2|^2 = L_{J_i}^2 = h_1^2 + h_2^2 + h_3^2 + h_4^2 + 2h_1 h_2 + 2h_1 h_3 + 2h_1 h_4 + 2h_2 h_3 + 2h_2 h_4 + 2h_3 h_4$.

Now we fix some JZ solid J_i for some i . Let m_i and n_i be the number of triangles and squares in J_i , respectively. Then we have $L_{J_i}^2 = m_i/4 + \sqrt{3}n_i/3$.

By the condition that $h_1, h_2, h_3, h_4 \in I_{+1/2}$, we can observe that

$$m_i = 4h_1^2 + 4h_2^2 + 4h_1 h_3 + 4h_2 h_4 + 13h_2^2 + 13h_3^2 \quad (1)$$

$$n_i = 6(h_1 h_2 + h_2 h_3 + h_3 h_4) \quad (2)$$

From the second equation, we can observe that n_i is a multiple of 3. Thus the JZ solids J1, J8, J10, J15, J16, J49, J50, J86, J87, J88, and J90 have no edge unfolding that is a net of a regular tetrahedron.

For the remaining JZ solids J12 ($n = 0, m = 6$), J13 ($n = 0, m = 10$), J14 ($n = 3, m = 6$), J51 ($n = 0, m = 14$), and J89 ($n = 3, m = 18$), we check them by a brute force. More precisely, we generate all possible $k_1, k_2, \dots, k_6 \in [-10..10] \cap I_{+1/2}$ with $\sum_{i=1}^6 |k_i| \leq 10$, and compute $h_1 = k_1 - k_6, h_2 = k_2 - k_5, h_3 = k_3 + k_6, h_4 = k_4 + k_5$, and n and m by the above equations. Then no 6-tuple (k_1, k_2, \dots, k_6) generates any pair of $(n = 0, m = 6)$, $(n = 0, m = 10)$, $(n = 3, m = 6)$, $(n = 0, m = 14)$, and $(n = 3, m = 18)$ ⁴.

Therefore, in \mathcal{J} , only J17 and J84 have feasible solutions $L_{J_{17}} = 2$ and $L_{J_{84}} = \sqrt{3}$ in the distances. \square

5 Convex Polyhedra with Regular Polygonal Faces

According to the classification in [2], there are 23 polyhedra with regular polygonal faces whose edge unfoldings allow tilings. Among them, 18 JZ solids have been discussed in Section 4, and four Platonic solids were discussed in [8]. The remaining one is hexagonal antiprism that consists of two regular hexagons and 12 unit triangles. By splitting each regular hexagon into six unit triangles, which is called *coplanar deltahedron*, we can show the following theorem using the same argument above:

Theorem 7. *The hexagonal antiprism has no edge unfolding that can fold into a regular tetrahedron.*

⁴ From the viewpoint of the programming, we introduce integer variables $k'_1 = 2k_1, k'_2 = 2k_2, \dots$, and $k'_6 = 2k_6$, and compute $4m$ and $4n$. Then all computation can be done on integers. Hence we can avoid computational errors, and the program runs in a second.

Thus we can conclude as follows:

Corollary 8 *Among convex polyhedra with regular polygonal faces, including the Platonic solids, the Archimedean solids, and the JZ solids, regular prisms, and regular anti-prisms, only the JZ solids J17 and J84 (and regular tetrahedron) admit to fold into regular tetrahedra from their edge unfoldings.*

6 Concluding Remarks

In this paper, we show that the JZ solids J17 and J84 are exceptionally in the sense that their edge unfoldings admit to fold into regular tetrahedra. Especially, some edge unfoldings can fold into a regular tetrahedron in two or three different ways. In this research, the characterization of nets by tiling (Theorem 4) plays an important role. In general, even the decision problem that asks if a polyhedron can be folded from a given polygon is quite difficult problem [5, Chapter 25]. More general framework to solve the problem is future work.

References

1. J. Akiyama. Tile-Makers and Semi-Tile-Makers. *The Mathematical Association of Amerika*, Monthly 114:602–609, August-September 2007.
2. J. Akiyama, T. Kuwata, S. Langerman, K. Okawa, I. Sato, and G. C. Shephard. Determination of All Tessellation Polyhedra with Regular Polygonal Faces. In *Proc. CGGA 2010*, pp. 1–11. LNCS 7033, 2011.
3. J. Akiyama and C. Nara. Developments of Polyhedra Using Oblique Coordinates. *J. Indonesia. Math. Soc.*, 13(1):99–114, 2007.
4. A. I. Bobenko and I. Izvestiev. Alexandrov’s theorem, weighted Delaunay triangulations, and mixed volumes. arXiv:math.DG/0609447, February 2008.
5. E. D. Demaine and J. O’Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007.
6. T. Horiyama and W. Shoji. Edge unfoldings of Platonic solids never overlap. In *Proc. CCCG 2011*, 65–70, 2011.
7. T. Horiyama and W. Shoji. The Number of Different Unfoldings of Polyhedra. In *Proc. ISAAC 2013*, pp. 623–633, LNCS 8283, 2013.
8. T. Horiyama and R. Uehara. Nonexistence of Common Edge Developments of Regular Tetrahedron and Other Platonic Solids. In *Proc. China-Japan Joint Conference on Computational Geometry, Graphs and Applications (CGGA 2010)*, pp. 56–57, 2010.
9. D. Kane, G. N. Price, and E. D. Demaine. A pseudopolynomial algorithm for Alexandrov’s Theorem. In *Proc. WADS 2009*, pp. 435–446, LNCS 5664, 2009.
10. J. O’Rourke. *How to Fold It: The Mathematics of Linkage, Origami and Polyhedra*. Cambridge University Press, 2011.
11. D. Schattschneider. The plane symmetry groups: their recognition and notation. *American Mathematical Monthly*, 85:439–450, 1978.
12. T. Shirakawa, T. Horiyama, and R. Uehara. Construct of Common Development of Regular Tetrahedron and Cube. In *Proc. EuroCG 2011*, pp. 47–50, 2011.
13. W. S. Strauss and A. Dürer. *The Painter’s Manual*. Abaris Books, 1977.