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<th><strong>Title</strong></th>
<th>Refined RC4 key correlations of internal states in WPA</th>
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<td><strong>Author(s)</strong></td>
<td>Ito, Ryoma; Miyaji, Atsuko</td>
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<td>Copyright (C) 2016 The Institute of Electronics, Information and Communication Engineers (IEICE). Ryoma Ito, Atsuko Miyaji, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Science, E99-A(6), 2016, 1132-1144. <a href="http://www.ieice.org/jpn/trans_online/">http://www.ieice.org/jpn/trans_online/</a></td>
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SUMMARY WPA is the security protocol for IEEE 802.11 wireless networks standardized as a substitute for WEP in 2003, and uses RC4 stream cipher for encryption. It improved a 16-byte RC4 key generation procedure, which is known as TKIP, from that in WEP. One of the remarkable features in TKIP is that the first 3-byte RC4 key is derived from the public parameter IV, and an analysis using this feature has been reported by Sen Gupta et al. at FSE 2014. They focused on correlations between the public parameter IV, and an analysis using this feature has been reported. Many attacks on WEP and WPA have been intensively analyzed over past 20 years. There are mainly two approaches to the cryptanalysis on RC4. One is to demonstrate the existence of events with non-randomness, which is known as bias, involving the RC4 key, the internal state variables and the keystream bytes. The other is to attack on RC4 using biases in order to recover the RC4 key (key recovery attacks) [18], [20], the internal state variables (state recovery attacks) [1], [10], [15] and the plaintexts (plaintext recovery attacks) [12], [14]. In addition, a number of analyses related to the security protocols have been reported such as the plaintext recovery attacks on SSL/TLS [6], [16], the key recovery attacks on WEP [3], [9] and the plaintext recovery attacks on WPA [4], [17]. Here, we refer to the event with the probability significantly higher or lower than $\frac{1}{N}$ (the probability of random association) as the positive bias or the negative bias, respectively.

WPA is the security protocol for IEEE 802.11 wireless networks standardized as a substitute for WEP in 2003. It improves a 16-byte RC4 key generation procedure from that in WEP, which is known as Temporal Key Integrity Protocol.
(TKIP). TKIP includes a key management scheme, a temporal key hash function [5] and a message integrity code function. The key management scheme after the authentication based on IEEE 802.1X generates a 16-byte Temporal Key (TK). Then, the TK, a 6-byte Transmitter Address and a 48-bit Initialization Vector (IV), which is a sequence counter, are given as the inputs to the temporal key hash function, and the function outputs a 16-byte RC4 key. In addition, TKIP uses MICHAEL [2] to ensure integrity of a message. One of the remarkable features in TKIP is that the first 3-byte RC4 key, $K[0]$, $K[1]$ and $K[2]$, are derived from the last 16-bit IV (IV16) as follows:

$$K[0] = (IV16 >> 8) \& 0xFF,$$

$$K[1] = ((IV16 >> 8) \& 0x20) \& 0x7F,$$

$$K[2] = IV16 \& 0xFF.$$

Note that these RC4 key bytes in WPA are known since IV can be obtained by observing a packet.

In 2014, Sen Gupta et al. demonstrated a probability distribution of a sum of the first 2-byte RC4 key, $K[0]$ and $K[1]$, in WPA. From Eqs. (1) and (2), the value of $K[1]$ depends on that of $K[0]$, and its range is limited to either from 32 to 63 or from 96 to 127 in order to avoid the known WEP attack by Fluhrer et al. [3]. In addition, $K[0] + K[1]$ must be always even. Therefore, such a relation between $K[0]$ and $K[1]$ induces biases of $K[0] + K[1]$ in WPA. Furthermore, they also showed some linear correlations between the keystream bytes and the known RC4 key bytes in WPA such as $Z_i = -K[0] - K[1]$, $Z_3 = K[0] + K[1] + K[2] + 3$, and so on. These correlations could be added to the known set of biases for the keystream bytes. Therefore, they could apply these correlations to the existing plaintext recovery attack on SSL/TLS [6] especially in WPA, and could improve its computational complexity necessary for the attack.

In this paper, we investigate new linear correlations including unknown internal state variables in both generic RC4 and WPA. Here, unknown internal state variables mean $S_i, S_{i+1}, S_{j+1}, j + t_{r+1}$ for $r \geq 0$. In addition, we also focus on the difference between generic RC4 and WPA, and discover some different correlations. These correlations exactly reflect difference of the distribution of $K[0] + K[1]$ between both generic RC4 and WPA. As a result, we discover more than 150 linear correlations newly and succeed to give proof of some of them. Our contributions can be summarized in the following 9 theorems:

- Theorems 1 and 2 show $Pr(S_0[i] = K[0])$ in generic RC4 and WPA, respectively. In particular, we stress that $Pr(S_0[i] = K[0]) = 0$ in WPA.
- Theorems 3 and 4 show $Pr(S_0[i] = K[0] - K[1] - 3)$, Theorems 5 and 6 show $Pr(S_0[i] = K[0] - K[1] - 1)$ in generic RC4 and WPA, respectively. Only WPA gives double probabilities of random association $\frac{1}{2}$.
- Theorem 7 shows $Pr(S_{255}[i_{256}] = K[0])$ is pretty high probability in comparison with the probability of random association $\frac{1}{256}$ in both generic RC4 and WPA. On the other hand, Theorem 8 shows $Pr(S_{255}[i_{256}] = K[1])$ is high probability only in WPA.
- Theorem 9 shows $Pr(S_{1}[i_{r+1}] = K[0] + K[1] + 1)$ in generic RC4 and WPA for $0 \leq r \leq N$, which is distributed in the same way as the distribution of $K[0] + K[1]$.

Some theoretical proofs of the preliminary version of this paper [7], [8] rather high relative errors, which are improved in this paper.

This paper is organized as follows: Section 2 summarizes the previous works necessary for both theoretical proofs and experiments such as Roos’ biases [18], [19], nested Roos’ biases [11], [13], the distribution of $K[0]+K[1]$ in WPA [4] and the number of samples necessary for distinguishing two distributions [14]. Section 3 shows the theoretical proofs of prominent linear correlations and the experimental results. Section 4 concludes this paper.

2. Preliminary

Let us summarize some previous results which will be used in both theoretical proofs and experiments as preliminary. Proposition 1 shows Roos’ biases [19], correlations between the RC4 key bytes and the initial state $S_0$ of the PRGA, proved by Paul and Maitra [18]. Propositions 2 shows nested Roos’ biases [11], correlations similar to Roos’ biases, proved by Maitra et al. [13]. Proposition 3 shows a distribution of $K[0] + K[1]$ based on a relation between $K[0]$ and $K[1]$ generated by the temporal key hash function in WPA, proved by Sen Gupta et al. [4]. Proposition 4 shows the number of samples necessary for distinguishing two distributions with a constant probability of success, proved by Mantin and Shamir [14].

**Proposition 1 ([18]):** In the initial state of the PRGA for $0 \leq y \leq N - 1$, we have

$$Pr(S_0[y] = \frac{y(q+1)}{2} + \sum_{x=0}^{y} K[x]) \approx (1 - \frac{y}{N}) \cdot (1 - \frac{1}{N})^{\frac{y(q+1)+N}{N}} + \frac{1}{N}.$$  

**Proposition 2 ([11]):** In the initial state of the PRGA for $0 \leq y \leq 31$, $Pr(S_0[S_0[y]] = f_y)$ is approximately

$$\left( \frac{y}{N} + \frac{1}{N} \right) \cdot (1 - \frac{1}{N})^{2-y} + (1 - \frac{y}{N}) \cdot (1 - \frac{1}{N})^{\frac{y(q+1)+2N-4}{N}},$$  

where $f_y = \frac{y(q+1)}{2} + \sum_{x=0}^{y} K[x]$.

**Proposition 3 ([4]):** For $0 \leq v \leq N - 1$, the distribution of the sum $v$ of $K[0]$ and $K[1]$ generated by the temporal key hash function in WPA is given as follows:

$$Pr(K[0] + K[1] = v) = 0 \quad \text{if } v \text{ is odd},$$

$$Pr(K[0] + K[1] = v) = 0 \quad \text{if } v \text{ is even and } v \text{ is odd, and } v \in [0, 31] \cup [128, 159],$$

$$Pr(K[0] + K[1] = v) = \frac{2}{256} \quad \text{if } v \text{ is even and } v \in [32, 63] \cup [96, 127] \cup [160, 191] \cup [224, 255],$$

$$Pr(K[0] + K[1] = v) = \frac{4}{256} \quad \text{if } v \text{ is even and } v \in [64, 95] \cup [192, 223].$$
3. Newly Discovered Linear Correlations

3.1 Experimental Observations

Let us investigate some correlations of the following unknown internal state variables in both generic RC4 and WPA: $S_r[i_{r+1}], S_r[j_{r+1}], j_{r+1}$ and $i_{r+1}$ for $r \geq 0$. Linear correlations of the keystream bytes $Z_r$ were investigated by Sen Gupta et al. in 2014 [4], which used a general linear form

$$Z_r = a \cdot K[0] + b \cdot K[1] + c \cdot K[2] + d$$  \hspace{1cm} (4)

for $a, b, c \in \{0, \pm1\}$ and $d \in \{0, \pm1, \pm2, \pm3\}$ for $r \geq 1$. Here, we further extend their linear form by Eq. (4) to

$$X_r = a \cdot Z_{r+1} + b \cdot K[0] + c \cdot K[1] + d \cdot K[2] + e,$$  \hspace{1cm} (5)

where $X_r \in \{S_r[i_{r+1}], S_r[j_{r+1}], j_{r+1}, i_{r+1}\}, a, b, c, d \in \{0, \pm1\}$ and $e \in \{0, \pm1, \pm2, \pm3\}$ for $r \geq 1$. Sen Gupta et al. could apply the biases by Eq. (4) to the plaintext recovery attack on WPA, and could improve its computational complexity necessary for the existing attack on SSL/TLS [6]. Similarly, we should apply the biases by Eq. (5) to the state recovery attack on WPA, and may reduce its computational complexity necessary for the existing attack on generic RC4 [11, 10], [15].

We have examined all $4 \cdot 3^4 \cdot 7$ equations defined by Eq. (5) in each round with $2^{32}$ randomly generated 16-byte RC4 keys in both generic RC4 and WPA. Some experimental results are presented in Tables 1 and A-1. We have summarized the correlations with more than 0.0048 or less than 0.0020 in either generic RC4 or WPA. Some correlations happen only in WPA although generic RC4 indicates neither positive nor negative bias. In particular, we stress that an event $S_0[i_1] = K[0]$ yields an impossible condition in WPA, and thus, the probability of the event is 0 (see Table 1). Then, the value of $S_0[i_1]$ is varied from 0 to $N - 1$ except $K[0]$.

We will prove these linear correlations theoretically shown in Table 1. In our proofs, we often use Roos’ biases (Proposition 1), nested Roos’ biases (Proposition 2) and the distribution of $K[0] + K[1]$ (Proposition 3), which are denoted by $\alpha_g = \Pr(S_0[g] = \frac{g - \beta_g}{2} + \frac{\beta_g}{2} + \sum_{x=0}^{g} K(x))$, $\beta_g = \Pr(S_0[g'] = \frac{g' - \beta_g}{2} + \sum_{x=0}^{g'} K(x))$ and $\gamma_e = \Pr(K[0] + K[1] = e)$, respectively.

We assume through proofs that the probability of certain events, confirmed experimentally1 that there are no significant biases, is the probability of random association $\frac{1}{N}$ (e.g. events related to the internal state). We also assume that the RC4 key $K$ is generated uniformly at random in both generic RC4 and WPA, except $K[0], K[1]$ and $K[2]$ in WPA since these are generated by IV using a sequence counter.

3.2 Bias in $S_0[i_1]$

In this section, we prove Theorems 1-6. Theorems 1 and 2 show that an event $S_0[i_1] = K[0]$ yields a negative bias in generic RC4 and never occurs in WPA, respectively. Theorems 3 and 4 show that an event $S_0[i_1] = K[0] - K[1] - 3$ yields a positive bias in generic RC4 and occurs with twice as frequently as the probability of random association $\frac{1}{N}$, respectively. Theorems 5 and 6 show that an event $S_0[i_1] = K[0] - K[1] - 1$ yields a slight bias in generic RC4 and occurs with twice as frequently as the probability of random association $\frac{1}{N}$, respectively. In addition, Theorems 4 and 6 are revised precisely form our preliminary version [8].

**Theorem 1**: In the initial state of the PRGA, we have

$$\Pr(S_0[i_1] = K[0])_{RC4} \approx \frac{1}{N} (1 - \frac{1}{N})^{N-2}.$$  \hspace{1cm}

**Proof**: Figure 2 shows a state transition diagram in the first 2 rounds of the KSA. From step 6 in Algorithm 1, both $j_0^k = j_0^h + S_0^k[0] + K[0] = 0 + 0 + K[0] = K[0]$ and $j_3^k = j_3^h + S_0^k[1] + K[1] = K[0] + K[1] + S_0^h[1]$ hold. The probability of event $S_0[i_1] = K[0]$ can be decomposed in three paths: $K[0] + K[1] = 0$ (Path 1), $K[0] + K[1] = 255$ (Path 2) and $K[0] + K[1] \neq 0, 255$ (Path 3). Both Paths 1 and 2 are further divided into two subpaths: $K[0] = 1$ (Paths 1-1 and 2-1) and $K[0] \neq 1$ (Paths 1-2 and 2-2), respectively. In the following proof, we use $S_0[i_1]$ instead of $S_0[i_1] (i_1 = 1)$ and $S_0^h[1]$ for simplicity.

**Path 1-1.** Figure 3 shows a state transition diagram in Path

<table>
<thead>
<tr>
<th>$X_r$</th>
<th>Linear correlations</th>
<th>RC4</th>
<th>WPA</th>
<th>Remarks</th>
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<tr>
<td>$S_0[i_1]$</td>
<td>$K[0] - K[1] - 3$</td>
<td>0.014320</td>
<td>0.003732</td>
<td>Theorems 1 and 2</td>
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<td></td>
<td>$K[0] - K[1] - 1$</td>
<td>0.003822</td>
<td>0.007587</td>
<td>Theorems 5 and 6</td>
</tr>
<tr>
<td>$S_{255}[i_1]$</td>
<td>$K[0]$</td>
<td>0.137294</td>
<td>0.138047</td>
<td>Theorem 7</td>
</tr>
<tr>
<td></td>
<td>$K[1]$</td>
<td>0.003911</td>
<td>0.051789</td>
<td>Theorem 8</td>
</tr>
<tr>
<td></td>
<td>$K[0] + K[1] + 1$</td>
<td>Fig. 1</td>
<td></td>
<td>Theorem 9</td>
</tr>
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</table>

1In order to use this assumption, the experiments are conducted with $2^{32}$ randomly generated RC4 keys of 16 bytes.
1-1. After the second round of the KSA, $S^k[1] = K[0]$ always holds since $j^r_k = K[0] = 1$ and $j^r_{2} = K[0] + K[1] + S^k[1] = 0 + 0 = 0$. Furthermore, $S^k[2] = S^k[1]$ for $3 \leq r \leq N$ if $j^r_k \neq 1$ during the subsequent $N - 2$ rounds, whose probability is $(1 - 1/N)^{N-2}$ approximately since we assume that $j^r_k = 1$ holds for each round with the probability of random association $1/N$. Therefore, we get

$$\Pr(S_0[1] = K[0] | \text{Path 1-1}) \approx (1 - 1/N)^{N-2}.$$  

**Path 1-2.** Figure 4 shows a state transition diagram in Path 1-2. After the second round of the KSA, $S^k[0] = K[0]$ always holds since $j^r_1 = K[0] \neq 1$ and $j^r_{2} = (K[0] + K[1]) + S^k[1] = 0 + 1 = 1$. Then, event $S_0[1] = K[0]$ never occurs because $S^k[1] \neq K[0]$ always holds for $r \geq 2$ from Algorithm 1. Therefore, we get

$$\Pr(S_0[1] = K[0] | \text{Path 1-2}) = 0.$$  

**Path 2-1.** Figure 5 shows a state transition diagram in Path 2-1. After the second round of the KSA, $S^k[0] = K[0]$ always holds in the same way as the case of Path 1-2. Then, event $S_0[1] = K[0]$ never occurs. Therefore, we get

$$\Pr(S_0[1] = K[0] | \text{Path 2-1}) = 0.$$  

**Path 2-2.** Figure 6 shows a state transition diagram in Path 2-2. After the second round of the KSA, $S^k[1] = K[0]$ always holds in the same way as the case of Path 1-1. Then, event $S_0[1] = K[0]$ occurs if $S_1[1] = S^k[1]$ for $3 \leq r \leq N$. Therefore, we get

$$\Pr(S_0[1] = K[0] | \text{Path 2-2}) \approx (1 - 1/N)^{N-2}.$$  

**Path 3.** Figure 2 shows a state transition diagram in Path 3. After the second round of the KSA, $S^k[0] = K[0]$ always holds in the same way as the cases of Paths 1-2 and 2-1. Then, event $S_0[1] = K[0]$ never occurs. Therefore, we get

$$\Pr(S_0[1] = K[0] | \text{Path 3}) = 0.$$  

In summary, event $S_0[i_1] = K[0]$ occurs only in either Paths 1-1 or 2-2. Therefore, since we assume that both $K[0]$ and $K[1]$ are generated uniformly at random, we get

$$\Pr(S_0[i_1] = K[0]) = \Pr(S_0[i_1] = K[0] | \text{Path 1-1}) \cdot \Pr(\text{Path 1-1})$$
$$+ \Pr(S_0[i_1] = K[0] | \text{Path 2-2}) \cdot \Pr(\text{Path 2-2})$$
$$\approx (1 - 1/N)^{N-2} \cdot 1/N + (1 - 1/N)^{N-2} \cdot 1/N(1 - 1/N) = 1/N(1 - 1/N)^{N-2}. 
$$

**Theorem 2:** In the initial state of the PRGA in WPA, we have

$$\Pr(S_0[i_1] = K[0])_{\text{WPA}} = 0.$$  

**Proof:** Note that event $S_0[i_1] = K[0]$ occurs if and only if either $K[0] + K[1] = 0$ or 255, and that Proposition 3 shows that neither $K[0] + K[1] = 0$ nor 255 holds in WPA. Therefore, we get

$$\Pr(S_0[i_1] = K[0])$$
$$= \Pr(S_0[i_1] = K[0] | \text{Path 1-1}) \cdot \Pr(\text{Path 1-1})$$
$$+ \Pr(S_0[i_1] = K[0] | \text{Path 2-2}) \cdot \Pr(\text{Path 2-2})$$
$$\approx (1 - 1/N)^{N-2} \cdot 0 + (1 - 1/N)^{N-2} \cdot 0 = 0.$$

$\square$
Theorem 3: In the initial state of the PRGA, we have
\[ \Pr(S[0][i_1] = K[0] - K[1] - 3)_{RC4} \approx \frac{2}{N} \alpha_1 + \frac{1}{N} (1 - \frac{2}{N})(1 - \alpha_1). \]

Proof: The probability of event \( S[0][i_1] = K[0] - K[1] - 3 \) can be decomposed in two paths: \( K[1] = 126, 254 \) (Path 1) and \( K[1] \neq 126, 254 \) (Path 2). In the following proof, we use \( S[0][1] \) instead of \( S[0][i_1] \) \((i_1 = 1)\) for simplicity.

Path 1. Since \( K[0] - K[1] - 3 = K[0] + K[1] + 1 \) if either \( K[1] = 126 \) or 254, event \( S[0][1] = K[0] - K[1] - 3 \) occurs if and only if \( S[0][0] = K[0] + K[1] + 1 \) under the condition of Path 1. Therefore, we get
\[ \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 1}) = \alpha_1. \]

Path 2. Since \( K[0] - K[1] - 3 \neq K[0] + K[1] + 1 \) if neither \( K[1] = 126 \) nor 254, event \( S[0][1] = K[0] - K[1] - 3 \) never occurs if \( S[0][0] = K[0] + K[1] + 1 \) under the condition of Path 2. If \( S[0][1] \neq K[0] + K[1] + 1 \) holds, then we assume that event \( S[0][1] = K[0] - K[1] - 3 \) occurs with the probability of random association \( \frac{1}{N} \). Therefore, we get
\[ \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 2}) = \frac{1}{N} \cdot (1 - \alpha_1). \]

In summary, since we assume that \( K[1] \) is generated uniformly at random, we get
\[ \Pr(S[0][i_1] = K[0] - K[1] - 3)_{RC4} \]
\[ = \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 1}) \cdot \Pr(\text{Path 1}) + \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 2}) \cdot \Pr(\text{Path 2}) \]
\[ \approx \frac{2}{N} \alpha_1 + \frac{1}{N} (1 - \frac{2}{N})(1 - \alpha_1), \]
\[ \text{where } \alpha_1 = \Pr(S[0][1] = K[0] + K[1] + 1) \approx \frac{(\frac{N}{N-1})^{N+2} + \frac{1}{N}}{N}. \]

Before showing Theorems 4 and 6, we prove Lemma 1. In the preliminary version [8], the relative errors of the events \( S[0][i_1] = K[0] - K[1] - 3 \) and \( S[0][i_1] = K[0] - K[1] - 1 \) in WPA are slightly large such as 4.589% and 4.212%, respectively. This is because we have proved them in the same way as the theoretical proofs of those in generic RC4. Furthermore, we could not discover the inherent feature in WPA, which is a probability distribution of \( K[0] - K[1] \). Lemma 1 shows the distribution of \( K[0] - K[1] \) in WPA. We may improve the relative errors by applying this feature to the theoretical proofs.

Lemma 1: For \( 0 \leq v \leq N - 1 \), the distribution of the difference \( v \) between \( K[0] \) and \( K[1] \) generated by the temporal key hash function in WPA is given as follows:
\[ \Pr(K[0] - K[1] = v) = \frac{1}{N} \text{ if } v \in \{0, 96, 128, 224\}, \]
\[ \Pr(K[0] - K[1] = v) = 0 \text{ otherwise.} \]

Proof: The value of \( K[0] - K[1] \) depends on the range of \( K[0] \) (see Table 2). Therefore, the probability distribution of \( K[0] - K[1] \) may be computed directly from the table. 

<table>
<thead>
<tr>
<th>( K[0] ) Range</th>
<th>( K[0] ) ( \mid ) ( K[1] ) value</th>
<th>( K[0] - K[1] ) Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 - 31 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 32 - 63 )</td>
<td>32</td>
<td>32 - 63</td>
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<tr>
<td>( 64 - 95 )</td>
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<tr>
<td>( 96 - 127 )</td>
<td>96</td>
<td>96 - 127</td>
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<td>( 128 - 159 )</td>
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<td>( 192 - 223 )</td>
<td>96</td>
<td>96 - 127</td>
</tr>
<tr>
<td>( 224 - 255 )</td>
<td>128</td>
<td>96 - 127</td>
</tr>
</tbody>
</table>

\[ \Pr(S[0][i_1] = K[0] - K[1] - 3)_{WPA} \approx \frac{4}{N} \alpha_1 + \frac{1}{N} (1 - \frac{N}{N-1})^{91} + (1 - \frac{1}{N})^{219} + (1 - \frac{1}{N})^{219}. \]

Proof: We note that the range of \( K[1] \) is limited to either from 32 to 63 or from 96 to 127 in WPA (see Table 2). Then, as with the discussion in the proof of Theorem 3, the probability of event \( S[0][i_1] = K[0] - K[1] - 3 \) in WPA can be decomposed in two paths: \( K[1] = 126 \) (Path 1) and \( K[1] \neq 126 \) (Path 2). In the following proof, we use \( S[0][1] \) instead of \( S[0][i_1] \) \((i_1 = 1)\) and \( S[N][1] \) for simplicity.

Path 1. Since \( K[0] - K[1] - 3 = K[0] + K[1] + 1 \) if either \( K[1] = 126 \), event \( S[0][1] = K[0] - K[1] - 3 \) occurs if and only if \( S[0][0] = K[0] + K[1] + 1 \) under the condition of Path 1. Therefore, we get
\[ \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 1}) = \alpha_1. \]

Path 2. Since \( K[0] - K[1] - 3 \neq K[0] + K[1] + 1 \) if \( K[1] \neq 126 \), event \( S[0][1] = K[0] - K[1] - 3 \) never occurs if \( S[0][0] = K[0] + K[1] + 1 \) under the condition of Path 2. We then focus on the distribution of \( K[0] - K[1] \) in WPA. Assuming that event \( S[0][1] = K[0] - K[1] - 3 \) occurs, \( S[0][1] \) can be one of the following values from Lemma 1: 93, 125, 221 or 253. Then, the probability in Path 2 can be further decomposed in four paths: \( K[0] - K[1] = 96 \) (Path 2-1), \( K[0] - K[1] = 128 \) (Path 2-2), \( K[0] - K[1] = 224 \) (Path 2-3) and \( K[0] - K[1] = 0 \) (Path 2-4).

Path 2-1. After the second round of the KSA, both \( S[2][1] = K[0] + K[1] + 1 \neq 93 \) (we can compute the sum of \( K[0] \) and \( K[1] \) from Table 2) and \( S[2][93] = 93 \) hold under the condition of Path 2-1 from Algorithm 1. After that, if \( S[3][93] \neq S[3][93] \) for \( 3 \leq r \leq 93 \), event \( S[0][1] = K[0] - K[1] - 3 = 93 \) never occurs in the same way as the discussion of Theorem 1 (Path 1-2 in the proof). If \( S[3][93] = S[3][93] = 93 \), whose probability is \( (1 - \frac{1}{N})^{91} \) approximately, since we assume that \( j^R = 93 \) holds for each round with the probability of random association \( \frac{1}{N} \), then we also assume that event \( S[0][1] = K[0] - K[1] - 3 \) occurs with the probability of random association \( \frac{1}{N} \). Therefore, we get
\[ \Pr(S[0][1] = K[0] - K[1] - 3 | \text{Path 2 \& Path 2-1}) = \frac{1}{N} (1 - \frac{1}{N})^{91}. \]
The probabilities of event $S_0[1] = K[0] - K[1] - 3$ under the conditions of Path 2-2, Path 2-3 and Path 2-4 can be computed in the same way as the discussion of Path 2-1. Therefore, we get

$$
\Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-2}) \approx \frac{1}{N^2}(1 - \frac{1}{N^2}),
$$

$$
\Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-3}) \approx \frac{1}{N^2}(1 - \frac{1}{N^2}),
$$

$$
\Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-4}) \approx \frac{1}{N^2}(1 - \frac{1}{N^2}).
$$

The probabilities of these subpaths are taken from Lemma 1. By substituting these probabilities, we get

$$
\Pr(S_0[1] = K[0] - K[1] - 3)
= \Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-1}) \cdot \Pr(\text{Path 2-1})
+ \Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-2}) \cdot \Pr(\text{Path 2-2})
+ \Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-3}) \cdot \Pr(\text{Path 2-3})
+ \Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2 } \land \text{ Path 2-4}) \cdot \Pr(\text{Path 2-4})
\approx \frac{1}{N^2}(1 - \frac{1}{N^2})^2 \cdot (1 - \frac{1}{N^2}),
$$

Note that the probability of $K[1] = 126$ in WPA is $\frac{1}{N}$ (see Table 2). In summary, we get

$$
\Pr(S_0[1] = K[0] - K[1] - 1)
= \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 1}) \cdot \Pr(\text{Path 1})
+ \Pr(S_0[1] = K[0] - K[1] - 3 \mid \text{Path 2}) \cdot \Pr(\text{Path 2})
\approx \frac{4}{N^2} \cdot \frac{1}{N^2}(1 - \frac{1}{N^2})^2 \cdot (1 - \frac{1}{N^2}).
$$

\[\square\]

**Theorem 5:** In the initial state of the PRGA, we have

$$
\Pr(S_0[i_1] = K[0] - K[1] - 1)_{\text{RC4}} \approx \frac{1}{N^2}(1 + \frac{1}{N^2}) \cdot (1 - \frac{1}{N^2}).
$$

**Proof:** The probability of event $S_0[i_1] = K[0] - K[1] - 1$ can be decomposed in four paths: $K[1] = 0$ (Path 1), $K[1] = 127$ (Path 2), $K[1] = 255$ (Path 3) and $K[1] \neq 0, 127, 255$ (Path 4). In the following proof, we use $S_0[i_1]$ instead of $S_0[i_1] (i_1 = 1)$ for simplicity.

**Path 1.** In the first 2 round of the KSA, both $j_1^K = K[0]$ and $j_2^K = K[0] + K[1] + S_0^K (i_1)$ hold (see Fig. 2). If $K[0] = 1$, then $S_0^K (i_1) = 0$ and $K[0] - K[1] - 1 = 0$ always hold since $j_1^K = 1$ and $S_0^K (i_1) = 0$. In this case, $S_0^K (i_1) = S_0^K (i_1)$ for $3 \leq r \leq N$ if $j_r^K$ is not the subsequence $N - 2$ rounds, whose probability is $(1 - \frac{1}{N^2})^{N-2}$ approximately since we assume that $j_r^K = 1$ holds for each round with the probability of random association $\frac{1}{N}$. Therefore, we get

$$
\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 1 } \land K[0] = 1) \approx (1 - \frac{1}{N})^{N-2}.
$$

On the other hand, if $K[0] \neq 1$, then $K[0] - K[1] - 1 = 0$. We then assume that event $S_0[1] = K[0] - K[1] - 1$ occurs with the probability of random association $\frac{1}{N}$. Therefore, we get

$$
\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 1 } \land K[0] = 0) \approx \frac{1}{N}.
$$

We assume that $K[0]$ is generated uniformly at random. By substituting these probabilities, we get

$$
\Pr(S_0[i_1] = K[0] - K[1] - 1 \mid \text{Path 1})
= \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid K[0] = 1) \cdot \Pr(K[0] = 1)
+ \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid K[0] = 0) \cdot \Pr(K[0] = 0)
\approx \frac{1}{N} \cdot \frac{1}{N}. \quad \text{Path 2.}
$$


$$
\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2}) = \alpha_1.
$$


$$
\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 3}) \approx \Pr(K[0] = 0, 1) \cdot \alpha_1.
$$


$$
\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 4}) \approx \frac{1}{N} \cdot (1 - \alpha_1).
$$

In summary, since we assume that $K[1]$ is generated uniformly at random, we get

$$
\Pr(S_0[i_1] = K[0] - K[1] - 1)
= \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid \text{Path 1}) \cdot \Pr(\text{Path 1})
+ \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid \text{Path 2}) \cdot \Pr(\text{Path 2})
+ \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid \text{Path 3}) \cdot \Pr(\text{Path 3})
+ \Pr(S_0[i_1] = K[0] - K[1] - 1 \mid \text{Path 4}) \cdot \Pr(\text{Path 4})
\approx \frac{1}{N} \cdot \frac{1}{N}. \quad \text{Path 6.}
$$

In the initial state of the PRGA in WPA, we have

$$
\Pr(S_0[i_1] = K[0] - K[1] - 1)_{\text{WPA}} \approx \frac{1}{N} \cdot (1 - \frac{1}{N} + \frac{1}{N} \cdot (1 - \frac{1}{N}))^2 + \frac{1}{N} \cdot (1 - \frac{1}{N} + \frac{1}{N} \cdot (1 - \frac{1}{N}))^2 + \frac{1}{N} \cdot (1 - \frac{1}{N} + \frac{1}{N} \cdot (1 - \frac{1}{N}))^2 + \frac{1}{N} \cdot (1 - \frac{1}{N})^2.
$$

**Proof:** The proof itself is similar to Theorem 4. We note that the range of $K[1]$ is limited to either from 32 to 63 or
from 96 to 127 in WPA (see Table 2). Then, as with the discussion in the proof of Theorem 5, the probability of event $S_0[i_1] = K[0] - K[1] - 1$ in WPA can be decomposed in two paths: $K[1] = 127$ (Path 1) and $K[1] \neq 127$ (Path 2). In the following proof, we use $S_0[1]$ instead of $S_0[i_1]$ ($i_1 = 1$) and $S_0[K]$ for simplicity.


$$\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 1}) = \alpha_1.$$


$$\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-2}) \approx \frac{1}{N}(1 - \frac{1}{N})^{93},$$
$$\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-2}) \approx \frac{1}{N}(1 - \frac{1}{N})^{125},$$
$$\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-3}) \approx \frac{1}{N}(1 - \frac{1}{N})^{221},$$
$$\Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-4}) \approx \frac{1}{N}(1 - \frac{1}{N})^{253}.$$

The probabilities of these subpaths are taken from Lemma 1. By substituting these probabilities, we get

$$\Pr(S_0[1] = K[0] - K[1] - 1)$$
$$= \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-1}) \cdot \Pr(\text{Path 2-1})$$
$$+ \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-2}) \cdot \Pr(\text{Path 2-2})$$
$$+ \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-3}) \cdot \Pr(\text{Path 2-3})$$
$$+ \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2} \land \text{Path 2-4}) \cdot \Pr(\text{Path 2-4})$$
$$\approx \frac{1}{N}(1 - \frac{1}{N})^{93} + \frac{1}{N}(1 - \frac{1}{N})^{125} + (1 - \frac{1}{N})^{221} + (1 - \frac{1}{N})^{253}.$$

Note that the probability of $K[1] = 126$ in WPA is $\frac{1}{2}$ (see Table 2). In summary, we get

$$\Pr(S_0[1] = K[0] - K[1] - 1)$$
$$= \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 1}) \cdot \Pr(\text{Path 1})$$
$$+ \Pr(S_0[1] = K[0] - K[1] - 1 \mid \text{Path 2}) \cdot \Pr(\text{Path 2})$$
$$\approx \frac{1}{N} \alpha_1 + \frac{1}{N}(1 - \frac{1}{N})^{93} + \frac{1}{N}(1 - \frac{1}{N})^{125} + (1 - \frac{1}{N})^{221} + (1 - \frac{1}{N})^{253}.$$

$$\square$$

### 3.3 Biases in $S_{255[256]}$

In this section, we prove Theorems 7 and 8. Theorem 7 shows that event $S_{255[256]} = K[0]$ occurs with high probability in both generic RC4 and WPA. On the other hand, Theorem 8 shows event $S_{255[256]} = K[1]$ occurs with high probability only in WPA.

**Theorem 7:** After the 255-th round of the PRGA, we have

$$\Pr(S_{255[256]} = K[0]) \approx \alpha_0(1 - \frac{1}{N})^{255} + \frac{1}{N}(1 - \alpha_0)(1 - \frac{1}{N})^{255}.$$

**Proof:** The probability of event $S_{255[256]} = K[0]$ can be decomposed in two paths: $S_0[0] = K[0]$ (Path 1) and $S_0[0] \neq K[0]$ (Path 2). In the following proof, we use $S_{255[0]}$ instead of $S_{255[256]}$ ($i_{256} = 0$) for simplicity.

**Path 1.** In $S_0[0] = K[0]$, event $S_{255[0]} = K[0]$ occurs if and only if $S_0[0] = S_0[0]$ for $1 \leq r \leq 255$, whose probability is $(1 - \frac{1}{N})^{255}$ approximately since we assume that $j_r = 0$ holds for each round with the probability of random association $\frac{1}{N}$. Therefore, we get

$$\Pr(S_{255[0]} = K[0] \mid \text{Path 1}) \approx \frac{1}{N}(1 - \frac{1}{N})^{255}.$$

**Path 2.** In $S_0[0] \neq K[0]$, event $S_{255[0]} = K[0]$ never occurs if $S_0[0] = S_0[0]$ for $1 \leq r \leq 255$. Except when $S_0[0] = S_0[0]$ for $1 \leq r \leq 255$, whose probability is $(1 - \frac{1}{N})^{255}$ approximately, we assume that event $S_{255[0]} = K[0]$ occurs with the probability of random association $\frac{1}{N}$. Therefore, we get

$$\Pr(S_{255[0]} = K[0] \mid \text{Path 2}) \approx \frac{1}{N}(1 - \frac{1}{N})^{255}.$$
that event $S_0[0] = K[1]$ occurs with the probability of random association $1/N$. Therefore, we get

$$\Pr(S_0[0] = K[1] \mid \text{Path 2}) \approx \frac{1}{N} \cdot (1 - \alpha_0).$$

In summary, we get

$$\Pr(S_0[0] = K[1] \mid \text{Path 1}) \cdot \Pr(\text{Path 1}) + \Pr(S_0[0] = K[1] \mid \text{Path 2}) \cdot \Pr(\text{Path 2}) \approx \frac{\alpha_0}{N} \cdot (1 - \frac{1}{N}) + \frac{1}{N} \cdot (1 - \frac{1}{N}) \cdot \frac{3}{4},$$

where $\alpha_0 = \Pr(S_0[0] = K[0]) \approx (1 - \frac{1}{N})^W + \frac{1}{N}$.

Lemma 2 reflects the probability of event $K[1] = K[0]$ in WPA. $\frac{3}{4}$ is higher than that in generic RC4, $\frac{1}{N}$.

**Theorem 8**: After the 255-th round of the PRGA, we have

$$\Pr(S_{255}[i_{256}] = K[1]) \approx \delta(1 - \frac{1}{N})^{255} + \frac{1}{N}(1 - \delta)(1 - (1 - \frac{1}{N})^{255}),$$

where $\delta = \Pr(S_0[0] = K[1])$ given as Lemma 2.

**Proof**: The proof itself is similar to Theorem 7, and is used the probability of event $S_0[0] = K[1]$ given as Lemma 2 instead of the probability of event $S_0[0] = K[0]$. Therefore, we get

$$\Pr(S_{255}[i_{256}] = K[1]) = \Pr(S_{255}[0] = K[1] \mid S_0[0] = K[1]) \cdot \Pr(S_0[0] = K[1]) + \Pr(S_{255}[0] = K[1] \mid S_0[0] \neq K[1]) \cdot \Pr(S_0[0] \neq K[1]) \approx \delta(1 - \frac{1}{N})^{255} + \frac{1}{N}(1 - \delta)(1 - (1 - \frac{1}{N})^{255}),$$

where $\delta = \Pr(S_0[0] = K[1])$ given as Lemma 2.

3.4 Bias in $S_r[i_{r+1}] (0 \leq r \leq N)$

In this section, we prove Theorem 9. Theorem 9 shows $\Pr(S_r[i_{r+1}] = K[0] + K[1] + 1)$ for $0 \leq r \leq N$, whose experimental result is listed Fig. 1 in Sect. 3.1. Before showing Theorem 9, Lemmas 3 and 4, distributions of the internal state in the first 2 rounds of the PRGA, are proved.

**Lemma 3**: In the initial state of the PRGA for $0 \leq x \leq N - 1$, we have

$$\Pr(S_0[x] = K[0] + K[1] + 1) \approx \begin{cases} (1 - \frac{1}{N})^{x+2} + \frac{1}{N} & \text{if } x = 1, \\ \frac{1}{N}(1 - \frac{1}{N})^2 & \text{if } x = 0 \text{ for WPA}, \\ \frac{1}{N}(1 - \frac{1}{N})(1 - \frac{x+2}{N^2}) + (1 - \frac{1}{N})^{x+2} & \text{otherwise}. \end{cases}$$

**Proof**: In the case of $x = 1$, the probability of event $S_0[1] = K[0] + K[1] + 1$ follows the result in Proposition 1. Therefore, we get

$$\Pr(S_0[1] = K[0] + K[1] + 1) \approx (1 - \frac{1}{N})^{x+2} + \frac{1}{N}.$$

On the other hand, the probability of event $S_0[x] = K[0] + K[1] + 1$ for $x \in [0, N]\{1\}$ can be decomposed in two paths:

$$S_0[x] = K[0] + K[1] + 1 \quad \text{for Path 1}$$

$$S_0[x] \neq K[0] + K[1] + 1 \quad \text{for Path 2}.$$

**Path 1**: From step 7 in Algorithm 1, $S_K^x[x] = K[0] + K[1] + 1$ always holds under the condition of Path 1 since $S_K^x[x]$ must be swapped from $S_K^j[x_{j+1}]$. In addition, if $S_K^x[x]$ $x \geq 2 \leq r \leq N$, whose probability is $(1 - \frac{1}{N})^{x-1}$ approximately since we assume that $j^-$ holds for each round with the probability of random association $\frac{1}{N}$, then event $S_0[x] = K[0] + K[1] + 1$ always occurs. Therefore, we get

$$\Pr(S_0[x] = K[0] + K[1] + 1 \mid \text{Path 1}) \approx (1 - \frac{1}{N})^{x-1}.$$

**Path 2**: Let $y$ be satisfied with $S_K^y[y] = K[0] + K[1] + 1$. In the same way as the discussion of Path 1, $S_K^x[x] = K[0] + K[1] + 1$ never holds under the condition of Path 2. After the second round of the PRGA, we have

$$\Pr(S_0[x] = K[0] + K[1] + 1 \mid \text{Path 2}) \approx \frac{1}{N}(1 - \frac{1}{N})^{x+1}.$$

We assume that event $S_K^x[j_{x+1}] = K[0] + K[1] + 1$ occurs with the probability of random association $\frac{1}{N}$. In summary, we get

$$S_0[x] = K[0] + K[1] + 1 \quad \text{for Path 2}.$$

$$\Pr(S_0[x] = K[0] + K[1] + 1 \mid \text{Path 2}) \approx (1 - \frac{1}{N})^{x-1}.$$
2 since \( S_1[1] = S_1[i_1] = S_0[j_1] = S_0[S_0[1]] \) from steps 4 and 5 in Algorithm 2. Therefore, we get
\[
\Pr(S_1[1] = K[0] + K[1] + 1) = \beta_1.
\]

On the other hand, the probability of event \( S_1[x] = K[0] + K[1] + 1 \) for \( x \in [0, N-1] \) can be decomposed in two paths: \( S_0[1] = K[0] + K[1] + 1 \) (Path 1) and \( S_0[x] = K[0] + K[1] + 1 \) (Path 2).

**Path 1.** From step 5 in Algorithm 2, event \( S_1[x] = K[0] + K[1] + 1 \) always occurs under the condition of Path 1 if and only if \( j_1 = x \) since \( S_1[j_1] \) must be swapped from \( S_0[i_1] = S_0[1] \). Although both \( S_0[1] = K[0] + K[1] + 1 \) and \( j_1 = x \) are not independent, both \( S_0[1] = K[0] + K[1] + 1 \) and \( K[0] + K[1] + 1 = x \) become independent by converting \( j_1 = x \) into \( j_1 = S_0[1] = K[0] + K[1] + 1 = x \). Therefore, we get
\[
\Pr(S_1[x] = K[0] + K[1] + 1 \mid \text{Path 1}) = \Pr(K[0] + K[1] + 1 = x - 1).
\]

**Path 2.** In the same way as the discussion of Path 1, event \( S_1[x] = K[0] + K[1] + 1 \) never occurs under the condition of Path 2. If \( j_1 \neq x \), then \( S_1[x] = S_0[x] = K[0] + K[1] + 1 \) always holds, and \( S_1[1] \neq K[0] + K[1] + 1 \) holds since \( S_1[1] = S_0[j_1] \neq S_0[x] \) from step 5 in Algorithm 2. So, we assume that both \( S_0[x] = K[0] + K[1] + 1 \) and \( S_1[1] \neq K[0] + K[1] + 1 \) are mutually independent. Therefore, we get
\[
\Pr(S_1[x] = K[0] + K[1] + 1 \mid \text{Path 2}) = \Pr(S_1[1] \neq K[0] + K[1] + 1).
\]

In summary, we get
\[
\Pr(S_1[x] = K[0] + K[1] + 1) = \Pr(S_1[x] = K[0] + K[1] + 1 \mid \text{Path 1}) \cdot \Pr(\text{Path 1}) + \Pr(S_1[x] = K[0] + K[1] + 1 \mid \text{Path 2}) \cdot \Pr(\text{Path 2}) = \alpha_1 \gamma_{x-1} + (1 - \beta_1) \varepsilon_x,
\]
where \( \alpha_1 = \Pr(S_0[1] = K[0] + K[1] + 1), \beta_1 = \Pr(S_0[S_0[1]] = K[0] + K[1] + 1), \gamma_{x-1} = \Pr(K[0] + K[1] + 1 = x - 1) \) and \( \varepsilon_x = \Pr(S_0[x] = K[0] + K[1] + 1) \) is given as Lemma 3.

**Theorem 9:** After the \( r \)-th round of the PRGA for \( 0 \leq x \leq N \), we have
\[
\Pr(S_1[i_{r+1}] = K[0] + K[1] + 1) = \left\{ \begin{array}{ll}
\alpha_1, & \text{if } r = 0, \\
\alpha_1 \gamma_1 + (1 - \beta_1) \varepsilon_2, & \text{if } r = 1, \\
\varepsilon_{N-1} \left(1 - \frac{1}{N} \right)^{N-1} + \frac{1}{N} (1 - \xi_0) \left(1 - \frac{1}{N} \right)^{N-1} & \text{if } r = N - 1, \\
\xi_1 \left(1 - \frac{1}{N} \right)^{N-1} + \frac{1}{N} \sum_{i=1}^{N-1} \eta_i \left(1 - \frac{1}{N} \right)^{N-1} & \text{if } r = N, \\
\zeta_{r+1} \left(1 - \frac{1}{N} \right)^{r-1} + \frac{1}{N} \sum_{i=1}^{N-1} \eta_i \left(1 - \frac{1}{N} \right)^{r-1} & \text{otherwise},
\end{array} \right.
\]
where \( \varepsilon_r = \Pr(S_0[r]) = K[0] + K[1] + 1 \) is given as Lemma 3, \( \zeta_r = \Pr(S_1[r] = K[0] + K[1] + 1) \) is given as Lemma 4 and \( \eta_r = \Pr(S_1[i_{r+1}] = K[0] + K[1] + 1) \) is given as this theorem.

**Proof:** In the cases of \( r = 0 \) and \( 1 \), the probability of events \( S_0[i] = K[0] + K[1] + 1 \) and \( S_1[i] = K[0] + K[1] + 1 \) follow the result in Lemmas 3 and 4, respectively. In the cases of \( r = N - 1 \) and \( N \), both events \( S_{N-1}[i_N] = K[0] + K[1] + 1 \) and \( S_N[i_{N+1}] = K[0] + K[1] + 1 \) can be proved in the same way as the proof of Theorem 7. In any other cases, the probability of event \( S_1[i_{r+1}] = K[0] + K[1] + 1 \) for \( 2 \leq r \leq N - 2 \) can be decomposed in two paths: \( S_1[i_{r+1}] = K[0] + K[1] + 1 \) (Path 1) and \( S_1[i_{r+1}] = K[0] + K[1] + 1 \) (Path 2).

**Path 1.** Event \( S_1[i_{r+1}] = K[0] + K[1] + 1 \) always occurs under the condition of Path 1 if \( S_0[i_{r+1}] = S_1[i_{r+1}] \), where \( \mu = \frac{1}{N} \) approximately since we assume that \( j_0 = i_{r+1} \) holds for each round with the probability of random association \( \frac{1}{N} \). Therefore, we get
\[
\Pr(S_1[i_{r+1}] = K[0] + K[1] + 1 \mid \text{Path 1}) \approx \left(1 - \frac{1}{N} \right)^{r-1}.
\]

**Path 2.** From step 5 in Algorithm 2, event \( S_1[i_{r+1}] = K[0] + K[1] + 1 \) always occurs under the condition of Path 2 if \( j_{r+1} = i_{r+1} \), where \( \mu = \frac{1}{N} \) approximately since we assume that \( j_0 = i_{r+1} \) holds for each round with the probability of random association \( \frac{1}{N} \). Therefore, we get
\[
\Pr(S_1[i_{r+1}] = K[0] + K[1] + 1 \mid \text{Path 2}) \approx \frac{1}{N} \left(1 - \frac{1}{N} \right)^{r-1}.
\]

Note that the range of \( x \) varies depending on the value of \( r \) in Path 2. In summary, we get
\[
\Pr(S_1[i_{r+1}] = K[0] + K[1] + 1) = \Pr(S_1[i_{r+1}] = K[0] + K[1] + 1 \mid \text{Path 1}) \cdot \Pr(\text{Path 1}) + \sum_{r=1}^{N-1} \Pr(S_1[i_{r+1}] = K[0] + K[1] + 1 \mid \text{Path 2}) \cdot \Pr(\text{Path 2}) \approx \zeta_{r+1} \left(1 - \frac{1}{N} \right)^{r-1} + \frac{1}{N} \sum_{i=1}^{N-1} \eta_i \left(1 - \frac{1}{N} \right)^{r-1},
\]
where \( \zeta_r = \Pr(S_1[r] = K[0] + K[1] + 1) \) and \( \eta_r = \Pr(S_1[i_{r+1}] = K[0] + K[1] + 1) \), which is recursive probability in this theorem.

**3.5 Experimental Results**

In order to confirm the accuracy of Theorems 1-9, we have conducted experiments in the following environment: Ubuntu 12.04 machine with 2.66 GHz CPU, 3.8 GiB memory, gcc 4.6.3 compiler and C language. The number of samples necessary for our experiments is at least \( O(N^2) \) according to Proposition 4. This is why each correlation has a relative bias with the probability of at least \( \frac{1}{N^2} \) with respect to a base event of the probability \( \frac{1}{N} \). Then, we have used \( N^5 \) randomly generated 16-byte RC4 keys in both generic RC4 and WPA. The number of these samples satisfies a condition to distinguish each correlation from random distribution with constant probability of success. We also evaluate the percentage of the relative error \( \epsilon \) of the experimental values compared with the theoretical values as follows:
Table 3 Comparison between the experimental and the theoretical values in Theorems 1-8.

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<th>Results</th>
<th>Experimental value</th>
<th>Theoretical value</th>
<th>error (%)</th>
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Fig. 7 Comparison between the experimental and the theoretical values in Theorem 9 for both generic RC4 and WPA.

\[ \epsilon = \frac{|\text{experimental value} - \text{theoretical value}|}{\text{experimental value}} \times 100(\%) \]

4. Conclusion

In this paper, we have investigated various linear correlations including unknown internal state variables as well as the keystream bytes and the first 3-byte RC4 key in both generic RC4 and WPA. Actually, those linear correlations may be effective for the state recovery attacks since they include the known (IV-related) RC4 key bytes in WPA. From the result, we have discovered more than 150 correlations with positive or negative biases. Then, We have proved some linear correlations theoretically, which are biases in \( S_0[i_1], S_{255}[i_{256}] \) and \( S_i[i_{i+1}] \) for \( 0 \leq r \leq N \). For example, the probability of event \( S_0[i_1] = 0 \) in WPA is 0 (Theorem 2 in Sect. 3.2). Thus, \( S_0[i_1] \) is varied from \([0, 255] \setminus K[0] \). Furthermore, we stress that the relative errors of the events \( S_0[i_1] = 0 \) in both generic RC4 and WPA. From the figure, these distributions almost match on the whole, but differences between the experimental and the theoretical values in both generic RC4 and WPA are slightly large. Let us investigate why such differences are produced in both generic RC4 and WPA. As far as we have confirmed experimentally, it became clear that there exist differences between the experimental and the theoretical values in Lemma 4. So, we need to prove Lemma 4 again precisely, which remains an open problem.

Acknowledgements

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References


## Appendix: Newly Obtained Linear Correlations

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<th>$P$</th>
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