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Revising a Labelled Sequent Calculus for Public Announcement Logic

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Abstract. We first show that a labelled sequent calculus **G3PAL** for Public Announcement Logic (PAL) by Maffezioli and Negri (2011) has been lacking rules for deriving an axiom of Hilbert-style axiomatization of PAL. Then, we provide our revised calculus **GPAL** to show that all the formulas provable by Hilbert-style axiomatization of PAL are also provable in **GPAL** together with the *cut* rule. We also establish that our calculus enjoys cut elimination theorem. Moreover, we show the soundness of our calculus for Kripke semantics with the notion of surviveness of possible worlds in a restricted domain. Finally, we provide a direct proof of the semantic completeness of **GPAL** for the link-cutting semantics of PAL.

1 Introduction

Public announcement logic (PAL) was first presented by Plaza [12], and it has been the basis of dynamic epistemic logic. PAL is a logic for formally expressing changes of human knowledge. Specifically, when we obtain some information through communication with others, our state of knowledge may change. For example, if 'John does not know if it will rain tomorrow or not' is true and he gets information from the weather forecast which says that 'it will not rain tomorrow,' then the state of John's knowledge changes and so 'John knows that it will not rain tomorrow' becomes true. While a Kripke model of the standard epistemic logic stands for the state of knowledge, the standard epistemic logic does not have any syntax for properly expressing changes of the state of knowledge. PAL was introduced for the purpose of dealing with flexibility of human knowledge; and dynamic epistemic logics based on PAL contain many possibilities to be applied to various fields such as artificial intelligence, epistemology in philosophy, formalizing law and so on.

A proof system for PAL has been provided in terms of Hilbert-style axiomatization (we call it **HPAL**) which is complete for Kripke semantics; however an easier system to calculate a theorem should be desirable, since Hilbert-style proof systems are, in general, hard to handle for proving a theorem. One possible candidate for such a proof system is a celebrated Gentzen-style sequent calculus [4], where a basic unit of a derivation is the notion of a *sequent*

$$\Gamma \Rightarrow \Delta,$$

which consists of two lists (or multi-sets, or sets) of formulas. How can we read $\Gamma \Rightarrow \Delta$ intuitively? There are at least two ways of reading it. First, we may read it as 'if all formulas in Γ hold, then some formula in Δ holds'. Second, we may also read it as 'It is not the case that all formulas in Γ hold and all formulas in Δ fail'. We may wonder if these two readings are equivalent with each other, but the equivalence depends on the underlying logic. For example, two readings are equivalent in the classical propositional logic, provided we understand that 'a formula A holds' by 'A is true in a given truth assignment' and 'A fails' by 'A is false under the assignment' (note that, under these readings, A does not holds if and only if A fails). One of the most uniform approaches for sequent calculus for modal logic is *labelled sequent calculus* (e.g., [9]), where each formula has a label corresponding to an element (sometimes called a possible world) of a domain in Kripke semantics for modal logic. The proof system we are concerned with in this paper is one of variants of labelled sequent calculus. An existing labelled sequent calculus for PAL, named **G3PAL**, was devised by Maffezioli and Negri [7]; however, a deficiency of **G3PAL** has been pointed out by Balbiani *et al* [1].[1] In this paper, we also suggest a different defect in it. In brief, because **G3PAL** does not have inference rules relating to accessibility relations, there exists a problem in case of proving one of axioms of **HPAL**. Therefore, we introduce a revised labelled sequent calculus **GPAL** (with the rule of cut, **GPAL**$^+$) to compensate for the deficiency by adding some rules for accessibility relations.

Moreover, we especially focus on the soundness theorem of **GPAL**, since there is a hidden factor behind the definition of validity of the sequent $\Gamma \Rightarrow \Delta$, of which the researchers of this field (e.g., [1, 7]) have not made a point. In particular, we notice that the above two readings of a sequent in our setting are not equivalent and that the notion of validity based on the first reading of a sequent is not sufficient to prove the soundness of our calculus for Kripke semantics; however, we employ the notion of validity based on the second reading of a sequent to establish soundness. One of the reasons why two notions of validity are not equivalent consists of deleting possible worlds by a (truthful) public announcement. In fact, we will show the completeness with PAL's another semantics, a version of the link-cutting semantics by van Benthem et al. [14] where only the accessibility relation is restricted in a model and two notions of validity become equivalent.

The outline of this paper is as follows: Section 2 provides definitions of syntax of PAL and Kripke semantics for it, then introduces one simple example of Kripke model that is used throughout the paper. Additionally, the existing Hilbert-style axiomatization **HPAL** of PAL and its semantic completeness is outlined. Section 3 reviews Maffezioli and Negri's labelled sequent calculus **G3PAL** and specifies which part of **G3PAL** is problematic. Section 4 introduces our calculus **GPAL**, a revised version of **G3PAL**, and we show that all the theorems of **HPAL** are provable in **GPAL**$^+$ (Theorem 1), and the cut elimination theorem of **GPAL**$^+$ (Theorem 2). Section 5 focuses on its soundness theorem (Theorem 3) in terms of two notions of validity based on the above two readings of a sequent. Section 6 ties the link-cutting semantics of PAL with its direct completeness result (Theorem 4). Finally, Section 7 concludes the paper.

[1] According to them, there are some valid formulas like $[A \wedge A]B \leftrightarrow [A]B$ which may be unprovable in **G3PAL**.

2 Kripke Semantics and Axiomatization of PAL

First of all, we will address the syntax of PAL. Let $\mathsf{Prop} = \{p, q, r, \ldots\}$ be a countably infinite set of propositional variables and $\mathsf{G} = \{a, b, c, \ldots\}$ a nonempty finite set with elements called agents. Then the set $\mathsf{Form} = \{A, B, C, \ldots\}$ of formulas of PAL is inductively defined as follows ($p \in \mathsf{Prop}$, $a \in \mathsf{G}$):

$$A ::= p \mid \neg A \mid (A \to A) \mid \mathsf{K}_a A \mid [A]A.$$

Other logical connectives ($\wedge$, $\vee$, etc.) are defined as usual. $\mathsf{K}_a A$ is read as 'agent a knows that A', and $[A]B$ is read as 'after public announcement of A, it holds that B'.

Example 1. Let us consider a propositional variable p to read 'it will rain tomorrow'. Then a formula $\neg(\mathsf{K}_a p \vee \mathsf{K}_a \neg p)$ means that a does not know whether it will rain tomorrow or not, and $[\neg p]\mathsf{K}_a \neg p$ means that after a public announcement (e.g., a weather report) of $\neg p$, a knows that it will not rain tomorrow.

2.1 Kripke Semantics of PAL

We should now consider the Kripke semantics of PAL. The sequent calculus introduced in the next section can be regarded as a formalized version of Kripke semantics of PAL. We mainly follow the semantics introduced in van Ditmarsch *et al.* [15]. We call $\mathfrak{M} = \langle W, (R_a)_{a\in\mathsf{G}}, V\rangle$ a *Kripke model* if W is a nonempty set, $R_a \subseteq W \times W$, and V is a valuation function which assigns an atomic formula to a subset of W. W is also called the *domain* of $\mathfrak{M}$, denoted by $\mathcal{D}(\mathfrak{M})$. Next, let us define the satisfaction relation.

Definition 1. *Given a Kripke model $\mathfrak{M}$, $w \in \mathcal{D}(\mathfrak{M})$, and $A \in \mathsf{Form}$, we define $\mathfrak{M}, w \Vdash A$ as follows:*

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \textit{iff} & w \in V(p), \\
\mathfrak{M}, w \Vdash \neg A & \textit{iff} & \mathfrak{M}, w \nVdash A, \\
\mathfrak{M}, w \Vdash A \to B & \textit{iff} & \mathfrak{M}, w \Vdash A \textit{ implies } \mathfrak{M}, w \Vdash B, \\
\mathfrak{M}, w \Vdash \mathsf{K}_a A & \textit{iff} & \textit{for all } v \in W : wR_a v \textit{ implies } \mathfrak{M}, v \Vdash A \ (a \in \mathsf{G}), \textit{ and} \\
\mathfrak{M}, w \Vdash [A]B & \textit{iff} & \mathfrak{M}, w \Vdash A \textit{ implies } \mathfrak{M}^A, w \Vdash B,
\end{array}$$

where the restriction $\mathfrak{M}^A$, at the definition of the announcement operator, is the restricted Kripke model to the truth set of A, defined as $\mathfrak{M}^A = \langle W^A, (R^A_a)_{a\in\mathsf{G}}, V^A\rangle$ with

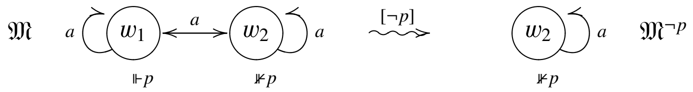
$$\begin{array}{ll}
W^A & := \{x \in W \mid \mathfrak{M}, x \Vdash A\}, \\
R^A_a & := R_a \cap (W^A \times W^A), \\
V^A(p) & := V(p) \cap W^A \ (p \in \mathsf{Prop}).
\end{array}$$

As above, the restriction of a Kripke model is based on the restriction of the set of possible worlds, so that this can be said to be the *world-deletion* semantics of PAL, and this will be distinguished from the *link-cutting* semantics in Section 6. In the semantics above, we do not assume any requirement on the accessibility relations $(R_a)_{a\in\mathsf{G}}$, while we usually assume that R_a is an equivalent relation in Kripke semantics for the standard epistemic logic; however, since the previous works [7, 1] also start with a Kripke model with an arbitrary accessibility relation, we also follow them in this respect.

Definition 2. *A formula A is* valid *in a Kripke model $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash A$ for all $w \in \mathcal{D}(\mathfrak{M})$.*

This is the definition of PAL's semantics, but readers who are not familiar with PAL may not easily see what it is, so the following example might help for understanding the heart of PAL.

Example 2. Example 1 can be semantically modeled as follows. Let us consider $\mathsf{G} = \{a\}$ and the following two models such as: $\mathfrak{M} = \langle \{w_1, w_2\}, \{w_1, w_2\}^2, V \rangle$ where $V(p) = \{w_1\}$, and $\mathfrak{M}^{\neg p} = \langle \{w_2\}, \{(w_2, w_2)\}, V^{\neg p} \rangle$ where $V^{\neg p}(p) = \emptyset$. These models can be shown in graphic forms as follows.



In $\mathfrak{M}$, agent a does not know whether p or $\neg p$ (i.e., $\neg(\mathsf{K}_a p \vee \mathsf{K}_a \neg p)$ is valid in $\mathfrak{M}$), but after announcement of $\neg p$, agent a comes to know $\neg p$ in the restricted model $\mathfrak{M}$ to $\neg p$.

2.2 Hilbert-style axiomatization of PAL

Hilbert-style axiomatization, **HPAL**, is defined in Table 1 below, and it includes some axioms with announcement operators as additional axioms to the axiomatization of **K**. These five additional axioms (from RA1 to RA5) are called *reduction axioms* (or sometimes, *recursion axioms*). They exist for reducing each of the theorems of **HPAL** into a theorem of modal logic **K**. The previous work [12] has shown the completeness theorem of **HPAL**.

Fact 1 (Completeness of PAL) *For any formula A, A is valid in all Kripke models iff A is provable in* **HPAL**.

Proof (Outline). In the case of the soundness theorem, it suffices to show validity of **HPAL**'s reduction axioms, which is straightforward. For the case of the completeness theorem, following [15, pp.186-7], the translation function t is defined as follows.

$t(p) = p$	$t([A]p) = t(A \to p)$
$t(\neg p) = \neg t(p)$	$t([A]B \to C) = t([A]B \to [A]C)$
$t(A \to B) = t(A) \to t(B)$	$t([A]\mathsf{K}_a B) = t(A \to \mathsf{K}_a[A]B)$
$t(\mathsf{K}_a A) = \mathsf{K}_a t(A)$	$t([A][B]C) = t([A \wedge [A]B]C)$

Here the underlying idea of this translation is that, with the help of reduction axioms, we can push each of the outermost occurrences of the announcement operator to a propositional variable up to equivalence. Then, suppose that A is shown to be valid on all Kripke models. Since $t(A) \leftrightarrow A$ is valid on all models, we obtain $t(A)$ is valid on all models. Since the Hilbert-style axiomatization of **K** is complete with respect to all Kripke models, $t(A)$ is provable in the Hilbert-style axiomatization **K**, hence also in **HPAL**. Note that $t(A) \leftrightarrow A$ is provable in **HPAL**, we conclude that A is provable in **HPAL**. □

Table 1. Hilbert-style axiomatization of PAL : **HPAL**

Modal Axioms
all instantiations of propositional tautologies
(K) $\mathsf{K}_a(A \to B) \to (\mathsf{K}_a A \to \mathsf{K}_a B)$
Reduction Axioms
(RA1) $[A]p \leftrightarrow (A \to p)$
(RA2) $[A](B \to C) \leftrightarrow ([A]B \to [A]C)$
(RA3) $[A]\neg B \leftrightarrow (A \to \neg[A]B)$
(RA4) $[A]\mathsf{K}_a B \leftrightarrow (A \to \mathsf{K}_a[A]B)$
(RA5) $[A][B]C \leftrightarrow [A \wedge [A]B]C$
Inference Rules
(MP) From A and $A \to B$, infer B
(Nec) From A, infer $\mathsf{K}_a A$

3 Sequent Calculus for PAL

As we have mentioned in the introduction, a labelled sequent calculus called **G3PAL** has been provided by Maffezioli and Negri [7] based on G3-style sequent calculus (or simply, G3-style) for modal logic **K**.[2]

3.1 G3PAL

In order to introduce **G3PAL**, as in [7], it is better to explicitly confirm the satisfaction relation with a list of formulas, that restricts a Kripke model, since the following inference rules of **G3PAL** are all obtained from those relations. We denote finite lists $(A_1, A_2, ..., A_n)$ of formulas by α,β, etc., and do the empty list by ϵ from here and after. As an abbreviation, for any list $\alpha = (A_1, A_2, ..., A_n)$ of formulas , we define $\mathfrak{M}^\alpha$ inductively as: $\mathfrak{M}^\alpha := \mathfrak{M}$ (if $\alpha = \epsilon$), and $\mathfrak{M}^\alpha := (\mathfrak{M}^\beta)^{A_n} = \langle W^{\beta,A_n}, (R_a^{\beta,A_n})_{a\in\mathsf{G}}, V^{\beta,A_n}\rangle$ (if $\alpha = \beta, A_n$). We may also denote $(\mathfrak{M}^\beta)^{A_n}$ by $\mathfrak{M}^{\beta,A_n}$ for simplicity. The satisfaction relation by restricting formulas is shown explicitly as follows:

$\mathfrak{M}^{\alpha,A}, w \Vdash p$	*iff*	$\mathfrak{M}^\alpha, w \Vdash A$ and $\mathfrak{M}^\alpha, w \Vdash p$,
$\mathfrak{M}^\alpha, w \Vdash \neg A$	*iff*	$\mathfrak{M}^\alpha, w \nVdash A$,
$\mathfrak{M}^\alpha, w \Vdash A \to B$	*iff*	$\mathfrak{M}^\alpha, w \Vdash A$ implies $\mathfrak{M}^\alpha, w \Vdash B$,
$\mathfrak{M}^\alpha, w \Vdash \mathsf{K}_a A$	*iff*	for all $v \in W : wR_a^\alpha v$ implies $\mathfrak{M}^\alpha, v \Vdash A$ ($a \in \mathsf{G}$), and
$\mathfrak{M}^\alpha, w \Vdash [A]B$	*iff*	$\mathfrak{M}^\alpha, w \Vdash A$ implies $\mathfrak{M}^{\alpha,A}, w \Vdash B$,

where $p \in \mathsf{Prop}, A, B \in \mathsf{Form}, \mathfrak{M}$ is any Kripke model, $w \in \mathcal{D}(\mathfrak{M})$, and α is any list of formulas. According to the Kripke semantics defined in Section 2, $\langle w, v\rangle \in R_a^{\alpha,A}$ is

[2] G3-style sequent calculus for modal logic **K** named **G3K** has been introduced in Negri [8]. And G3-style sequent calculus is a calculus that does not have any structural rules and the most outstanding feature of this calculus is the contraction rules are admissible. The specific introduction of G3-style sequent calculus (or G3-system) itself can be found in Negri and Plato [9] and Troelstra and Schwichtenberg [13].

equivalent to the three conjuncts as follows:

$$\langle w, v\rangle \in R_a^{\alpha,A} \textit{ iff } \langle w, v\rangle \in R_a^{\alpha} \text{ and } \mathfrak{M}^{\alpha}, w \Vdash A \text{ and } \mathfrak{M}^{\alpha}, v \Vdash A.$$

A point to notice here is that from an accessibility relation with restricting formulas, we may obtain three conjuncts; thus we can obtain sound inference rules for relational atoms.

Now we will introduce **G3PAL**. Let $\mathsf{Var} = \{x, y, z, \ldots\}$ be a countably infinite set of variables. Then, given any $x, y \in \mathsf{Var}$, any list of formulas α and any formula A, we say $x{:}^{\alpha}A$ is a *labelled formula*, and that, for any agent $a \in \mathsf{G}$, $x\mathsf{R}_a^{\alpha}y$ is a *relational atom*. Intuitively, the labelled formula $x{:}^{\alpha}A$ corresponds to '$\mathfrak{M}^{\alpha}, x \Vdash A$' and is to read 'after a sequence α of public announcements, x still survives[3] and A holds at x', and the relational atom $x\mathsf{R}_a^{\alpha}y$ is to read 'after a sequence α of public announcements both x and y survive and we can still access from x to y'. We also use the term, *labelled expressions* to indicate that they are either labelled formulas or relational atoms, and we denote them by $\mathfrak{A}$, $\mathfrak{B}$, etc. A *sequent* $\Gamma \Rightarrow \Delta$ is a pair of finite multi-sets of labelled expressions. The set of inference rules of **G3PAL** is shown in Table 2. Hereinafter, for any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is provable in **G3PAL**, we write **G3PAL** $\vdash \Gamma \Rightarrow \Delta$. The rules of (*Lat*) and (*Rat*) are obtained from the above satisfaction relation, hence if there is an announcement A and a propositional variable p, we get p with the restricting formula A. In the case of (L[.]) and (R[.]), although the satisfaction relation of the announcement operator is the same as that of implication only with the exception of restricting formulas, the rules, (L[.]) and (R[.]), are (probably) modified for G3-style. The last two rules (L_{cmp}) and (R_{cmp}) are for dealing with the proof of (RA5) of **HPAL** (we will discuss them shortly afterwards). Other inference rules result naturally from the semantics. As we have referred in the previous paragraph, while we could have sound inference rules corresponding to restricted relational atoms, there is, actually, no rule of relational atoms in **G3PAL**, and due to this fact, **G3PAL** may not have an ability to prove one of the reduction axioms, (RA4).

3.2 Problems of G3PAL

Maffezioli and Negri stated, in Section 5 of [7], that **G3PAL** may prove all inference rules and axioms of **HPAL**, namely if **HPAL** $\vdash A$, then **G3PAL** $\vdash \Rightarrow x{:}^{\epsilon}A$ (for any A and x). Nevertheless, there are, in fact, some problems in proving (RA4):

$$[A]\mathsf{K}_a B \leftrightarrow (A \to \mathsf{K}_a[A]B).$$

This axiom seemingly cannot be proven in **G3PAL**. Let us look at possible but plausible attempts to derive both directions of (RA4). First, a possible attempt of deriving the direction from right to left is given as follows:

[3] The notion of *surviveness* will be referred in Section 5 more specifically.

Table 2. G3PAL

(Initial Sequent)

$$x{:}^{\epsilon} p, \Gamma \Rightarrow \Delta, x{:}^{\epsilon} p$$

(Rules for propositional connectives)

$$\frac{}{x{:}^{\alpha}\bot, \Gamma \Rightarrow \Delta}\ (L\bot)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}{x{:}^{\alpha}\neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\neg A}\ (R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L\to) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B}\ (R\to)$$

(Rules for knowledge operators)

$$\frac{y{:}^{\alpha}A, x{:}^{\alpha}\mathsf{K}_aA, x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\mathsf{K}_aA, x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta}\ (L\mathsf{K}_a) \quad \frac{x\mathsf{R}^{\alpha}_a y, \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\mathsf{K}_aA}\ (R\mathsf{K}_a)\dagger$$

† y does not appear in the lower sequent.

(Rules for PAL)

$$\frac{x{:}^{\alpha}A, x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat)$$

$$\frac{x{:}^{\alpha,A}B, x{:}^{\alpha}[A]B, x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}\ (L[.]) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha,A,B}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B}C, \Gamma \Rightarrow \Delta}\ (L_{cmp}) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B}C}\ (R_{cmp})$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_1\\ x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{A}\mathsf{K}_aB\end{array} \quad \dfrac{\begin{array}{c}\vdots\ ?\\ x{:}^{\epsilon}A, x{:}^{\epsilon}\mathsf{K}_a[A]B, x\mathsf{R}^{A}_a y \Rightarrow y{:}^{A}B\end{array}}{x{:}^{\epsilon}A, x{:}^{\epsilon}\mathsf{K}_a[A]B \Rightarrow x{:}^{A}\mathsf{K}_aB}\ (R\mathsf{K}_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \mathsf{K}_a[A]B \Rightarrow x{:}^{A}\mathsf{K}_aB}\ (L\to)}{x{:}^{\epsilon}A \to \mathsf{K}_a[A]B \Rightarrow x{:}^{\epsilon}[A]\mathsf{K}_aB}\ (R[.])}{\Rightarrow x{:}^{\epsilon}(A \to \mathsf{K}_a[A]B) \to [A]\mathsf{K}_aB}\ (R\to) \qquad (*)$$

Starting from the bottom sequent, the bottom sequent of $\mathcal{D}_1$ is clearly derivable, but it is difficult to find the way to go step forward from the right uppermost sequent of the derivation. The problem here is that A in $x\mathsf{R}^{A}_a y$ and ϵ in $x{:}^{\epsilon}\mathsf{K}_a[A]B$ on the left side of the sequent do not match, and therefore we cannot apply the rule $(L\mathsf{K}_a)$.

Secondly, the other direction of (RA4) also seemingly cannot be proven by **G3PAL**. A possible attempt to derive it may be as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ ?}{y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_{a}y, x{:}^{A}\mathsf{K}_{a}B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\mathsf{K}_{a}B \Rightarrow y{:}^{A}B}}{x\mathsf{R}^{\epsilon}_{a}y, x{:}^{A}\mathsf{K}_{a}B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\mathsf{K}_{a}B \Rightarrow y{:}^{\epsilon}[A]B}\ (R[.])}{x{:}^{A}\mathsf{K}_{a}B, x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\mathsf{K}_{a}B \Rightarrow x{:}^{\epsilon}\mathsf{K}_{a}[A]B}\ (R\mathsf{K}_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\mathsf{K}_{a}B \Rightarrow x{:}^{\epsilon}\mathsf{K}_{a}[A]B}\ (L[.])}{x{:}^{\epsilon}[A]\mathsf{K}_{a}B \Rightarrow x{:}^{\epsilon}A \to \mathsf{K}_{a}[A]B}\ (R\to)}{\Rightarrow x{:}^{\epsilon}[A]\mathsf{K}_{a}B \to (A \to \mathsf{K}_{a}[A]B)}\ (R\to) \qquad (**)$$

The derivation also comes to a dead end (in fact, the rule ($L[.]$) is applicable infinitely many times, but no new labelled expression is obtained by the application). The problem here is also that ϵ in $x\mathsf{R}^{\epsilon}_{a}y$ and A in $x{:}^{A}\mathsf{K}_{a}B$ on the left side of the left uppermost sequent do not match, and again the rule ($L\mathsf{K}_a$) cannot be applied.

In brief, for applying the rule ($L\mathsf{K}_a$), α in $x\mathsf{R}^{\alpha}_{a}y$, and β in $x{:}^{\beta}\mathsf{K}_{a}B$ must be the same and ($L\mathsf{K}_a$) is indispensable for proving both directions of (RA4); however there seems no way to make them equal in **G3PAL**. To settle the problems, we introduce rules for relational atoms for decomposing $x\mathsf{R}^{A}_{a}y$ into $x\mathsf{R}^{\epsilon}_{a}y$ and related labelled formulas.

4 Revising G3PAL

In this section, we revise **G3PAL** to make it possible to cope with (RA4) of **HPAL**. Let us examine the problem of (∗) first. To overcome the dead end of the derivation, we introduce rules of the relational atom with a list of formulas, i.e., ($Lrel_a1$), ($Lrel_a2$), ($Lrel_a3$) and ($Rrel_a$), and it is not trivial if these rules are derivable in **G3PAL**. Here are our additional rules:

$$\dfrac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}^{\alpha,A}_{a}y, \Gamma \Rightarrow \Delta}\ (Lrel_a1) \qquad \dfrac{y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}^{\alpha,A}_{a}y, \Gamma \Rightarrow \Delta}\ (Lrel_a2) \qquad \dfrac{x\mathsf{R}^{\alpha}_{a}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}^{\alpha,A}_{a}y, \Gamma \Rightarrow \Delta}\ (Lrel_a3)$$

$$\dfrac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x\mathsf{R}^{\alpha}_{a}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}^{\alpha,A}_{a}y}\ (Rrel_a)$$

These inference rules are obtained in PAL's Kripke semantics. Namely, as we have already seen in Section 3.1, any restricted accessibility relation $wR^{\alpha,A}_{a}v$ is equivalent to three conjuncts such as: $wR^{\alpha}_{a}v$, $\mathfrak{M}^{\alpha}, w \Vdash A$ and $\mathfrak{M}^{\alpha}, v \Vdash A$. These three conjuncts correspond to three ($Lrel_ai$) rules and three uppersequents of ($Rrel_a$). If we use ($Lrel_a3$) to the dead end of (∗), $x\mathsf{R}^{\epsilon}_{a}y$ which we desire is obtained and it is obvious that the new emerged sequent is provable.

However, in the case of (∗∗), the additional inference rules are not sufficient to make the branch reach initial sequent(s), though the problem here is the same as that of (∗) where the restricting formulas of the relational atom and the labelled expression with knowledge operator do not match for the application of ($L\mathsf{K}_a$), the labelled expression is more restricted than the relational atom unlike (∗∗). To settle the problem, we reformulate the rule of ($L\mathsf{K}_a$) in a semantically natural way. Our reformulated rule ($L\mathsf{K}'_a$) is then defined as follows.

$$\frac{\Gamma \Rightarrow \Delta, x\mathsf{R}^{\alpha}_a y \quad y{:}^{\alpha} A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\mathsf{K}_a A, \Gamma \Rightarrow \Delta}\ (L\mathsf{K}'_a)$$

It is necessary to note that, by this change of the rule, we need to depart from G3-style. [4] Although a solution with keeping G3-style might be a better solution than ours, we choose the semantically natural way to reformulate the rule ($L\mathsf{K}_a$) first, and at the same time we reformulate the rule ($L[.]$) in a natural form.

4.1 Revised Sequent Calculus GPAL

Now, we introduce our revised calculus, **GPAL**. The definition of **GPAL** is presented in Table 3. For drawing simpler derivations, we prepare the following lemma.

Lemma 1. *For any labelled expression* $\mathfrak{A}$ *and multi-sets of labelled expressions* Γ *and* Δ, **GPAL** $\vdash \mathfrak{A}, \Gamma \Rightarrow \Delta, \mathfrak{A}$.

Proof. It is obvious by applying (Rw) and/or (Lw) a finite number of times. □

Let us now show the derivations of (RA4) of **HPAL**.

Proposition 1. GPAL $\vdash \Rightarrow x{:}^{\epsilon}[A]\mathsf{K}_a B \leftrightarrow (A \to \mathsf{K}_a[A]B)$

Proof. We may find a derivation of $x{:}^{\epsilon}[A]\mathsf{K}_a B \to (A \to \mathsf{K}_a[A]B)$ in **GPAL** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}A, x{:}^{\epsilon}\mathsf{K}_a[A]B} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, x\mathsf{R}^{A}_a y} \quad \dfrac{\text{Lemma 1}}{y{:}^{A}B, x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x{:}^{A}\mathsf{K}_a B, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B}\,(L\mathsf{K}'_a)}{x{:}^{\epsilon}A, x{:}^{A}\mathsf{K}_a B, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{\epsilon}[A]B}\,(R[.])}{x{:}^{\epsilon}A, x{:}^{A}\mathsf{K}_a B \Rightarrow x{:}^{\epsilon}\mathsf{K}_a[A]B}\,(R\mathsf{K}_a)}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A]\mathsf{K}_a B \Rightarrow x{:}^{\epsilon}\mathsf{K}_a[A]B}\,(L[.]')}{x{:}^{\epsilon}[A]\mathsf{K}_a B \Rightarrow x{:}^{\epsilon}A \to \mathsf{K}_a[A]B}\,(R\to)}{\Rightarrow x{:}^{\epsilon}[A]\mathsf{K}_a B \to (A \to \mathsf{K}_a[A]B)}\,(R\to)$$

The derivation $\mathcal{D}_1$ is given as follows:

$$\dfrac{\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, x{:}^{\epsilon}A} \quad \dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, y{:}^{\epsilon}A} \quad \dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, x\mathsf{R}^{\epsilon}_a y}}{x{:}^{\epsilon}A, y{:}^{\epsilon}A, x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, x\mathsf{R}^{A}_a y}\,(Rrel)$$

[4] Of course, there might still exist a possibility to keep G3-style with the additional rules for relational atoms.

We may also find a derivation of $x{:}^{\epsilon}(A \to \mathsf{K}_a[A]B) \to [A]\mathsf{K}_a B$ in **HPAL** as follows:

$$
\dfrac{
\dfrac{
\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A \Rightarrow x{:}^{A}\mathsf{K}_aB, x{:}^{\epsilon}A}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\text{Lemma 1}}{x\mathsf{R}^{\epsilon}_a y \Rightarrow y{:}^{A}B, x\mathsf{R}^{\epsilon}_a y}}{x\mathsf{R}^{A}_a y \Rightarrow y{:}^{A}B, x\mathsf{R}^{\epsilon}_a y}\,(Lrel_a3)
\quad
\dfrac{
\dfrac{\dfrac{\text{Lemma 1}}{y{:}^{\epsilon}A \Rightarrow y{:}^{A}B, y{:}^{\epsilon}A}}{x\mathsf{R}^{A}_a y \Rightarrow y{:}^{A}B, y{:}^{\epsilon}A}\,(Lrel_a2)
\quad
\dfrac{\text{Lemma 1}}{y{:}^{A}B, x\mathsf{R}^{A}_a y \Rightarrow y{:}^{A}B}
}{y{:}^{\epsilon}[A]B, x\mathsf{R}^{A}_a y \Rightarrow y{:}^{A}B}\,(L[.]')
}{x\mathsf{R}^{A}_a y, x{:}^{\epsilon}\mathsf{K}_a[A]B \Rightarrow y{:}^{A}B}\,(L\mathsf{K}'_a)
}{x{:}^{\epsilon}\mathsf{K}_a[A]B \Rightarrow x{:}^{A}\mathsf{K}_aB}\,(R\mathsf{K}_a)
}{x{:}^{\epsilon}\mathsf{K}_a[A]B, x{:}^{\epsilon}A \Rightarrow x{:}^{A}\mathsf{K}_aB}\,(Lw)
}{x{:}^{\epsilon}A, x{:}^{\epsilon}A \to \mathsf{K}_a[A]B \Rightarrow x{:}^{A}\mathsf{K}_aB}\,(L\to)
}{x{:}^{\epsilon}A \to \mathsf{K}_a[A]B \Rightarrow x{:}^{\epsilon}[A]\mathsf{K}_aB}\,(R[.])
}{\Rightarrow (x{:}^{\epsilon}A \to \mathsf{K}_a[A]B) \to [A]\mathsf{K}_aB}\,(R\to)
$$

□

As we can see above, the proof of (RA4) in **GPAL** can be done thanks to the rules of relational atoms.

Moreover, **GPAL**$^+$ is defined to be **GPAL** with the following rule (Cut),

$$\dfrac{\Gamma \Rightarrow \Delta, \mathfrak{A} \quad \mathfrak{A}, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Cut).$$

$\mathfrak{A}$ in (Cut) is called a *cut expression*, and we use the term *principal expression* of an inference rule of **GPAL**$^+$ if a labelled expression is newly introduced on the left uppersequent or the right uppersequent by the rule of **GPAL**$^+$.

Let us briefly summarize our revised calculus in order. **GPAL** is different from **G3PAL** in respect to the following features:

1. It is based on Gentzen's standard sequent calculus [4] but not in G3-style, and so it contains structural rules.
2. It includes rules for relational atoms which **G3PAL** lacks.
3. (L[.]) and ($L\mathsf{K}_a$) are redefined in a semantically natural way, and each of them is denoted by (L[.]$'$) and ($L\mathsf{K}'_a$).
4. It does not contain (L_{cmp}) and (R_{cmp}) of **G3PAL**, but without them **GPAL** can prove (RA5). These rules are also derivable in **GPAL**$^+$.
5. (Lat) and (Rat) are redefined taking into account of the notion of surviveness, and each of them is denoted by (Lat') and (Rat').

The last two features have not been mentioned so far, and the last feature of **GPAL** will be considered at the beginning of Section 6. In this paragraph, we focus on feature 4. According to [7], the following rules

$$\dfrac{x{:}^{\alpha,A,B}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B}C, \Gamma \Rightarrow \Delta}\ (L_{cmp}) \qquad \dfrac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B}C}\ (R_{cmp})$$

are required to prove (RA5) of **HPAL**:

$$[A][B]C \leftrightarrow [A \wedge [A]B]C.$$

Table 3. Gentzen-style Sequent Calculus **GPAL**

(Initial Sequents)

$$x{:}^{\alpha}A \Rightarrow x{:}^{\alpha}A \quad x\mathsf{R}_a^{\alpha}v \Rightarrow x\mathsf{R}_a^{\alpha}v$$

(Structural Rules)

$$\frac{\Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lw) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rw)$$

$$\frac{\mathfrak{A}, \mathfrak{A}, \Gamma \Rightarrow \Delta}{\mathfrak{A}, \Gamma \Rightarrow \Delta}\ (Lc) \quad \frac{\Gamma \Rightarrow \Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma \Rightarrow \Delta, \mathfrak{A}}\ (Rc)$$

(Rules for propositional connectives)

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A}{x{:}^{\alpha}\neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\neg A}\ (R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}A \to B, \Gamma \Rightarrow \Delta}\ (L \to) \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \to B}\ (R \to)$$

(Rules for knowledge operators)

$$\frac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y \quad y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}\mathsf{K}_a A, \Gamma \Rightarrow \Delta}\ (L\mathsf{K}_a') \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta, y{:}^{\alpha}A}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}\mathsf{K}_a A}\ (R\mathsf{K}_a)\dagger$$

† y does not appear in the lower sequent.

(Rules for PAL)

$$\frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat') \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat')$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad x{:}^{\alpha,A}B, \Gamma \Rightarrow \Delta}{x{:}^{\alpha}[A]B, \Gamma \Rightarrow \Delta}\ (L[.]') \quad \frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta, x{:}^{\alpha,A}B}{\Gamma \Rightarrow \Delta, x{:}^{\alpha}[A]B}\ (R[.])$$

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a1) \quad \frac{y{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a2) \quad \frac{x\mathsf{R}_a^{\alpha}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,A}y, \Gamma \Rightarrow \Delta}\ (Lrel_a3)$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, y{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A}y}\ (Rrel_a)$$

In what follows, however, we reveal that these rules of (L_{cmp}) and (R_{cmp}) are not necessary in the set of inference rules of **GPAL**. Let us see the details. First, let us define the length of a labelled expression $\mathfrak{A}$ in advance.

Definition 3. *For any formula A,* $\mathsf{len}(A)$ *is equal to the number of the propositional variables and the logical connectives in A.*

$$\mathsf{len}(\alpha) = \begin{cases} 0 & \textit{if } \alpha = \epsilon \\ \mathsf{len}(\beta) + \mathsf{len}(A) & \textit{if } \alpha = \beta, A \end{cases}$$

$$\mathsf{len}(\mathfrak{A}) = \begin{cases} \mathsf{len}(\alpha) + \mathsf{len}(A) & \text{if } \mathfrak{A} = x{:}^{\alpha}A \\ \mathsf{len}(\alpha) + 1 & \text{if } \mathfrak{A} = x\mathsf{R}_a^{\alpha}y \end{cases}$$

Then, let us show Lemma 2.

Lemma 2. *For any* $A, B \in \mathsf{Form}$*,* $x, y \in \mathsf{Var}$ *and for any list* α, β *of formulas,*
(i) **GPAL** $\vdash x{:}^{\alpha,A,B,\beta}C \Rightarrow x{:}^{\alpha,A\wedge[A]B,\beta}C$*,*
(ii) **GPAL** $\vdash x{:}^{\alpha,A\wedge[A]B,\beta}C \Rightarrow x{:}^{\alpha,A,B,\beta}C$*,*
(iii) **GPAL** $\vdash x\mathsf{R}_a^{\alpha,A,B,\beta}y \Rightarrow x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y$*,*
(iv) **GPAL** $\vdash x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y \Rightarrow x\mathsf{R}_a^{\alpha,A,B,\beta}y$*.*

Proof. The proofs of (i), (ii), (iii) and (iv) are done simultaneously by double induction on C and β. We only see the case where C is the form of K_aD and the case where C is the form of $[D]E$, because the provability of the other sequents (ii), (iii) and (iv) can also be shown similarly. First, let us consider the case where C is the form of K_aD. Let γ be (α, A, B, β) and θ be $(\alpha, A \wedge [A]B, \beta)$.

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{x\mathsf{R}_a^{\theta}y \Rightarrow x\mathsf{R}_a^{\gamma}y}}{x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D, x\mathsf{R}_a^{\gamma}y}\,(Rw) \qquad \dfrac{\dfrac{\vdots\ \mathcal{D}_2}{y{:}^{\gamma}D \Rightarrow y{:}^{\theta}D}}{y{:}^{\gamma}D, x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D}\,(Lw)}{\dfrac{x{:}^{\gamma}\mathsf{K}_aD, x\mathsf{R}_a^{\theta}y \Rightarrow y{:}^{\theta}D}{x{:}^{\gamma}\mathsf{K}_aD \Rightarrow x{:}^{\theta}\mathsf{K}_aD}\,(R\mathsf{K}_a)}\,(L\mathsf{K}'_a)$$

Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are obtained by induction hypothesis, since the length of the labelled expressions is reduced. We may need to pay attention to the length of the labelled expression at the bottom sequent of $\mathcal{D}_1$, and according to Definition 3, $\mathsf{len}(x{:}^{\gamma}\mathsf{K}_aD) > \mathsf{len}(x\mathsf{R}_a^{\gamma}y)$ (for any γ).

Second, let us consider the case where C is the form of $[D]E$. γ be (α, A, B, β) and θ be $(\alpha, A \wedge [A]B, \beta)$.

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_3}{x{:}^{\theta}D \Rightarrow x{:}^{\gamma}D}}{x{:}^{\theta}D \Rightarrow x{:}^{\gamma}D, x{:}^{\theta,D}E}\,(Rw) \qquad \dfrac{\dfrac{\vdots\ \mathcal{D}_4}{x{:}^{\gamma,D}E \Rightarrow x{:}^{\theta,D}E}}{x{:}^{\gamma,D}E, x{:}^{\theta}D \Rightarrow x{:}^{\theta,D}E}\,(Lw)}{\dfrac{x{:}^{\gamma}[D]E, x{:}^{\theta}D \Rightarrow x{:}^{\theta,D}E}{x{:}^{\gamma}[D]E \Rightarrow x{:}^{\theta}[D]E}\,(R[.])}\,(L[.]')$$

The derivations $\mathcal{D}_3$ and $\mathcal{D}_4$ are obtained by induction hypotheses. □

Now with the help of the rule (Cut), we can also show the derivability of more general rules than (L_{cmp}) and (R_{cmp}) of **G3PAL** as follows:

Proposition 2. *The following rules* (L'_{cmp}) *and* (R'_{cmp}) *are derivable in* **GPAL**$^{+}$.

$$\dfrac{x{:}^{\alpha,A,B,\beta}C, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A\wedge[A]B,\beta}C, \Gamma \Rightarrow \Delta}\,(L'_{cmp}) \qquad \dfrac{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A,B,\beta}C}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A\wedge[A]B,\beta}C}\,(R'_{cmp})$$

where $a \in \mathsf{G}, A, B, C \in \mathsf{Form}$ *and* α *and* β *are arbitrary lists of formulas.*

Proof. It is shown immediately from Lemma 2 and (*Cut*). [5] □

4.2 All theorems of HPAL are provable in GPAL⁺

We first define the substitution of variables in labelled expressions.

Definition 4. *Let $\mathfrak{A}$ be any labelled expression. Then the substitute of x for y in $\mathfrak{A}$, denoted by $\mathfrak{A}[x/y]$, is defined by*

$$\begin{array}{lll} z[x/y] & := z & (\text{if } y \neq z) \\ z[x/y] & := x & (\text{if } y = z) \\ (z{:}^{\alpha}A)[x/y] & := (z[x/y]){:}^{\alpha}A & \\ (z\mathsf{R}_a^{\alpha}w)[x/y] & := (z[x/y])\mathsf{R}_a^{\alpha}(w[x/y]) & \end{array}$$

Substitution $[x/y]$ to a multi-set Γ of labelled expressions is defined as

$$\Gamma[x/y] := \{\mathfrak{A}[x/y] \mid \mathfrak{A} \in \Gamma\}.$$

Next, for a preparation of Theorem 1, we show the next lemma.

Lemma 3.
(i) **GPAL** $\vdash \Gamma \Rightarrow \Delta$ *implies* **GPAL** $\vdash \Gamma[x/y] \Rightarrow \Delta[x/y]$ *for any* $x, y \in \mathsf{Var}$.
(ii) **GPAL⁺** $\vdash \Gamma \Rightarrow \Delta$ *implies* **GPAL⁺** $\vdash \Gamma[x/y] \Rightarrow \Delta[x/y]$ *for any* $x, y \in \mathsf{Var}$.

Proof. By induction on the height of the derivation, we go through almost the same procedure in the proof in Negri and von Plato [10, p.194]. □

Finally, let us show the following theorem:

Theorem 1. *For any formula A, if* **HPAL** $\vdash A$, *then* **GPAL⁺** $\vdash \Rightarrow x{:}^{\epsilon}A$ (*for any x*).

Proof. The proof is carried out by the height of the derivation in **HPAL**. Since the case of reduction axiom (RA4) has been shown in Proposition 1, let us prove one direction of (RA5) $[A][B]C \leftrightarrow [A \wedge [A]B]C$ of **HPAL** for one of the base cases (the derivation height of **HPAL** is equal to 0).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A, x{:}^{A,B}C} \quad \dfrac{\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{A}B, x{:}^{A,B}C}}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}[A]B, x{:}^{A,B}C}\,(R[.])}{x{:}^{\epsilon}A, x{:}^{A}B \Rightarrow x{:}^{\epsilon}A \wedge [A]B, x{:}^{A,B}C}\,(R\wedge) \quad \dfrac{\dfrac{\text{Lemma 2}}{x{:}^{A\wedge[A]B}C \Rightarrow x{:}^{A,B}C}}{x{:}^{\epsilon}A, x{:}^{A}B, x{:}^{A\wedge[A]B}C \Rightarrow x{:}^{A,B}C}\,(Lw)}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A \wedge [A]B]C, x{:}^{A}B \Rightarrow x{:}^{A,B}C}\,(L[.]')}{x{:}^{\epsilon}A, x{:}^{\epsilon}[A \wedge [A]B]C \Rightarrow x{:}^{A}[B]C}\,(R[.])}{x{:}^{\epsilon}[A \wedge [A]B]C \Rightarrow x{:}^{\epsilon}[A][B]C}\,(R[.])}{\Rightarrow x{:}^{\epsilon}[A \wedge [A]B]C \to [A][B]C}\,(R\to)$$

[5] The following rules are also derivable in **GPAL⁺**.

$$\dfrac{x\mathsf{R}_a^{\alpha,A,B,\beta}y, \Gamma \Rightarrow \Delta}{x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y, \Gamma \Rightarrow \Delta}\,(L_{cmpr}) \qquad \dfrac{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,A,B,\beta}y}{\Gamma \Rightarrow \Delta, x\mathsf{R}_a^{\alpha,(A\wedge[A]B),\beta}y}\,(R_{cmpr})$$

In the inductive step, we show the inference rules, (MP) and (Nec), by **GPAL**. The former is shown with (Cut).

$$\dfrac{\dfrac{\text{Assumption}}{\Rightarrow x{:}^{\epsilon}A} \quad \dfrac{\dfrac{\text{Assumption}}{\Rightarrow x{:}^{\epsilon}A \to B} \quad \dfrac{\dfrac{\text{Lemma 1}}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B, x{:}^{\epsilon}A} \quad \dfrac{\text{Lemma 1}}{x{:}^{\epsilon}B, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}}{x{:}^{\epsilon}A \to B, x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}\,(L\to)}{x{:}^{\epsilon}A \Rightarrow x{:}^{\epsilon}B}\,(Cut)}{\Rightarrow x{:}^{\epsilon}B}\,(Cut)$$

The latter is shown by $(R\mathsf{K}_a)$, (Lw) and Lemma 3. □

4.3 Cut Elimination of GPAL$^+$

Here we prove an important theorem of the paper, the (syntactic) cut elimination theorem of **GPAL**$^+$.

Theorem 2 (Cut elimination theorem of GPAL$^+$). *For any sequent $\Gamma \Rightarrow \Delta$, if* **GPAL**$^+ \vdash \Gamma \Rightarrow \Delta$, *then* **GPAL** $\vdash \Gamma \Rightarrow \Delta$.

Proof. The proof is carried out using Ono and Komori's method [11] introduced in the reference [6] by Kashima where we employ the following rule $(Ecut)$ instead of the usual method of 'mix cut'. We denote the n-copies of the same labelled expression $\mathfrak{A}$ by $\mathfrak{A}^n$, and $(Ecut)$ is defined as follows:

$$\dfrac{\Gamma \Rightarrow \Delta, \mathfrak{A}^n \quad \mathfrak{A}^m, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

where $n, m \geq 0$. The theorem is proven by double induction on the height of the derivation and the length of the cut expression $\mathfrak{A}$ of $(Ecut)$. The proof is divided into four cases. In brief, 1) at least one of uppersequents of $(Ecut)$ is an initial sequent; 2) the last inference rule of either uppersequents of $(Ecut)$ is a structural rule; 3) the last inference rule of either uppersequents of $(Ecut)$ is a non-structural rule, and the principal expression introduced by the rule is not cut expression; and 4) the last inference rules of two uppersequents of $(Ecut)$ are both non-structural rules, and the principal expressions introduced by the rules used on the uppersequents of $(Ecut)$ are both cut expressions. We look at one of significant subcases of 4) in which principal expressions introduced by non-structural rules are both cut expressions.

Let us consider one of the cases 4) where both sides of $\mathfrak{A}$ are $x\mathsf{R}_a^{\alpha,A}y$ and principal expressions. When we obtain the following derivation:

$$\dfrac{\dfrac{\overset{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x{:}^{\alpha}A} \quad \overset{\vdots\ \mathcal{D}_2}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, y{:}^{\alpha}A} \quad \overset{\vdots\ \mathcal{D}_3}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x\mathsf{R}_a^{\alpha}y}}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n}}\,(Rrel_a) \quad \dfrac{\overset{\vdots\ \mathcal{D}_4}{x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta'}}{(x\mathsf{R}_a^{\alpha,A}y)^{m}, \Gamma' \Rightarrow \Delta'}\,(Lrel_a3)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

,

it is transformed into the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_1}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n\text{-}1}, x{:}^{\alpha}A} \quad \dfrac{\vdots\ \mathcal{D}_4'}{(x\mathsf{R}_a^{\alpha,A}y)^{m}, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x{:}^{\alpha}A}\,(Ecut) \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_{123}'}{\Gamma \Rightarrow \Delta, (x\mathsf{R}_a^{\alpha,A}y)^{n}} \quad \dfrac{\vdots\ \mathcal{D}_4}{x{:}^{\alpha}A, (x\mathsf{R}_a^{\alpha,A}y)^{m\text{-}1}, \Gamma' \Rightarrow \Delta'}}{x{:}^{\alpha}A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)\ \text{height-1}}{\dfrac{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Rc/Lc)}\,(Ecut)\ \text{length-1} ,$$

where (*Ecut*) to the two uppersequents is applicable by induction hypothesis, since the derivation height of (*Ecut*) is reduced by comparison with the original derivation. Additionally, the application of (*Ecut*) to the lowersequents is also allowed by induction hypothesis, since the length of the cut expression is reduced, namely $\mathsf{len}(x{:}^{\alpha}A) < \mathsf{len}(x\mathsf{R}_a^{\alpha,A}y)$. □

As a corollary of Theorem 2, the consistency of **GPAL**$^+$ is shown.

Corollary 1 (Consistency of GPAL). *The empty sequent* $\Rightarrow$ *cannot be proven in* **GPAL**$^+$.

Proof. Suppose for contradiction that $\Rightarrow$ is derivable in **GPAL**$^+$. By Theorem 2, $\Rightarrow$ is derivable in **GPAL**; however, there is no inference rule in **GPAL** which can derive the empty sequent. This is a contradiction. □

5 Soundness of GPAL

Now, we switch the subject to the soundness theorem of **GPAL**. For the theorem, we extend Kripke semantics of PAL to cover the labelled expressions. Given any Kripke model $\mathfrak{M}$, we say that $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ is an *assignment*.

Definition 5. *Let* $\mathfrak{M}$ *be a Kripke model and* $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ *an assignment.*

$\mathfrak{M}, f \Vdash x{:}^{\alpha}A$	*iff*	$\mathfrak{M}^{\alpha}, f(x) \Vdash A$ *and* $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$
$\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\epsilon}y$	*iff*	$\langle f(x), f(y)\rangle \in R_a$
$\mathfrak{M}, f \Vdash x\mathsf{R}_a^{\alpha,A}y$	*iff*	$\langle f(x), f(y)\rangle \in R_a^{\alpha}$ *and* $\mathfrak{M}^{\alpha}, f(x) \Vdash A$ *and* $\mathfrak{M}^{\alpha}, f(y) \Vdash A$

Here we have to be careful of the fact that $f(x)$ and $f(y)$ above must be defined in $\mathcal{D}(\mathfrak{M}^{\alpha})$. In the clause $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$, for example, $f(x)$ should survive (well-defined) in the restricted Kripke model $\mathfrak{M}^{\alpha}$. Taking into account of this fact, it is essential that we pay attention to the negation of $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$.

Proposition 3. $\mathfrak{M}, f \nVdash x{:}^{\alpha}A$ *iff* $f(x) \notin \mathcal{D}(\mathfrak{M}^{\alpha})$ *or* $(f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha})$ *and* $\mathfrak{M}^{\alpha}, f(x) \nVdash A)$.

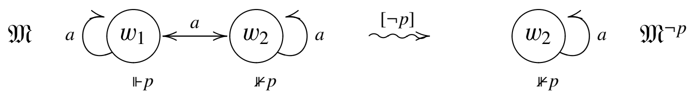
As far as the authors know, this point has not been suggested in previous works [1, 7]. Then, the reader may wonder if the following 'natural' definition of the validity for sequents (which we call *s-valid*) also works. The following notion can be regarded as an implementation of the reading of a sequent $\Gamma \Rightarrow \Delta$ as 'if all of the antecedent Γ hold, then some of the consequents Δ hold'.

Definition 6 (***s*-validity).** *$\Gamma \Rightarrow \Delta$ is s-valid in $\mathfrak{M}$ if, for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \Vdash \mathfrak{B}$.*

However, following this natural definition of validity of sequents, we come to a deadlock on the way to prove the soundness theorem, especially in the case of rules for logical negation, as we can see the following proposition.

Proposition 4. *There is a Kripke model $\mathfrak{M}$ such that $(R\neg)$ of* **GPAL** *does not preserve s-validity in $\mathfrak{M}$.*

Proof. Let $\mathsf{G} = \{a\}$ for simplicity. We use the same model as Example 2, that is, we consider a Kripke model $\mathfrak{M} = \langle \{w_1, w_2\}, \{w_1, w_2\}^2, V \rangle$ where $V(p) = \{w_1\}$.



And the particular instance of the application of $(R\neg)$ is as follows:

$$\frac{x{:}^{\neg p} p \Rightarrow}{\Rightarrow x{:}^{\neg p} \neg p}\ (R\neg)$$

We show that the uppersequent is s-valid in $\mathfrak{M}$ but the lowersequent is not s-valid in $\mathfrak{M}$, and so $(R\neg)$ does not preserve s-validity in this case. Note that w_1 does not survive after $\neg p$, i.e., $w_1 \notin \mathcal{D}(\mathfrak{M}^{\neg p}) = \{w_2\}$.

First, we show that $x{:}^{\neg p} p \Rightarrow$ is s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ for any assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. So, we fix any $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$. We divide our argument into: $f(x) = w_1$ or $f(x) = w_2$. If $f(x) = w_1$, $f(x)$ does not survive after $\neg p$, and so $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ by Proposition 3. If $f(x) = w_2$, $f(x)$ survives after $\neg p$ but $f(x) \notin \emptyset = V(p) \cap \mathcal{D}(\mathfrak{M}^{\neg p})$, which implies $\mathfrak{M}^{\neg p}, f(x) \nVdash p$ hence $\mathfrak{M}, f \nVdash x{:}^{\neg p} p$ by Proposition 3.

Second, we show that $\Rightarrow x{:}^{\neg p} \neg p$ is not s-valid in $\mathfrak{M}$, i.e., $\mathfrak{M}, f \nVdash x{:}^{\neg p} \neg p$ for some assignment $f : \mathsf{Var} \to W$. We fix $f : \mathsf{Var} \to W$ such that $f(x) = w_1$. Since $f(x) \notin \mathcal{D}(\mathfrak{M}^{\neg p})$ ($f(x)$ does not survive after $\neg p$), $\mathfrak{M}, f \nVdash x{:}^{\neg p} \neg p$ by Proposition 3, as desired. □

Proposition 4 forces us to abandon the notion of s-validity and have an alternative notion of validity. Here we recall the second intuitive reading of sequent $\Gamma \Rightarrow \Delta$ as 'It is not the case that all of the antecedents Γ hold and all of the consequents fail.' In order to realize the idea of 'failure', we first introduce the syntactic notion of the negated form $\overline{\mathfrak{A}}$ of a labelled expression $\mathfrak{A}$ and then provide the semantics $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha} A}$ with such negated forms, where we may read $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha} A}$ as '$\mathfrak{A}$ fails in $\mathfrak{M}$ under f.' Moreover, with this definition, our second notion of validity of a sequent, which we call *t-valid*,[6] is defined.

Definition 7 (***t*-validity).** *Let $\mathfrak{M}$ be a Kripke model and $f : \mathsf{Var} \to D(\mathfrak{M})$ an assignment. Then,*

$$\begin{array}{lll}
\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha} A} & \textit{iff} & \mathfrak{M}^{\alpha}, f(x) \Vdash \neg A \textit{ and } f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha}), \\
\mathfrak{M}, f \Vdash \overline{x \mathsf{R}_a^{\epsilon} y} & \textit{iff} & \langle f(x), f(y) \rangle \notin R_a, \\
\mathfrak{M}, f \Vdash \overline{x \mathsf{R}_a^{\alpha, A} y} & \textit{iff} & \mathfrak{M}, f \Vdash \overline{x \mathsf{R}_a^{\alpha} y} \textit{ or } \mathfrak{M}, f \Vdash \overline{x{:}^{\alpha} A} \textit{ or } \mathfrak{M}, f \Vdash \overline{y{:}^{\alpha} A}.
\end{array}$$

[6] We note that t-validity is close to the validity in the tableaux method of PAL [2].

We say that $\Gamma \Rightarrow \Delta$ is t-valid in $\mathfrak{M}$ if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$.

In this definition, we explicitly gave a condition of surviveness that $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$, e.g., in $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha A}$. Therefore, '$x :^\alpha A$ fails in $\mathfrak{M}$ under f' means that $f(x)$ survives after α but A is false at $f(x)$ in $\mathfrak{M}^\alpha$. The following proposition shows that the clauses for relational atoms and their negated forms characterize what they intend to capture.

Proposition 5. *For any Kripke model $\mathfrak{M}$, assignment f, $a \in \mathsf{G}$ and $x, y \in \mathsf{Var}$,*

(i) $\mathfrak{M}, f \Vdash x\mathsf{R}^\alpha_a y$ *iff* $\langle f(x), f(y)\rangle \in R^\alpha_a$,
(ii) $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^\alpha_a y}$ *iff* $\langle f(x), f(y)\rangle \notin R^\alpha_a$.

Proof. Both are easily shown by induction of α. Let us consider the case of $\alpha = \alpha', A$ in the proof of (ii).

We show $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}^{\alpha,A}_a y}$ iff $\langle f(x), f(y)\rangle \in R^{\alpha,A}_a$. $\mathfrak{M}, f \nVdash \overline{x\mathsf{R}^{\alpha,\mathsf{A}}_a y}$ is, by Definition 7 and the induction hypothesis, equivalent to $\langle f(x), f(y)\rangle \in R^\alpha_a$ and $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $\mathfrak{M}^\alpha, f(y) \Vdash A$. That is also equivalent to $\langle f(x), f(y)\rangle \in R^{\alpha,A}_a$. □

Following this, we may prove the soundness of **GPAL** properly.

Theorem 3 (Soundness of GPAL). *Given any sequent $\Gamma \Rightarrow \Delta$ in* **GPAL**, *if* **GPAL** $\vdash \Gamma \Rightarrow \Delta$, *then $\Gamma \Rightarrow \Delta$ is t-valid in every Kripke model $\mathfrak{M}$.*

Proof. The proof is carried out by induction of the height of the derivation of $\Gamma \Rightarrow \Delta$ in **GPAL**. We only confirm one of base cases of relational atoms and some cases in the inductive step.

Base case: we show that $x\mathsf{R}^\alpha_a v \Rightarrow x\mathsf{R}^\alpha_a v$ is *t*-valid. Suppose for contradiction that $\mathfrak{M}, f \Vdash x\mathsf{R}^\alpha_a v$ and $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^\alpha_a v}$. By Proposition 5, this is impossible.

The case where the last applied rule is $(R\neg)$: We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha \neg A}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash x{:}^\alpha A$. Then, $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha \neg A}$ iff $\mathfrak{M}^\alpha, f(x) \nVdash \neg A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$, which is equivalent to: $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$. By Definition 5, $\mathfrak{M}, f \Vdash x{:}^\alpha A$. So, the contraposition has been shown.

The case where the last applied rule is $(L\mathsf{K}')$: We show the contraposition. Suppose that there is some $f : \mathsf{Var} \to W$ such that $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$ and $\mathfrak{M}, f \Vdash x^\alpha{:}\mathsf{K}_a A$ and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^\alpha_a y}$ or $\mathfrak{M}, f \Vdash y{:}^\alpha A$. Then, from $\mathfrak{M}, f \Vdash x{:}^\alpha \mathsf{K}_a A$, we obtain $\langle f(x), f(y)\rangle \notin R^\alpha_a$ or $\mathfrak{M}^\alpha, f(y) \Vdash A$. Suppose the former disjunct, i.e., $\langle f(x), f(y)\rangle \notin R^\alpha_a$, which is, by Proposition 5, $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}^\alpha_a y}$. Then, suppose the latter disjunct $\mathfrak{M}^\alpha, f(y) \Vdash A$. By definition, this is equivalent to $\mathfrak{M}, f \Vdash y :^\alpha A$. Then, the contraposition has been shown.

The case where the last applied rule is (Rat'): Similar to the above, we show the contraposition. Suppose there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A} p}$. Fix such f. It suffices to show $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha p}$. By Definition 7, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A} p}$ is equivalent to $\mathfrak{M}^{\alpha,A}, f(x) \Vdash \neg p$ and $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$. By $f(x) \in \mathcal{D}(\mathfrak{M}^{\alpha,A})$, we obtain $f(x) \in \mathcal{D}(\mathfrak{M}^\alpha)$ and $\mathfrak{M}^\alpha, f(x) \Vdash A$. It follows from $\mathfrak{M}^\alpha, f(x) \Vdash A$ and $\mathfrak{M}^{\alpha,A}, f(x) \Vdash \neg p$ that $f(x) \notin V^\alpha(p)$. This is equivalent to $\mathfrak{M}, f \Vdash \overline{x{:}^\alpha p}$. Then, the contraposition has been shown.

The case where the last applied rule is (*Rrel*)**:** As before, we show the contraposition. Suppose there is some $f : \mathsf{Var} \to W$ such that, $\mathfrak{M}, f \Vdash \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \Vdash \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$, and $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha,A}y}$. Fix such f. By Definition 7, $\overline{x\mathsf{R}_a^{\alpha,A}y}$ is equivalent to $\mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y}$ or $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha}A}$ or $\mathfrak{M}, f \Vdash \overline{y{:}^{\alpha}A}$. This is what we want to show. □

For the following corollary, we prepare the next proposition.

Proposition 6. *If* $\Rightarrow x{:}^{\epsilon}A$ *is t-valid in a Kripke model* $\mathfrak{M}$*, then A is valid in* $\mathfrak{M}$*.*

Proof. Suppose that $\Rightarrow x{:}^{\epsilon}A$ is t-valid. So, it is not the case that there exists some assignment f such that $\mathfrak{M}, f \Vdash \overline{x{:}^{\epsilon}A}$. Equivalently, for all assignments f, $\mathfrak{M}, f \nVdash \overline{x{:}^{\epsilon}A}$. For any assignment f, $\mathfrak{M}, f \nVdash \overline{x{:}^{\epsilon}A}$ is equivalent to $\mathfrak{M}, f(x) \Vdash A$ because $f(x) \in \mathcal{D}(\mathfrak{M})$. So, it follows that $\mathfrak{M}, f(x) \Vdash A$ for all assignments f. Then, it is immediate to see that A is valid in $\mathfrak{M}$, as required. □

Then an indirect proof of completeness of **GPAL** can be provided as follows:

Corollary 2. *Given any formula A and label* $x \in \mathsf{Var}$*, the following are equivalent.*

(i) *A is valid on all Kripke models.*
(ii) **HPAL** $\vdash A$
(iii) **GPAL**$^+$ $\vdash \Rightarrow x{:}^{\epsilon}A$
(iv) **GPAL** $\vdash \Rightarrow x{:}^{\epsilon}A$

Proof. The direction from (i) to (ii) is established by Fact 1 and the direction from (ii) to (iii) is shown by Theorem 1. Then, the direction from (iii) to (iv) is established by the admissibility of (*Cut*) (Theorem 2). Finally, the direction from (iv) to (i) is shown by Theorem 3 and Proposition 6. □

6 Completeness of GPAL for Link-cutting semantics

Let us denote by **GPALw** as the resulting sequent calculus of replacing (*Lat′*) and (*Rat′*) of **GPAL** with the following modified version of (*Lat*) and (*Rat*) in **G3PAL**:

$$\frac{x{:}^{\alpha}A, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat1) \quad \frac{x{:}^{\alpha}p, \Gamma \Rightarrow \Delta}{x{:}^{\alpha,A}p, \Gamma \Rightarrow \Delta}\ (Lat2) \quad \frac{\Gamma \Rightarrow \Delta, x{:}^{\alpha}A \quad \Gamma \Rightarrow \Delta, x{:}^{\alpha}p}{\Gamma \Rightarrow \Delta, x{:}^{\alpha,A}p}\ (Rat).$$

We checked that all results needed to show Corollary 2 hold also for **GPALw**, and so we can establish the similar result to Corollary 2 also for **GPALw**. While (*Rat*) *does* preserve t-validity in a Kripke model $\mathfrak{M}$ by the similar argument to the proof of Theorem 3, we remark that one premise $\Gamma \Rightarrow \Delta, x{:}^{\alpha}A$ of (*Rat*) becomes redundant when we prove that (*Rat*) preserves t-validity in a Kripke model. This is because, for any assignment f, $\mathfrak{M}, f \Vdash \overline{x{:}^{\alpha,A}p}$ already implies that A holds at $f(x)$ after α, i.e., $\mathfrak{M}, f \Vdash x{:}^{\alpha}A$. We realize that this difference between **GPALw** and **GPAL** comes from the difference between the (standard) world-deletion semantics of PAL and the link-cutting semantics of PAL (see also Remark 1). In this section, we introduce our version of link-cutting semantics of

PAL and provide a direct proof of completeness of **GPAL** for link-cutting semantics. [7] The specific definition of the link-cutting version of PAL's semantics is given as follows, where we keep the symbol $\Vdash$ for the previous world-deletion semantics of PAL and use the new symbol '$\models$' for the satisfaction relation for the link-cutting semantics.

Definition 8 (Link-cutting semantics of PAL). *Given a Kripke model $\mathfrak{M}$, $w \in \mathcal{D}(\mathfrak{M})$ and a formula A, $\mathfrak{M}, w \models A$ is defined by*

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \textit{iff} & w \in V(p), \\
\mathfrak{M}, w \models \neg A & \textit{iff} & \mathfrak{M}, w \not\models A, \\
\mathfrak{M}, w \models A \to B & \textit{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}, w \models B, \\
\mathfrak{M}, w \models \mathsf{K}_a A & \textit{iff} & \textit{for all } v \in W : wR_a v \textit{ implies } \mathfrak{M}, v \models A (a \in \mathsf{G}), \textit{and} \\
\mathfrak{M}, w \models [A]B & \textit{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}^{A!}, w \models B,
\end{array}$$

where the restriction $\mathfrak{M}^{A!}$ is defined by triple $\langle W, (R_a^{A!})_{a \in \mathsf{G}}, V \rangle$ with

$$R_a^{A!} := R_a \cap (\llbracket A \rrbracket_{\mathfrak{M}} \times \llbracket A \rrbracket_{\mathfrak{M}}), \quad \textit{where } \llbracket A \rrbracket_{\mathfrak{M}} := \{x \in W \mid \mathfrak{M}, x \models A\}.$$

Remark 1. As far as the authors know, van Benthem et al. [14, p.166] first provides an idea of link-cutting semantics of public announcement logic. Their underlying idea is: cutting the links (pairs in an accessibility relation) between A-zone and $\neg A$-zone. Then, they state that all valid formulas in the resulting semantics are also the same as those in the world-deletion semantics [14, Fact 1]. Their semantics is similar but different to our semantics above. Hansen [5, p.145] touches on the same link-cutting semantics as ours in the public announcement extension of hybrid logic (an extended modal logic), but he does not investigate the semantics in detail there. A variant of our link-cutting semantics is also explained for logic of belief in [15], though the notion of public announcement there is not truthful and this is why the announcement there is called the 'introspective announcement.'

According to this definition, only the accessibility relation is restricted to A in $\mathfrak{M}^{A!}$, and the set of possible worlds and valuation stay as they were. Similar to the world-deletion semantics, we can also define the notion of validity in a Kripke model. The following soundness of **HPAL** for the link-cutting semantics is straightforward.

Proposition 7. *If A is a theorem of* **HPAL**, *A is valid in every Kripke model $\mathfrak{M}$ for the link-cutting semantics.*

As before, for any list $\alpha = (A_1, A_2, ..., A_n)$ of formulas , we define $\mathfrak{M}^{\alpha!}$ inductively as: $\mathfrak{M}^{\alpha!} := \mathfrak{M}$ (if $\alpha = \epsilon$), and $\mathfrak{M}^{\alpha!} := (\mathfrak{M}^{\beta!})^{A_n!} = \langle W, (R_a^{\beta!, A_n!})_{a \in \mathsf{G}}, V \rangle$ (if $\alpha = \beta, A_n$). Now we can show that the corresponding notions to s- and t-validity become equivalent under our link-cutting semantics.

Definition 9. *Let $\mathfrak{M}$ be a Kripke model and $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ an assignment.*

$$\begin{array}{lll}
\mathfrak{M}, f \models x{:}^{\alpha} A & \textit{iff} & \mathfrak{M}^{\alpha!}, f(x) \models A \\
\mathfrak{M}, f \models x \mathsf{R}_a^{\epsilon} y & \textit{iff} & \langle f(x), f(y) \rangle \in R_a \\
\mathfrak{M}, f \models x \mathsf{R}_a^{\alpha, A} y & \textit{iff} & \langle f(x), f(y) \rangle \in R_a^{\alpha!} \textit{ and } \mathfrak{M}^{\alpha!}, f(x) \models A \textit{ and } \mathfrak{M}^{\alpha!}, f(y) \models A
\end{array}$$

[7] Thanks to a comment from Makoto Kanazawa in the annual meeting of MLG2014, we noticed that link-cutting semantics may be suitable for our labelled sequent calculus of PAL.

By this definition, the next proposition immediately follows.

Proposition 8. *For any Kripke model* $\mathfrak{M}$*, assignment* f*,* $a \in \mathsf{G}$ *and* $x, y \in \mathsf{Var}$*,*

$$\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha} y \quad \textit{iff} \quad \langle f(x), f(y) \rangle \in R_a^{\alpha!}$$

The semantics of the negated form of a labelled expression $\overline{\mathfrak{A}}$ is also defined as before.

Definition 10. *Let* $\mathfrak{M}$ *be a Kripke model and* $f : \mathsf{Var} \to D(\mathfrak{M})$ *an assignment. Then,*

$$\begin{array}{lll} \mathfrak{M}, f \models \overline{x{:}^{\alpha}A} & \textit{iff} & \mathfrak{M}^{\alpha!}, f(x) \not\models A, \\ \mathfrak{M}, f \models \overline{x\mathsf{R}_a^{\epsilon}y} & \textit{iff} & \langle f(x), f(y) \rangle \notin R_a, \\ \mathfrak{M}, f \models \overline{x\mathsf{R}_a^{\alpha,A}y} & \textit{iff} & \mathfrak{M}, f \Vdash \overline{x\mathsf{R}_a^{\alpha}y} \text{ or } \mathfrak{M}, f \not\models x{:}^{\alpha}A \text{ or } \mathfrak{M}, f \not\models y{:}^{\alpha}A \end{array}$$

Now we may confirm that, based on the semantics, t-validity and s-validity are equivalent since $\mathfrak{M}, f \not\models \overline{\mathfrak{B}}$ is equivalent to $\mathfrak{M}, f \models \mathfrak{B}$ in this semantics.

Proposition 9. *Under the link-cutting semantics, a sequent* $\Gamma \Rightarrow \Delta$ *is s-valid in a Kripke model* $\mathfrak{M}$ *iff it is t-valid in* $\mathfrak{M}$*.*

Proof. Suppose $\Gamma \Rightarrow \Delta$ is t-valid. In other words, if there is no assignment $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$ such that $\mathfrak{M}, f \models \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, and $\mathfrak{M}, f \models \overline{\mathfrak{B}}$ for all $\mathfrak{B} \in \Delta$. Equivalently, for all assignments $f : \mathsf{Var} \to \mathcal{D}(\mathfrak{M})$, $\mathfrak{M}, f \models \mathfrak{A}$ for all $\mathfrak{A} \in \Gamma$, there exists $\mathfrak{B} \in \Delta$ such that $\mathfrak{M}, f \models \mathfrak{B}$. □

Because the notion of surviveness is expelled, the definition of the satisfaction of labelled expressions becomes wholly natural. Thus, we do not need to worry about the notion of surviveness of possible worlds in this link-cutting semantics.

Hereafter in this section we consider possibly infinite multi-sets of labelled expressions. That is, we call $\Gamma \Rightarrow \Delta$ an infinite sequent if Γ or Δ are infinite multi-sets. We use the notation **GPAL** $\vdash \Gamma \Rightarrow \Delta$ to mean that there are finite multi-sets Γ' and Δ' of labelled expressions such that **GPAL** $\vdash \Gamma' \Rightarrow \Delta'$ in the ordinary sense and $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. To establish the completeness result of **GPAL** for the link-cutting semantics, we first introduce the notion of saturation as follows.

Definition 11. *A possibly infinite sequent* $\Gamma \Rightarrow \Delta$ *is* saturated *if it satisfies the following:*

($unprov$) *$\Gamma \Rightarrow \Delta$ is not derivable in* **GPAL**,
($\to l$) *if* $x{:}^{\alpha}A \to B \in \Gamma$*, then* $x{:}^{\alpha}A \in \Delta$ *or* $x{:}^{\alpha}B \in \Gamma$*,*
($\to r$) *if* $x{:}^{\alpha}A \to B \in \Delta$*, then* $x{:}^{\alpha}A \in \Gamma$ *and* $x{:}^{\alpha}B \in \Delta$*,*
($\neg l$) *if* $x{:}^{\alpha}\neg A \in \Gamma$*, then* $x{:}^{\alpha}A \in \Delta$*,*
($\neg r$) *if* $x{:}^{\alpha}\neg A \in \Delta$*, then* $x{:}^{\alpha}A \in \Gamma$*,*
($\mathsf{K}_a l$) *if* $x{:}^{\alpha}\mathsf{K}_a A \in \Gamma$*, then* $x\mathsf{R}_a^{\alpha}y \in \Delta$ *or* $y{:}^{\alpha}A \in \Gamma$ *for any label* y*,*
($\mathsf{K}_a r$) *if* $x{:}^{\alpha}\mathsf{K}_a A \in \Delta$*, then* $x\mathsf{R}_a^{\alpha}y \in \Gamma$ *and* $y{:}^{\alpha}A \in \Delta$ *for some label* y*,*
($[.]l$) *if* $x{:}^{\alpha}[A]B \in \Gamma$*, then* $x{:}^{\alpha}A \in \Delta$ *or* $x{:}^{\alpha,A}B \in \Gamma$*,*
($[.]r$) *if* $x{:}^{\alpha}[A]B \in \Delta$*, then* $x{:}^{\alpha}A \in \Gamma$ *and* $x{:}^{\alpha,A}B \in \Delta$*,*
(atl) *if* $x{:}^{\alpha,A}p \in \Gamma$*, then* $x{:}^{\alpha}p \in \Gamma$*,*
(atr) *if* $x{:}^{\alpha,A}p \in \Delta$*, then* $x{:}^{\alpha}p \in \Delta$*,*
($rell$) *if* $x\mathsf{R}_a^{\alpha,A}y \in \Gamma$*, then* $x{:}^{\alpha}A \in \Gamma$ *and* $y{:}^{\alpha}A \in \Gamma$*, and* $x\mathsf{R}_a^{\alpha}y \in \Gamma$*, and*

(*relr*) *if* $x\mathsf{R}_a^{\alpha,A}y \in \Delta$, *then* $x{:}^{\alpha}A \in \Delta$ *or* $y{:}^{\alpha}A \in \Delta$, *or* $x\mathsf{R}_a^{\alpha}y \in \Delta$.

We show the next lemma which states that any unprovable sequent in **GPAL** can be extended to a (possibility infinite) saturated sequent.

Lemma 4. *Let* $\Gamma \Rightarrow \Delta$ *be a finite sequent. If* **GPAL** $\nvdash \Gamma \Rightarrow \Delta$, *then there exists a possibility infinite saturated sequent* $\Gamma^+ \Rightarrow \Delta^+$ *where* $\Gamma \subseteq \Gamma^+$ *and* $\Delta \subseteq \Delta^+$.

Proof. Suppose that there is a finite sequent $\Gamma \Rightarrow \Delta$ such that **GPAL** $\nvdash \Gamma \Rightarrow \Delta$. Let $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ be an enumeration of all labelled expressions such that each labelled expression appears infinitely many times. We inductively construct an infinite sequence $(\Gamma_i \Rightarrow \Delta_i)_{i\in\mathbb{N}}$ of finite sequents such that **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$ at each $i \in \mathbb{N}$ as follows and define $\Gamma^+ \Rightarrow \Delta^+$ as the 'limit' of such sequence.

Let $\Gamma_0 \Rightarrow \Delta_0$ be $\Gamma \Rightarrow \Delta$ as the basis of $\Gamma_i \Rightarrow \Delta_i$, and by the supposition **GPAL** $\nvdash \Gamma_0 \Rightarrow \Delta_0$. The $i+1$-th step consists of the procedures to define an underivable $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ from $\Gamma_i \Rightarrow \Delta_i$ depending on the shape of the labelled expression $\mathfrak{A}_i$. In the $i+1$-th step, one of the following operations is executed.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha}A \to B$ **and** $\mathfrak{A}_i \in \Gamma_i$**:** Because $\Gamma_i \Rightarrow \Delta_i$ is unprovable, either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $x{:}^{\alpha}B, \Gamma_i \Rightarrow \Delta_i$ is also unprovable by $(L \to)$. Then we choose one unprovable sequent as $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha}A \to B$ **and** $\mathfrak{A}_i \in \Delta_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}B$. By $(R \to)$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha}\neg A$ **and** $\mathfrak{A}_i \in \Gamma_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$. Because of $(L\neg)$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha}\neg A$ **and** $\mathfrak{A}_i \in \Delta_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i$. Because of $(R\neg)$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x :^{\alpha} [A]B$ **and** $\mathfrak{A}_i \in \Gamma_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ as either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $x{:}^{\alpha,A}B, \Gamma_i \Rightarrow \Delta_i$. Because of $(L[.])$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x :^{\alpha} [A]B$ **and** $\mathfrak{A}_i \in \Delta_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha,A}B$. Because of $(R[.])$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha,A}p$ **and** $\mathfrak{A}_i \in \Gamma_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}p, \Gamma_i \Rightarrow \Delta_i$. Because of (Lat') and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha,A}p$ **and** $\mathfrak{A}_i \in \Delta_i$**:** We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}p$. Because of (Rat') and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ **is** $x{:}^{\alpha}\mathsf{K}_aA$ **and** $\mathfrak{A}_i \in \Gamma_i$**:** Let $\{y_1, ..., y_n\}$ be the set of all labels appearing in $\Gamma_i \Rightarrow \Delta_i$. Suppose we have constructed $(\Gamma_i^{(k)} \Rightarrow \Delta_i^{(k)})_{1\leq k\leq \ell}$ such that $(\Gamma_i^{(k)} \Rightarrow \Delta_i^{(k)})$ is unprovable, $\Gamma_i^{(k)} \subseteq \Gamma_i^{(k+1)}$, and $\Delta_i^{(k)} \subseteq \Delta_i^{(k+1)}$. Because of $(L\mathsf{K}_a)$ and **GPAL** $\nvdash \Gamma_i \Rightarrow \Delta_i$, either $\Gamma_i \Rightarrow \Delta_i, x\mathsf{R}_a^{\alpha}y_{\ell+1}$ or $y_{\ell+1}{:}A, \Gamma_i \Rightarrow \Delta_i$ is unprovable, and we choose one unprovable sequent as $\Gamma_i^{(\ell+1)} \Rightarrow \Delta_i^{(\ell+1)}$. Then we define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i^{(n)} \Rightarrow \Delta_i^{(n)}$, and $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is unprovable by construction.

The case where $\mathfrak{A}_i$ is $x{:}^{\alpha}\mathsf{K}_a A$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x\mathsf{R}_a^{\alpha}y, \Gamma_i \Rightarrow \Delta_i, y{:}^{\alpha}A$, where y is a fresh variable that does not appear in $\Gamma_i \Rightarrow \Delta_i$. Because of $(R\mathsf{K}_a)$ and $\mathbf{GPAL} \nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ is $x\mathsf{R}_a^{\alpha,A}y$ and $\mathfrak{A}_i \in \Gamma_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := x{:}^{\alpha}A, y{:}^{\alpha}A, x\mathsf{R}_a^{\alpha}y, \Gamma_i \Rightarrow \Delta_i$. Because of $(Lrel)$ and $\mathbf{GPAL} \nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

The case where $\mathfrak{A}_i$ is $x\mathsf{R}_a^{\alpha,A}y$ and $\mathfrak{A}_i \in \Delta_i$: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ as either $\Gamma_i \Rightarrow \Delta_i, x{:}^{\alpha}A$ or $\Gamma_i \Rightarrow \Delta_i, y{:}^{\alpha}A$ or $\Gamma_i \Rightarrow \Delta_i, x\mathsf{R}_a^{\alpha}y$. Because of $(Rrel)$ and $\mathbf{GPAL} \nvdash \Gamma_i \Rightarrow \Delta_i$, the sequent $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is also unprovable.

Otherwise: We define $\Gamma_{i+1} \Rightarrow \Delta_{i+1} := \Gamma_i \Rightarrow \Delta_i$.

Finally, let $\Gamma^+ \Rightarrow \Delta^+$ be the union $\bigcup_{i\in\mathbb{N}} \Gamma_i \Rightarrow \bigcup_{i\in\mathbb{N}} \Delta_i$. Then, it is routine to check that $\Gamma^+ \Rightarrow \Delta^+$ is saturated and $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$. □

We now prove the completeness of **GPAL** for the link-cutting semantics.

Theorem 4. *If a formula A is valid in every Kripke model $\mathfrak{M}$ for the link-cutting semantics, then* $\mathbf{GPAL} \vdash\ \Rightarrow x{:}^{\epsilon}A$.

Proof. We show its contraposition, and so suppose $\mathbf{GPAL} \nvdash\ \Rightarrow x{:}^{\epsilon}A$. By Lemma 4, there exists a saturated sequent $\Gamma^+ \Rightarrow \Delta^+$ such that $\{x{:}^{\epsilon}A\} \subseteq \Delta^+$. Using the saturated sequent, we construct the derived Kripke model $\mathfrak{M} = \langle W, (R_a)_{a\in\mathsf{G}}, V\rangle$ from the saturated sequent $\Gamma^+ \Rightarrow \Delta^+$.

- W is a set of all labels appearing in $\Gamma^+ \Rightarrow \Delta^+$,
- $xR_a^{\epsilon}y$ iff $x\mathsf{R}_a^{\epsilon}y \in \Gamma^+$,
- $x \in V(p)$ iff $x{:}p \in \Gamma^+$.

In addition to this, let $f : \mathsf{Var} \to W$ such that $f(x) = x$ (if x is in W), and otherwise $f(x)$ is an arbitrary label.

(i) $\mathfrak{A} \in \Gamma^+$ implies $\mathfrak{M}, f \models \mathfrak{A}$,
(ii) $\mathfrak{A} \in \Delta^+$ implies $\mathfrak{M}, f \not\models \mathfrak{A}$.

The second item implies that $\mathfrak{M}, f(x) \not\models A$ hence A is not valid in the derived model $\mathfrak{M}$. The proof for these two items is conducted by induction on the length of $\mathfrak{A}$. Here we only look at the cases where $\mathfrak{A}$ is $x{:}^{\alpha,A}p$ or $x{:}^{\alpha}\mathsf{K}_a A$.

The case where $\mathfrak{A}$ is $x{:}^{\alpha,A}p$: (i) If $x{:}^{\alpha,A}p \in \Gamma^+$, then by saturatedness, we have $x{:}^{\alpha}p \in \Gamma^+$. Then by induction hypothesis, $\mathfrak{M}, f \models x{:}^{\alpha}p$ is obtained. This is equivalent to $\mathfrak{M}^{\alpha}, f(x) \models p$, i.e., $f(x) \in V(p)$. Hence $\mathfrak{M}, f \models x{:}^{\alpha,A}p$.
(ii) If $x{:}^{\alpha,A}p \in \Delta^+$, then by the saturatedness, we have $x{:}^{\alpha}p \in \Delta^+$. Then by induction hypothesis, $\mathfrak{M}, f \not\models x{:}^{\alpha}p$ is obtained. This is equivalent to $f(x) \notin V(p)$, and so $\mathfrak{M}, f \not\models x{:}^{\alpha,A}p$.

The case where $\mathfrak{A}$ is $x{:}^{\alpha}\mathsf{K}_a A$: (i) Suppose $x{:}^{\alpha}\mathsf{K}_a A \in \Gamma^+$. What we show is $\mathfrak{M}, f \models x{:}^{\alpha}\mathsf{K}_a A$, i.e., for all $y \in \mathcal{D}(\mathfrak{M})$, $xR_a^{\alpha!}y$ implies $\mathfrak{M}^{\alpha!}, y \models A$. So, fix any $y \in \mathcal{D}(\mathfrak{M})$ such that $xR_a^{\alpha!}y$. Now it suffices to show $\mathfrak{M}^{\alpha!}, y \models A$. By Proposition 8, we have $\mathfrak{M}, f \models x\mathsf{R}_a^{\alpha}y$. Suppose for contradiction that $x\mathsf{R}_a^{\alpha}y \in \Delta^+$. By induction hypothesis, $\mathfrak{M}, f \not\models x\mathsf{R}_a^{\alpha}y$. A contradiction. Therefore, $x\mathsf{R}_a^{\alpha}y \notin \Delta^+$. Since $\Gamma^+ \Rightarrow \Delta^+$ is saturated

and $x{:}^{\alpha}\mathsf{K}_a A \in \Gamma^+$, we have $x\mathsf{R}^{\alpha}_a y \in \Delta^+$ or $y{:}^{\alpha}A \in \Gamma^+$. It follows that $y{:}^{\alpha}A \in \Gamma^+$, hence $\mathfrak{M}^{\alpha!}, y \models A$ by induction hypothesis.
(ii) Suppose $x{:}^{\alpha}\mathsf{K}_a A \in \Delta^+$. By Definition 11, $x\mathsf{R}^{\alpha}_a y \in \Gamma^+$ and $y{:}^{\alpha}A \in \Delta^+$, for some y. By induction hypothesis, $\mathfrak{M}, f \models x\mathsf{R}^{\alpha}_a y$ and $\mathfrak{M}, f \not\models y{:}^{\alpha}A$, for some y. By Proposition 8, the definition of f and Definition 5, $\langle x, f(y)\rangle \in R^{\alpha!}_a$ and $\mathfrak{M}^{\alpha!}, f(y) \not\models A$, for some y. Then, we get the goal: $\mathfrak{M}, f \not\models x{:}^{\alpha}\mathsf{K}_a A$. □

Corollary 3. *Given any formula A and label $x \in \mathsf{Var}$, the following are equivalent.*

(i) *A is valid on all Kripke models for the world-deletion semantics.*
(ii) **HPAL** $\vdash A$
(iii) **GPAL**$^+$ $\vdash\Rightarrow x{:}^{\epsilon}A$
(iv) **GPAL** $\vdash\Rightarrow x{:}^{\epsilon}A$
(v) *A is valid on all Kripke models for the link-cutting semantics.*

Proof. The direction from (v) to (iv) is established by Theorem 4 and the direction from (ii) to (v) is shown by Propostion 7. Then, Corollary 2 implies the equivalence between five items. □

7 Conclusion

We found that rules related with relational atoms were missing in the existing labelled sequent calculus of **G3PAL**, and that (RA4) was not provable by the system, although it should be if it is complete for Kripke semantics. Therefore, we have revised **G3PAL** by reformulating and adding some rules to it and named our calculus **GPAL**. During this revision, we also make the notion of *surviveness* explicit. According to this revision, we can show that our **GPAL** is sound for Kripke semantics. Moreover, by carefully considering the notion of surviveness, we found the link-cutting version of PAL's semantics is more applicable to our labelled sequent calculus than the standard semantics i.e., the world-deletion semantics, and then we have shown **GPAL** is complete for the link-cutting semantics. Lastly, we would like to stress that the consideration of surviveness in the the restricted domain may be significant not only to PAL but also to other dynamic epistemic logics, such as Action Model Logic (cf. [3, 15]), in general where we need a restriction on possible worlds.

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