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Doctoral Dissertation

Logic-based Analysis of Belief Change in Judgment

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Abstract

This study aims to propose a formalization of a judge's belief change in terms of dynamic epistemic logic (DEL). Belief revision is an important concept for a judge to decide if he/she will believe the received information or not. Reliability among witnesses is usually considered to be a key issue for judgment. That is, when a judge receives a piece of information from a witness, he/she has to decide if such witness is reliable or not. If the judge considers such witness to be reliable, he/she will accept the received information. In order to formalize this situation, we apply the notions of signed information and reliability relation to represent an information source and its reliability, respectively. Furthermore, the judge may change his/her reliability for the other witnesses, when he/she receives a new piece of information from one of them. This process is called reliability change. This shows that the connection between belief change and reliability change is an important aspect. In order to capture changing of both belief and reliability, six dynamic operators are proposed. Three operators including upgrade, downgrade and joint downgrade are used to change the reliability of some agents with respect to a specific agent's perspective. That is, the upgrade operator is employed for making some agents more reliable, while downgrade and joint downgrade operators are applied for downgrading all of them. Belief change can be handled by private announcement, private permission and careful policy. The first operator is used to remove some beliefs, while the second one is used to restore the former beliefs. The careful policy aims to derive an agent's belief from the received signed information.

Since our goal of this study is to realize a judge's changing of belief and reliability in a judgment process by DEL, we need to consider two difficulties for applying DEL to a legal case. First, since a key feature of DEL is that possibilities in an agent's belief can be represented by a Kripke model, our question is how we can construct the model from a legal case. Second, since this study employs several dynamic operators, our question is how we can decide which operators are to be applied for changing belief and reliability. In order to solve these difficulties, we propose an analysis method and then implement a computer system which provides two functions. First, the system can generate a Kripke model from a legal case. Second, the system provides an inconsistency management policy which can automatically perform several operations in order to reduce the effort needed to decide which operators are to be applied. By our analysis method and implementation, the above questions can be adequately solved. In addition, six legal cases are analyzed to demonstrate our implementation.

Keywords: Belief change, Belief revision, Reliability change, Legal case, Judgment, Dynamic epistemic logic

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Chapter 1

Introduction

1.1 Background and Motivation

In legal proceedings, a judge has to reconsider his/her belief in light of new evidence in order to reach his/her own decision.[1] That is, when a judge receives a piece of information, he/she has to decide if he/she will believe the received information or not. This is the same concept as *belief revision* which is a study of rationally revising beliefs in light of new information. That is, if an agent considers the received information to be inconsistent with his/her belief, he/she has to revise his/her belief. On the other hand, the agent will accept and might believe the received information if it is considered to be consistent with his/her belief. Let us briefly introduce an example taken from a legal case (the second legal case in Appendix C).

> "In the inquiry stage, two witnesses gave the statements to the police, and then the defendant was decided to be guilty. After that, in the Youth Court, those witnesses recanted their statements. Finally, the judge decided that the defendant was innocent."

This example shows that the judge gives firstly a decision, but after then, he/she overturned the first. We could successfully depict this process by revising belief state in multiple times. Although, a legal case can describe how a judge decides to changes his/her decision by the reasoning process, it cannot describe how a judge changes his/her belief. This process can be demonstrated by belief revision. Nowadays, belief revision is one of the areas that has been studied widely in the context of artificial intelligence (AI) such as in [1, 2, 3, 4, 5]. *Dynamic epistemic logic* (DEL) [6], which is a branch of modal logic for studying belief change, has been applied to formalize belief revision such as in [7, 8, 9, 10].

A key question of belief revision is how an agent decides which information he/she should believe. A common criterion is to consider the *reliability* of an information source. If the agent considers a source of received information to be reliable, he/she would accept and might believe the received information. On the other hand, the agent may reject the received information if he/she considers its source to be unreliable. Since a consideration of reliability has a strong influence on a judge's decision, the judge also needs a concept of the reliability of an information source. That is, when the judge receives a piece of

[1] In this study, we suppose that a judge has not any mental problems. Although a judges mental problem can affect his/her belief change and decision, it is difficult to find an example in a legal case.

information from a witness, he/she should consider if the witness is reliable or not. In addition, when the judge receives new information, he/she might change his/her reliability ordering between the witnesses. This situation is called *reliability change*.

The reliability of an information source has been addressed by [11, 12, 13, 14]. Among of them, Lorini et al. [13] introduced a modal framework for reasoning about *signed information*. In their framework, an agent can keep track of the information source by using a notion of signed information. They also considered the reliability relation over information sources. However, Their works cannot capture any dynamics of reliability. Therefore, the first motivation for this study is to overcome this limitation by addressing the following question:

(Q1) How can we handle reliability change of an agent ?

Since logic can be used to clarify the meaning and soundness of reasoning, several studies [15, 16, 17] have presented logical approaches in the legal systems. An application of DEL to law was initially presented by [18]. They introduced a dynamic operator representing an agent's commitment for formalizing belief change of a judge. In their work, the formalization provided only the process of removing accessible links for an agent to believe an announced statement but did not include the process of restoring links. In other words, their work only dealt with monotonic changes of an agent's belief but cannot cover *non-monotonic* changes of them. In order to handle non-monotonic changes of an agent's belief, we introduce a notion of *belief re-revision* which is not only a sequence of multiple belief revisions but also a restoration of former beliefs. Therefore, the second motivation for this study is addressed by the following question:

(Q2) How can we formalize belief re-revision of an agent ?

From [18], we found that there are two difficulties for applying DEL to a legal case: First, since DEL provides a Kripke model which can be used to demonstrate possibilities, it is required to construct such model. There are several ways for generating all possibilities and representing them by a Kripke model. In the previous work, they proposed only one way to construct the model for formalizing belief change of a judge in a judgment process. However, since a legal case is full of variety, their method may not be suitable for some legal cases. Thus, our question is how we can find a general way for constructing the model. Second, the previous work proposed a logical operator in terms of DEL for formalizing belief change of an agent. In addition to such operator, this study requires the other dynamic operators for formalizing both belief re-revision and reliability of an agent in order to satisfy the above questions (Q1) and (Q2). When there are several dynamic operators, these operators can be applied in different ways, that is, different sequences or combinations of such operators. Thus, our question is how such operators are to be applied for changing an agent's belief and/or reliability. For this reason, the third motivation for this study is to solve these difficulties by addressing the following questions:

(Q3) How can we construct a model from a legal case ?

(Q4) How can we decide which operators are to be applied for analyzing an agent's changing of belief and reliability ?

1.2 Former Formalisms

Non-monotonic reasoning is a viable tool for AI and can be described as a theory of reasoning. Human reasoning is *non-monotonic* because we have to draw conclusions from our knowledge in our daily life. Thus far, default logic, predicate completion, circumscription, autoepistemic logic and so on have been proposed as non-monotonic logic in [19, 20, 21]. Furthermore, argumentation theory or argumentation [22] has been used to provide non-monotonic reasoning in law. In order to deal with the dynamics of such human reasoning, two prominent formalisms based on non-monotonic reasoning including defeasible logic and belief revision are discussed in this section.

1.2.1 Defeasible logic

Defeasible logic (DL) [23] is a simple and flexible non-monotonic formalism, based on deductive reasoning. The goal of this logic is to derive plausible conclusions from the knowledge base. There are five different kinds of features: facts, strict rules, defeasible rules, defeaters and a superiority relation among rules. Strict rules cannot be defeated, while defeasible rules can be defeated by the contrary evidence. Defeaters are rules that cannot be used to draw any conclusions but to prevent some conclusions. Essentially, the superiority relation provides the priority orderings between rules where one rule may override the conclusion of another rule.

1.2.2 Belief revision

Belief revision [24] is a study of how an agent should revise his/her belief when he/she receives new information without generating inconsistencies. Belief revision composes of five basic operations, i.e., revision, contraction, expansion, consolidation and merging. Contraction is an operation of removing some beliefs, while expansion aims to adding beliefs without a consideration of inconsistency. The difference between revision and merging operations can be described as follows: For revision, the new belief is considered to be more reliable than the old ones. When there is an inconsistency, some old beliefs are removed. For merging, the priority among the beliefs is considered to be the same. However, revision can be performed by first incorporating the new belief and then restoring the consistency by a consolidation operation. This can be considered as merging rather than revision because the new belief is not always treated as more reliable than the old ones.

1.2.3 Limitations

According to the above former formalisms, they cannot adequately handle belief re-revision and reliability change of an agent because of the following limitations:

- Both of them do not provide a process of restoring the former beliefs. DL provides only a process of retracting beliefs. Although belief revision provides a process of revising belief including removing and adding beliefs, it cannot restore the former beliefs. Therefore, both of them cannot deal with belief re-revision. In addition, belief re-revision in our study is different from belief contraction into three aspects. First, belief re-revision is formulated based on Kripke semantics in modal logic, while belief contraction does not in Kripke semantics. Second, belief contraction focuses on belief sets, while belief re-revision focuses on accessible links in Kripke semantics. Third, belief contraction captures only a process of removing, while belief re-revision covers processes of removing and adding.

- Both of them do not consider reliability change of an agent. Although DL and belief revision consider the reliability by providing a concept of the priority, they do not consider it in terms of dynamics. That is, the priority ordering in DL or the reliability of beliefs in belief revision is static, i.e., it cannot be changed. Although both of them can be used to realize reliability change, DEL has a clearer formal semantics than DL and belief revision.

1.3 Research Methodology

This study aims to propose a logical formalization for analyzing belief change of an agent by demonstrating in a judgment process. From Section 1.1, our motivation of this study is addressed by the following questions:

(Q1) How can we handle reliability change of an agent ?

(Q2) How can we formalize belief re-revision of an agent ?

(Q3) How can we construct a model from a legal case ?

(Q4) How can we decide which operators are to be applied for analyzing an agent's changing of belief and reliability ?

In order to answer the above questions, this study requires three components as follows:

Logical tool for reliability change (in Chapter 3)

A goal of this part is to solve the first question (Q1). First, we formalize the reliability of information sources by applying two notions. The first one is a notion of signed statement which is used to represent a source of information, i.e., an agent. The second one is a reliability relation between agents. Then, three operators including *downgrade*, *upgrade* and *joint downgrade* are formulated for changing the grade of agents' reliability. The downgrade operator is used to downgrade some specified agents less reliable than other agents in terms of the degree of reliability, while the upgrade operator is used for upgrading. Moreover, the joint downgrade operator is introduced to cover a different kind of downgrading. A target of this operator is to make such agents in a specific group equally reliable first and then downgrade them less reliable than the agents in the other groups. In this study, we focus only on the reliability of agents but do not consider the reliability of statements. In order to cover this limitation, it is required to state the first hypothesis as follows:

(H1) When agent a receives statement φ from agent b, he/she has already decided if statement φ is reliable or not. If agent a considers statement φ to be reliable, then he/she believes that agent b who gives statement φ will be reliable.

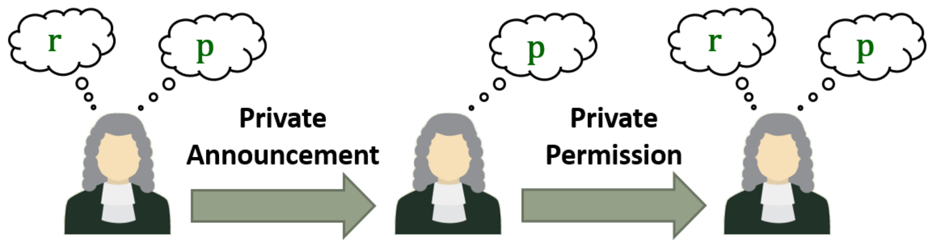

Figure 1.1: Example of belief re-revision

Logical tool for belief re-revision (in Chapter 4)

In order to solve the second question (Q2), we introduce three dynamic logical operators including *private announcement*, *careful policy* and *private permission*[2] for formalizing belief re-revision. In the conventional settings, belief revision simply abandons former belief states and we cannot revive those former states in the later stage. In this study, however, we intend to get back to the former states. Belief re-revision of an agent can be handled by both private announcement and private permission operators as shown in Fig. 1.1. That is, a judge first believes statements p and r. Then, when the judge applies the private announcement operator for removing some beliefs, he/she will believe only statement p. After that, the judge can restore the possibility of statement r by employing the private permission. As a result, the judge will believe statements p and r as in the initial stage. In addition, we require the careful policy for performing a process of information aggregation. That is, when a judge receives several signed information from witnesses, he/she needs an operation for switching from the received signed information to beliefs. In order to satisfy our goal of this study, our logical formalization is constructed by combining two logical tools for reliability change and belief re-revision. This logical formalization will be used in the next part for analyzing an agent's changing of belief and reliability.

Dynamic logical analysis of legal cases (in Chapter 5)

This part aims to propose a method for analyzing a legal case by our logical formalization as mentioned above. First, we introduce our proposed method for capturing how we can analyze a judge's changing of belief and reliability in a judgment process. There are two main steps as follows: (1) constructing a model for analyzing a judge's changing of belief and reliability from a legal case, and (2) applying our dynamic operators for formalizing belief re-revision and reliability change of agent. In the second step, we propose an approach for applying our dynamic operators in Section 5.1. With the help of this approach, we can decide which operators should be applied for analyzing a judge's changing of belief and reliability. That is, the fourth question (Q4) can be partially solved. By this method and our logical formalization, we develop an implementation in a computer system for realizing an agent's changing of belief and reliability (in Appendix

[2]Permission in this study does not refer to an approval in a legal procedure, but an admission of a possibility in beliefs.

B). Our implementation provides two main functions as follows:

- The system can generate a model for analyzing a judge's changing of belief and reliability from a legal case by our proposed method in Section 5.1. This function can solve the third question (Q3).

- The system can automatically perform an inconsistency management policy consisting of three steps. First, when the system can detect that there is an agent giving inconsistent statements, it will downgrade such agent less reliable than the other agents by the joint downgrade operator because such agent is considered to be unreliable. Second, the system will apply the private permission operators for restoring the former beliefs. Third, if there is the received information which is not inconsistent with the existing belief of an agent and is signed by the most reliable agent, the system will apply the private announcement operator for admitting such information. With this function and the application approach of dynamic operators as mentioned above, we can solve the fourth question (Q4). In addition, this function can help an agent to reduce the effort to apply dynamic operators because the system can perform some operators automatically instead of the agent.

From the above inconsistency management policy, the second hypothesis of reliability is:

(H2) When agent b gives inconsistent statements to agent a, he/she will be considered to be unreliable from agent a's perspective.

Consequently, we can analyze a judge's changing of belief and reliability in a judgment process by three components as mentioned above. In addition, we can investigate an interaction between belief change and reliability change. That is, when a judge changed his/her reliability ordering between some witnesses, he/she may change his/her beliefs about information from those witnesses. This example demonstrates an effect of reliability change on belief change. On the other hand, belief change may affect reliability change.

1.4 Thesis Structure

The rest of this thesis is organized as follows: Chapter 2 provides backgrounds and theories of four logics including modal logic in Section 2.1, propositional dynamic logic (PDL) in Section 2.2, dynamic epistemic logic (DEL) in Section 2.3 including public announcement logic (PAL) in Section 2.3.1 and action models (AM) in Section 2.3.2, and logic for signed information in Section 2.4. Next, our logical formalization for analyzing an agent's changing of belief and reliability is described in Chapters 3 and 4. Chapter 3 introduces our logical tool for reliability change including two parts. First, the static logic of agents' beliefs equipped with the notions of signed information and the reliability relation is stated in Section 3.1. Second, three dynamic operators including downgrade, upgrade and joint downgrade for capturing an agent's changing of reliability ordering between agents are presented in Section 3.2. Chapter 4 introduces our logical tool for belief re-revision of an agent including three parts. First, an extension of the static logic of agents' beliefs from Section 3.1 is presented in Section 4.1. Second, three kinds of dynamic operators including private announcement, careful policy and private permission for capturing belief re-revision of an agent are presented in Section 4.2. Third, our logical formalism which is a combination of two logical tools for reliability change and belief re-revision is proposed in Section 4.3. In Chapter 5, our proposed method for handling how to analyze a legal case by our logical formalism is introduced in Section 5.1. With this method, six target legal cases are analyzed by our logical formalization in Section 5.2. In this section, six target legal cases are first summarized in Section 5.2.1, and then their analysis results are presented in Section 5.2.2. From six target legal cases, an analysis process of the second legal case is demonstrated in Section 5.2.3. Our analysis results are discussed in Section 5.2.4. Finally, Chapter 6 concludes this study and states our further directions.

Chapter 2

Preliminaries

In this chapter, we first describe the technical details of three logics including modal logic, propositional dynamic logic (PDL) and dynamic epistemic logic (DEL). In Section 2.1, modal logic is presented as an introduction to logical concepts and conventions used throughout this study. Then, Section 2.2 states PDL which is a branch of modal logic for reasoning about programs. After that, Section 2.3 provides DEL which is used for studying changing of knowledge and belief. In this section, two kinds of logics including public announcement logic (PAL) and action models (AM) are included. In each section, we provide the Hilbert-style proof systems for each logic. Finally, we give an example of modal framework for deriving an agent's belief by a consideration of reliability relations over information sources in Section 2.4. Since our first motivation of this study focuses on the reliability of information sources (mentioned in Section 1), our logical formalism is proposed based on this framework.

2.1 Modal Logic

Modal logic is extended from classical logic with new operators called *modalities* which are used to model intensional notions such as necessity, possibility, belief, knowledge, obligation and so on. An intuitive semantics of this logic is based on possible worlds called *Kripke semantics*. This section presents a normal modal logic including the basic language and its semantics. In addition, we provide the proof system for this logic.

2.1.1 Syntax and Kripke Semantics

Definition 1. *A language $\mathcal{L}_{ML}$ consists of the following vocabulary: (i) a countably infinite set* $\mathsf{Prop} = \{ p, q, r, \dots \}$ *of propositional letters, (ii) Boolean connectives: $\neg$, $\rightarrow$, and (iii) a finite set* $\mathsf{Mod} = \{ \Box, \Box', \Box'', \dots \}$ *of modal operators. A set* Form_{ML} *of formulas of $\mathcal{L}_{ML}$ is inductively defined as follows:*

$$\mathsf{Form}_{ML} \ni A ::= p \,|\, \neg A \,|\, (A \rightarrow A) \,|\, \Box A$$

where $p \in \mathsf{Prop}$ *and* $\Box \in \mathsf{Mod}$.

Note that the outer parentheses of a formula can normally be omitted. We say that $\Diamond A$ which is the duality of $\Box$ is defined as $\neg\Box\neg A$. The abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ are given in the following definition.

Definition 2. *Let A and B be any formulas in $\mathcal{L}_{ML}$. The ordinary abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ can be defined as follows:*

$$\begin{array}{lcllcl} A \vee B & := & \neg A \rightarrow B & \qquad A \wedge B & := & \neg(A \rightarrow \neg B) \\ \top & := & A \rightarrow A & \qquad \bot & := & \neg\top \\ A \leftrightarrow B & := & (A \rightarrow B) \wedge (B \rightarrow A) & & & \end{array}$$

We read $\Box A$ as "it is necessary that A" and $\Diamond A$ as "it is possible that A." In order to define the semantics of this language, we first specify a *frame* which is a relational structure by the following definition.

Definition 3. *A Kripke frame $\mathfrak{F}$ is a pair $(W, (R_\Box)_{\Box \in \mathsf{Mod}})$ where:*

- *W is a non-empty set of worlds or states, called the domain.*
- *$R_\Box \subseteq W \times W$ is an accessibility relation.*

If $\mathfrak{F} = (W, (R_\Box)_{\Box \in \mathsf{Mod}})$ is a frame, then we say "w is in $\mathfrak{F}$" to mean $w \in W$.

With the above definition, we can construct a Kripke model based on a frame as follows:

Definition 4. *A Kripke model $\mathfrak{M}$ is a tuple $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V is a valuation function. Therefore, the Kripke model $\mathfrak{M}$ can be defined as $\mathfrak{M} = (W, (R_\Box)_{\Box \in \mathsf{Mod}}, V)$ where:*

- *W is a non-empty set of worlds or states.*
- *$R_\Box \subseteq W \times W$ is an accessibility relation. We read $wR_\Box v$ as "world v is accessible from world w." We write $wR_\Box v$ for $(w, v) \in R_\Box$.*
- *$V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function that specifies a truth value of propositional letters at worlds in W. We say that $V(p)$ is a set of worlds where p is true.*

Definition 5. *Given any model $\mathfrak{M}$, any state $w \in W$ and any formula A, we define the* satisfaction relation *$\mathfrak{M}, w \models A$ inductively as follows:*

$$\begin{array}{lll} \mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\ \mathfrak{M}, w \models \neg A & \text{iff} & \mathfrak{M}, w \not\models A \\ \mathfrak{M}, w \models A \rightarrow B & \text{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}, w \models B \\ \mathfrak{M}, w \models \Box A & \text{iff} & \textit{for all } v \textit{ such that } wR_\Box v \textit{ implies } \mathfrak{M}, v \models A \end{array}$$

We can define the validity of a formula by the following definition.

Definition 6. *Let A be any formulas in $\mathcal{L}_{ML}$. The notions of validity are defined as follows:*

- *$\mathfrak{M} \models A$ means that A is valid on a model $\mathfrak{M}$ if $\mathfrak{M}, w \models A$ for all worlds $w \in W$.*
- *$\mathfrak{F} \models A$ means that A is valid on a frame $\mathfrak{F}$ if $(\mathfrak{F}, V) \models A$ for all valuations V on $\mathfrak{F}$.*
- *$\mathbb{M} \models A$ means that A is valid on a class $\mathbb{M}$ of models if $\mathfrak{M} \models A$ for all models $\mathfrak{M} \in \mathbb{M}$.*
- *$\mathbb{F} \models A$ means that A is valid on a class $\mathbb{F}$ of frames if $\mathfrak{F} \models A$ for all frames $\mathfrak{F} \in \mathbb{F}$.*

These validities correspond to various frame properties which are related to modal formulas as in Table 2.1. A frame $\mathfrak{F} = (W, (R_\Box)_{\Box \in \mathsf{Mod}})$ is called reflexive if its accessibility relation $R_\Box$ is reflexive, and similarly for the other properties. For example, a frame $\mathfrak{F} \models \Box p \rightarrow p$ if and only if $\mathfrak{F}$ is reflexive.

Modal formula	Property of relation	Frame condition
$\mathrm{T}_\Box := \Box p \to p$	Reflexive	$wR_\Box w$ for all $w \in W$
$\mathrm{B}_\Box := p \to \Box\Diamond p$	Symmetric	$wR_\Box v$ implies $vR_\Box w$ for all $w, v \in W$
$\mathrm{D}_\Box := \Box p \to \Diamond p$	Serial	$wR_\Box v$ for all $w \in W$ for some $v \in W$
$4_\Box := \Box p \to \Box\Box p$	Transitive	$wR_\Box v$ and $vR_\Box u$ imply $wR_\Box u$ for all $w, v, u \in W$
$5_\Box := \Diamond p \to \Box\Diamond p$	Euclidean	$wR_\Box v$ and $wR_\Box u$ imply $vR_\Box u$ for all $w, v, u \in W$

Table 2.1: Five correspondences between modal formulas and frame properties

All instances of propositional tautologies		
$(K_\Box)$	$\Box(A \to B) \to (\Box A \to \Box B)$	$(\Box \in \mathsf{Mod})$
(MP)	From A and $A \to B$, infer B	
$(Nec_\Box)$	From A, infer $\Box A$	$(\Box \in \mathsf{Mod})$

Table 2.2: Hilbert-style system **HK** for $\mathcal{L}_{ML}$

2.1.2 Hilbert-style Axiomatization HK

The Hilbert-style system **HK** for $\mathcal{L}_{ML}$ is presented in Table 2.2. Note that K, MP and Nec refer to a distribution axiom, modus ponens which is a rule of inference, and a necessitation rule, respectively. In addition, we can construct the extensions of **HK** by adding modal formulas from Table 2.1 as additional axioms as follows:

$$\begin{array}{lll} \mathrm{HT} & := & \{(K_\Box), (T_\Box) \mid \Box \in \mathsf{Mod}\} \\ \mathrm{HS4} & := & \{(K_\Box), (T_\Box), (4_\Box) \mid \Box \in \mathsf{Mod}\} \\ \mathrm{HS5} & := & \{(K_\Box), (T_\Box), (5_\Box) \mid \Box \in \mathsf{Mod}\} \\ \mathrm{HB} & := & \{(K_\Box), (B_\Box) \mid \Box \in \mathsf{Mod}\} \\ \mathrm{HD} & := & \{(K_\Box), (D_\Box) \mid \Box \in \mathsf{Mod}\} \end{array}$$

From the above extensions of **HK**, we can define the Hilbert-style system **HKΣ** is an axiomatic extension of **HK**. Then, we will show the proof of soundness and completeness for **HKΣ**. Before describing the details of this proof, let us define a notion of derivability ($\vdash$) by the following definition.

Definition 7. *Let $\Gamma \cup \{ A \} \subseteq \mathsf{Form}_{ML}$. A* derivation *in* **HKΣ** *is a finite sequence of formulas such that each of them is either an axiom of* **HKΣ** *or a result derived by a rule of* **HKΣ** *to preceding formulas. A formula A is* derivable *in* **HKΣ** *(denoted $\vdash_{\mathbf{HK\Sigma}} A$) if A is the last formula of a derivation in* **HKΣ**. *We can write $\vdash_{\mathbf{HK\Sigma}} A$ to mean that A is a theorem of* **HKΣ**. *Then, A is* derivable *from Γ in* **HKΣ** *(denoted $\Gamma \vdash_{\mathbf{HK\Sigma}} A$) if there is some finite $\Gamma' \subseteq \Gamma$ such that $\vdash_{\mathbf{HK\Sigma}} (\bigwedge \Gamma') \to A$.*

Then, the soundness theorem is given as follows:

Theorem 1 (Soundness)**.** *Let $\mathbb{F}_\Sigma$ be the class of all frames in Table 2.1 and $\mathbb{M}_\Sigma$ be the class of models such that $\mathbb{M}_\Sigma := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_\Sigma$ and V is a valuation V on $\mathfrak{F}\}$. For all $A \in \mathsf{Form}_{ML}$,*

$$\text{if } \vdash_{\mathbf{HK\Sigma}} A, \text{ then } \mathbb{M}_\Sigma \models A.$$

From the above theorem, we will prove that if a formula A is a theorem of $\mathbf{HK\Sigma}$, then A is valid on all models in a class $\mathbb{M}_\Sigma$ according to the semantics. This proof is straightforward. Therefore, we will move to the completeness proof by considering the reverse direction. That is, the completeness proof aims to prove that if a formula A is valid on all models in a class $\mathbb{M}_\Sigma$, then A is a theorem of $\mathbf{HK\Sigma}$. We will show this proof by contraposition. That is, if a formula A is not a theorem of $\mathbf{HK\Sigma}$, then A is not valid on some model in $\mathbb{M}_\Sigma$. Thus, it suffices to find a counter model $\mathfrak{M}$ such that a formula A is false at some state of $\mathfrak{M}$, i.e., $\mathfrak{M}, w \models \neg A$. This model can be called *canonical model*. In order to construct this canonical model, we need to define the maximally consistent set which is used as the set of states in the canonical model as follows:

Definition 8. *Let* $\Gamma \subseteq \mathsf{Form}_{ML}$. *We say that* Γ *is* $\mathbf{HK\Sigma}$-inconsistent *if* $\Gamma \vdash_{\mathbf{HK\Sigma}} \bot$. Γ *is a* maximally $\mathbf{HK\Sigma}$-consistent set (*or* $\mathbf{HK\Sigma}$-*MCS, in short*) iff

(i) Γ *is* $\mathbf{HK\Sigma}$-consistent *if* Γ *is not* $\mathbf{HK\Sigma}$-*inconsistent, i.e.,* $\Gamma \nvdash_{\mathbf{HK\Sigma}} \bot$.

(ii) Γ *is* maximal *if* $A \in \Gamma$ *or* $\neg A \in \Gamma$ *for all formulas* $A \in \mathsf{Form}_{ML}$.

Lemma 1 (Lindenbaum)**.** *Given any* $\mathbf{HK\Sigma}$-*consistent set* Γ, *there exists an* $\mathbf{HK\Sigma}$-*MCS* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$.

Then, we can define the properties of the maximally consistent as follows:

Proposition 9. *If* Γ *be an* $\mathbf{HK\Sigma}$-*MCS, then the following hold.*

(i) $\Gamma \vdash_{\mathbf{HK\Sigma}} A$ iff $A \in \Gamma$.

(ii) if $A \in \Gamma$ *and* $\vdash_{\mathbf{HK\Sigma}} A \to B$, *then* $B \in \Gamma$.

(iii) $\neg A \in \Gamma$ iff $A \notin \Gamma$.

(iv) $A \to B \in \Gamma$ iff $A \in \Gamma$ *implies* $B \in \Gamma$.

(v) if $\Box A \notin \Gamma$, *then* $\{\neg A\} \cup \{B \mid \Box B \in \Gamma\} \nvdash_{\mathbf{HK\Sigma}} \bot$.

With the help of all the above structures, we can define the canonical model as follows:

Definition 10. *For any* $\mathbf{HK\Sigma}$, *the* canonical model *for* $\mathbf{HK\Sigma}$: $\mathfrak{M}^{\mathbf{HK\Sigma}} = (W^{\mathbf{HK\Sigma}}, (R_\Box^{\mathbf{HK\Sigma}})_{\Box \in \mathsf{Mod}}, V^{\mathbf{HK\Sigma}})$ *is defined as follows:*

- $W^{\mathbf{HK\Sigma}} := \{\Gamma \mid \Gamma$ *is an* $\mathbf{HK\Sigma}$-*MCS*$\}$, *i.e.,* $W^{\mathbf{HK\Sigma}}$ *is a set of all* $\mathbf{HK\Sigma}$-*MCSs.*
- $\Gamma R_\Box^{\mathbf{HK\Sigma}} \Delta$ iff $\Box A \in \Gamma$ *implies* $A \in \Delta$ *for all formulas* A.
- $\Gamma \in V^{\mathbf{HK\Sigma}}(p)$ iff $p \in \Gamma$.

By the properties of the maximally consistent set in Proposition 9, we can prove the following Truth Lemma.

Lemma 2 (Truth)**.** *Let* Γ *be any* $\mathbf{HK\Sigma}$-*MCS. For all* $A \in \mathsf{Form}_{ML}$,

$$\mathfrak{M}^{\mathbf{HK\Sigma}}, \Gamma \models A \quad \text{iff} \quad A \in \Gamma.$$

By the above canonical model, we can define the following lemma.

Lemma 3. *Given the canonical model* $\mathfrak{M}^{\mathbf{HK\Sigma}} = (W^{\mathbf{HK\Sigma}}, (R_{\Box}^{\mathbf{HK\Sigma}})_{\Box\in\mathsf{Mod}}, V^{\mathbf{HK\Sigma}})$,

$$\Gamma R_{\Box}^{\mathbf{HK\Sigma}} \Delta \ \text{ iff } \ A \in \Delta \ \textit{ implies } \ \Diamond A \in \Gamma \textit{ for all formulas } \ A.$$

In order to show that $\mathbf{HK\Sigma}$ is complete with respect to $\mathbb{M}_{\Sigma}$, we need to show that the canonical model $\mathfrak{M}^{\mathbf{HK\Sigma}}$ satisfies the following constraints.

Lemma 4. *Given the canonical model* $\mathfrak{M}^{\mathbf{HK\Sigma}} = (W^{\mathbf{HK\Sigma}}, (R_{\Box}^{\mathbf{HK\Sigma}})_{\Box\in\mathsf{Mod}}, V^{\mathbf{HK\Sigma}})$,

(i) If $\vdash_{\mathbf{HK\Sigma}} \Box A \to A$ *for all formulas* A*, then* $R_{\Box}^{\mathbf{HK\Sigma}}$ *is reflexive.*

(ii) If $\vdash_{\mathbf{HK\Sigma}} A \to \Box\Diamond A$ *for all formulas* A*, then* $R_{\Box}^{\mathbf{HK\Sigma}}$ *is symmetric.*

(iii) If $\vdash_{\mathbf{HK\Sigma}} \Box A \to \Diamond A$ *for all formulas* A*, then* $R_{\Box}^{\mathbf{HK\Sigma}}$ *is serial.*

(iv) If $\vdash_{\mathbf{HK\Sigma}} \Box A \to \Box\Box A$ *for all formulas* A*, then* $R_{\Box}^{\mathbf{HK\Sigma}}$ *is transitive.*

(v) If $\vdash_{\mathbf{HK\Sigma}} \Diamond A \to \Box\Diamond A$ *for all formulas* A*, then* $R_{\Box}^{\mathbf{HK\Sigma}}$ *is Euclidean.*

Now, we can provide the completeness proof for $\mathbf{HK\Sigma}$ as follows:

Theorem 2 (Completeness)**.** *Let* $\mathbb{F}_{\Sigma}$ *be the class of all frames in Table 2.1 and* $\mathbb{M}_{\Sigma}$ *be the class of models such that* $\mathbb{M}_{\Sigma} := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_{\Sigma}$ *and* V *is a valuation* V *on* $\mathfrak{F}\}$*. For all* $A \in \mathsf{Form}_{ML}$*,*

$$\textit{if } \mathbb{M}_{\Sigma} \models A, \ \textit{then } \vdash_{\mathbf{HK\Sigma}} A.$$

Proof. By contraposition. Suppose $\nvdash_{\mathbf{HK\Sigma}} A$. Our goal is to show $\mathbb{M}_{\Sigma} \not\models A$. It suffices to find a counter model $\mathfrak{M}$ such that $\mathfrak{M}, w \not\models A$ for some w of $\mathfrak{M}$. By our supposition, we obtain that $\{\neg A\}$ is an $\mathbf{HK\Sigma}$-consistent set, i.e., $\{\neg A\} \nvdash_{\mathbf{HK\Sigma}} \bot$. By Lemma 1, there exists an $\mathbf{HK\Sigma}$-MCS Γ such that $\{\neg A\} \subseteq \Gamma$, i.e., $\neg A \in \Gamma$. By Lemma 2, we obtain that $\mathfrak{M}^{\mathbf{HK\Sigma}}, \Gamma \models \neg A$, i.e., $\mathfrak{M}^{\mathbf{HK\Sigma}}, \Gamma \not\models A$. Finally, we need to show that the canonical model $\mathfrak{M}^{\mathbf{HK\Sigma}}$ belongs to $\mathbb{M}_{\Sigma}$. That is, it suffices to show that $\mathfrak{M}^{\mathbf{HK\Sigma}}$ satisfies the properties of all frames in $\mathbb{M}_{\Sigma}$. This is shown by Lemma 4. □

2.2 Propositional Dynamic Logic

Propositional dynamic logic (PDL) [25] describes the interaction between programs and propositions. In this section, we will present two kinds of PDL including basic PDL and PDL without iteration. The later version of PDL will be used for building the dynamic logic of relation changers in Section 4.1.

2.2.1 Syntax and Kripke Semantics

This section provides two languages $\mathcal{L}_{PDL}$ for PDL and $\mathcal{L}_{PDL^{-}}$ for PDL without iteration. Let us first define $\mathcal{L}_{PDL}$ by the following definition.

Definition 11. *A language* $\mathcal{L}_{PDL}$ *consists of the following vocabulary: (i) a countably infinite set* $\mathsf{Prop} = \{p, q, r, \dots\}$ *of propositional letters,(ii) a countably infinite set* $\mathsf{AP} = \{a, b, c, \dots\}$ *of atomic programs, (iii) Boolean connectives:* $\neg$, $\to$*, (iv) program operators:* $\cup$ *(non-deterministic choice),* ; *(sequential composition),* $*$ *(iteration), and (v) mixed*

$\pi;\pi'$	:	first execute π, then execute π'.
$\pi \cup \pi'$	:	choose either π or π' nondeterministically and execute it.
$A?$	:	test A; proceed if true, fail if false.
π^*	:	execute π a nondeterministically chosen finite number of times (zero or more).
$[\pi]A$	:	it is necessary that after executing π, A is true.

Table 2.3: Examples of logical formalization of $\mathcal{L}_{PDL}$

operators: ? (test), $[\cdot]$ (necessity). A set Form_{PDL} *of formulas A of $\mathcal{L}_{PDL}$ and a set* Prog *of programs π of $\mathcal{L}_{PDL}$ are inductively defined as follows:*

$$\mathsf{Form}_{PDL} \ni A ::= p \mid \neg A \mid (A \to A) \mid [\pi]A$$

$$\mathsf{Prog} \ni \pi ::= a \mid \pi \cup \pi \mid \pi;\pi \mid A? \mid \pi^*$$

where $p \in \mathsf{Prop}$ and $a \in \mathsf{AP}$.

The intuitive readings of formulas are shown in Table 2.3. The abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ can be defined in the same way as in Definition 2. In addition, the dual operator $\langle\pi\rangle$ of $[\pi]$ is defined by $\langle\pi\rangle A := \neg[\pi]\neg A$. Before giving the semantics, let us define the language $\mathcal{L}_{PDL^-}$ for PDL without iteration. Based on the above definition, $\mathcal{L}_{PDL^-}$ is $\mathcal{L}_{PDL}$ which does not contain the iteration operator $*$, and Form_{PDL^-} is Form_{PDL} which does not contain $[\pi^*]A$. Next, the semantics of PDL can be defined by the following definitions.

Definition 12. *A Kripke* model $\mathfrak{M}$ *is a tuple* $\mathfrak{M} = (W, (R_\pi)_{\pi \in \mathsf{Prog}}, V)$*, where*

- *W is a non-empty set of states, called the* domain*,*
- *$R_\pi \subseteq W \times W$ is an accessibility relation for program π,*
- *$V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.*

Definition 13. *Given any model $\mathfrak{M}$, any state $w \in W$ and any formula A, we define the* satisfaction relation $\mathfrak{M}, w \models A$ *inductively as follows:*

$$\begin{array}{lll} \mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\ \mathfrak{M}, w \models \neg A & \text{iff} & \mathfrak{M}, w \not\models A \\ \mathfrak{M}, w \models A \to B & \text{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}, w \models B \\ \mathfrak{M}, w \models [\pi]A & \text{iff} & \mathfrak{M}, v \models A \textit{ for all } v \textit{ such that } wR_\pi v, \end{array}$$

where R_π can be defined as follows:

$$\begin{array}{lll} R_{\pi\cup\pi'} & = & R_\pi \cup R_{\pi'} \\ & = & \{ (w,v) \mid wR_\pi v \textit{ or } wR_{\pi'} v \} \\ R_{\pi;\pi'} & = & R_\pi \circ R_{\pi'} \\ & = & \{ (w,v) \mid wR_\pi u \textit{ and } uR_{\pi'} v \textit{ for some } u \in W \} \\ R_{A?} & = & \{ (w,v) \mid w = v \textit{ and } \mathfrak{M}, v \models A \} \\ R_{\pi^*} & = & (R_\pi)^* \\ & = & \{ (w,v) \mid wR_{\pi^n} v \textit{ for some } 0 \leqslant n \}, \end{array}$$

where π^n can be defined as $\pi^0 = \top?$ and $\pi^{n+1} = \pi;\pi^n$.

We can define the notions of validity in the same way as in Definition 6.

All instances of propositional tautologies	
$(K_{[\pi]})$	$[\pi](A \to B) \to ([\pi]A \to [\pi]B)$
$(RA1)$	$[\pi \cup \pi']A \leftrightarrow [\pi]A \wedge [\pi']A$
$(RA2)$	$[\pi; \pi']A \leftrightarrow [\pi][\pi']A$
$(RA3)$	$[B?]A \leftrightarrow (B \to A)$
$(RA4)$	$A \wedge [\pi][\pi^*]A \leftrightarrow [\pi^*]A$
(Ind)	$A \wedge [\pi^*](A \to [\pi]A) \to [\pi^*]A$
(MP)	From A and $A \to B$, infer B
$(Nec_{[\pi]})$	From A, infer $[\pi]A$

Table 2.4: Hilbert-style system **HPDL** for $\mathcal{L}_{PDL}$

2.2.2 Hilbert-style Axiomatization HPDL

Table 2.4 presents the Hilbert-style system **HPDL** of $\mathcal{L}_{PDL}$. Note that RA refers to recursion axioms. We can construct the Hilbert-style system $\mathbf{HPDL}^-$ of $\mathcal{L}_{PDL^-}$ by removing axioms $(RA4)$ and (Ind) from **HPDL**. Since $\mathcal{L}_{PDL^-}$ will be used in this study (in Section 4.1), we will focus on the proof of soundness and completeness for $\mathbf{HPDL}^-$ that is similar to the proof for $\mathbf{HK\Sigma}$ in Section 2.1.2. Therefore, we first define a derivation in $\mathbf{HPDL}^-$ in the same manner as in Definitions 7. As a result, we obtain that A is a theorem of $\mathbf{HPDL}^-$, denoted $\vdash_{\mathbf{HPDL}^-} A$. Next, we will show the soundness proof as follows:

Theorem 3 (Soundness)**.** *Let $\mathbb{M}$ be the class of all models. For all $A \in \mathsf{Form}_{PDL^-}$,*

$$\text{if } \vdash_{\mathbf{HPDL}^-} A, \text{ then } \mathbb{M} \models A.$$

By the above theorem, $\mathbf{HPDL}^-$ is sound if all axioms in $\mathbf{HPDL}^-$ are valid and all the inference rules in $\mathbf{HPDL}^-$ preserve validity. This is straightforward. For the completeness proof for $\mathbf{HPDL}^-$, we use the same manner in Section 2.1.2 as the following steps. First, we will define a maximally $\mathbf{HPDL}^-$-consistent set $\mathbf{HPDL}^-$-MCS by Definitions 8. Then, we will give the following lemma.

Lemma 5 (Lindenbaum)**.** *Given any $\mathbf{HPDL}^-$-consistent set Γ, there exists an $\mathbf{HPDL}^-$-MCS Γ^+ such that $\Gamma \subseteq \Gamma^+$.*

Next, the properties of $\mathbf{HPDL}^-$-MCS are defined by the following proposition.

Proposition 14. *Let Γ be an $\mathbf{HPDL}^-$-MCS. Then, the following hold.*

(i) $\Gamma \vdash_{\mathbf{HPDL}^-} A$ iff $A \in \Gamma$.

(ii) *if* $A \in \Gamma$ *and* $\vdash_{\mathbf{HPDL}^-} A \to B$*, then* $B \in \Gamma$.

(iii) $\neg A \in \Gamma$ iff $A \notin \Gamma$.

(iv) $A \to B \in \Gamma$ iff $A \in \Gamma$ *implies* $B \in \Gamma$.

(v) *if* $[\pi]A \notin \Gamma$*, then* $\{ \neg A \} \cup \{ B \mid [\pi]B \in \Gamma \} \nvdash_{\mathbf{HPDL}^-} \bot$.

(vi) $[\pi \cup \pi']A \in \Gamma$ iff $[\pi]A \in \Gamma$ *and* $[\pi']A \in \Gamma$.

(vii) $[\pi;\pi']A \in \Gamma$ iff $[\pi][\pi']A \in \Gamma$.

(viii) $[B?]A \in \Gamma$ iff $(B \to A) \in \Gamma$.

By the above preparation, the canonical model for $\mathbf{HPDL^-}$ is constructed by the following definition.

Definition 15. *The* canonical model $\mathfrak{M}^{\mathbf{HPDL^-}} = (W^{\mathbf{HPDL^-}}, (R_\pi^{\mathbf{HPDL^-}})_{\pi\in\mathsf{Prog}}, V^{\mathbf{HPDL^-}})$ *for any* $\mathbf{HPDL^-}$ *is defined by:*

- $W^{\mathbf{HPDL^-}} := \{\,\Gamma \mid \Gamma$ *is an* $\mathbf{HPDL^-}$*-MCS*$\}$.
- $\Gamma R_\pi^{\mathbf{HPDL^-}} \Delta$ iff $[\pi]A \in \Gamma$ *implies* $A \in \Delta$ *for all formulas* A.
- $\Gamma \in V^{\mathbf{HPDL^-}}(p)$ iff $p \in \Gamma$.

Lemma 6 (Truth)**.** *Let* Γ *be any* $\mathbf{HPDL^-}$*-MCS. The following is true for all* $A \in \mathsf{Form}_{PDL^-}$ *and all* $\pi \in \mathsf{Prog}$*:*

(i) $\mathfrak{M}^{\mathbf{HPDL^-}}, \Gamma \models A$ iff $A \in \Gamma$.

(ii) $R_\pi^{\mathbf{HPDL^-}} = Q_\pi^{\mathbf{HPDL^-}}$*, where* $\Gamma Q_\pi^{\mathbf{HPDL^-}} \Delta$ iff $[\pi]A \in \Gamma$ *implies* $A \in \Delta$ *for all* A.

Now, we are ready to provide the completeness proof for $\mathbf{HPDL^-}$ as follows:

Theorem 4 (Completeness)**.** *Let* $\mathbb{M}$ *be the class of all models. For all* $A \in \mathsf{Form}_{PDL^-}$*,*

$$\text{if } \mathbb{M} \models A, \text{ then } \vdash_{\mathbf{HPDL^-}} A.$$

Proof. We demonstrate the proof by contrapositive implication. Suppose that $\nvdash_{\mathbf{HPDL^-}} A$. Our goal is to show $\mathbb{M} \not\models A$. It suffices to find a counter model $\mathfrak{M}$ such that $\mathfrak{M}, w \not\models A$ for some w of $\mathfrak{M}$. By our supposition, we obtain that $\{\neg A\}$ is an $\mathbf{HPDL^-}$-consistent set, i.e., $\{\neg A\} \nvdash_{\mathbf{HPDL^-}} \bot$. By Lemma 5, there exists an $\mathbf{HPDL^-}$-MCS Γ such that $\{\neg A\} \subseteq \Gamma$, i.e., $\neg A \in \Gamma$. By Lemma 6, we obtain that $\mathfrak{M}^{\mathbf{HPDL^-}}, \Gamma \models \neg A$, i.e., $\mathfrak{M}^{\mathbf{HPDL^-}}, \Gamma \not\models A$, as desired. □

2.3 Dynamic Epistemic Logic

Dynamic epistemic logic (DEL) is used to cover an extension of epistemic logic by adding dynamic modalities for expressing changing of knowledge and belief. In order to formalize such changes, this study focuses on two kinds of models including epistemic models by using public announcement logic (PAL) and action models (AM).

2.3.1 Public Announcement Logic

Public announcement logic (PAL) which is the first and the simplest version of DEL and is employed to numerous applications. PAL [26] is used to describe the dynamics of agents' informational states when true information is publicly announced.

Syntax and Kripke Semantics

Definition 16. *Let G be a* finite *set of agents. A language $\mathcal{L}_{PAL}$ consists of the following vocabulary: (i) a countably infinite set* $\mathsf{Prop} = \{ p, q, r, \dots \}$ *of propositional letters, (ii) Boolean connectives: $\neg$, $\to$, (iii) knowledge operators $[\mathsf{K}_a]$ ($a \in G$), and (iv) public announcement operators $[!A]$. A set* Form_{PAL} *of formulas of $\mathcal{L}_{PAL}$ is inductively defined as follows:*

$$\mathsf{Form}_{PAL} \ni A ::= p \mid \neg A \mid (A \to A) \mid [\mathsf{K}_a]A \mid [!A]A$$

where $p \in \mathsf{Prop}$ and $a \in G$. The outer parentheses of a formula will normally be omitted.

The intuitive reading of $[\mathsf{K}_a]A$ is "agent a knows that A" and $[!A]B$ is read as "after the truthful announcement that A, B holds." The abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ can be defined as in Definition 2. In addition, the dual operator $\langle \mathsf{K}_a \rangle$ of $[\mathsf{K}_a]$ is defined by $\langle \mathsf{K}_a \rangle A := \neg [\mathsf{K}_a] \neg A$. Next, the semantics for this language can be defined by the following definition.

Definition 17. *A Kripke* model $\mathfrak{M}$ *is a tuple $\mathfrak{M} = (W, (R_a)_{a \in G}, V)$, where*

- *W is a non-empty set of states, called the* domain*,*
- *$R_a \subseteq W \times W$ is an accessibility relation for agent a,*
- *$V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation.*

Definition 18. *Given any model $\mathfrak{M}$, any state $w \in W$ and any formula A, we define the* satisfaction relation *$\mathfrak{M}, w \models A$ inductively as follows:*

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \neg A & \text{iff} & \mathfrak{M}, w \not\models A \\
\mathfrak{M}, w \models A \to B & \text{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}, w \models B \\
\mathfrak{M}, w \models [\mathsf{K}_a]A & \text{iff} & \textit{for all } v \textit{ such that } wR_av \textit{ implies } \mathfrak{M}, v \models A \\
\mathfrak{M}, w \models [!A]B & \text{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}^{!A}, w \models B,
\end{array}$$

where $\mathfrak{M}^{!A} = (W', (R'_a)_{a \in G}, V')$ is defined by:

$$\begin{array}{lll}
W' & := & \{ w \in W \mid \mathfrak{M}, w \models A \} \\
R'_a & := & R_a \cap (W' \times W') \\
V'(p) & := & V(p) \cap W'
\end{array}$$

According to the above semantics, the updated model $\mathfrak{M}^{!A}$ can be interpreted as a model which deletes the states where A is false. The notions of validity can be defined similarly as in Definition 6.

Hilbert-style Axiomatization HPAL

The Hilbert-style system **HPAL** for $\mathcal{L}_{PAL}$ is presented in Table 2.5. Note that $(T_{[\mathsf{K}_a]})$, $(4_{[\mathsf{K}_a]})$ and $(5_{[\mathsf{K}_a]})$ are called truth, positive introspective and negative introspective axioms, respectively. In order to prove soundness and completeness for **HPAL**, we first define a derivation in **HPAL** in the same manner as in Definition 7 in Section 2.1.2. Then, the completeness proof for **HPAL** can be captured by a translation method as in [27].

All instances of propositional tautologies	
$(K_{[\mathsf{K}_a]})$	$[\mathsf{K}_a](A \to B) \to ([\mathsf{K}_a]A \to [\mathsf{K}_a]B)$
$(T_{[\mathsf{K}_a]})$	$[\mathsf{K}_a]A \to A$
$(4_{[\mathsf{K}_a]})$	$[\mathsf{K}_a]A \to [\mathsf{K}_a][\mathsf{K}_a]A$
$(5_{[\mathsf{K}_a]})$	$\neg[\mathsf{K}_a]A \to [\mathsf{K}_a]\neg[\mathsf{K}_a]A$
$(RA1)$	$[!A]p \leftrightarrow (A \to p)$
$(RA2)$	$[!A]\neg B \leftrightarrow (A \to \neg[!A]B)$
$(RA3)$	$[!A]B \to C \leftrightarrow ([!A]B \to [!A]C)$
$(RA4)$	$[!A][\mathsf{K}_a]B \leftrightarrow (A \to [\mathsf{K}_a][!A]B)$
$(RA5)$	$[!A][!B]C \leftrightarrow [A \wedge [!A]B]C$
(MP)	From A and $A \to B$, infer B
$(Nec_{[\mathsf{K}_a]})$	From A, infer $[\mathsf{K}_a]A$

Table 2.5: Hilbert-style system **HPAL** for $\mathcal{L}_{PAL}$

Definition 19. *The translation* $t : \mathsf{Form}_{PAL} \to \mathsf{Form}_{ML}$ *is defined as follows:*

$$
\begin{array}{lcl}
t(p) & = & p \\
t(\neg A) & = & \neg t(A) \\
t(A \to B) & = & t(A) \to t(B) \\
t([\mathsf{K}_a]A) & = & [\mathsf{K}_a]t(A) \\
t([!A]p) & = & t(A \to p) \\
t([!A]\neg B) & = & t(A \to \neg[!A]B) \\
t([!A](B \to C)) & = & t([!A]B \to [!A]C) \\
t([!A][\mathsf{K}_a]B) & = & t(A \to [\mathsf{K}_a][!A]B) \\
t([!A][!B]C) & = & t([!(A \wedge [!A]B)]C)
\end{array}
$$

Lemma 7. *For all formulas* $A \in \mathsf{Form}_{PAL}$,

$$\vdash_{\mathbf{HPAL}} A \leftrightarrow t(A).$$

Theorem 5 (Soundness)**.** *Let* $\mathbb{F}_{PAL}$ *be the class of frames such that* $\mathbb{F}_{PAL} := \{\mathrm{T}_{[\mathsf{K}_a]}, 4_{[\mathsf{K}_a]}, 5_{[\mathsf{K}_a]} \mid a \in \mathsf{G}$ *and* $[\mathsf{K}_a] \in \mathsf{Mod}\}$ *and* $\mathbb{M}_{PAL}$ *be the class of models such that* $\mathbb{M}_{PAL} := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_{PAL}$ *and* V *is a valuation* V *on* $\mathfrak{F}\}$*. For all* $A \in \mathsf{Form}_{PAL}$,

$$\textit{if } \vdash_{\mathbf{HPAL}} A, \textit{ then } \mathbb{M}_{PAL} \models A.$$

Proof. Suppose that $\vdash_{\mathbf{HPAL}}$. Our goal is to show that $\mathbb{M}_{PAL} \models A$ for all A. It suffices to show that all axioms and all rules in **HPAL** are valid on all models in a class $\mathbb{M}_{PAL}$ with respect to the semantics of $\mathcal{L}_{PAL}$. This is straightforward. □

Theorem 6 (Completeness)**.** *Let* $\mathbb{F}_{PAL}$ *be the class of frames such that* $\mathbb{F}_{PAL} := \{\mathrm{T}_{[\mathsf{K}_a]}, 4_{[\mathsf{K}_a]}, 5_{[\mathsf{K}_a]} \mid a \in \mathsf{G}$ *and* $[\mathsf{K}_a] \in \mathsf{Mod}\}$ *and* $\mathbb{M}_{PAL}$ *be the class of models such that* $\mathbb{M}_{PAL} := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_{PAL}$ *and* V *is a valuation* V *on* $\mathfrak{F}\}$*. For all* $A \in \mathsf{Form}_{PAL}$,

$$\textit{if } \mathbb{M}_{PAL} \models A, \textit{ then } \vdash_{\mathbf{HPAL}} A.$$

Proof. Suppose that $\mathbb{M}_{PAL} \models A$. Our goal is to show $\vdash_{\mathbf{HPAL}} A$ for all formulas A. By the soundness theorem (Theorem 5) and Lemma 7, we obtain that $\mathbb{M}_{PAL} \models A \leftrightarrow t(A)$. By this and our supposition, we get $\mathbb{M}_{PAL} \models t(A)$. Since the formula $t(A)$ does not contain any announcement operators, we can reduce the completeness of **HPAL** to that of **HS5** (Theorem 2 in Section 2.1.2). Therefore, we obtain $\vdash_{\mathbf{HS5}} t(A)$ by the completeness of **HS5** (Theorem 2). Since **HS5** is a sub system of **HPAL**, we have that $\vdash_{\mathbf{HPAL}} t(A)$. By this and Lemma 7, we obtain $\vdash_{\mathbf{HPAL}} A$, as desired. □

2.3.2 Action Models

From PAL in Section 2.3.1, agents do not know whether some actions happened or not. In order to capture this situation, Baltag et al. [28] introduced relational structures called *action models* (AM) to model actions or events.

Syntax and Kripke Semantics

Definition 20. *Let G be a* finite *set of agents and S be a finite set of action points. A language $\mathcal{L}_{AM}$ consists of the following vocabulary: (i) a countably infinite set Prop $= \{p, q, r, \dots\}$ of propositional letters, (ii) Boolean connectives: $\neg$, $\to$, (iii) knowledge operators $[\mathsf{K}_a]$ ($a \in G$), and (iv) a pointed action model (M, s) with $\mathsf{s} \in \mathsf{S}$. An* ***S5*** action model M *is a tuple $\mathsf{M} = (\mathsf{S}, (\sim_a)_{a \in G}, \mathrm{pre})$, where S is a non-empty set of action points, $\sim_a$ is an equivalence relation on S, and $\mathrm{pre} : \mathsf{S} \to \mathcal{L}_{AM}$ is a preconditions function that assigns a formula $\mathrm{pre}(\mathsf{s}) \in \mathcal{L}_{AM}$ to each action $s \in \mathsf{S}$. A set Form_{AM} of formulas of $\mathcal{L}_{AM}$ is inductively defined as follows:*

$$\mathsf{Form}_{AM} \ni A ::= p \,|\, \neg A \,|\, (A \to A) \,|\, [\mathsf{K}_a]A \,|\, [\mathsf{M}, \mathsf{s}]A$$

where $p \in \mathsf{Prop}$, $a \in G$ and $\mathsf{s} \in \mathsf{S}$. The outer parentheses of a formulas will normally be omitted.

We read $[\mathsf{K}_a]A$ as "agent a knows that A" and $[\mathsf{M}, \mathsf{s}]A$ as "after an action s happens, A holds." We can define the abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ as in Definition 2. Before giving the semantics of this language, let us define the composition of two action models.

Definition 21. *Let $\mathsf{M} = (\mathsf{S}, (\sim_a)_{a \in G}, \mathrm{pre})$ and $\mathsf{M}' = (\mathsf{S}', (\sim'_a)_{a \in G}, \mathrm{pre}')$ be two action models. Then, the composition of $(\mathsf{M}; \mathsf{M}')$ is a tuple $(\mathsf{S}'', (\sim''_a)_{a \in G}, \mathrm{pre}'')$ such that*

$$\begin{array}{lcl} \mathsf{S}'' & = & \mathsf{S} \times \mathsf{S}' \\ (\mathsf{s}, \mathsf{s}') \sim''_a (\mathsf{t}, \mathsf{t}') & \text{iff} & \mathsf{s} \sim_a \mathsf{t} \textit{ and } \mathsf{s}' \sim'_a \mathsf{t}' \\ \mathrm{pre}''((\mathsf{s}, \mathsf{s}')) & = & \mathrm{pre}(\mathsf{s}) \wedge [\mathsf{M}, \mathsf{s}]\mathrm{pre}'(\mathsf{s}') \end{array}$$

In order to define the Kripke semantics, a Kripke model $\mathfrak{M} = (W, (R_a)_{a \in G}, V)$ is constructed in the same way as in Definition 17 of PAL.

Definition 22. *Given any model $\mathfrak{M}$, any state $w \in W$ and any formula A, we define the* satisfaction relation *$\mathfrak{M}, w \models A$ inductively as follows:*

$$\begin{array}{lll} \mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\ \mathfrak{M}, w \models \neg A & \text{iff} & \mathfrak{M}, w \not\models A \\ \mathfrak{M}, w \models A \to B & \text{iff} & \mathfrak{M}, w \models A \textit{ implies } \mathfrak{M}, w \models B \\ \mathfrak{M}, w \models [\mathsf{K}_a]A & \text{iff} & \textit{for all } v \textit{ such that } wR_av \textit{ implies } \mathfrak{M}, v \models A \\ \mathfrak{M}, w \models [\mathsf{M}, \mathsf{s}]A & \text{iff} & \mathfrak{M}, w \models \mathrm{pre}(\mathsf{s}) \textit{ implies } \mathfrak{M}^{\otimes \mathsf{M}}, (w, \mathsf{s}) \models A, \end{array}$$

All instances of propositional tautologies	
$(K_{[\mathsf{K}_a]})$	$[\mathsf{K}_a](A \to B) \to ([\mathsf{K}_a]A \to [\mathsf{K}_a]B)$
$(T_{[\mathsf{K}_a]})$	$[\mathsf{K}_a]A \to A$
$(4_{[\mathsf{K}_a]})$	$[\mathsf{K}_a]A \to [\mathsf{K}_a][\mathsf{K}_a]A$
$(5_{[\mathsf{K}_a]})$	$\neg[\mathsf{K}_a]A \to [\mathsf{K}_a]\neg[\mathsf{K}_a]A$
$(RA1)$	$[\mathsf{M},\mathsf{s}]p \leftrightarrow (\mathrm{pre}(\mathsf{s}) \to p)$
$(RA2)$	$[\mathsf{M},\mathsf{s}]\neg A \leftrightarrow (\mathrm{pre}(\mathsf{s}) \to \neg[\mathsf{M},\mathsf{s}]A)$
$(RA3)$	$[\mathsf{M},\mathsf{s}](A \to B) \leftrightarrow ([\mathsf{M},\mathsf{s}]A \to [\mathsf{M},\mathsf{s}]B)$
$(RA4)$	$[\mathsf{M},\mathsf{s}][\mathsf{K}_a]A \leftrightarrow (\mathrm{pre}(\mathsf{s}) \to \bigwedge_{\mathsf{s}\sim_a\mathsf{t}}[\mathsf{K}_a][\mathsf{M},\mathsf{t}]A)$
$(RA5)$	$[\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']A \leftrightarrow [(\mathsf{M},\mathsf{s});(\mathsf{M}',\mathsf{s}')]A$
(MP)	From A and $A \to B$, infer B
$(Nec_{[\mathsf{K}_a]})$	From A, infer $[\mathsf{K}_a]A$

Table 2.6: Hilbert-style system **HAM** for $\mathcal{L}_{AM}$

where (w,s) *is the updated state of* $\mathfrak{M}^{\otimes\mathsf{M}}$ *and* $\mathfrak{M}^{\otimes\mathsf{M}} = (W', (R'_a)_{a\in G}, V')$ *is defined by:*

$$\begin{array}{lll} W' & := & \{ (w,\mathsf{s}) \in W \times \mathsf{S} \mid \mathfrak{M}, w \models \mathrm{pre}(\mathsf{s}) \} \\ (w,\mathsf{s})R'_a(v,\mathsf{t}) & \text{iff} & wR_a v \textit{ and } \mathsf{s} \sim_a \mathsf{t} \\ (w,\mathsf{s}) \in V'(p) & \text{iff} & w \in V(p) \end{array}$$

A formula A is valid *in a model* $\mathfrak{M}$ *if* $\mathfrak{M}, w \models A$ *for all states* w *of* $\mathfrak{M}$.

According to the above semantics, $\mathfrak{M}^{\otimes\mathsf{M}}$ is called the *product update model* which is the updated model by a product update mechanism.

Hilbert-style Axiomatization HAM

Table 2.6 presents the Hilbert-style system **HAM** for $\mathcal{L}_{AM}$. The proof of soundness and completeness for **HAM** can be captured by a similar argument in Section 2.3.1. First, we use the same manner in Section 2.1.2 (Definition 7) to define a derivation in **HAM**. Then, we provide the following translation.

Definition 23. *The translation* $t : \mathsf{Form}_{AM} \to \mathsf{Form}_{ML}$ *is defined as follows:*

$$\begin{array}{lll} t(p) & = & p \\ t(\neg A) & = & \neg t(A) \\ t(A \to B) & = & t(A) \to t(B) \\ t([\mathsf{K}_a]A) & = & [\mathsf{K}_a]t(A) \\ t([\mathsf{M},\mathsf{s}]p) & = & t(\mathrm{pre}(\mathsf{s}) \to p) \\ t([\mathsf{M},\mathsf{s}]\neg A) & = & t(\mathrm{pre}(\mathsf{s}) \to \neg[\mathsf{M},\mathsf{s}]A) \\ t([\mathsf{M},\mathsf{s}](A \to B)) & = & t([\mathsf{M},\mathsf{s}]A \to [\mathsf{M},\mathsf{s}]B) \\ t([\mathsf{M},\mathsf{s}][\mathsf{K}_a]A) & = & \bigwedge_{\mathsf{s}\sim_a\mathsf{t}} t(\mathrm{pre}(\mathsf{s}) \to [\mathsf{K}_a][\mathsf{M},\mathsf{t}]A) \\ t([\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']A) & = & t([\mathsf{M},\mathsf{s};\mathsf{M}',\mathsf{s}']A) \end{array}$$

Lemma 8. *Given any formula* $A \in \mathsf{Form}_{AM}$,

$$\vdash_{\mathbf{HAM}} A \leftrightarrow t(A).$$

Theorem 7 (Soundness)**.** *Let* $\mathbb{F}_{AM}$ *be the class of frames such that* $\mathbb{F}_{AM} := \{\mathrm{T}_{[\mathsf{K}_a]}, 4_{[\mathsf{K}_a]}, 5_{[\mathsf{K}_a]} \mid a \in \mathsf{G}$ *and* $[\mathsf{K}_a] \in \mathsf{Mod}\}$ *and* $\mathbb{M}_{AM}$ *be the class of models such that* $\mathbb{M}_{AM} := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_{AM}$ *and* V *is a valuation* V *on* $\mathfrak{F}\}$*. For all* $A \in \mathsf{Form}_{AM}$*,*

$$\text{if } \vdash_{\mathbf{HAM}} A, \text{ then } \mathbb{M}_{AM} \models A.$$

Proof. Suppose that $\vdash_{\mathbf{HAM}}$. Our goal is to show that $\mathbb{M}_{AM} \models A$ for all A. It suffices to show that all axioms and all rules in **HAM** are valid on all models in a class $\mathbb{M}_{AM}$ with respect to the semantics of $\mathcal{L}_{AM}$. This is straightforward. □

Theorem 8 (Completeness)**.** *Let* $\mathbb{F}_{AM}$ *be the class of frames such that* $\mathbb{F}_{AM} := \{\mathrm{T}_{[\mathsf{K}_a]}, 4_{[\mathsf{K}_a]}, 5_{[\mathsf{K}_a]} \mid a \in \mathsf{G}$ *and* $[\mathsf{K}_a] \in \mathsf{Mod}\}$ *and* $\mathbb{M}_{AM}$ *be the class of models such that* $\mathbb{M}_{AM} := \{(\mathfrak{F}, V) \mid \mathfrak{F} \in \mathbb{F}_{AM}$ *and* V *is a valuation* V *on* $\mathfrak{F}\}$*. For all* $A \in \mathsf{Form}_{AM}$*,*

$$\text{if } \mathbb{M}_{AM} \models A, \text{ then } \vdash_{\mathbf{HAM}} A.$$

Proof. Suppose that $\mathbb{M}_{AM} \models A$. Our goal is to show $\vdash_{\mathbf{HAM}} A$ for all formulas A. By the soundness theorem (Theorem 7) and Lemma 8, we obtain that $\mathbb{M}_{AM} \models A \leftrightarrow t(A)$. By this and our supposition, we get $\mathbb{M}_{AM} \models t(A)$. Since the formula $t(A)$ does not contain any action models, we can reduce the completeness of **HAM** to that of **HS5** (Theorem 2 in Section 2.1.2). Therefore, we obtain $\vdash_{\mathbf{HS5}} t(A)$ by the completeness of **HS5** (Theorem 2). Since **HS5** is a sub system of **HAM**, we have that $\vdash_{\mathbf{HAM}} t(A)$. By this and Lemma 8, we obtain $\vdash_{\mathbf{HAM}} A$, as desired. □

2.4 Logic for Signed Information

This section presents a logical framework proposed by Lorini et al. [13] for formalizing an agent's belief based on signed information. First, they provide the notions of an agent's belief, a signed statement and a reliability relation. Then, they propose two ways for constructing an agent's belief from static and dynamic perspectives. From a static perspective, they apply policies for aggregating signed information. The tell-action is introduced for capturing a process of belief construction from a dynamic perspective. In this section, we simplify their framework by removing the universal quantifier for agents. This is because we realized that most of the ideas in [13] are done without quantifiers for agents when a set of agents is finite, i.e., the universal quantifier for a finite domain is just reduced to the conjunction of finite conjuncts. This idea will be applied for constructing our logical formalism in Chapter 3.

2.4.1 Formal Framework for Beliefs, Signatures and Preferences

Firstly, we give a language $\mathcal{L}_{BSP}$ for formalizing beliefs, signatures and preferences. There are three notions including belief and signature operators, and reliability orderings. The first modal operator represents agents' beliefs, while the second one is used to represent signed statements for handling sources of information. An agent's preference over information sources is represented by a notion of reliability orderings.

Definition 24. *Let* G *be a* finite *set of agents. The language* $\mathcal{L}_{BSP}$ *consists of the following vocabulary: (i) a finite set* $\mathsf{Prop} = \{p, q, r, \dots\}$ *of propositional letters, (ii) Boolean* *connectives:* $\neg$, $\rightarrow$, *(iii) the belief operators* $\mathsf{Bel}(a, \cdot)$ *($a \in G$), (iv) the signature operators* $\mathsf{Sign}(a, \cdot)$ *($a \in G$), and (v) the constants for reliability ordering $a \leqslant b$ ($a, b \in G$). A set* Form_{BSP} *of formulas of* $\mathcal{L}_{BSP}$ *is inductively defined as follows:*

$$\mathsf{Form}_{BSP} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid \mathsf{Bel}(a, \varphi) \mid \mathsf{Sign}(a, \varphi) \mid a \leqslant b,$$

where $p \in \mathsf{Prop}$ *and* $a, b \in G$.

From the above definition, we read $\mathsf{Bel}(a, \varphi)$ as "agent a believes that φ", $\mathsf{Sign}(a, \varphi)$ as "agent a signs statement φ," and $b \leqslant c$ as "agent b is at least as reliable as agent c." For the reliability orderings, we can define by the following notions: $b < c$ stands for agent b is strictly more reliable than agent c, i.e., $(b \leqslant c) \wedge \neg(c \leqslant b)$, and $b \approx c$ which stands for agents b and c are equally reliable can be defined as $(b \leqslant c) \wedge (c \leqslant b)$. The abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, $\top$ and $\bot$ can be defined as in Definition 2. The semantics of this language is given by the following definitions.

Definition 25. *A Kripke* model $\mathfrak{M}$ *is a tuple* $\mathfrak{M} = (W, (B_a)_{a \in G}, (S_a)_{a \in G}, \preccurlyeq, V)$*, where:*

- W *is a non-empty set of possible states.*
- $B_a \subseteq W \times W$ *is an accessibility relation representing beliefs.*
- $S_a \subseteq W \times W$ *is an accessibility relation representing signatures.*
- $\preccurlyeq$ *is a function which maps from* W *to* $\mathcal{P}(G \times G)$ *representing reliability ordering between agents.*
- V *is a function of the propositional letters with respect to each possible state.*

Definition 26. *Given any model* $\mathfrak{M}$*, any state* $w \in W$ *and any formula* φ*, we define the* satisfaction relation $\mathfrak{M}, w \models \varphi$ *inductively as follows:*

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \mathfrak{M}, w \not\models \varphi \\
\mathfrak{M}, w \models \varphi \rightarrow \psi & \text{iff} & \mathfrak{M}, w \models \varphi \textit{ implies } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models a \leqslant b & \text{iff} & (a, b) \in \preccurlyeq (w) \\
\mathfrak{M}, w \models \mathsf{Sign}(a, \varphi) & \text{iff} & \mathfrak{M}, v \models \varphi \textit{ for all } v \textit{ such that } (w, v) \in S_a \\
\mathfrak{M}, w \models \mathsf{Bel}(a, \varphi) & \text{iff} & \mathfrak{M}, v \models \varphi \textit{ for all } v \textit{ such that } (w, v) \in B_a
\end{array}$$

A formula φ is *valid* in a model $\mathfrak{M}$ if $\mathfrak{M}, w \models \varphi$ for all states w of $\mathfrak{M}$.

Hilbert-style Axiomatization HBSP

The Hilbert-style system **HBSP** for $\mathcal{L}_{BSP}$ is presented in Table 2.7. This table states the axioms and inference rules for belief and signature operators and reliability orderings. Belief operators follow **K45** logic, while signature operators follow **KD45** logic. We can regard both belief and signature operators as a **K45** operator which means that B_a and S_a are transitive and euclidean. The axiom (D_{S}) means that S_a are serial, that is, a signature has to be consistent. However, the axiom D cannot hold for belief operators because of the tell-action which is described in Section 2.4.2. For reliability orderings $\preccurlyeq$, it is a total preorder between agents because it is reflexive (by $(R_{\leqslant})$), transitive (by $(Tr_{\leqslant})$) and comparable (by $(T_{\leqslant})$). In addition, the axiom $(To_{\leqslant})$ means that $\preccurlyeq$ has to be believed as total. Based on [13], the proof of soundness and completeness for **HBSP** can be provided by Theorem 9.

All instances of propositional tautologies	
(K_{B})	$\mathsf{Bel}(a, \varphi \to \psi) \to \big(\mathsf{Bel}(a, \varphi) \to \mathsf{Bel}(a, \psi)\big)$
(4_{B})	$\mathsf{Bel}(a, \varphi) \to \mathsf{Bel}(a, \mathsf{Bel}(a, \varphi))$
(5_{B})	$\neg\mathsf{Bel}(a, \varphi) \to \mathsf{Bel}(a, \neg\mathsf{Bel}(a, \varphi))$
(K_{S})	$\mathsf{Sign}(a, \varphi \to \psi) \to \big(\mathsf{Sign}(a, \varphi) \to \mathsf{Sign}(a, \psi)\big)$
(D_{S})	$\mathsf{Sign}(a, \varphi) \to \neg\mathsf{Sign}(a, \neg\varphi)$
(4_{S})	$\mathsf{Sign}(a, \varphi) \to \mathsf{Sign}(a, \mathsf{Sign}(a, \varphi))$
(5_{S})	$\neg\mathsf{Sign}(a, \varphi) \to \mathsf{Sign}(a, \neg\mathsf{Sign}(a, \varphi))$
$(R_{\leqslant})$	$a \leqslant a$
$(Tr_{\leqslant})$	$(a \leqslant b \wedge b \leqslant c) \to a \leqslant c$
$(T_{\leqslant})$	$a \leqslant b \vee b \leqslant a$
$(To_{\leqslant})$	$\mathsf{Bel}(a, b \leqslant c) \vee \mathsf{Bel}(a, c \leqslant b)$
(MP)	From φ and $\varphi \to \psi$ infer ψ
(Nec_{S})	From φ infer $\mathsf{Sign}(a, \varphi)$
(Nec_{B})	From φ infer $\mathsf{Bel}(a, \varphi)$

Table 2.7: Hilbert-style system **HBSP** for $\mathcal{L}_{BSP}$

Theorem 9 (Completeness [13]). *Let $\mathbb{M}_{BSP}$ be the class of models where B_a satisfies the properties of transitivity and Euclideanness, S_a satisfies the properties of seriality, transitivity and Euclideanness, and $\preccurlyeq$ satisfies the properties of reflexivity, transitivity and totality as shown in Table 2.7. For all $\psi \in \mathsf{Form}_{BSP}$,*

$$\mathbb{M}_{BSP} \models \psi \quad \text{iff} \quad \vdash_{\mathbf{HBSP}} \psi.$$

2.4.2 Logical Tools for Formalizing Belief based on Signed Information

This section describes two ways for formalizing an agent's belief based on signed information. The first way provides policies of information aggregation for constructing an agent's belief from a static point of view. The second way focuses a dynamic perspective by proposing a tell-action in a similar way to private announcements in DEL.

Information Aggregation

Lorini et al. [13] introduced several policies, as *meta-logical* principles, in order to decide which pieces of signed information an agent should believe. A common and rational policy is called a *careful policy* that will be used in Chapter 4. An idea of this policy is to accept, as beliefs, the statements which are universally signed by a group of agents who are equally reliable. Before providing the details of the careful policy, let us describe how to rank agents.

From Section 2.4.1, since the reliability orderings $\preccurlyeq$ is a total preorder, agents can be ranked by giving a partition (C_i) as follows: Let $(\mathsf{C}_i)_{i \leq M}$ to G be a partition, where M is a natural number representing the maximum rank (such M always exists because G is finite). We define $(\mathsf{C}_i)_{i \leq M}$ inductively as follows. First, C_1 which stands for "a group of agents which is the most reliable" can be defined by the following formula:

$$c \in \mathsf{C}_1 := \bigwedge_{b\in G} (c \leqslant b),$$

where we recall that G is a finite set of agents and $a, b, c \in G$. Then, we can rank the group of agents C_i such that $i > 1$ as follows:

$$c \in \mathsf{C}_i := \Bigg(\Bigg(\bigwedge_{1\leq j\leq i-1} \neg(c \in \mathsf{C}_j)\Bigg) \wedge \Bigg(\bigwedge_{b\in G} \Bigg(\Big(\bigwedge_{1\leq j\leq i-1} \neg(b \in \mathsf{C}_j)\Big) \rightarrow (c \leqslant b)\Bigg)\Bigg)\Bigg).$$

This implies that all agents in C_i are equally reliable, and if $i <_{\mathbb{N}} j$ then $c < b$ for all agents $c \in \mathsf{C}_i$ and $b \in \mathsf{C}_j$. Note that $c \in \mathsf{C}_i$ is read as "a rank of agent c is i." From the above definition, we obtain that agents who are equally reliable are categorized in the same group. Next, we define $\mathsf{Sign}(\mathsf{C}_i, \varphi)$ which stands for "all agents who are in C_i sign statement φ" by the following definition.

$$\mathsf{Sign}(\mathsf{C}_i, \varphi) := \bigwedge_{c\in \mathsf{C}_i} \big(\mathsf{Sign}(c, \varphi)\big)$$

After that, the careful policy is given by the following definition.

$$\mathsf{Careful}(a, \varphi) := \bigvee_{i\leq M} \left(\begin{array}{c} \mathsf{Bel}\big(a, \mathsf{Sign}(\mathsf{C}_i, \varphi)\big) \wedge \\ \mathsf{Bel}\big(a, \bigwedge_{1\leq j\leq i-1} \neg \mathsf{Sign}(\mathsf{C}_j, \neg\varphi)\big) \end{array} \right) \rightarrow \mathsf{Bel}(a, \varphi),$$

where M is the maximum natural number of $\{i \leq \#G \mid \mathsf{C}^a_i \neq \emptyset\}$ and $\mathsf{Careful}(a, \varphi)$ can be read as "agent a aggregates information about φ by the careful policy."

Tell Action

In the previous section, the careful policy is stated for deriving an agent's belief from a static point of view. Let us consider a dynamic perspective by introducing a notion of tell-action $[\mathsf{Tell}(b, a, \varphi)]$ (whose reading is "agent b tells to agent a that φ is true"). An underlying idea of tell-action is that agent b *privately* tells φ to agent a, that is, the other agents than a would not notice this action. As a result, only agent a would change his/her belief by φ but the other agents than a would not change their beliefs. After $[\mathsf{Tell}(b, a, \varphi)]$, agent a believes that agent b signs φ as the following proposition.

Proposition 27 (Successful Telling [13]). *$[\mathsf{Tell}(b, a, \varphi)]\mathsf{Bel}\big(a, \mathsf{Sign}(b, \varphi)\big)$ is valid on all models $\mathfrak{M}$.*

This proposition is an essential aspect of the tell-action. That is, after agent b tells to agent a information φ, agent a believes that agent b signs φ. After that, agent a might believe that φ if he/she considers agent b to be reliable based on the aggregation policy as mentioned in the previous section.

Definition 28. *Given a Kripke model $\mathfrak{M} = (W, (B_c)_{c\in G}, (S_c)_{c\in G}, \preccurlyeq, V)$, a semantic clause for $[\mathsf{Tell}(b, a, \varphi)]$ on $\mathfrak{M}$ and $w \in W$ is defined as follows:*

$$\mathfrak{M}, w \models [\mathsf{Tell}(b, a, \varphi)]\psi \quad \text{iff} \quad \mathfrak{M}^{\mathsf{Tell}(b,a,\varphi)}, w' \models \psi,$$

where $\mathfrak{M}^{\mathsf{Tell}(b,a,\varphi)} = (W^*, (B^*_c)_{c\in G}, (S^*_c)_{c\in G}, \preccurlyeq^*, V^*)$ is defined by:

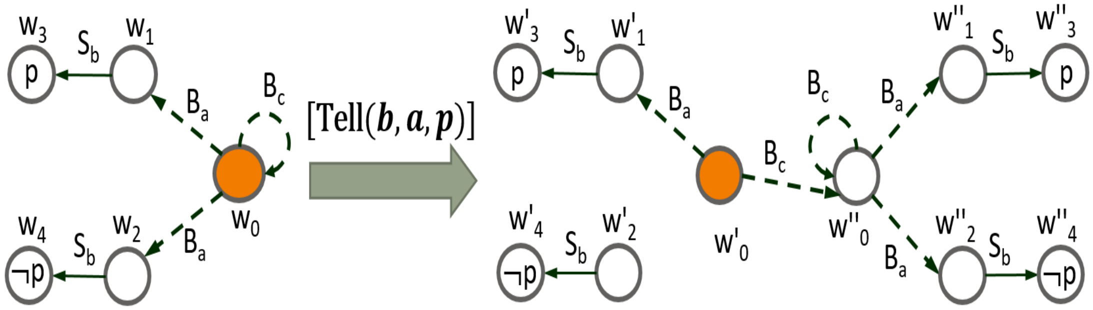

Figure 2.1: Update operation of $[\mathsf{Tell}(b, a, p)]$

- $W^* := W' \cup W''$ *such that* $W' = \{ w' \mid w \in W \}$ *and* $W'' = \{ w'' \mid w \in W \}$.
- $B^*_a = \{ (w', v') \mid (w, v) \in B_a$ and $\mathfrak{M}, v \models \mathsf{Sign}(b, \varphi) \} \cup \{ (w'', v'') \mid (w, v) \in B_a \}$.
- $B^*_c = \{ (w', v'') \mid (w, v) \in B_c \} \cup \{ (w'', v'') \mid (w, v) \in B_c \}$ *(for all* $c \in G$ *such that* $c \neq a$*)*.
- $S^*_c = \{ (w', v') \mid (w, v) \in S_c \} \cup \{ (w'', v'') \mid (w, v) \in S_c \}$ *(for all* $c \in G$*)*.
- $\preccurlyeq^* (w') = \preccurlyeq^* (w'') = \preccurlyeq (w)$.
- $w' \in V^*(p)$ iff $w \in V(p)$ *and* $w'' \in V^*(p)$ iff $w \in V(p)$.

From the above definition, the process of the tell-action can be demonstrated by Fig. 2.1. The concept of this process consists of two steps. First, all states $w \in W$ are duplicated in W' and W''. Second, the set of possible belief states of agent a is restricted to W' representing agent a's belief after $[\mathsf{Tell}(b, a, p)]$. In order to capture the private action, we add the links representing relations B_c for the other agents except agent a from W' to W'' in order to represent that the other agents than agent a do not change their beliefs after $[\mathsf{Tell}(b, a, p)]$. Fig. 2.1 illustrates that when agent b tells to agent a information p ($[\mathsf{Tell}(b, a, p)]$), agent a will believe $\mathsf{Sign}(b, p)$ by removing all the links from state w_0 into the states where $\mathsf{Sign}(b, p)$ is false. That is, state w_2 should not be a possible belief state for agent a that is represented by state w'_2. However, belief change of agent a should have no influence on agent c's belief. That is, we should keep state w_2 as a possible belief state for agent a that is represented by state w''_2. Thus, agent c does not change his/her belief after $[\mathsf{Tell}(b, a, p)]$. Note that the agents' signatures and the reliability orderings between agents will not be affected by the tell-action. That is, S^*_c and $\preccurlyeq^*$ are full copies of initial S_c and $\preccurlyeq$. We define $\mathcal{L}_{BSPT}$ which is an extension of $\mathcal{L}_{BSP}$ by adding $[\mathsf{Tell}(b, a, \varphi)]$. A set of formulas of $\mathcal{L}_{BSPT}$ is denoted as Form_{BSPT}.

Table 2.8 presents the Hilbert-style system **HBSPT** for $\mathcal{L}_{BSPT}$. An axiom ($T_{\mathsf{B}\neq}$) means that after $[\mathsf{Tell}(b, a, \varphi)]$, the other agents than agent a would not change their beliefs. The axioms (T_{S}) and ($T_{\preccurlyeq}$) describe the permanence of agents' signatures and the reliability orderings between agents. Based on [13], the soundness and completeness for **HBSPT** can be proved by Theorem 10.

All axioms and rules of **HSP**			
(T_{atom})	$[\mathsf{Tell}(b,a,\varphi)]p$	$\leftrightarrow$	p
$(T_{\neg})$	$[\mathsf{Tell}(b,a,\varphi)]\neg\psi$	$\leftrightarrow$	$\neg[\mathsf{Tell}(b,a,\varphi)]\psi$
$(T_{\wedge})$	$[\mathsf{Tell}(b,a,\varphi)](\psi_1 \to \psi_2)$	$\leftrightarrow$	$([\mathsf{Tell}(b,a,\varphi)]\psi_1 \to [\mathsf{Tell}(b,a,\varphi)]\psi_2)$
(T_{B})	$[\mathsf{Tell}(b,a,\varphi)]\mathsf{Bel}(a,\psi)$	$\leftrightarrow$	$\mathsf{Bel}(a,(\mathsf{Sign}(b,\varphi) \to [\mathsf{Tell}(b,a,\varphi)]\psi))$
$(T_{\mathsf{B}\neq})$	$[\mathsf{Tell}(b,a,\varphi)]\mathsf{Bel}(c,\psi)$	$\leftrightarrow$	$\mathsf{Bel}(c,\psi)\ \ (c \neq a)$
(T_{S})	$[\mathsf{Tell}(b,a,\varphi)]\mathsf{Sign}(c,\psi)$	$\leftrightarrow$	$\mathsf{Sign}(c,\psi)$
$(T_{\leqslant})$	$[\mathsf{Tell}(b,a,\varphi)](a \leqslant b)$	$\leftrightarrow$	$a \leqslant b$
(Nec_{T})	From ψ, infer $[\mathsf{Tell}(b,a,\varphi)]\psi$		

Table 2.8: Hilbert-style system **HBSPT** for $\mathcal{L}_{BSPT}$

Theorem 10 (Completeness [13]). *Let $\mathbb{M}_{BSP}$ be the class of models where B_a satisfies the properties of transitivity and Euclideanness, S_a satisfies the properties of seriality, transitivity and Euclideanness, and $\preccurlyeq$ satisfies the properties of reflexivity, transitivity and totality as shown in Table 2.7. For all $\psi \in \mathsf{Form}_{BSPT}$,*

$$\mathbb{M}_{BSP} \models \psi \quad \text{iff} \quad \vdash_{\mathbf{HBSPT}} \psi.$$

Chapter 3

Logical Tool for Reliability Change

This chapter provides a formal tool for handling reliability change of an agent from a logical point of view. First, Section 3.1 presents the logic of agents' beliefs based on the modal framework proposed by Lorini et al. [13] as mentioned in Section 2.4. An information source is formalized by a notion of singed statement and its reliability is represented by a notion of reliability ordering. However, our logical formalism is different from Lorini's framework [13] in three aspects. First, the universal quantifier for agents is not introduced (mentioned in Section 2.4). Second, the notion of reliability ordering is relativized to a specific agent (described in Section 3.1). Third, the careful policy and the tell-action in [13] can be captured in terms of dynamic operator by the private announcement which is our dynamic operator for formalizing belief re-revision of an agent that will be described in Chapter 4. By this logic, three dynamic operators consisting of upgrade, downgrade and joint downgrade are formulated in Section 3.2. A target of these operators is to change the reliability of the other agents from a specific agent's perspective. With these operators, we can demonstrate how a judge changes his/her reliability ordering between witnesses.

3.1 Static Logic of Agents' Beliefs for Signed Information

In this section, we introduce a language $\mathcal{L}_{BSR}$ in order to formalize an agent's belief, signed information and the reliability of information sources. This language can be defined in a similar way to $\mathcal{L}_{BSP}$ as mentioned in Section 2.4.

3.1.1 Syntax and Semantics

Definition 29. *Let G be a fixed* finite *set of agents. The language $\mathcal{L}_{BSR}$ consists of the following vocabulary: (i) a countably infinite set $\mathsf{Prop} = \{ p, q, r, \dots \}$ of propositional letters, (ii) Boolean connectives: $\neg$, $\rightarrow$, (iii) the belief operators $\mathsf{Bel}(a, \cdot)$ $(a \in G)$, (iv) the signature operators $\mathsf{Sign}(a, \cdot)$ $(a \in G)$, and (v) the constants for reliability orderings $b \leqslant_a c$ $(a, b, c \in G)$. A set Form_{BSR} of formulas of $\mathcal{L}_{BSR}$ is inductively defined as follows:*

$$\mathsf{Form}_{BSR} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid \mathsf{Bel}(a, \varphi) \mid \mathsf{Sign}(a, \varphi) \mid b \leqslant_a c,$$

where $p \in \mathsf{Prop}$ and $a, b, c \in G$.

$\mathsf{Bel}(a, \varphi)$	:	agent a believes that φ.
$\mathsf{Sign}(a, \varphi)$	:	agent a signs statement φ.
$\mathsf{Sign}(a, \mathsf{Sign}(b, \varphi))$	:	agent a signs statement that agent b signs statement φ.
$b \leqslant_a c$	:	from agent a's perspective, agent b is at least as reliable as agent c.
$\mathsf{Bel}\big(a, \mathsf{Sign}(b, \varphi)\big)$	:	agent a believes that agent b signs statement φ.
$\mathsf{Bel}\big(a, b \leqslant_a c\big)$	:	agent a believes that from agent a's perspective, agent b is at least as reliable as agent c.

Table 3.1: Examples of static logical formalization of $\mathcal{L}_{BSR}$

The intuitive readings of the above formulas are presented in Table 3.1. We can define the abbreviations for $\wedge$, $\vee$, $\leftrightarrow$, $\top$ and $\bot$ as in Definition 2 in Section 2.1. Let us describe two important operators, i.e., signature and belief. First, signature operators are information sources by a notion of signed information. For example, $\mathsf{Sign}(a, \varphi)$ represents information φ is given by agent a. This allows us to keep track of information sources. From Table 3.1, $\mathsf{Bel}\big(a, \mathsf{Sign}(b, \varphi)\big)$ can be interpreted by the context of information gathering in the judicial process as follows: When witness b gives information φ to judge a, judge a cannot believe that such witness tells the truth. That is, judge a cannot believe information φ, but judge a can believe that witness b signs or provides information φ.

This language $\mathcal{L}_{BSR}$ is different from $\mathcal{L}_{BSP}$ of [13] in Section 2.4 in at least two respects. First, we do not introduce the universal quantifier for agents (mentioned in Section 2.4) because we realized that the use of universal quantifier over agents in [13] is *redundant*. Second, we relativize the notion of reliability ordering $\leqslant$ to each agent. In order to analyze our example from a logical perspective, we need to formalize belief change of a judge of the Court and we regard that belief change is induced by reliability change. However, there is no need for us to change the reliability ordering of the other agents other than the judge of the court. This is why we propose the notion of reliability ordering between agents depending on a particular agent's perspective.

Definition 30. *A* model $\mathfrak{M}$ *is a tuple* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_a)_{a\in G}, V)$*, where*

- W *is a non-empty set of states, called the* domain*,*
- $R_a \subseteq W \times W$ *is an accessibility relation representing beliefs,*
- $S_a \subseteq W \times W$ *is an accessibility relation representing signatures,*
- $\preccurlyeq_a$ *is a function which maps from* W *to* $\mathcal{P}(G \times G)$ *representing agent* a*'s reliability ordering between agents,*
- $V : \mathsf{Prop} \to \mathcal{P}(W)$ *is a valuation.*

In what follows, we simply write $b \preccurlyeq_a^w c$ for $(b, c) \in \preccurlyeq_a (w)$. Following [13], $\preccurlyeq_a^w$ is always required to be a *total preordering* between agents, i.e., $\preccurlyeq_a^w$ is reflexive ($b \preccurlyeq_a^w b$ for all b), transitive ($b \preccurlyeq_a^w c$ and $c \preccurlyeq_a^w d$ jointly imply $b \preccurlyeq_a^w d$ for all b, c, d), and comparable (for any b and c, $b \preccurlyeq_a^w c$ or $c \preccurlyeq_a^w b$). For any binary relation X on W and any state $w \in W$, we write $X(w)$ to mean $\{ v \in W \mid (w, v) \in X \}$.

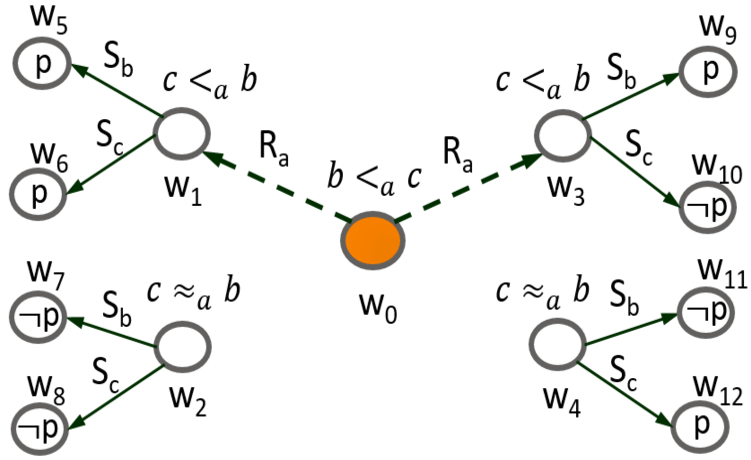

Figure 3.1: Kripke model representing a relationship between belief and signature states

Definition 31. *Given any model $\mathfrak{M}$, any state $w \in W$ and any formula φ, we define the* satisfaction relation $\mathfrak{M}, w \models \varphi$ *inductively as follows:*

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \rightarrow \psi$	iff	$\mathfrak{M}, w \models \varphi$ *implies* $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models b \leqslant_a c$	iff	$b \preccurlyeq_a^w c$
$\mathfrak{M}, w \models \mathsf{Sign}(a, \varphi)$	iff	$\mathfrak{M}, v \models \varphi$ *for all v such that wS_av*
$\mathfrak{M}, w \models \mathsf{Bel}(a, \varphi)$	iff	$\mathfrak{M}, v \models \varphi$ *for all v such that wR_av*

A formula φ is valid *in a model $\mathfrak{M}$ if $\mathfrak{M}, w \models \varphi$ for all states w of $\mathfrak{M}$.*

Example 32. *Fig. 3.1 illustrates how agent a can build his/her belief using information which is signed by two agents (i.e., b and c) and their reliability. Signed statements can be represented by one relation S per source of information. For example, S_b represents that agent b signs statement p or $\neg p$. From Fig. 3.1, the semantics of belief and signature operators, and reliability ordering can be described as follows:*

- *The semantics of signature operators can be described as follows: Following S_b from state w_1, statement p holds at state w_5. Thus, $\mathsf{Sign}(b, p)$ is true at state w_1.*

- *We can describe the semantics of belief operators as follows: In order to check if $\mathsf{Bel}\big(a, \mathsf{Sign}(b, p)\big)$ is true or not, we need to check that $\mathsf{Sign}(b, p)$ is true at all states that have R_a from state w_0. Following R_a from state w_0, $\mathsf{Sign}(b, p)$ is true at states w_1 and w_3. Therefore, we obtain that $\mathsf{Bel}\big(a, \mathsf{Sign}(b, p)\big)$ is true at state w_0.*

- *For the semantics of the reliability orderings, $\preccurlyeq_a^w$ represents the reliability ordering of agent a is specific to each state w as shown in Fig. 3.1. For example, $b <_a c$ is true at state w_0, while $c \approx_a b$ is true at states w_2 and w_4. This means that "from agent a's perspective, agent b is more reliable than agent c at state w_0, while agents b and c are equally reliable at states w_2 and w_4."*

All instances of propositional tautologies	
(K_{B})	$\mathsf{Bel}(a, \varphi \to \psi) \to (\mathsf{Bel}(a, \varphi) \to \mathsf{Bel}(a, \psi))$
(K_{S})	$\mathsf{Sign}(a, \varphi \to \psi) \to (\mathsf{Sign}(a, \varphi) \to \mathsf{Sign}(a, \psi))$
$(R_{\leqslant_a})$	$b \leqslant_a b$
$(Tr_{\leqslant_a})$	$(b \leqslant_a c \wedge c \leqslant_a d) \to b \leqslant_a d$
$(To_{\leqslant_a})$	$b \leqslant_a c \vee c \leqslant_a b$
(MP)	From φ and $\varphi \to \psi$, infer ψ
(Nec_{B})	From φ, infer $\mathsf{Bel}(a, \varphi)$
(Nec_{S})	From φ, infer $\mathsf{Sign}(a, \varphi)$

Table 3.2: Hilbert-style system **HBSR** for $\mathcal{L}_{BSR}$

By using the total preordering $\preccurlyeq_a^w$, we can rank agents in a similar way to the idea of [13] mentioned in Section 2.4 but use a notion C_i^a instead of C_i as follows:

$$c \in \mathsf{C}_1^a := \bigwedge\nolimits_{b \in G} (c \leqslant_a b),$$

where C_1^a stands for "a group of agents which is the most reliable from a's perspective," and we recall that G is a finite set of agents and $a, b, c \in G$. Then, we can rank the group of agents C_i^a such that $i > 1$ as follows:

$$c \in \mathsf{C}_i^a := \left(\left(\bigwedge_{1 \leq j \leq i-1} \neg (c \in \mathsf{C}_j^a) \right) \wedge \left(\bigwedge_{b \in G} \left(\Big(\bigwedge_{1 \leq j \leq i-1} \neg (b \in \mathsf{C}_j^a) \Big) \to (c \leqslant_a b) \right) \right) \right).$$

This implies that all agents in C_i^a are equally reliable, and if $i <_{\mathbb{N}} j$ then $c <_a b$ for all agents $c \in \mathsf{C}_i^a$ and agent $b \in \mathsf{C}_j^a$. This means that we relativize the notion C_i^a to a specific agent a because our notion of reliability ordering $\leqslant_a$ depends on agent a. This point differs from the framework in Section 2.4 that does not consider C_i with respect to a specific agent.

3.1.2 Hilbert-style Axiomatization HBSR

The Hilbert-style system **HBSR** for $\mathcal{L}_{BSR}$ is presented in Table 3.2. For the reliability orderings, we regard that $\preccurlyeq_a$ is a total preordering between agents, i.e., $\preccurlyeq_a$ is reflexive (by $(R_{\leqslant_a})$), transitive (by $(Tr_{\leqslant_a})$) and comparable (by $(To_{\leqslant_a})$). From Lorini's framework [13] mentioned in Section 2.4, S_a has three properties of relations including serial, transitive and Euclidean. That is, we ensures that an agent never signs a contradiction (due to the serial property of S_a) and has both positive and negative introspection of his/her signed information (due to the transitive and Euclidean properties of S_a). However, in this study, there is no need to assume these properties of S_a for agents in a legal case. For example, a witness first gave statement p in the inquiry stage, but after then he/she gave statement $\neg p$ in the court. Thus, the judge came to notice that the witness gave both p and $\neg p$. This example shows that the witness in a legal case can sign a contradiction. For belief operators $\mathsf{Bel}(a, \cdot)$, we suppose that R_a has no properties of relations because of the private announcement and the private permission (described in Section 4.2.3). In this study, the properties of R_a and S_a are supposed in a different way from Lorini's framework [13] (in Section 2.4). Since **HBSR** is regarded as an extension of **HKΣ**, the proof of soundness and completeness can be captured in a similar way as in Section 2.1.2. First, a derivation in **HBSR** and a maximally **HBSR**-consistent set **HBSR**-MCS are defined in the same manner as in Definitions 7 and 8, respectively.

Lemma 9 (Lindenbaum)**.** *Given any* **HBSR**-*consistent set* Γ*, there exists an* **HBSR**-*MCS* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$*.*

Then, the canonical model for **HBSR** is constructed by the following definition.

Definition 33. *The* canonical model *for* **HBSR***:* $\mathfrak{M}^{\mathbf{HBSR}} = \{W^{\mathbf{HBSR}}, (R_a^{\mathbf{HBSR}})_{a \in \mathsf{G}}, (S_a^{\mathbf{HBSR}})_{a \in \mathsf{G}}, (\preccurlyeq_a^{\mathbf{HBSR}})_{a \in \mathsf{G}}, V^{\mathbf{HBSR}}\}$ *is defined by:*

- $W^{\mathbf{HBSR}} := \{\, \Gamma \mid \Gamma$ *is an* **HBSR**-*MCS*$\}$.
- $\Gamma R_a^{\mathbf{HBSR}} \Delta$ iff $\mathsf{Bel}(a, \psi) \in \Gamma$ *implies* $\psi \in \Delta$ *for all formulas* ψ.
- $\Gamma S_a^{\mathbf{HBSR}} \Delta$ iff $\mathsf{Sign}(a, \psi) \in \Gamma$ *implies* $\psi \in \Delta$ *for all formulas* ψ.
- $b \preccurlyeq_a^{\Gamma} c$ iff $b \leqslant_a c \in \Gamma$.
- $\Gamma \in V^{\mathbf{HBSR}}(p)$ iff $p \in \Gamma$.

Next, we can show the following equivalence as in Lemma 2.

Lemma 10 (Truth)**.** *Let* Γ *be any* **HBSR**-*MCS. For all* $\psi \in \mathsf{Form}_{BSR}$*,*

$$\mathfrak{M}^{\mathbf{HBSR}}, \Gamma \models \psi \quad \text{iff} \quad \psi \in \Gamma.$$

Theorem 11 (Soundness)**.** *Let* $\mathbb{M}_{BSR}$ *be the class of all models where* $\preccurlyeq_a$ *satisfies the properties of reflexivity, transitivity and totality as shown in Table 3.2. For all* $\psi \in \mathsf{Form}_{BSR}$*,*

$$\textit{if } \vdash_{\mathbf{HBSR}} \psi, \ \textit{then } \mathbb{M}_{BSR} \models \psi.$$

Since the soundness proof is to show the validity of recursion axioms of **HBSR** that is straightforward, we will focus on the completeness proof.

Theorem 12 (Completeness)**.** *Let* $\mathbb{M}_{BSR}$ *be the class of all models where* $\preccurlyeq_a$ *satisfies the properties of reflexivity, transitivity and totality as shown in Table 3.2. For all* $\psi \in \mathsf{Form}_{BSR}$*,*

$$\textit{if } \mathbb{M}_{BSR} \models \psi, \ \textit{then } \vdash_{\mathbf{HBSR}} \psi.$$

Proof. The proof is by contrapositive implication. Suppose that $\nvdash_{\mathbf{HBSR}} \psi$. Our goal is to show $\mathbb{M}_{BSR} \not\models \psi$. It suffices to find a counter model $\mathfrak{M}$ such that $\mathfrak{M}, w \not\models \psi$ for some w of $\mathfrak{M}$. By our supposition, we obtain that $\{\neg\psi\}$ is an **HBSR**-consistent set, i.e., $\{\neg\psi\} \nvdash_{\mathbf{HBSR}} \bot$. By Lemma 9, there exists an **HBSR**-MCS Γ such that $\{\neg\psi\} \subseteq \Gamma$, i.e., $\neg\psi \in \Gamma$. By Lemma 10, we obtain that $\mathfrak{M}^{\mathbf{HBSR}}, \Gamma \models \neg\psi$, i.e., $\mathfrak{M}^{\mathbf{HBSR}}, \Gamma \not\models \psi$, as desired. □

3.2 Dynamic Operators for Reliability Change

This section provides three dynamic logical operators including upgrade, downgrade and joint downgrade for changing a reliability ordering between agents from a particular agent's perspective. The first operator is used to upgrade some agents more reliable, while the two later operators are used to downgrade some agents less reliable.

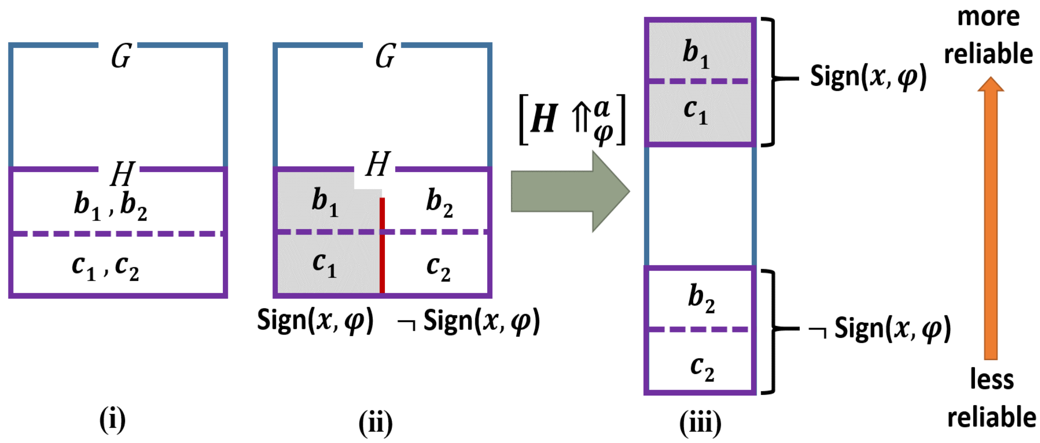

Figure 3.2: Model for upgrading by $[H \Uparrow^a_\varphi]$

3.2.1 Upgrade

This section introduces an upgrade operator $[H \Uparrow^a_\varphi]$, where $H \subseteq G$ is a set of agents. Our goal of this operator is to change the reliability of some specific agents to be more reliable. We can read $[H \Uparrow^a_\varphi]\psi$ as "after agent a upgraded such agents who sign statement φ in H, ψ holds." Semantically speaking, $[H \Uparrow^a_\varphi]$ makes such agents who sign φ in H more reliable than all the other agents.

Before giving the detailed semantics, let us demonstrate the effects of $[H \Uparrow^a_\varphi]$ by figures. Firstly, we assume that a rectangle G of Fig. 3.2(i) represents a fixed finite set of agents. Secondly, we will select a specified set of agents in order to change their reliability ordering that can be represented by a rectangle H, and we assume that $b_1 \approx_a b_2 <_a c_1 \approx_a c_2$ holds, i.e., agents b_1 and b_2 which are equally reliable are more reliable than agents c_1 and c_2 which are equally reliable from agent a's perspective. In this sense, b_1, b_2, c_1 and c_2 are situated as in Fig. 3.2(i). Then, if we focus on agents who sign statement φ, H is divided into two equal vertical parts by $\mathsf{Sign}(x, \varphi)$ as in Fig. 3.2(ii), namely by the set $\{x \in H \mid \mathfrak{M}, w \models \mathsf{Sign}(x, \varphi)\}$ and the set $\{x \in H \mid \mathfrak{M}, w \models \neg\mathsf{Sign}(x, \varphi)\}$. Next, if agent a *upgrades* all the agents signing statement φ in H, we upgrade all of them more reliable than the other agents as in Fig.3.2(iii). Based on this idea, the semantics of $[H \Uparrow^a_\varphi]$ is given by the following definition.

Definition 34. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_d)_{d\in G}, V)$, *a semantic clause for* $[H \Uparrow^a_\varphi]$ *on* $\mathfrak{M}$ *and* $w \in W$ *is defined by:*

$$\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\psi \quad \text{iff} \quad \mathfrak{M}^{H\Uparrow^a_\varphi}, w \models \psi,$$

where $\mathfrak{M}^{H\Uparrow^a_\varphi} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq'_d)_{d\in G}, V)$ *and* $\preccurlyeq'_d$ *is defined as: for all* $u \in W$*:*

- *if* $d \neq a$*, we put* $\preccurlyeq'^u_d = \preccurlyeq^u_d$.

- *otherwise (if $d = a$), we define* $b \preccurlyeq_a'^u c$ iff

  $\big(b, c \in H$ *and* $\mathfrak{M}, u \models \mathsf{Sign}(b, \varphi) \wedge \mathsf{Sign}(c, \varphi)$ *and* $b \preccurlyeq_a^u c\big)$ *or*
  $\big(b, c \in (G \setminus H) \cup \{x \in H \mid \mathfrak{M}, u \models \neg\mathsf{Sign}(x, \varphi)\}$ *and* $b \preccurlyeq_a^u c\big)$ *or*
  $\big(c \in (G \setminus H) \cup \{x \in H \mid \mathfrak{M}, u \models \neg\mathsf{Sign}(x, \varphi)\}$ *and* $b \in H$ *and* $\mathfrak{M}, u \models \mathsf{Sign}(b, \varphi)\big)$.[1]

Note the upgrade operator can preserve the property of total preordering of $(\preccurlyeq_d)_{d \in G}$.

Proposition 35 (Recursive Validities)**.** *The following are valid on all models. Moreover, if ψ is valid on all models, then $[H \Uparrow_\varphi^a]\psi$ is also valid on all models.*

$[H \Uparrow_\varphi^a]p$	$\leftrightarrow$	p	
$[H \Uparrow_\varphi^a](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$[H \Uparrow_\varphi^a](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$[H \Uparrow_\varphi^a](b \leqslant_a c)$	$\leftrightarrow$	$\big(\mathsf{Sign}(b, \varphi) \wedge \mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)\big) \vee$ $\big(\neg\mathsf{Sign}(b, \varphi) \wedge \neg\mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)\big) \vee$ $\big(\mathsf{Sign}(b, \varphi) \wedge \neg\mathsf{Sign}(c, \varphi)\big)$	$(b, c \in H)$
$[H \Uparrow_\varphi^a](b \leqslant_a c)$	$\leftrightarrow$	$\neg\mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)$	$(c \in H, b \in G \setminus H)$
$[H \Uparrow_\varphi^a](b \leqslant_a c)$	$\leftrightarrow$	$\mathsf{Sign}(b, \varphi) \vee \big(\neg\mathsf{Sign}(b, \varphi) \wedge (b \leqslant_a c)\big)$	$(b \in H, c \in G \setminus H)$
$[H \Uparrow_\varphi^a]\neg\psi$	$\leftrightarrow$	$\neg[H \Uparrow_\varphi^a]\psi$	
$[H \Uparrow_\varphi^a](\psi_1 \rightarrow \psi_2)$	$\leftrightarrow$	$[H \Uparrow_\varphi^a]\psi_1 \rightarrow [H \Uparrow_\varphi^a]\psi_2$	
$[H \Uparrow_\varphi^a]\mathsf{Sign}(b, \psi)$	$\leftrightarrow$	$\mathsf{Sign}(b, [H \Uparrow_\varphi^a]\psi)$	
$[H \Uparrow_\varphi^a]\mathsf{Bel}(b, \psi)$	$\leftrightarrow$	$\mathsf{Bel}(b, [H \Uparrow_\varphi^a]\psi)$	

Proof. Our goal is to show that all axioms are valid with respect to the semantics of $[H \Uparrow_\varphi^a]$ (defined in Definition 34) that is straightforward. We will show only the proof of five axioms of reliability ordering as follows:

Case $(d \neq a)$:

$$[H \Uparrow_\varphi^a](b \leqslant_d c) \leftrightarrow b \leqslant_d c$$

Suppose that $d \neq a$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_d c) \text{ iff } \mathfrak{M}, w \models b \leqslant_d c.$$

From $\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_d c)$,

$$\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_d c) \text{ iff } \mathfrak{M}^{H \Uparrow_\varphi^a}, w \models b \leqslant_d c \text{ iff } b \preccurlyeq_d'^w c \text{ iff } b \preccurlyeq_d^w c \text{ iff } \mathfrak{M}, w \models b \leqslant_d c$$

Case $(b, c \in G \setminus H)$:

$$[H \Uparrow_\varphi^a](b \leqslant_a c) \leftrightarrow b \leqslant_a c$$

Suppose that $b, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models b \leqslant_a c.$$

From $\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \Uparrow_\varphi^a](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H \Uparrow_\varphi^a}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq_a'^w c \text{ iff } b \preccurlyeq_a^w c \text{ iff } \mathfrak{M}, w \models b \leqslant_a c$$

[1]Also in this case, since there is no relation between agents b and c, $b \preccurlyeq_a^u c$ is omitted.

Case $(b, c \in H)$:

$$[H \Uparrow^a_\varphi]\,(b \leqslant_a c) \leftrightarrow \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ \big(\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi)\big)$$

Suppose that $b, c \in H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models (b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ \big(\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi)\big).$$

From $\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\,(b \leqslant_a c)$,

$$\begin{aligned}
\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\,(b \leqslant_a c) &\text{ iff } \mathfrak{M}^{H\Uparrow^a_\varphi}, w \models b \leqslant_a c \\
&\text{ iff } b \preccurlyeq'^w_a c \\
&\text{ iff } \big(\mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c\big) \text{ or} \\
&\quad\ \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c\big) \text{ or} \\
&\quad\ \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \text{ and } \mathfrak{M}, w \models \mathsf{Sign}(b,\varphi)\big) \\
&\text{ iff } \big(\mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \text{ or} \\
&\quad\ \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \text{ or} \\
&\quad\ \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \wedge \mathsf{Sign}(b,\varphi)\big) \\
&\text{ iff } \mathfrak{M}, w \models \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\
&\qquad\qquad \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\
&\qquad\qquad \big(\neg\mathsf{Sign}(c,\varphi) \wedge \mathsf{Sign}(b,\varphi)\big)
\end{aligned}$$

Case $(c \in H, b \in G \setminus H)$:

$$[H \Uparrow^a_\varphi]\,(b \leqslant_a c) \leftrightarrow \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)$$

Suppose that $c \in H, b \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\,(b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c).$$

From $\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\,(b \leqslant_a c)$,

$$\begin{aligned}
\mathfrak{M}, w \models [H \Uparrow^a_\varphi]\,(b \leqslant_a c) &\text{ iff } \mathfrak{M}^{H\Uparrow^a_\varphi}, w \models b \leqslant_a c \\
&\text{ iff } b \preccurlyeq'^w_a c \\
&\text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c \\
&\text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)
\end{aligned}$$

Case $(b \in H, c \in G \setminus H)$:

$$[H \Uparrow^a_\varphi]\,(b \leqslant_a c) \leftrightarrow \mathsf{Sign}(b,\varphi) \vee \big(\neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)\big)$$

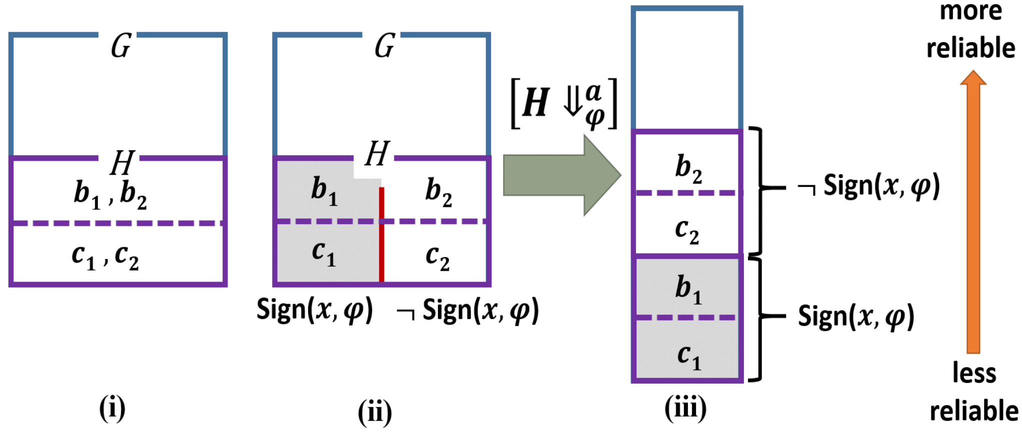

Figure 3.3: Model for downgrading by $[H \Downarrow^a_\varphi]$

Suppose that $b \in H, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Uparrow^a_\varphi] (b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \vee \big(\neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)\big).$$

From $\mathfrak{M}, w \models [H \Uparrow^a_\varphi] (b \leqslant_a c)$,

$$\begin{aligned}
\mathfrak{M}, w \models [H \Uparrow^a_\varphi] (b \leqslant_a c) &\text{ iff } \mathfrak{M}^{H\Uparrow^a_\varphi}, w \models b \leqslant_a c \\
&\text{ iff } b \preccurlyeq'^w_a c \\
&\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \text{ or } \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \text{ and } b \preccurlyeq^w_a c\big) \\
&\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \text{ or } \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)\big) \\
&\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \vee \big(\neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)\big)
\end{aligned}$$

□

3.2.2 Downgrade

This section propose a downgrade operator $[H \Downarrow^a_\varphi]$, where $H \subseteq G$ is a set of agents. This operator aims at changing the reliability of some specific agents to be less reliable. $[H \Downarrow^a_\varphi]\psi$ can be read as "after agent a downgraded such agents who sign statement φ in H, ψ holds." Semantically speaking, $[H \Downarrow^a_\varphi]$ makes such agents who sign φ in H less reliable than all the other agents.

Before giving the detailed semantics, let us demonstrate the effects of $[H \Downarrow^a_\varphi]$ by figures. Firstly, we assume that a rectangle G of Fig. 3.3(i) represents a fixed finite set of agents. Secondly, we will select a specified set of agents in order to change their reliability ordering that can be represented by a rectangle H, and we assume that $b_1 \approx_a b_2 <_a c_1 \approx_a c_2$ holds, i.e., agents b_1 and b_2 which are equally reliable are more reliable than agents c_1 and c_2 which are equally reliable from agent a's perspective. In this sense, b_1, b_2, c_1 and c_2 are situated as in Fig. 3.3(i). Then, if we focus on agents who sign statement φ, H is divided into two equal vertical parts by $\mathsf{Sign}(x,\varphi)$ as in Fig. 3.3(ii), namely by the set

$\{x \in H \mid \mathfrak{M}, w \models \mathsf{Sign}(x, \varphi)\}$ and the set $\{x \in H \mid \mathfrak{M}, w \models \neg\mathsf{Sign}(x, \varphi)\}$. Next, if agent a *downgrades* all the agents signing statement φ in H, we downgrade all of them less reliable than the other agents as in Fig.3.3(iv). Based on this idea, the semantics of $[H \Downarrow^a_\varphi]$ is given by the following definition.

Definition 36. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_d)_{d\in G}, V)$*, a semantic clause for* $[H \Downarrow^a_\varphi]$ *on* $\mathfrak{M}$ *and* $w \in W$ *is defined by:*

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi]\psi \quad \text{iff} \quad \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models \psi,$$

where $\mathfrak{M}^{H\Downarrow^a_\varphi} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq'_d)_{d\in G}, V)$ *and* $\preccurlyeq'_d$ *is defined as: for all* $u \in W$*:*

- *if* $d \neq a$*, we put* $\preccurlyeq'^u_d = \preccurlyeq^u_d$*.*
- *otherwise (if* $d = a$*), we define* $b \preccurlyeq'^u_a c$ iff

$$\big(b, c \in H \textit{ and } \mathfrak{M}, u \models \mathsf{Sign}(b, \varphi) \wedge \mathsf{Sign}(c, \varphi) \textit{ and } b \preccurlyeq^u_a c\big) \textit{ or}$$
$$\big(b, c \in (G \setminus H) \cup \{x \in H \mid \mathfrak{M}, u \models \neg\mathsf{Sign}(x, \varphi)\} \textit{ and } b \preccurlyeq^u_a c\big) \textit{ or}$$
$$\big(b \in (G \setminus H) \cup \{x \in H \mid \mathfrak{M}, u \models \neg\mathsf{Sign}(x, \varphi)\} \textit{ and } c \in H \textit{ and } \mathfrak{M}, u \models \mathsf{Sign}(c, \varphi)\big).^2$$

Note that downgrade operator can preserve the property of total preordering of $(\preccurlyeq_d)_{d\in G}$.

Proposition 37 (Recursive Validities)**.** *The following are valid on all models. Moreover, if* ψ *is valid on all models, then* $[H \Downarrow^a_\varphi]\psi$ *is also valid on all models.*

$[H \Downarrow^a_\varphi]p$	$\leftrightarrow$	p	
$[H \Downarrow^a_\varphi](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$[H \Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$[H \Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\big(\mathsf{Sign}(b, \varphi) \wedge \mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)\big) \vee$ $\big(\neg\mathsf{Sign}(b, \varphi) \wedge \neg\mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)\big) \vee$ $\big(\neg\mathsf{Sign}(b, \varphi) \wedge \mathsf{Sign}(c, \varphi)\big)$	$(b, c \in H)$
$[H \Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\mathsf{Sign}(c, \varphi) \vee \big(\neg\mathsf{Sign}(c, \varphi) \wedge (b \leqslant_a c)\big)$	$(c \in H, b \in G \setminus H)$
$[H \Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\neg\mathsf{Sign}(b, \varphi) \wedge (b \leqslant_a c)$	$(b \in H, c \in G \setminus H)$
$[H \Downarrow^a_\varphi]\neg\psi$	$\leftrightarrow$	$\neg[H \Downarrow^a_\varphi]\psi$	
$[H \Downarrow^a_\varphi](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[H \Downarrow^a_\varphi]\psi_1 \to [H \Downarrow^a_\varphi]\psi_2$	
$[H \Downarrow^a_\varphi]\mathsf{Sign}(b, \psi)$	$\leftrightarrow$	$\mathsf{Sign}(b, [H \Downarrow^a_\varphi]\psi)$	
$[H \Downarrow^a_\varphi]\mathsf{Bel}(b, \psi)$	$\leftrightarrow$	$\mathsf{Bel}(b, [H \Downarrow^a_\varphi]\psi)$	

Proof. Our goal is to show that all axioms are valid with respect to the semantics of $[H \Downarrow^a_\varphi]$ (defined in Definition 36) that is straightforward. We will show only the proof of five axioms of reliability ordering as follows:

Case $(d \neq a)$:

$$[H \Downarrow^a_\varphi](b \leqslant_d c) \leftrightarrow b \leqslant_d c$$

Suppose that $d \neq a$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_d c) \text{ iff } \mathfrak{M}, w \models b \leqslant_d c.$$

[2] In this case, since there is no relation between agents b and c, $b \preccurlyeq^u_a c$ is omitted.

From $\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_d c)$,

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_d c) \text{ iff } \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models b \leqslant_d c \text{ iff } b \preccurlyeq'^w_d c \text{ iff } b \preccurlyeq^w_d c \text{ iff } \mathfrak{M}, w \models b \leqslant_d c$$

Case $(b, c \in G \setminus H)$:

$$[H \Downarrow^a_\varphi](b \leqslant_a c) \leftrightarrow b \leqslant_a c$$

Suppose that $b, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models b \leqslant_a c.$$

From $\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq'^w_a c \text{ iff } b \preccurlyeq^w_a c \text{ iff } \mathfrak{M}, w \models b \leqslant_a c$$

Case $(b, c \in H)$:

$$\begin{aligned}[H \Downarrow^a_\varphi](b \leqslant_a c) \leftrightarrow & \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \big(\neg\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi)\big)\end{aligned}$$

Suppose that $b, c \in H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\begin{aligned}\mathfrak{M}, w \models (b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models & \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \big(\neg\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi)\big).\end{aligned}$$

From $\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c)$,

$$\begin{aligned}\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c) & \text{ iff } \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models b \leqslant_a c \\ & \text{ iff } b \preccurlyeq'^w_a c \\ & \text{ iff } \big(\mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c\big) \text{ or} \\ & \quad \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c\big) \text{ or} \\ & \quad \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \text{ and } \mathfrak{M}, w \models \mathsf{Sign}(c,\varphi)\big) \\ & \text{ iff } \big(\mathfrak{M}, w \models \mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \text{ or} \\ & \quad \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \text{ or} \\ & \quad \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi)\big) \\ & \text{ iff } \mathfrak{M}, w \models \big(\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \qquad \big(\neg\mathsf{Sign}(b,\varphi) \wedge \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big) \vee \\ & \qquad \big(\neg\mathsf{Sign}(b,\varphi) \wedge \mathsf{Sign}(c,\varphi)\big)\end{aligned}$$

Case $(c \in H, b \in G \setminus H)$:

$$[H \Downarrow^a_\varphi](b \leqslant_a c) \leftrightarrow \mathsf{Sign}(c,\varphi) \vee \big(\neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big)$$

Suppose that $c \in H, b \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(c,\varphi) \vee \big(\neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big).$$

From $\mathfrak{M}, w \models [H \Downarrow^a_\varphi] (b \leqslant_a c)$,

$$\begin{aligned}\mathfrak{M}, w \models [H \Downarrow^a_\varphi] (b \leqslant_a c) &\text{ iff } \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models b \leqslant_a c\\ &\text{ iff } b \preccurlyeq'^w_a c\\ &\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(c,\varphi) \text{ or } \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \text{ and } b \preccurlyeq^w_a c\big)\\ &\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(c,\varphi) \text{ or } \big(\mathfrak{M}, w \models \neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big)\\ &\text{ iff } \mathfrak{M}, w \models \mathsf{Sign}(c,\varphi) \vee \big(\neg\mathsf{Sign}(c,\varphi) \wedge (b \leqslant_a c)\big)\end{aligned}$$

Case ($b \in H, c \in G \setminus H$):

$$[H \Downarrow^a_\varphi] (b \leqslant_a c) \leftrightarrow \neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)$$

Suppose that $b \in H, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \Downarrow^a_\varphi] (b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c).$$

From $\mathfrak{M}, w \models [H \Downarrow^a_\varphi] (b \leqslant_a c)$,

$$\begin{aligned}\mathfrak{M}, w \models [H \Downarrow^a_\varphi] (b \leqslant_a c) &\text{ iff } \mathfrak{M}^{H\Downarrow^a_\varphi}, w \models b \leqslant_a c\\ &\text{ iff } b \preccurlyeq'^w_a c\\ &\text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \text{ and } b \preccurlyeq^w_a c\\ &\text{ iff } \mathfrak{M}, w \models \neg\mathsf{Sign}(b,\varphi) \wedge (b \leqslant_a c)\end{aligned}$$

□

3.2.3 Joint Downgrade

From Section 3.2.2, when a judge considers a witness to be unreliable, he/she downgrades such witness less reliable than other witnesses by the downgrade operator $[H \Downarrow^a_\varphi]$. Nevertheless, this downgrade operator cannot be applied in some cases. For example, when judge j receives inconsistent statements from the first witness w_1, he/she considers w_1 to be unreliable and then downgrades w_1 less reliable than other witnesses. Next, if the second witness w_2 gives inconsistent statements, j also downgrades w_2 less reliable than other witnesses. As a result, w_2 is less reliable than w_1. In fact, j cannot determine if the reliability of w_2 is less than w_1 or not. The judge only believes that w_2 is as unreliable as w_1, i.e., both w_1 and w_2 should be equally reliable.

For this reason, we introduce a new kind of downgrade operator, namely *joint downgrade*. The joint downgrade operator $[H \downdownarrows^a]$ allows an agent to downgrade the agents in the specific group equally reliable and less reliable than the agents in the other groups. For $[H \downdownarrows^a]$, $H \subseteq G$ is a set of agents. The reading of $[H \downdownarrows^a]\psi$ is "after such agents in H are downgraded jointly by agent a, ψ holds." Semantically speaking, $[H \downdownarrows^a]$ makes such agents in H equally reliable and less reliable than the agents in the other groups. Note that the joint downgrade operator $[H \downdownarrows^a]$ is different from the downgrade operator $[H \Downarrow^a_\varphi]$ in two respects. First, $[H \downdownarrows^a]$ focuses only on the agents in H without consideration of information, while $[H \Downarrow^a_\varphi]$ considers both the agents in H and their signed information, that is, $[H \Downarrow^a_\varphi]$ focuses on the agents who sign information φ in H. The second respect is the result of downgrading, that is, $[H \downdownarrows^a]$ makes the reliability ordering between agents in H equal, while $[H \Downarrow^a_\varphi]$ keeps the same reliability ordering between agents in H.

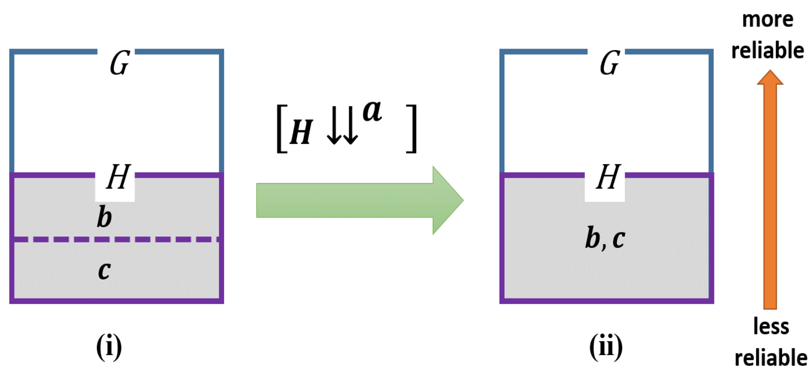

Figure 3.4: Model for jointly downgrading by $[H \downdownarrows^a]$

Before giving the detailed semantics, let us demonstrate the effects of $[H \downdownarrows^a]$ by figures. Firstly, we assume that a rectangle G of Fig. 3.4(i) represents a fixed finite set of agents. Secondly, we will select a specified set of agents in order to jointly downgrade the reliability ordering that can be represented by a rectangle H, and we assume that $b <_a c$ holds, i.e., agent b is more reliable than agent c from agent a's perspective. In this sense, b and c are situated as in Fig. 3.4(i). Then, if the agents in H are *downgraded jointly* by agent a, all of them will be made to be equally reliable and less reliable than all the other agents as in Fig. 3.4(ii). Based on this idea, the semantics of $[H \downdownarrows^a]$ is given by the following definition.

Definition 38. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_d)_{d\in G}, V)$*, a semantic clause for* $[H \downdownarrows^a]$ *on* $\mathfrak{M}$ *and* $w \in W$ *is defined by:*

$$\mathfrak{M}, w \models [H \downdownarrows^a]\psi \quad \text{iff} \quad \mathfrak{M}^{H\downdownarrows^a}, w \models \psi,$$

where $\mathfrak{M}^{H\downdownarrows^a} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq'_d)_{d\in G}, V)$ *and* $\preccurlyeq'_d$ *is defined as: for all* $u \in W$*:*

- *if* $d \neq a$*, we put* $\preccurlyeq'^u_d \; = \; \preccurlyeq^u_d$*.*
- *otherwise (if* $d = a$*), we define* $b \preccurlyeq'^u_a c$ iff

$$\big(b, c \in H\big) \text{ or } \big(b, c \in (G \setminus H) \text{ and } b \preccurlyeq^u_a c\big) \text{ or } \big(b \in (G \setminus H) \text{ and } c \in H\big).$$

Note that this joint downgrade can preserve the property of total preordering of $(\preccurlyeq_d)_{d\in G}$.

Proposition 39 (Recursive Validities)**.** *The following are valid on all models. Moreover, if* ψ *is valid on all models, then* $[H \downdownarrows^a]\psi$ *is also valid on all models.*

$[H \downdownarrows^a]p$	$\leftrightarrow$	p	
$[H \downdownarrows^a](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$[H \downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$[H \downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\top$	$(b, c \in H)$
$[H \downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\top$	$(c \in H, b \in G \setminus H)$
$[H \downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\bot$	$(b \in H, c \in G \setminus H)$
$[H \downdownarrows^a]\neg\psi$	$\leftrightarrow$	$\neg[H \downdownarrows^a]\psi$	
$[H \downdownarrows^a](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[H \downdownarrows^a]\psi_1 \to [H \downdownarrows^a]\psi_2$	
$[H \downdownarrows^a]\mathsf{Sign}(b, \psi)$	$\leftrightarrow$	$\mathsf{Sign}(b, [H \downdownarrows^a]\psi)$	
$[H \downdownarrows^a]\mathsf{Bel}(b, \psi)$	$\leftrightarrow$	$\mathsf{Bel}(b, [H \downdownarrows^a]\psi)$	

Proof. Our goal is to show that all axioms are valid with respect to the semantics of $[H \downdownarrows^a]$ (defined in Definition 38) that is straightforward. We will show only the proof of five axioms of reliability ordering as follows:

Case ($d \neq a$):

$$[H \downdownarrows^a](b \leqslant_d c) \leftrightarrow b \leqslant_d c$$

Suppose that $d \neq a$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_d c) \text{ iff } \mathfrak{M}, w \models b \leqslant_d c.$$

From $\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_d c)$,

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_d c) \text{ iff } \mathfrak{M}^{H\downdownarrows^a}, w \models b \leqslant_d c \text{ iff } b \preccurlyeq_d'^w c \text{ iff } b \preccurlyeq_d^w c \text{ iff } \mathfrak{M}, w \models b \leqslant_d c$$

Case ($b, c \in G \setminus H$):

$$[H \downdownarrows^a](b \leqslant_a c) \leftrightarrow b \leqslant_a c$$

Suppose that $b, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models b \leqslant_a c.$$

From $\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H\downdownarrows^a}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq_a'^w c \text{ iff } b \preccurlyeq_a^w c \text{ iff } \mathfrak{M}, w \models b \leqslant_a c$$

Case ($b, c \in H$):

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \top$$

Suppose that $b, c \in H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \top.$$

From $\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H\downdownarrows^a}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq_a'^w c \text{ iff } \mathfrak{M}, w \models \top$$

Case ($c \in H, b \in G \setminus H$):

$$[H \downdownarrows^a](b \leqslant_a c) \leftrightarrow \top$$

Suppose that $c \in H, b \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \top.$$

From $\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H\downdownarrows^a}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq_a'^w c \text{ iff } \mathfrak{M}, w \models \top$$

Case ($b \in H, c \in G \setminus H$):

$$[H \downdownarrows^a](b \leqslant_a c) \leftrightarrow \bot$$

Suppose that $b \in H, c \in G \setminus H$. Fix any model $\mathfrak{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}, w \models \bot.$$

From $\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c)$,

$$\mathfrak{M}, w \models [H \downdownarrows^a](b \leqslant_a c) \text{ iff } \mathfrak{M}^{H\downdownarrows^a}, w \models b \leqslant_a c \text{ iff } b \preccurlyeq_a'^w c \text{ iff } \mathfrak{M}, w \models \bot$$

□

Chapter 4

Logical Tool for Belief Re-revision

This chapter provides a formal tool for analyzing an agent's belief re-revision from a logical point of view. In Section 4.1, the static logic of agents' belief for signed information as mentioned in Section 3.1 is extended with a relation changer based on PDL. With this logic, we introduce a new version of update mechanism by means of action product. This update mechanism will be used for constructing the private announcement and the private permission in Section 4.2. From Lorini's framework [13] in Section 2.4, they propose two logical operators including the careful policy and the tell-action in different settings. That is, the careful policy is employed in a static viewpoint, while the tell-action is captured in the sense of DEL. A main idea of the tell-action is to handle a private informing between two agents (i.e., a sender and a receiver). Based on this idea, we introduce the private announcement operator in Section 4.2.1. By this private announcement, we can reformulate the careful policy in terms of dynamic operator that is different from Lorini's framework [13]. Thus, we can regard that our private announcement operator can capture both the careful policy and the tell-action from Lorini's framework [13] in a unified setting. In order to cover belief re-revision of an agent, the private permission operator is proposed for dealing with a restoration process of an agent's belief in Section 4.2.3. Finally, we provide a logical formalization by integrating our dynamic operators for reliability change and belief re-revision in Section 4.3.

4.1 PDL-extension of Static Logic of Agents' Beliefs for Signed Information

4.1.1 Syntax and Semantics

In this section, we introduce a language $\mathcal{L}_{RC}$ which is a PDL-extension of $\mathcal{L}_{BSR}$ as mentioned in Section 3.1.

Definition 40. *Let G be a fixed* finite *set of agents. The language $\mathcal{L}_{RC}$ consists of the following vocabulary: (i) a countably infinite set* $\mathsf{Prop} = \{p, q, r, \dots\}$ *of propositional letters, (ii) Boolean connectives: $\neg$, $\rightarrow$, (iii) the constants for reliability ordering $b \leqslant_a c$ ($a, b, c \in G$), (iv) atomic programs:* $\mathsf{1}$, B_a *($a \in G$),* S_a *($a \in G$), (v) program operators: $\cup$ (non-deterministic choice), ; (sequential composition), and (vi) mixed operators: ? (test), $[\cdot]$ (necessity). A set* Form_{RC} *of formulas φ of $\mathcal{L}_{RC}$ and a set* Prog *of programs π of $\mathcal{L}_{RC}$*

are inductively defined as follows:

$$\mathsf{Form}_{RC} \ni \varphi ::= p \,|\, \neg\varphi \,|\, \varphi \to \varphi \,|\, b \leqslant_a c \,|\, [\pi]\varphi$$

$$\mathsf{Prog} \ni \pi ::= \mathsf{1} \,|\, \mathsf{B}_a \,|\, \mathsf{S}_a \,|\, \pi \cup \pi \,|\, \pi ; \pi \,|\, \varphi ?$$

where $p \in \mathsf{Prop}$ *and* $a, b, c \in G$. *Note that* $[\mathsf{B}_a]$ *and* $[\mathsf{S}_a]$ *correspond to the belief operator* $\mathsf{Bel}(a, \cdot)$ *and the signature operator* $\mathsf{Sign}(a, \cdot)$*, respectively. In what follows, we can write* $\mathsf{Bel}(a, \cdot)$ *for* $[\mathsf{B}_a]$ *and* $\mathsf{Sign}(a, \cdot)$ *for* $[\mathsf{S}_a]$*. The abbreviations for* $\wedge$*,* $\vee$*,* $\leftrightarrow$*,* $\top$ *and* $\bot$ *can be defined as shown in Definition 2. The dual operator* $\langle\pi\rangle$ *of* $[\pi]$ *is defined in the same manner in PDL (in Section 2.2). For the semantics of this language, a Kripke model is defined by Definition 30 mentioned in Section 3.1 and the satisfaction relation* $\mathfrak{M}, w \models \varphi$ *is defined in a standard way for modal logic and PDL by Definition 41.*

Definition 41. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_a)_{a\in G}, V)$*, any state* $w \in W$ *and any formula* φ*, we define the* satisfaction relation $\mathfrak{M}, w \models \varphi$ *inductively as follows:*

$$\begin{array}{lll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \mathfrak{M}, w \not\models \varphi \\
\mathfrak{M}, w \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w \models \varphi \textit{ implies } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models b \leqslant_a c & \text{iff} & b \preccurlyeq_a^w c \\
\mathfrak{M}, w \models [\pi]\varphi & \text{iff} & \mathfrak{M}, v \models \varphi \textit{ for all } v \textit{ such that } wR_\pi v,
\end{array}$$

where R_π *can be defined as follows:*

$$\begin{array}{lll}
R_{\mathsf{1}} & = & W \times W \\
R_{\mathsf{B}_a} & = & R_a \\
R_{\mathsf{S}_a} & = & S_a \\
R_{\pi\cup\pi'} & = & R_\pi \cup R_{\pi'} \\
 & = & \{ (w, v) \mid wR_\pi v \textit{ or } wR_{\pi'} v \} \\
R_{\pi;\pi'} & = & R_\pi \circ R_{\pi'} \\
 & = & \{ (w, v) \mid wR_\pi u \textit{ and } uR_{\pi'} v \textit{ for some } u \in W \} \\
R_{\varphi?} & = & \{ (w, v) \mid w = v \textit{ and } \mathfrak{M}, v \models \varphi \}
\end{array}$$

Hilbert-style Axiomatization HRC

Table 4.1 presents the Hilbert-style system **HRC** of $\mathcal{L}_{RC}$. Note that *Incl* refers to an inclusion axiom and [1] can be regarded as an **S5** operator.

Theorem 13 (Soundness)**.** *Let* $\mathbb{M}$ *be the class of all models. For all* $\psi \in \mathsf{Form}_{RC}$*,*

$$\textit{if } \vdash_{\mathbf{HRC}} \psi, \textit{ then } \mathbb{M} \models \psi.$$

Proof. Suppose that $\vdash_{\mathbf{HRC}}$. Our goal is to show that $\mathbb{M} \models \psi$ for all ψ. It suffices to show that all axioms and all rules in **HRC** are valid on all models in a class $\mathbb{M}$ with respect to the semantics of $\mathcal{L}_{RC}$. This is straightforward. □

For the completeness proof for **HRC**, we use the same manner in Section 2.2.2 as the following steps. First, we will define a maximally **HRC**-consistent set **HRC**-MCS by Definition 8. Then, we will give the following lemma.

All instances of propositional tautologies	
$(K_{[\pi]})$	$[\pi](\varphi \to \psi) \to ([\pi]\varphi \to [\pi]\psi)$
$(T_{[1]})$	$[1]\varphi \to \varphi$
$(B_{[1]})$	$\varphi \to [1]\langle 1\rangle\varphi$
$(4_{[1]})$	$[1]\varphi \to [1][1]\varphi$
$(Incl_{[1]})$	$[1]\varphi \to [\pi]\varphi$
$(RA1)$	$[\pi \cup \pi']\varphi \quad \leftrightarrow \quad [\pi]\varphi \wedge [\pi']\varphi$
$(RA2)$	$[\pi; \pi']\varphi \quad \leftrightarrow \quad [\pi][\pi']\varphi$
$(RA3)$	$[\psi?]\varphi \quad \leftrightarrow \quad (\psi \to \varphi)$
$(R_{\leqslant})$	$b \leqslant_a b$
$(Tr_{\leqslant})$	$(b \leqslant_a c \wedge c \leqslant_a d) \to b \leqslant_a d$
$(To_{\leqslant})$	$b \leqslant_a c \vee c \leqslant_a b$
(MP)	From φ and $\varphi \to \psi$, infer ψ
$(Nec_{[\pi]})$	From φ, infer $[\pi]\varphi$

Table 4.1: Hilbert-style system **HRC** for $\mathcal{L}_{RC}$

Lemma 11 (Lindenbaum)**.** *Given any* **HRC***-consistent set* Γ*, there exists an* **HRC***-MCS* Γ^+ *such that* $\Gamma \subseteq \Gamma^+$*.*

Next, the properties of **HRC**-MCS are defined by the following proposition.

Proposition 42. *Let* Γ*,* Σ *and* Δ *be any* **HRC***-MCS. Then, the following hold.*

(i) $\Gamma \vdash_{\mathbf{HRC}} \varphi$ iff $\varphi \in \Gamma$.

(ii) if $\varphi \in \Gamma$ *and* $\vdash_{\mathbf{HRC}} \varphi \to \psi$*, then* $\psi \in \Gamma$.

(iii) $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$.

(iv) $\varphi \to \psi \in \Gamma$ iff $\varphi \in \Gamma$ *implies* $\psi \in \Gamma$.

(v) if $[\pi]\varphi \notin \Gamma$ *and* $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$*, then* $\{\neg\varphi\} \cup \{\psi \mid [\pi]\psi \in \Gamma\} \cup \{\theta \mid [1]\theta \in \Sigma\} \nvdash_{\mathbf{HRC}} \bot$.

(vi) if $\{\langle\alpha;\beta\rangle\varphi \mid \varphi \in \Delta\} \subseteq \Gamma$ *and* $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$*, then* $\{\theta \mid [1]\theta \in \Sigma\} \cup \{\varphi \mid [\alpha]\varphi \in \Gamma\} \cup \{\langle\beta\rangle\psi \mid \psi \in \Delta\} \nvdash_{\mathbf{HRC}} \bot$.

(vii) $[pi \cup \pi']\varphi \in \Gamma$ iff $[\pi]\varphi \in \Gamma$ *and* $[\pi']\varphi \in \Gamma$.

(viii) $[\pi; \pi']\varphi \in \Gamma$ iff $[\pi][\pi']\varphi \in \Gamma$.

(ix) $[\psi?]\varphi \in \Gamma$ iff $(\psi \to \varphi) \in \Gamma$.

Proof. We will show only two items including (v) and (vi) as follows:

(v) if $[\pi]\varphi \notin \Gamma$ and $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$, then $\{\neg\varphi\} \cup \{\psi \mid [\pi]\psi \in \Gamma\} \cup \{\theta \mid [1]\theta \in \Sigma\} \nvdash_{\mathbf{HRC}} \bot$.
First, we suppose that Γ and Σ are **HRC**-MCSs. Then, we also assume that $[\pi]\varphi \notin \Gamma$ and $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$. Our goal is to show:

$$\{\neg\varphi\} \cup \{\psi \mid [\pi]\psi \in \Gamma\} \cup \{\theta \mid [1]\theta \in \Sigma\} \nvdash_{\mathbf{HRC}} \bot$$

Suppose for a contradiction that

$$\{ \neg\varphi \} \cup \{ \psi \mid [\pi]\psi \in \Gamma \} \cup \{ \theta \mid [1]\theta \in \Sigma \} \vdash_{\mathbf{HRC}} \bot$$

This is equivalent to

$$\exists\Delta \subseteq \{ \neg\varphi \} \cup \{ \psi \mid [\pi]\psi \in \Gamma \} \cup \{ \theta \mid [1]\theta \in \Sigma \}(\Delta \text{ is finite and } \vdash_{\mathbf{HRC}} \bigwedge \Delta \to \bot)$$

Fix such $\Delta \subseteq \{ \neg\varphi \} \cup \{ \psi \mid [\pi]\psi \in \Gamma \} \cup \{ \theta \mid [1]\theta \in \Sigma \}$. Then, we suppose that

$$\vdash_{\mathbf{HRC}} (\neg\varphi \wedge \psi_1 \wedge \ldots \wedge \psi_n \wedge \theta_1 \wedge \ldots \wedge \theta_m) \to \bot$$

where $[\pi]\psi_1, \ldots, [\pi]\psi_n \in \Gamma$ and $[1]\theta_1, \ldots, [1]\theta_m \in \Sigma$. It follows that:

$$\begin{aligned}
&\vdash_{\mathbf{HRC}} (\neg\varphi \wedge \psi_1 \wedge \ldots \wedge \psi_n \wedge \theta_1 \wedge \ldots \wedge \theta_m) \to \bot \\
&\text{iff } \vdash_{\mathbf{HRC}} (\psi_1 \wedge \ldots \wedge \psi_n \wedge \theta_1 \wedge \ldots \wedge \theta_m) \to (\neg\varphi \to \bot) \\
&\text{iff } \vdash_{\mathbf{HRC}} (\psi_1 \wedge \ldots \wedge \psi_n \wedge \theta_1 \wedge \ldots \wedge \theta_m) \to \varphi
\end{aligned}$$

By axioms $Nec_{[\pi]}$ and $K_{[\pi]}$, we get

$$\begin{aligned}
&\vdash_{\mathbf{HRC}} ([\pi]\psi_1 \wedge \ldots \wedge [\pi]\psi_n \wedge [\pi]\theta_1 \wedge \ldots \wedge [\pi]\theta_m) \to [\pi]\varphi \\
&\text{iff } \vdash_{\mathbf{HRC}} ([\pi]\psi_1 \wedge \ldots \wedge [\pi]\psi_n \wedge [\pi]\theta_1 \wedge \ldots \wedge [\pi]\theta_m) \to (\neg[\pi]\varphi \to \bot) \\
&\text{iff } \vdash_{\mathbf{HRC}} ([\pi]\psi_1 \wedge \ldots \wedge [\pi]\psi_n \wedge [\pi]\theta_1 \wedge \ldots \wedge [\pi]\theta_m \wedge \neg[\pi]\varphi) \to \bot
\end{aligned}$$

By axiom $Incl_{[1]}$ and (ii), we get that

$$\text{if } [1]\theta \in \Sigma, \text{ then } [1][\pi]\theta \in \Sigma.$$

By $[1][\pi]\theta \in \Sigma$ and our assumption of $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$, we get that $[\pi]\theta_1, \ldots, [\pi]\theta_m \in \Gamma$. Thus, we have $\Gamma' := \{ [\pi]\psi_1, \ldots, [\pi]\psi_n, [\pi]\theta_1, \ldots, [\pi]\theta_m, \neg[\pi]\varphi \} \subseteq \Gamma$ and $\vdash_{\mathbf{HRC}} \bigwedge \Gamma' \to \bot$. Therefore, Γ is **HRC**-inconsistent, but we assume that Γ is an **HRC**-MCS. This is a contradiction.

(vi) if $\{ \langle\alpha;\beta\rangle\varphi \mid \varphi \in \Delta \} \subseteq \Gamma$ and $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$, then $\{ \theta \mid [1]\theta \in \Sigma \} \cup \{ \varphi \mid [\alpha]\varphi \in \Gamma \} \cup \{ \langle\beta\rangle\psi \mid \psi \in \Delta \} \nvdash_{\mathbf{HRC}} \bot$.
First, we suppose that Γ, Σ and Δ are **HRC**-MCSs. Then, we assume $\{ \langle\alpha;\beta\rangle\varphi \mid \varphi \in \Delta \} \subseteq \Gamma$ and $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$. Our goal is to show:

$$\{ \theta \mid [1]\theta \in \Sigma \} \cup \{ \varphi \mid [\alpha]\varphi \in \Gamma \} \cup \{ \langle\beta\rangle\psi \mid \psi \in \Delta \} \nvdash_{\mathbf{HRC}} \bot$$

Suppose for a contradiction that

$$\{ \theta \mid [1]\theta \in \Sigma \} \cup \{ \varphi \mid [\alpha]\varphi \in \Gamma \} \cup \{ \langle\beta\rangle\psi \mid \psi \in \Delta \} \vdash_{\mathbf{HRC}} \bot$$

This is equivalent to

$$\exists\Lambda \subseteq \{ \theta \mid [1]\theta \in \Sigma \} \cup \{ \varphi \mid [\alpha]\varphi \in \Gamma \} \cup \{ \langle\beta\rangle\psi \mid \psi \in \Delta \}(\Lambda \text{ is finite and } \vdash_{\mathbf{HRC}} \bigwedge \Lambda \to \bot)$$

Fix such $\Lambda \subseteq \{ \theta \mid [1]\theta \in \Sigma \} \cup \{ \varphi \mid [\alpha]\varphi \in \Gamma \} \cup \{ \langle\beta\rangle\psi \mid \psi \in \Delta \}$. Then, we suppose that

$$\vdash_{\mathbf{HRC}} (\theta_1 \wedge \ldots \wedge \theta_n \wedge \varphi_1 \wedge \ldots \wedge \varphi_m \wedge \langle\beta\rangle\psi_1 \wedge \ldots \wedge \langle\beta\rangle\psi_k) \to \bot$$

where $[1]\theta_1, \ldots, [1]\theta_n \in \Sigma$, $[\alpha]\varphi_1, \ldots, [\alpha]\varphi_m \in \Gamma$, and $\psi_1, \ldots, \psi_k \in \Delta$. It follows that:

$$\begin{aligned}
&\vdash_{\mathbf{HRC}} \bigwedge\{\, \theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_m \,\} \wedge \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\} \to \bot \\
&\text{iff } \vdash_{\mathbf{HRC}} \bigwedge\{\, \theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_m \,\} \to \left(\bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\} \to \bot\right) \\
&\text{iff } \vdash_{\mathbf{HRC}} \bigwedge\{\, \theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_m \,\} \to \neg \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\}
\end{aligned}$$

By axioms $Nec_{[\alpha]}$ and $K_{[\alpha]}$, we get that:

$$\vdash_{\mathbf{HRC}} \bigwedge\{\, [\alpha]\theta_1, \ldots, [\alpha]\theta_n, [\alpha]\varphi_1, \ldots, [\alpha]\varphi_m \,\} \to [\alpha]\neg \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\}.$$

By axiom $Incl_{[1]}$ and (ii), we get that:

$$\text{if } [1]\theta \in \Sigma, \text{ then } [1][\alpha]\theta \in \Sigma.$$

By $[1][\alpha]\theta \in \Sigma$ and our assumption of $\{\varphi \mid [1]\varphi \in \Sigma\} \subseteq \Gamma$, we get that $[\alpha]\theta_1, \ldots, [\alpha]\theta_n \in \Gamma$. Thus, we have $[\alpha]\theta_1, \ldots, [\alpha]\theta_n, [\alpha]\varphi_1, \ldots, [\alpha]\varphi_m \in \Gamma$. By this, we obtain that:

$$\text{if } \bigwedge\{\, [\alpha]\theta_1, \ldots, [\alpha]\theta_n, [\alpha]\varphi_1, \ldots, [\alpha]\varphi_m \,\} \in \Gamma, \text{ then } [\alpha]\neg \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\} \in \Gamma.$$

From our assumption of $\{\, \langle\alpha;\beta\rangle\varphi \mid \varphi \in \Delta \,\} \subseteq \Gamma$,

$$\{\, \langle\alpha;\beta\rangle\varphi \mid \varphi \in \Delta \,\} \subseteq \Gamma \text{ iff } \{\, \langle\alpha\rangle\langle\beta\rangle\varphi \mid \varphi \in \Delta \,\} \subseteq \Gamma$$

By this and $\psi_1, \ldots, \psi_k \in \Delta$, we get $\langle\alpha\rangle\langle\beta\rangle\psi_1, \ldots, \langle\alpha\rangle\langle\beta\rangle\psi_k \in \Gamma$. Therefore, we have:

$$\langle\alpha\rangle \left(\neg \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\} \wedge \bigwedge\{\, \langle\beta\rangle\psi_1, \ldots, \langle\beta\rangle\psi_k \,\}\right) \in \Gamma.$$

This is equivalent to $\langle\alpha\rangle\bot \in \Gamma$. Since $\langle\alpha\rangle\bot$ iff $\bot$, we obtain $\bot \in \Gamma$. This means that Γ is **HRC**-inconsistent, but we assume that Γ is an **HRC**-MCS. This is a contradiction. □

Definition 43. *The model $\mathfrak{M}$ is a tuple $\mathfrak{M} = (W, U, V)$ is defined by:*

- $W := \{\Gamma \mid \Gamma$ *is an* **HRC**-*MCS*$\}$.
- $\Gamma U \Delta$ iff $\{\varphi \mid [1]\varphi \in \Gamma\} \subseteq \Delta$ *for all formulas* φ.
- $\Gamma \in V(p)$ iff $p \in \Gamma$.

In addition, U of Definition 43 is an equivalent relation by the following lemma.

Lemma 12. *Given the model $\mathfrak{M} = (W, U, V)$ of Definition 43,*

(i) If $\vdash_{\mathbf{HRC}} [1]\varphi \to \varphi$ *for all formulas* φ*, then* U *is reflexive.*

(ii) If $\vdash_{\mathbf{HRC}} \varphi \to [1]\langle 1\rangle\varphi$ *for all formulas* φ*, then* U *is symmetric.*

(iii) If $\vdash_{\mathbf{HRC}} [1]\varphi \to [1][1]\varphi$ *for all formulas* φ*, then* U *is transitive.*

By the above preparation, the canonical model for **HRC** is constructed by the following definition.

Definition 44. *Let* Σ *be an* **HRC**-*MCS. The* canonical model $\mathfrak{M}^{\mathbf{HRC}}$ *is a tuple* $\mathfrak{M} = (W^{\mathbf{HRC}}, (R_a^{\mathbf{HRC}})_{a\in\mathsf{G}}, (S_a^{\mathbf{HRC}})_{a\in\mathsf{G}}, (\preccurlyeq_a^{\mathbf{HRC}})_{a\in\mathsf{G}}, V^{\mathbf{HRC}})$ *for any* **HRC** *is defined by:*

- $W^{\mathbf{HRC}} := \{\Gamma \mid \Gamma$ *is an* **HRC**-*MCS and* $\Sigma U \Gamma\}$.
- $\Gamma R_a^{\mathbf{HRC}} \Delta$ iff $[\mathsf{B}_a]\psi \in \Gamma$ *implies* $\psi \in \Delta$ *for all formulas* ψ.
- $\Gamma S_a^{\mathbf{HRC}} \Delta$ iff $[\mathsf{S}_a]\psi \in \Gamma$ *implies* $\psi \in \Delta$ *for all formulas* ψ.
- $b \preccurlyeq_a^{\Gamma} c$ iff $b \leqslant_a c \in \Gamma$.
- $\Gamma \in V^{\mathbf{HRC}}(p)$ iff $p \in \Gamma$.

By the properties of the maximally consistent set in Proposition 42, we can prove the following Truth Lemma.

Lemma 13 (Truth)**.** *Let* Γ *be any* **HRC**-*MCS. The following is true for all* $\psi \in \mathsf{Form}_{RC}$ *and all* $\pi \in \mathsf{Prog}$*:*

(i) $\mathfrak{M}^{\mathbf{HRC}}, \Gamma \models \psi$ iff $\psi \in \Gamma$.

(ii) $\Gamma R_\pi^{\mathbf{HRC}} \Delta$ iff $[\pi]\psi \in \Gamma$ *implies* $\psi \in \Delta$ *for all* ψ.

Now, we are ready to provide the completeness proof for **HRC** as follows:

Theorem 14 (Completeness)**.** *Let* $\mathbb{M}$ *be the class of all models. For all* $\psi \in \mathsf{Form}_{RC}$*,*

$$\textit{if } \mathbb{M} \models \psi, \textit{ then } \vdash_{\mathbf{HRC}} \psi.$$

Proof. The proof is by contrapositive implication. Suppose that $\nvdash_{\mathbf{HRC}} \psi$. Our goal is to show $\mathbb{M} \not\models \psi$. It suffices to find a counter model $\mathfrak{M}$ such that $\mathfrak{M}, w \not\models \psi$ for some w of $\mathfrak{M}$. By our supposition, we obtain that $\{\neg\psi\}$ is an **HRC**-consistent set, i.e., $\{\neg\psi\} \nvdash_{\mathbf{HRC}} \bot$. By Lemma 11, there exists an **HRC**-MCS Γ such that $\{\neg\psi\} \subseteq \Gamma$, i.e., $\neg\psi \in \Gamma$. By Lemma 13, we obtain that $\mathfrak{M}^{\mathbf{HRC}}, \Gamma \models \neg\psi$, i.e., $\mathfrak{M}^{\mathbf{HRC}}, \Gamma \not\models \psi$, as desired. □

4.1.2 Action Model Update

From Section 2.4, Lorini et al. [13] propose the tell-action for capturing a private action, i.e., when agent b privately tells φ to agent a, only agent a will change his/her belief by φ but the other agents than a will not change their beliefs. Based on this private action, the private announcement and the private permission are captured in terms of *action model* (cf. [27, 6]) in DEL because the action model can be used for modeling a variety of events involving communication including public and private messages. In this section, we introduce the action models which are applied to construct an update operation for the private announcement in Section 4.2.1 and the private permission in Section 4.2.3. First, the action models are defined by Definition 45. Then, the semantics of update models are provided in Definition 46.

Definition 45. *An* action model $\mathbb{E}$ *is a tuple* $\mathbb{E} = (E, (D_c)_{c\in G}, (U_a)_{a\in G}, \mathrm{pre}, \Pi)$ *such that*

- E *is a finite domain of action points,*
- $D_c \subseteq E \times E$ *is an accessibility relation representing beliefs,*

- $U_a \subseteq E \times E$ *is an accessibility relation representing signatures,*
- pre *is a preconditions function that assigns a precondition to each action,*
- $\Pi = (\pi_c)_{c\in G}$ *is a family of programs where* π_c *can be defined as follows:*

$$\pi_c ::= \mathsf{1} \mid \mathsf{B}_c \mid \mathsf{S}_c \mid \pi \cup \pi \mid \pi;\pi \mid \mathrm{pre}(\star_1)? \mid \mathrm{pre}(\star_2)?$$

where $c \in G$*, and* $\star_1$ *and* $\star_2$ *are variables ranging over* $\mathbb{E}$*. Moreover, we use* $\pi_c(e,f)$ *to mean the result of replacing* $\star_1$ *and* $\star_2$ *with actions* e *and* f *in* $\pi_c(e,f)$*, respectively.*

Definition 46. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_a)_{a\in G}, V)$*, a semantic clause for* $[\mathbb{E},\Pi,e]$ *on* $\mathfrak{M}$ *and* $w \in W$ *is defined by:*

$$\mathfrak{M}, w \models [\mathbb{E},\Pi,e]\psi \quad \text{iff} \quad \mathfrak{M}^{\otimes\mathbb{E},\Pi}, (w,e) \models \psi,$$

where (w,e) *is the updated state of* $\mathfrak{M}^{\otimes\mathbb{E},\Pi}$ *(defined just below) by the action model of Definition 45, and* $\mathfrak{M}^{\otimes\mathbb{E},\Pi} = (W', (R'_c)_{c\in G}, (S'_c)_{c\in G}, (\preccurlyeq'_c)_{c\in G}, V')$ *is the updated model by the action model of Definition 45. The updated model* $\mathfrak{M}^{\otimes\mathbb{E},\Pi}$ *which is constructed with an operation called a product update [27] is defined by:*

- $W' := W \times E$.
- $(w,e)R'_c(v,f)$ iff $(e,f) \in D_c$ *and* $wR_{\pi_c(e,f)}v$ *(for all* $c \in G$*).*
- $(w,e)S'_c(v,f)$ iff $wR_{\mathsf{S}_c}v$ *and* $(e,f) \in U_c$ *(for all* $c \in G$*).*
- $d \preccurlyeq_c'^{(w,e)} d'$ iff $d \preccurlyeq_c^w d'$.
- $(w,e) \in V'(p)$ iff $w \in V(p)$.

Note that $[\mathbb{E},\Pi,e]$ states that if action $e \in \mathbb{E}$ occurs, then ψ is true at state w in the result.

Proposition 47 (Recursive Validities)**.** *The following are valid on all models. Moreover, if* ψ *is valid on all models, then* $[\mathbb{E},\Pi,e]\psi$ *is also valid on all models.*

$$\begin{array}{lcl}
[\mathbb{E},\Pi,e]p & \leftrightarrow & p \\
[\mathbb{E},\Pi,e]d \leqslant_c d' & \leftrightarrow & d \leqslant_c d' \\
[\mathbb{E},\Pi,e]\neg\psi & \leftrightarrow & \neg[\mathbb{E},\Pi,e]\psi \\
[\mathbb{E},\Pi,e](\psi_1 \to \psi_2) & \leftrightarrow & [\mathbb{E},\Pi,e]\psi_1 \to [\mathbb{E},\Pi,e]\psi_2 \\
[\mathbb{E},\Pi,e][\mathsf{1}]\psi & \leftrightarrow & \bigwedge_{f\in E}[\mathsf{1}][\mathbb{E},\Pi,f]\psi \\
[\mathbb{E},\Pi,e][\mathsf{S_a}]\psi & \leftrightarrow & \bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi \\
[\mathbb{E},\Pi,e][\mathsf{B_a}]\psi & \leftrightarrow & \bigwedge_{f\in D_a(e)}[\pi_a(e,f)][\mathbb{E},\Pi,f]\psi \\
[\mathbb{E},\Pi,e][\pi\cup\pi']\psi & \leftrightarrow & [\mathbb{E},\Pi,e][\pi]\psi \wedge [\mathbb{E},\Pi,e][\pi']\psi \\
[\mathbb{E},\Pi,e][\pi;\pi']\psi & \leftrightarrow & [\mathbb{E},\Pi,e][\pi][\pi']\psi \\
[\mathbb{E},\Pi,e][\varphi?]\psi & \leftrightarrow & [\mathbb{E},\Pi,e](\varphi \to \psi)
\end{array}$$

Proof. Our goal is to show that all axioms are valid with respect to the semantics of $[\mathbb{E},\Pi,e]$ (defined in Definition 46) that is straightforward. We will show only the proof of three axioms as follows:

- $[\mathbb{E},\Pi,e][1]\psi \leftrightarrow \bigwedge_{f\in E}[1][\mathbb{E},\Pi,f]\psi$

Our goal is to show $\mathbb{M} \models [\mathbb{E},\Pi,e][1]\psi \leftrightarrow \bigwedge_{f\in E}[1][\mathbb{E},\Pi,f]\psi$. Fix any model $\mathfrak{M} \in \mathbb{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [\mathbb{E},\Pi,e][1]\psi \text{ iff } \mathfrak{M}, w \models \bigwedge_{f\in E}[1][\mathbb{E},\Pi,f]\psi.$$

From $\mathfrak{M}, w \models [\mathbb{E},\Pi,e][1]\psi$,

$$\begin{aligned}
\mathfrak{M}, w \models [\mathbb{E},\Pi,e][1]\psi \text{ iff } & \mathfrak{M}^{\otimes\mathbb{E},\Pi}, (w,e) \models [1]\psi \\
\text{iff } & \forall_{(v,f)} \big((w,e)R'_1(v,f) \Rightarrow \mathfrak{M}^{\otimes\mathbb{E},\Pi}, (v,f) \models \psi\big) \\
\text{iff } & \forall_{(v,f)} ((w,e)R'_1(v,f) \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \forall_{(v,f)} ((v \in W \;\; and \;\; f \in E) \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \forall_v \forall_f (f \in E \Rightarrow (v \in W \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi)) \\
\text{iff } & \forall_f (f \in E \Rightarrow \forall_v (v \in W \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi)) \\
\text{iff } & \forall_f (f \in E \Rightarrow \mathfrak{M}, w \models [1][\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \mathfrak{M}, w \models \bigwedge_{f\in E}[1][\mathbb{E},\Pi,f]\psi
\end{aligned}$$

- $[\mathbb{E},\Pi,e][\mathsf{S_a}]\psi \leftrightarrow \bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi$

Our goal is to show $\mathbb{M} \models [\mathbb{E},\Pi,e][\mathsf{S_a}]\psi \leftrightarrow \bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi$. Fix any model $\mathfrak{M} \in \mathbb{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [\mathbb{E},\Pi,e][\mathsf{S_a}]\psi \text{ iff } \mathfrak{M}, w \models \bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi.$$

From $\mathfrak{M}, w \models [\mathbb{E},\Pi,e][\mathsf{S_a}]\psi$,

$$\begin{aligned}
\mathfrak{M}, w \models [\mathbb{E},\Pi,e][\mathsf{S_a}]\psi \text{ iff } & \mathfrak{M}^{\otimes\mathbb{E},\Pi}, (w,e) \models [\mathsf{S_a}]\psi \\
\text{iff } & \forall_{(v,f)} \big((w,e)S'_a(v,f) \Rightarrow \mathfrak{M}^{\otimes\mathbb{E},\Pi}, (v,f) \models \psi\big) \\
\text{iff } & \forall_{(v,f)} ((w,e)S'_a(v,f) \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \forall_{(v,f)} ((wR_{\mathsf{S_a}}v \;\; and \;\; (e,f) \in U_a) \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \forall_v \forall_f ((e,f) \in U_a \Rightarrow (wR_{\mathsf{S_a}}v \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi)) \\
\text{iff } & \forall_f ((e,f) \in U_a \Rightarrow \forall_v (wR_{\mathsf{S_a}}v \Rightarrow \mathfrak{M}, v \models [\mathbb{E},\Pi,f]\psi)) \\
\text{iff } & \forall_f ((e,f) \in U_a \Rightarrow \mathfrak{M}, w \models [\mathsf{S_a}][\mathbb{E},\Pi,f]\psi) \\
\text{iff } & \mathfrak{M}, w \models \bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi,
\end{aligned}$$

where $U_a(e) = \{f \in E \mid (e,f) \in U_a\}$

- $[\mathbb{E},\Pi,e][\mathsf{B_a}]\psi \leftrightarrow \bigwedge_{f\in D_a(e)}[\pi_a(e,f)][\mathbb{E},\Pi,f]\psi$

Our goal is to show $\mathbb{M} \models [\mathbb{E},\Pi,e][\mathsf{B_a}]\psi \leftrightarrow \bigwedge_{f\in D_a(e)}[\pi_a(e,f)][\mathbb{E},\Pi,f]\psi$. Fix any model $\mathfrak{M} \in \mathbb{M}$ and any state $w \in W$. It suffices to show:

$$\mathfrak{M}, w \models [\mathbb{E},\Pi,e][\mathsf{B_a}]\psi \text{ iff } \mathfrak{M}, w \models \bigwedge_{f\in D_a(e)}[\pi_a(e,f)][\mathbb{E},\Pi,f]\psi.$$

From $\mathfrak{M}, w \models [\mathbb{E}, \Pi, e][\mathsf{B_a}]\psi$,

$$
\begin{aligned}
\mathfrak{M}, w \models [\mathbb{E}, \Pi, e][\mathsf{B_a}]\psi \text{ iff } & \mathfrak{M}^{\otimes \mathbb{E},\Pi}, (w,e) \models [\mathsf{B_a}]\psi \\
\text{iff } & \forall_{(v,f)} \big((w,e)R'_a(v,f) \Rightarrow \mathfrak{M}^{\otimes \mathbb{E},\Pi}, (v,f) \models \psi\big) \\
\text{iff } & \forall_{(v,f)} ((w,e)R'_a(v,f) \Rightarrow \mathfrak{M}, v \models [\mathbb{E}, \Pi, f]\psi) \\
\text{iff } & \forall_{(v,f)} \big((wR_{\pi_a(e,f)}v \ \ and \ \ (e,f) \in D_a) \Rightarrow \mathfrak{M}, v \models [\mathbb{E}, \Pi, f]\psi\big) \\
\text{iff } & \forall_v \forall_f \big(wR_{\pi_a(e,f)}v \Rightarrow ((e,f) \in D_a \Rightarrow \mathfrak{M}, v \models [\mathbb{E}, \Pi, f]\psi)\big) \\
\text{iff } & \forall_f \big((e,f) \in D_a \Rightarrow \forall_v (wR_{\pi_a(e,f)}v \Rightarrow \mathfrak{M}, v \models [\mathbb{E}, \Pi, f]\psi)\big) \\
\text{iff } & \forall_f ((e,f) \in D_a \Rightarrow \mathfrak{M}, w \models [\pi_a(e,f)][\mathbb{E}, \Pi, f]\psi) \\
\text{iff } & \mathfrak{M}, w \models \bigwedge_{f \in D_a(e)} [\pi_a(e,f)][\mathbb{E}, \Pi, f]\psi,
\end{aligned}
$$

where $D_a(e) = \{f \in E \mid (e,f) \in D_a\}$ □

4.2 Dynamic Operators for Belief Re-revision

This section presents three operators including private announcement, careful policy and private permission for formalizing belief re-revision of an agent. In Section 4.2.1, the private announcement is proposed for handling an agent's commitment. For example, when a judge receives signed information from a witness and considers it to be reliable, he/she will accept and believe the received signed information by applying the private announcement. Section 4.2.2 presents the careful policy which aims to deal with an information aggregation based on Lorini's framework [13] (in Section 2.4). That is, when a judge receives many signed information from witnesses, he/she needs to derive his/her belief from the received information. However, the careful policy in our formalism is captured in terms of dynamic operator by the help of the private announcement. This is different from Lorini's framework [13] which employs the careful policy in a static viewpoint. Section 4.2.3 addresses the private permission providing a process of belief restoration. Both private announcement and private permission operators have the same concept of a private action. That is, when there is an announcement, only an agent who knows about such announcement will change his/her belief. In order to deal with this private action, the action model update operation as mentioned in Section 4.1.2 is applied for both private announcement and private permission operators.

4.2.1 Private Announcement

From Lorini's framework [13] (mentioned in Section 2.4), they provide the tell-action for capturing a private action which enables an agent to restrict his/her belief as received information. Based on this idea, the *private announcement* operator $[\varphi \rightsquigarrow a]$ (whose reading is "a private announcement of φ to agent a") is introduced. The first concept of this operator is to restrict a's attention to the φ's states. With this operator, an agent can remove some possibilities from his/her belief that can be described by Example 48.

Example 48. *Fig. 4.1 illustrates a process of a private announcement of* $\mathsf{Sign}(b,p)$ *to agent a. This private announcement can be interpreted as agent b privately tells information p to agent a. We can represent "agent b tells information p" by* $\mathsf{Sign}(b,p)$*. When*

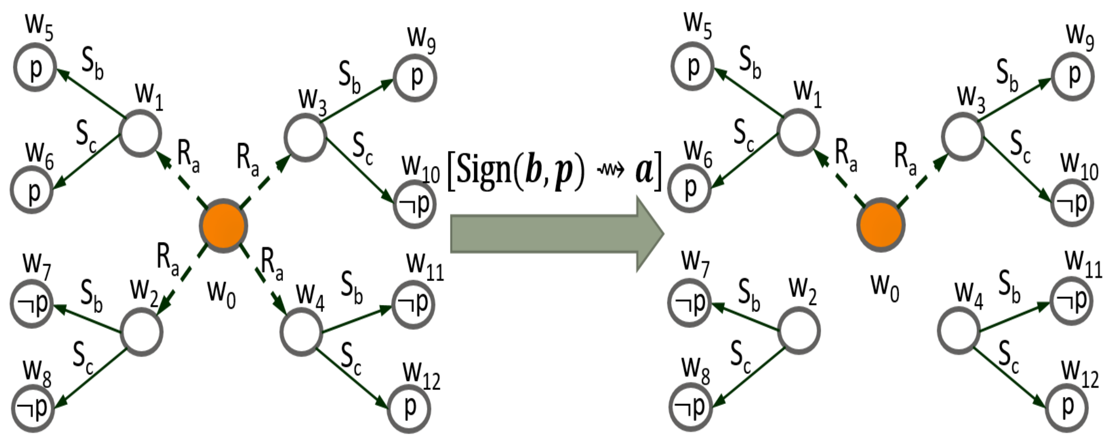

Figure 4.1: Update operation of [$\mathsf{Sign}(b, p) \rightsquigarrow a$]

agent a receives $\mathsf{Sign}(b, p)$*, he/she commits him/herself to* $\mathsf{Sign}(b, p)$ *by* [$\mathsf{Sign}(b, p) \rightsquigarrow a$]*. Let us describe a process of this private announcement. Firstly, agent a does not believe* $\mathsf{Sign}(b, p)$*, i.e.,* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *as in the left-hand side of Fig. 4.1. By the update of* [$\mathsf{Sign}(b, p) \rightsquigarrow a$]*, we delete all the links from state w_0 into the states where* $\mathsf{Sign}(b, p)$ *is false. That is, the links into states w_2 and w_4 will be eliminated. After this, agent a now believes* $\mathsf{Sign}(b, p)$*, i.e.,* $\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *as shown in the right-hand side of Fig. 4.1.*

The second concept of this private announcement is that the other agent than a will not notice a's belief change.[1] One of the merits of this operator is that a sender of message φ is not specified, while a recipient is defined as agent a. This means that we may use this operator also for self-decision of agent a, i.e., the sender and the recipient are the same. In order to capture this private action, we introduce the following action model.

Definition 49. *An action model $\mathbb{E}_{!^a_\varphi}$ for a private announcement of φ to agent a is defined as a tuple $(E, (D_c)_{c \in G}, (U_a)_{a \in G}, \text{pre}, \Pi)$ (based on Definition 45), where E is a finite domain of action points consisting of two actions: φ-announcing action $!^a_\varphi$ to agent a and non-announcing action $\top$,*[2] *D_c is an accessibility relation representing beliefs such that $D_a = \{ (!^a_\varphi, !^a_\varphi), (\top, \top) \}$ and $D_c = \{ (!^a_\varphi, \top), (\top, \top) \}$ if $c \neq a$, U_c is an accessibility relation representing signatures such that $U_c = \{ (!^a_\varphi, \top), (\top, \top) \}$ for all $c \in G$,* pre *is a preconditions function that assigns a precondition to each action by $\text{pre}(!^a_\varphi) = \varphi$ and* $\text{pre}(\top) = \top$*, and $\Pi = (\pi_c)_{c \in G}$ is a family of programs where π_c can be defined as:*

$$\pi_c = \mathsf{B}_c \,;\, pre(\star_2)?$$

Example 50. *Given a Kripke model $\mathfrak{M} = (W, (R_a)_{a \in G}, (S_a)_{a \in G}, (\preccurlyeq_a)_{a \in G}, V)$ (defined in Definition 30) and an action model $\mathbb{E}_{!^a_{\mathsf{Sign}(b,p)}}$ for a private announcement of* $\mathsf{Sign}(b, p)$ *to*

[1] [$\varphi \rightsquigarrow a$] captures that the action of a's privately receiving message φ will not affect of the other agents' beliefs than a. Thus, this work considers only the case that the other agents than a do not know about such event.

[2] The φ-announcing action $!^a_\varphi$ is an action where there is an announcement of φ, while the non-announcing action is an action where nothing happens.

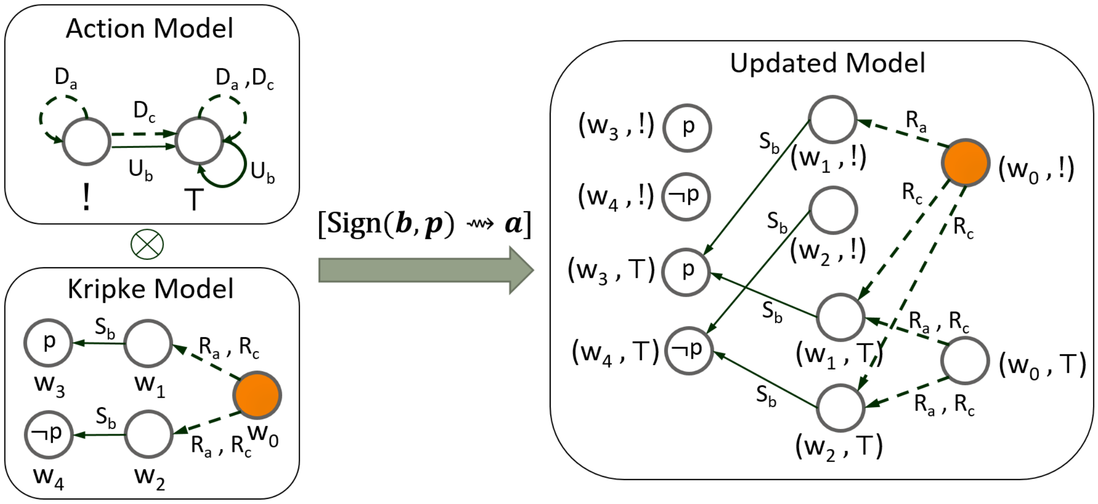

Figure 4.2: Product update operation of $[\mathsf{Sign}(b,p) \rightsquigarrow a]$ (! represents the $\mathsf{Sign}(b,p)$-announcing action $!^a_{\mathsf{Sign}(b,p)}$ to agent a)

agent a (defined in Definition 49), we can define an updated model $\mathfrak{M}^{\otimes \mathbb{E}_{!^a_{\mathsf{Sign}(b,p)}},\Pi}$ which is a tuple $(W', (R'_c)_{c\in G}, (S'_c)_{c\in G}, (\preccurlyeq'_c)_{c\in G}, V')$ (see the right-hand side of Fig. 4.2) by Definition 46 as follows:

$$
\begin{array}{lll}
W' & = & \{(w_0, !^a_{\mathsf{Sign}(b,p)}), (w_1, !^a_{\mathsf{Sign}(b,p)}), (w_2, !^a_{\mathsf{Sign}(b,p)}), (w_3, !^a_{\mathsf{Sign}(b,p)}), (w_4, !^a_{\mathsf{Sign}(b,p)}), \\
 & & (w_0, \top), (w_1, \top), (w_2, \top), (w_3, \top), (w_4, \top)\} \\
R'_a & = & \{\big((w_0, !^a_{\mathsf{Sign}(b,p)}), (w_1, !^a_{\mathsf{Sign}(b,p)})\big), \big((w_0, \top), (w_1, \top)\big), \big((w_0, \top), (w_2, \top)\big)\} \\
R'_c & = & \{\big((w_0, !^a_{\mathsf{Sign}(b,p)}), (w_1, \top)\big), \big((w_0, !^a_{\mathsf{Sign}(b,p)}), (w_2, \top)\big), \big((w_0, \top), (w_1, \top)\big), \\
 & & \big((w_0, \top), (w_2, \top)\big)\} \\
S'_b & = & \{\big((w_1, !^a_{\mathsf{Sign}(b,p)}), (w_3, \top)\big), \big((w_2, !^a_{\mathsf{Sign}(b,p)}), (w_4, \top)\big), \big((w_1, \top), (w_3, \top)\big), \\
 & & \big((w_2, \top), (w_4, \top)\big)\} \\
\preccurlyeq'_c & = & \preccurlyeq_c \\
V'(p) & = & \{(w_3, !^a_{\mathsf{Sign}(b,p)}), (w_3, \top)\}
\end{array}
$$

Let us describe how a Kripke model is updated with the action model $\mathbb{E}_{!^a_{\mathsf{Sign}(b,p)}}$ by Fig. 4.2. First, agents a and c do not believe $\mathsf{Sign}(b,p)$, i.e., $\neg\mathsf{Bel}(a, \mathsf{Sign}(b,p))$ and $\neg\mathsf{Bel}(c, \mathsf{Sign}(b,p))$ (see Kripke model in the left-hand side of Fig. 4.2). By the product update operation, agent a believes $\mathsf{Sign}(b,p)$, i.e., $\mathsf{Bel}(a, \mathsf{Sign}(b,p))$, but agent c still does not believe $\mathsf{Sign}(b,p)$, i.e., $\neg\mathsf{Bel}(c, \mathsf{Sign}(b,p))$ (see the right-hand side of Fig. 4.2). The updated model can be explained as follows: When we focus on the action $!^a_{\mathsf{Sign}(b,p)}$ representing "there is an announcement of $\mathsf{Sign}(b,p)$ to agent a", we obtain that only agent a changes his/her belief, and his/her belief change will not be noticed by agent c. On the other hand, when we focus on the action $\top$, we obtain that both agents a and c do not change their belief because there is no announcement.

With this private announcement, we can capture both the tell-action and the careful policy from Lorini's framework [13] (in Section 2.4). In this section, we only describe how to capture the tell-action by the private announcement, while the description for the careful policy will be provided in Section 4.2.2. For the tell-action $[\mathsf{Tell}(b, a, \varphi)]$, a concept is that agent a would update his/her belief by the signed statement $\mathsf{Sign}(b, \varphi)$ after $[\mathsf{Tell}(b, a, \varphi)]$. Thus, we can define:

$$[\mathsf{Tell}(b, a, \varphi)]\psi := [\mathsf{Sign}(b, \varphi) \rightsquigarrow a]\psi.$$

4.2.2 Careful Policy

From Lorini's framework [13] as mentioned in Section 2.4, a *careful policy* which is one of aggregation policies aims to derive an agent's beliefs from the received signed information. The policy is that only statements which are universally signed by a group of agents who are equally reliable are accepted as beliefs. Since Lorini et al. [13] did not propose a logical treatment from dynamic epistemic viewpoints for the careful policy, this study proposes to capture the careful policy by the private announcement[3] as follows: Firstly, the careful policy is defined in terms of dynamic operators by $[\mathsf{Careful}(a, \varphi)]$, whose reading is "agent a aggregates information about φ." Then, we can obtain $\mathsf{UniSign}(\varphi, a)$, whose reading is "agent a believes that φ is universally signed by a group of agents who are equally reliable," by using the definition in Section 2.4 as follows:

$$\mathsf{UniSign}(\varphi, a) := \bigvee_{i \leq M} \left(\begin{array}{c} \mathsf{Bel}\big(a, \mathsf{Sign}(\mathsf{C}_i^a, \varphi)\big) \wedge \\ \mathsf{Bel}\big(a, \bigwedge_{1 \leq j \leq i-1} \neg \mathsf{Sign}(\mathsf{C}_j^a, \neg\varphi)\big) \end{array} \right),$$

where M is the maximum natural number of $\{i \leq \#G \mid \mathsf{C}_i^a \neq \emptyset\}$ and C_i^a is a group of agents who are equally reliable from agent a's perspective (defined in Section 3.1). Then, Lorini et al. [13]'s definition of careful policy is introduced as the following implication:

$$\mathsf{UniSign}(\varphi, a) \rightarrow \mathsf{Bel}(a, \varphi).$$

However, Lorini et al. did not discuss how we can handle the idea of careful policy in terms of dynamic operators, while they used the policy as a meta-logical principle. With the help of our private announcement operator $[\varphi \rightsquigarrow a]$, we now define the careful policy as a dynamic operator as follows:

$$[\mathsf{Careful}(a, \varphi)]\psi := \mathsf{UniSign}(\varphi, a) \rightarrow [\varphi \rightsquigarrow a]\psi,$$

where $[\mathsf{Careful}(a, \varphi)]\psi$ can be read as "after agent a aggregates information about φ by the careful policy, ψ holds."

Example 51. *Fig. 4.3 illustrates how agent a aggregates information about $\neg p$. At state w_0, agent a believes that agent b is more reliable than agent c, i.e.,* $\mathsf{Bel}(a, b <_a c)$. *In the initial situation, agent a believes* $\mathsf{Sign}(b, \neg p)$ *but does not believe* $\mathsf{Sign}(c, \neg p)$, *i.e.,* $\mathsf{Bel}(a, \mathsf{Sign}(b, \neg p))$ *and* $\neg\mathsf{Bel}(a, \mathsf{Sign}(c, \neg p))$ *as shown in the left-hand side of Fig. 4.3. By* $[\mathsf{Careful}(a, \neg p)]$, *agent a will aggregate information from agent b who is more reliable than agent c. Finally, agent a believes $\neg p$, i.e.,* $\mathsf{Bel}(a, \neg p)$ *as shown in the right-hand side of Fig. 4.3.*

[3] Note that the sender and the recipient are regarded as the same to capture the careful policy.

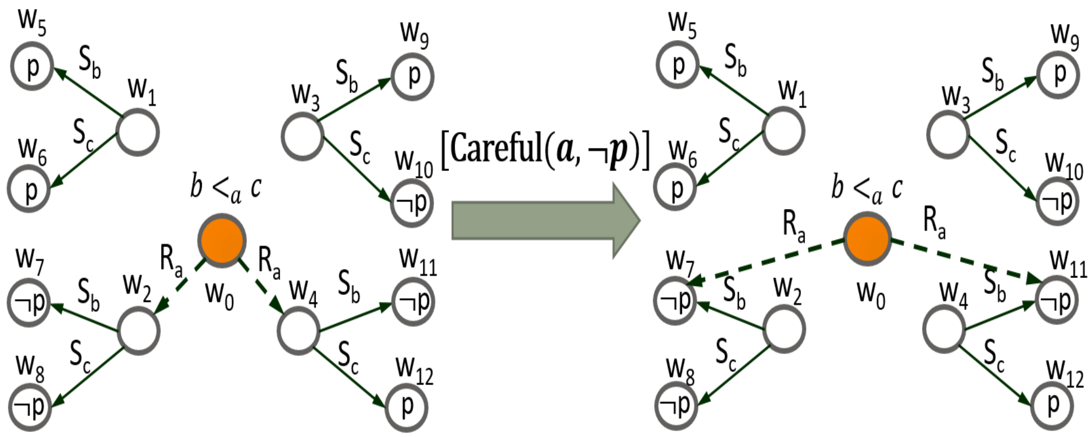

Figure 4.3: Agent a aggregates information about $\neg p$ by [$\mathsf{Careful}(a, \neg p)$].

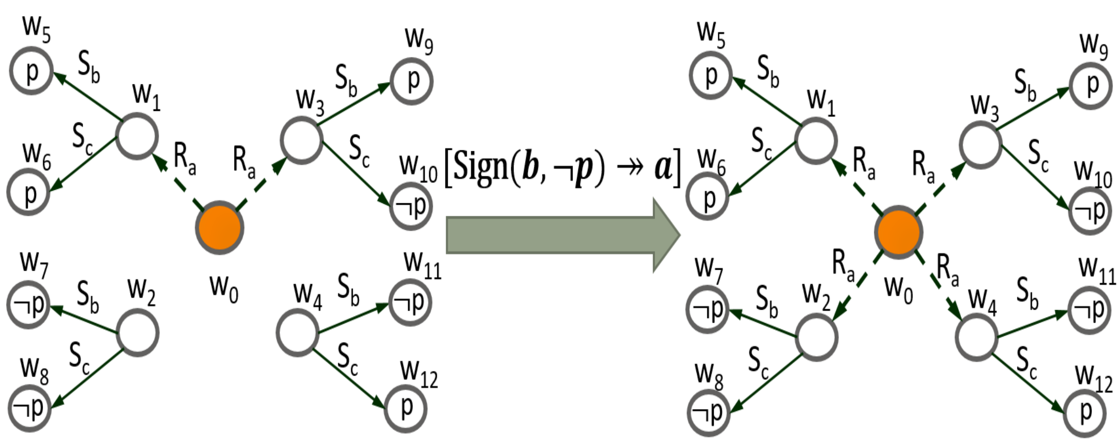

Figure 4.4: Update operation of [$\mathsf{Sign}(b, \neg p) \twoheadrightarrow a$]

4.2.3 Private Permission

In order to capture belief re-revision of an agent, it requires two processes including removing and restoring. The first process can be done by the private announcement as mentioned in Section 4.2.1. In order to cover the second process, the *private permission* $[\varphi \twoheadrightarrow a]$ is introduced. Our intended reading of $[\varphi \twoheadrightarrow a]\psi$ is "after the agent a permitted φ to be the case, ψ." Semantically speaking, $[\varphi \twoheadrightarrow a]$ enlarges a's attention to cover all the φ's states. In this study, we assume that an effect of this private permission is applied globally for all $w \in W$. The first concept of this operator is to restore the former possibilities to an agent's belief that can be described by Example 52.

Example 52. *Fig. 4.4 illustrates a process of a private permission of* $\mathsf{Sign}(b, \neg p)$ *to agent* a*.* $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$ *can be interpreted as agent* a *privately permits the possibility of* $\mathsf{Sign}(b, \neg p)$ *to his/her belief. Firstly, agent* a *believes* $\mathsf{Sign}(b, p)$*, i.e.,* $\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *as shown in the left-hand side of Fig. 4.4. Then,* $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$ *allows us to restore all* *the former links to the states where* $\mathsf{Sign}(b, \neg p)$ *is true. That is, the links into states* w_2 *and* w_4 *will be restored as shown in the right-hand side of Fig. 4.4. At this stage, agent* a *becomes undetermined on* $\mathsf{Sign}(b, \neg p)$*, i.e.,* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *and* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, \neg p))$*. Note that the right-hand side of Fig. 4.4 represents the result of* $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$ *when we focus on a current viewpoint of agent* a *representing by state* w_0*, that is, it shows only the links from state* w_0*.*

From Fig. 4.1 (mentioned in Section 4.2.1) and Fig. 4.4, let us describe a restoration process of the former links. First, agent a accepts $\mathsf{Sign}(b, p)$ by applying $[\mathsf{Sign}(b, p) \rightsquigarrow a]$. As a result, we delete all the links into the states where $\mathsf{Sign}(b, p)$ is false (see Fig. 4.1). Next, agent a reconsiders his/her decision and decides to reject $\mathsf{Sign}(b, p)$ but accept $\mathsf{Sign}(b, \neg p)$ instead. In order to overturn the decision, agent a first needs to permit the possibility of $\mathsf{Sign}(b, \neg p)$ by applying $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$. By the update of $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$, we add all the links into the states where $\mathsf{Sign}(b, \neg p)$ is true (see Fig. 4.4). Note that the states where $\mathsf{Sign}(b, p)$ is false are the same as the states where $\mathsf{Sign}(b, \neg p)$ is true. Therefore, we can regard that all the links into $\mathsf{Sign}(b, \neg p)$'s states (*i.e.*, states where $\mathsf{Sign}(b, \neg p)$ is true), which are deleted by $[\mathsf{Sign}(b, \neg p) \rightsquigarrow a]$, can be restored by $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$.

Similar to the private announcement, the second concept of $[\varphi \twoheadrightarrow a]$ is to allow only agent a to notice his/her belief change after he/she permitted φ to be the case.[4] In order to capture this private action, we introduce the following action model.

Definition 53. *An action model* $\mathbb{E}_{¡^a_\varphi}$ *for a private permission of* φ *to agent* a *is defined as a tuple* $(E, (D_c)_{c\in G}, (U_a)_{a\in G}, \mathrm{pre}, \Pi)$ *(based on Definition 45), where* E *is a finite domain of action points consisting of two actions:* φ*-announcing action* $¡^a_\varphi$ *to agent* a *and non-announcing action* $\bot$*,*[5] D_c *is an accessibility relation representing beliefs such that* $D_a = \{ (¡^a_\varphi, ¡^a_\varphi), (\bot, \bot) \}$ *and* $D_c = \{ (¡^a_\varphi, \bot), (\bot, \bot) \}$ *if* $c \neq a$*,* U_c *is an accessibility relation representing signatures such that* $U_c = \{ (¡^a_\varphi, \bot), (\bot, \bot) \}$ *for all* $c \in G$*,* pre *is a preconditions function that assigns a precondition to each action by* $\mathrm{pre}(¡^a_\varphi) = \varphi$ *and* $\mathrm{pre}(\bot) = \bot$*, and* $\Pi = (\pi_c)_{c\in G}$ *is a family of programs where* π_c *can be defined as:*

$$\pi_c \;=\; \mathsf{B}_c \cup \big(\mathsf{1}\,;\, \mathit{pre}(\star_2)?\big)$$

According to Definition 46, we can describe how a Kripke model is updated with the action model of Definition 53 by Example 54.

Example 54. *Given a Kripke model* $\mathfrak{M} = (W, (R_a)_{a\in G}, (S_a)_{a\in G}, (\preccurlyeq_a)_{a\in G}, V)$ *(defined in Definition 30) and an action model* $\mathbb{E}_{¡^a_{\mathsf{Sign}(b,\neg p)}}$ *for a private permission of* $\mathsf{Sign}(b, \neg p)$ *to agent* a *(defined in Definition 53), we can define an updated model* $\mathfrak{M}^{\otimes \mathbb{E}_{¡^a_{\mathsf{Sign}(b,\neg p)}}, \Pi}$ *which is a tuple* $(W', (R'_c)_{c\in G}, (S'_c)_{c\in G}, (\preccurlyeq'_c)_{c\in G}, V')$ *(see the right-hand side of Fig. 4.5) by Definition*

[4]Based on the idea of the private action, $[\varphi \twoheadrightarrow a]$ captures that the action of a's privately permitting the possibility of φ will not affect of the other agents' beliefs than a. Thus, this work considers only the case that the other agents than a do not know about such event.

[5]The φ-announcing action $¡_\varphi$ is an action where there is an announcement of φ, while the non-announcing action is an action where nothing happens.

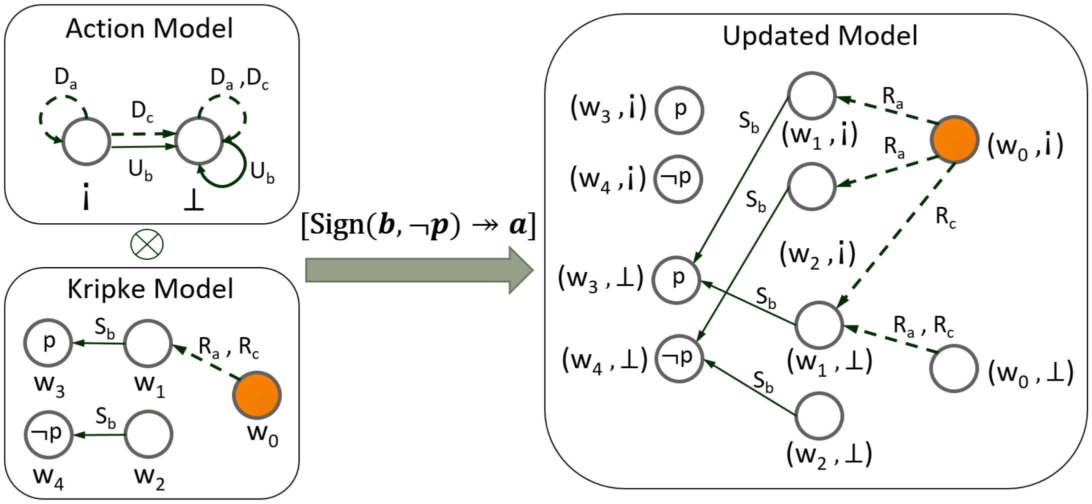

Figure 4.5: Product update operation of $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$ (¡ represents the $\mathsf{Sign}(b, \neg p)$-announcing action $\text{¡}^a_{\mathsf{Sign}(b,\neg p)}$ to agent a)

46 as follows:

$$
\begin{array}{lcl}
W' & = & \{(w_0, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_1, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_2, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_3, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_4, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), \\
& & (w_0, \bot), (w_1, \bot), (w_2, \bot), (w_3, \bot), (w_4, \bot)\} \\
R'_a & = & \{\big((w_0, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_1, \text{¡}^a_{\mathsf{Sign}(b,\neg p)})\big), \big((w_0, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_2, \text{¡}^a_{\mathsf{Sign}(b,\neg p)})\big), \\
& & \big((w_0, \bot), (w_1, \bot)\big)\} \\
R'_c & = & \{\big((w_0, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_1, \bot)\big), \big((w_0, \bot), (w_1, \bot)\big)\} \\
S'_b & = & \{\big((w_1, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_3, \bot)\big), \big((w_2, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_4, \bot)\big), \big((w_1, \bot), (w_3, \bot)\big), \\
& & \big((w_2, \bot), (w_4, \bot)\big)\} \\
\preccurlyeq'_c & = & \preccurlyeq_c \\
V'(p) & = & \{(w_3, \text{¡}^a_{\mathsf{Sign}(b,\neg p)}), (w_3, \bot)\}
\end{array}
$$

The result of the product update operation can be described by Fig. 4.5. First, agents a and c believe $\mathsf{Sign}(b, p)$*, i.e.,* $\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *and* $\mathsf{Bel}(c, \mathsf{Sign}(b, p))$ *(see Kripke model in the left-hand side of Fig. 4.5). By the product update operation, agent a does not believe both* $\mathsf{Sign}(b, p)$ *and* $\mathsf{Sign}(b, \neg p)$*, i.e.,* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *and* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, \neg p))$*, but agent c still believe* $\mathsf{Sign}(b, p)$*, i.e.,* $\mathsf{Bel}(c, \mathsf{Sign}(b, p))$ *as shown in the right-hand side of Fig. 4.5. The updated model in this figure can be described as follows: When we focus on the action* $\text{¡}^a_{\mathsf{Sign}(b,\neg p)}$*, we obtain that only agent a changes his/her belief from* $\mathsf{Bel}(a, \mathsf{Sign}(b, p))$ *into* $\neg\mathsf{Bel}(a, \mathsf{Sign}(b, p))$*. This belief change of agent a can be noticed only by him/herself. On the other hand, when we focus on the action $\bot$, we obtain that beliefs of agents a and b are the same as the initial situation before* $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$*. Note that the right-hand side of Fig. 4.5 (the updated model* $\mathfrak{M}^{\otimes \mathbb{E}_{\text{¡}^a_{\mathsf{Sign}(b,\neg p)}}, \Pi}$*) represents the result of* $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow a]$ *when we focus on a current viewpoint of agent a representing by state $(w_0, \text{¡})$, that is, it shows only the links from state $(w_0, \text{¡})$.*

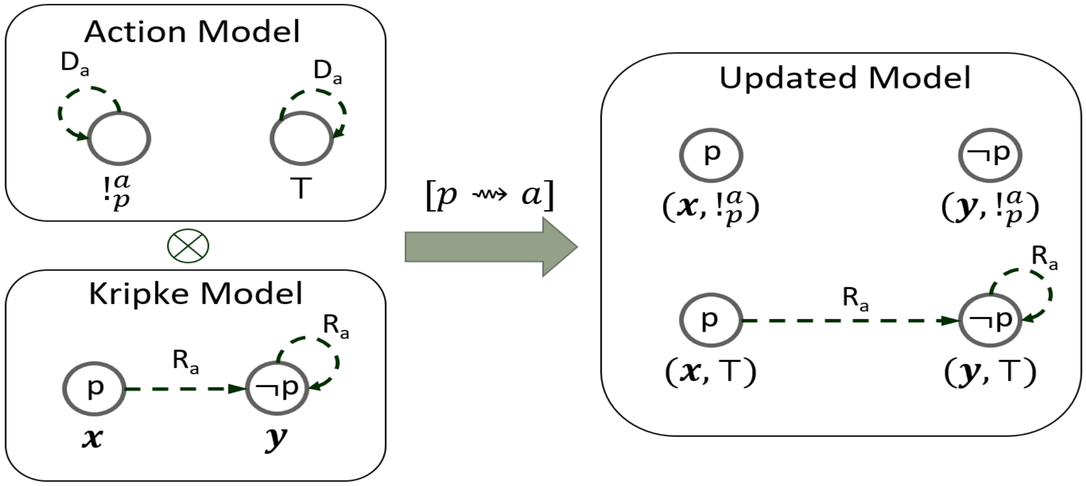

Figure 4.6: Product update operation of $[p \rightsquigarrow a]$

Interaction between Private Announcement and Private Permission

This section describes how we need both private announcement and private permission. For R_c, we suppose that R_c has no properties of relations in order to capture the non-monotonic change of an agent's belief. That is, we allow an agent to change his/her belief from $\mathsf{Bel}(a, \varphi)$ into $\neg\mathsf{Bel}(a, \varphi)$. For example, we assume that agent a first does not believe φ, i.e., $\neg\mathsf{Bel}(a, \varphi)$. By the private announcement, agent a changes his/her belief from $\neg\mathsf{Bel}(a, \varphi)$ into $\mathsf{Bel}(a, \varphi)$ by removing the links into $\neg\varphi$'s states. If there is no link, we can regard that agent a believes both φ and $\neg\varphi$, i.e., $\mathsf{Bel}(a, \bot)$. In this sense, the property of seriality cannot be preserved. However, the private announcement can preserve the properties of transitivity and Euclideanness. Next, the private permission may be applied in order to change agent a's belief from $\mathsf{Bel}(a, \varphi)$ into $\neg\mathsf{Bel}(a, \varphi)$ by adding the links into $\neg\varphi$'s states. This process may break the properties of transitivity and Euclideanness. Nevertheless, the property of seriality can be preserved by the private permission. For this reason, we may regard that the repetitive application of the private announcement and the private permission could retrieve the properties of seriality, transitivity and Euclideanness.

Proposition 55. *The private announcement $[\varphi \rightsquigarrow a]$ cannot preserve the property of seriality.*

Proof. It is easily proved by counterexample. Suppose that $G = \{a\}$. Given a Kripke model $\mathfrak{M} = (W, R_a, V)$, where $W = \{x, y\}$, $R_a = \{(x, y), (y, y)\}$ and $V(p) = \{x\}$ (see a Kripke model in Fig. 4.6). Let $\mathbb{E}$ be an action model for a private announcement of φ to agent a which is defined as a tuple (E, D_a, pre), where $E = \{!^a_\varphi, \top\}$, $D_a = \{(!^a_\varphi, !^a_\varphi), (\top, \top)\}$, and $\mathrm{pre}(!^a_\varphi) = \varphi$ and $\mathrm{pre}(\top) = \top$. We fix φ as p. After $[p \rightsquigarrow a]$, the updated model $\mathfrak{M}^{\otimes\mathbb{E}} = (W', R'_a, V')$ (see the right-hand side of Fig. 4.6) can be defined by Definition 46 as follows:

$$\begin{array}{lcl} W' & = & \{(x, !^a_p), (y, !^a_p), (x, \top), (y, \top)\} \\ R'_a & = & \{\big((x, \top), (y, \top)\big), \big((y, \top), (y, \top)\big)\} \\ V'(p) & = & \{(x, !^a_p), (x, \top)\} \end{array}$$

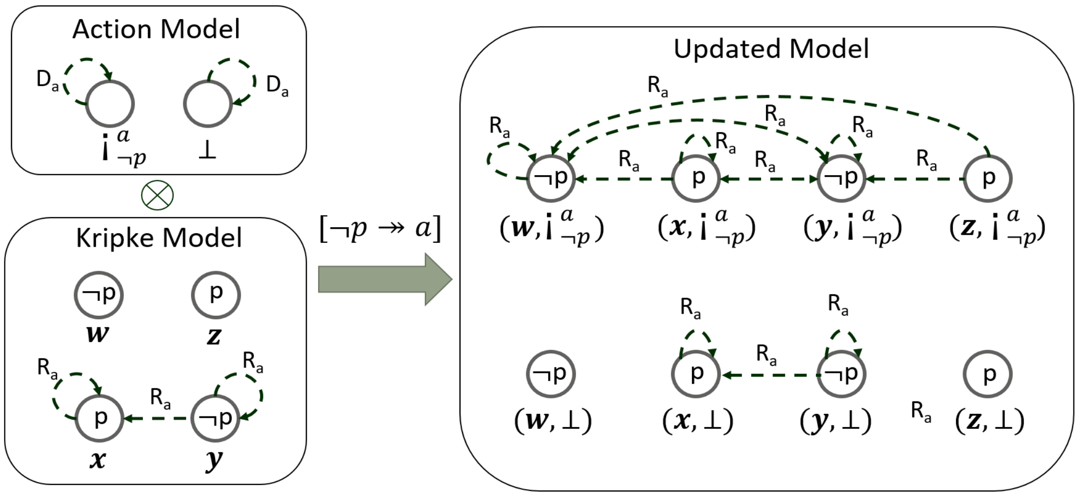

Figure 4.7: Product update operation of $[\neg p \twoheadrightarrow a]$

When a precondition is $\text{pre}(!^a_p)$, we can find that some state does not have a successor. Thus, R'_a is not serial. □

Proposition 56. *The private permission $[\varphi \twoheadrightarrow a]$ cannot preserve the properties of transitivity and Euclideanness.*

Proof. It is easily proved by counterexample. Suppose that $G = \{ a \}$. Given a Kripke model $\mathfrak{M} = (W, R_a, V)$, where $W = \{ w, x, y, z \}$, $R_a = \{(y, x), (x, x), (y, y)\}$ and $V(p) = \{ x, z \}$ (see a Kripke model in Fig. 4.7). Let $\mathbb{E}$ be an action model for a private permission of φ to agent a which is defined as a tuple (E, D_a, pre), where $E = \{ ¡^a_\varphi, \bot \}$, $D_a = \{ (¡^a_\varphi, ¡^a_\varphi), (\bot, \bot) \}$, and $\text{pre}(¡^a_\varphi) = \varphi$ and $\text{pre}(\bot) = \bot$. We fix φ as $\neg p$. After $[\neg p \twoheadrightarrow a]$, the updated model $\mathfrak{M}^{\otimes \mathbb{E}} = (W', R'_a, V')$ (see the right-hand side of Fig. 4.7) can be defined by Definition 46 as follows:

$$
\begin{array}{lcl}
W' & = & \{(w, ¡^a_{\neg p}), (x, ¡^a_{\neg p}), (y, ¡^a_{\neg p}), (z, ¡^a_{\neg p}), (w, \bot), (x, \bot), (y, \bot), (z, \bot)\} \\
R'_a & = & \{\big((w, ¡^a_{\neg p}), (w, ¡^a_{\neg p})\big), \big((w, ¡^a_{\neg p}), (y, ¡^a_{\neg p})\big), \big((x, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big), \big((x, ¡^a_{\neg p}), (w, ¡^a_{\neg p})\big), \\
 & & \big((x, ¡^a_{\neg p}), (y, ¡^a_{\neg p})\big), \big((y, ¡^a_{\neg p}), (y, ¡^a_{\neg p})\big), \big((y, ¡^a_{\neg p}), (w, ¡^a_{\neg p})\big), \big((y, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big), \\
 & & \big((z, ¡^a_{\neg p}), (w, ¡^a_{\neg p})\big), \big((z, ¡^a_{\neg p}), (y, ¡^a_{\neg p})\big), \big((x, \bot), (x, \bot)\big), \big((y, \bot), (y, \bot)\big), \\
 & & \big((y, \bot), (x, \bot)\big)\} \\
V'(p) & = & \{ (x, ¡^a_{\neg p}), (x, \bot), (z, ¡^a_{\neg p}), (z, \bot) \}
\end{array}
$$

When a precondition is $\text{pre}(¡^a_{\neg p})$, we can find that $\big((y, ¡^a_{\neg p}), (w, ¡^a_{\neg p})\big) \in R'_a$ and $\big((y, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big) \in R'_a$, but $\big((w, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big) \notin R'_a$. Thus, R'_a is not Euclidean. In a similar way, we can find that $\big((z, ¡^a_{\neg p}), (y, ¡^a_{\neg p})\big) \in R'_a$ and $\big((y, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big) \in R'_a$, but $\big((z, ¡^a_{\neg p}), (x, ¡^a_{\neg p})\big) \notin R'_a$. Thus, R'_a is not transitive. □

4.3 Logical Formalism for Belief Re-revision and Reliability Change

Since this study aims to construct a logical formalism for analyzing an agent's changing of belief and reliability, it requires to combine two logical tools for reliability change and belief re-revision into a unified system. From Section 3.2, we have three dynamic operators for reliability change including upgrade $[H \Uparrow^a_\varphi]$, downgrade $[H \Downarrow^a_\varphi]$ and joint downgrade $[H \downdownarrows^a]$. From Section 4.1.2, we have an action model modality $[\mathbb{E}, \Pi, e]$ for capturing an update operation of private announcement and private permission operators which is used to formalize belief re-revision. In order to define our logical formalism for reliability change and belief re-revision, we introduce a language $\mathcal{L}_{BRRC}$ which is an extension of $\mathcal{L}_{RC}$ by adding the action model modality $[\mathbb{E}, \Pi, e]$ and three dynamic operators for reliability change including upgrade $[H \Uparrow^a_\varphi]$, downgrade $[H \Downarrow^a_\varphi]$ and joint downgrade $[H \downdownarrows^a]$.

4.3.1 Syntax and Semantics

Definition 57. *Let G be a fixed* finite *set of agents and E be a finite set of action points. The language $\mathcal{L}_{BRRC}$ consists of the following vocabulary: (i) a countably infinite set* Prop $= \{ p, q, r, ... \}$ *of propositional letters, (ii) Boolean connectives: $\neg$, $\to$, (iii) the constants for reliability ordering $b \leqslant_a c$ ($a, b, c \in G$), (iv) atomic programs:* $\mathsf{1}$, B_a *($a \in G$),* S_a *($a \in G$), (v) program operators: $\cup$ (non-deterministic choice), ; (sequential composition), (vi) mixed operators: ? (test), $[\cdot]$ (necessity), (vii) an action model modality: $[\mathbb{E}, \Pi, e]$ with $e \in E$ ($\mathbb{E} = (E, (D_c)_{c \in G}, (U_a)_{a \in G}, \mathrm{pre}, \Pi)$ is an action model, where E is a finite domain of action points, D_c is an accessibility relation representing beliefs, U_a is an accessibility relation representing signatures,* pre *is a preconditions function, and Π is a family of programs), and (viii) dynamic operators for reliability change: $[H \Uparrow^a_\varphi]$ (upgrade), $[H \Downarrow^a_\varphi]$ (downgrade), $[H \downdownarrows^a]$ (joint downgrade). A set* Form_{BRRC} *of formulas φ of $\mathcal{L}_{RC}$ and a set* Prog *of programs π of $\mathcal{L}_{RC}$ are inductively defined as follows:*

$$\mathsf{Form}_{BRRC} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid b \leqslant_a c \mid [\pi]\varphi \mid [\mathbb{E}, \Pi, e]\varphi \mid [H \Uparrow^a_\varphi]\varphi \mid [H \Downarrow^a_\varphi]\varphi \mid [H \downdownarrows^a]\varphi$$

$$\mathsf{Prog} \ni \pi ::= \mathsf{1} \mid \mathsf{B}_a \mid \mathsf{S}_a \mid \pi \cup \pi \mid \pi ; \pi \mid \varphi ?$$

where $p \in$ Prop*, $e \in E$, $H \subseteq G$ and $a, b, c \in G$.*

The abbreviations for $\wedge$, $\vee$, $\leftrightarrow$, $\top$ and $\bot$ can be defined as shown in Definition 2. For the semantics of this language, the definition of models and the satisfaction relation $\mathfrak{M}, w \models \varphi$ is defined in a similar way to Definition 41 (in Section 4.1).

4.3.2 Hilbert-style Axiomatization HBRRC

The Hilbert-style system **HBRRC** for $\mathcal{L}_{BRRC}$ is presented in Table 4.2 including all recursion axioms for $[\mathbb{E}, \Pi, e]$ (action model modality in Section 4.1.2), $[H \Uparrow^a_\varphi]$ (upgrade in Section 3.2.1), $[H \Downarrow^a_\varphi]$ (downgrade in Section 3.2.2) and $[H \downdownarrows^a]$ (joint downgrade in Section 3.2.3). Since **HBRRC** is regarded as an extension of **HRC**, we can provide the proof of completeness by a translation method.

All axioms and rules of **HRC**				
$(RA1_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e]p$	$\leftrightarrow$	p	
$(RA2_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e]d \leqslant_c d'$	$\leftrightarrow$	$d \leqslant_c d'$	
$(RA3_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e]\neg\psi$	$\leftrightarrow$	$\neg[\mathbb{E},\Pi,e]\psi$	
$(RA4_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[\mathbb{E},\Pi,e]\psi_1 \to [\mathbb{E},\Pi,e]\psi_2$	
$(RA5_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][1]\psi$	$\leftrightarrow$	$\bigwedge_{f\in E}[1][\mathbb{E},\Pi,f]\psi$	
$(RA6_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][\mathsf{S_a}]\psi$	$\leftrightarrow$	$\bigwedge_{f\in U_a(e)}[\mathsf{S_a}][\mathbb{E},\Pi,f]\psi$	
$(RA7_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][\mathsf{B_a}]\psi$	$\leftrightarrow$	$\bigwedge_{f\in D_a(e)}[\pi_a(e,f)][\mathbb{E},\Pi,f]\psi$	
$(RA8_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][\pi\cup\pi']\psi$	$\leftrightarrow$	$[\mathbb{E},\Pi,e][\pi]\psi \wedge [\mathbb{E},\Pi,e][\pi']\psi$	
$(RA9_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][\pi;\pi']\psi$	$\leftrightarrow$	$[\mathbb{E},\Pi,e][\pi][\pi']\psi$	
$(RA10_{[\mathbb{E},\Pi,e]})$	$[\mathbb{E},\Pi,e][\varphi?]\psi$	$\leftrightarrow$	$[\mathbb{E},\Pi,e](\varphi \to \psi)$	
$(RA1_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi]p$	$\leftrightarrow$	p	
$(RA2_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$(RA3_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$(RA4_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\big([\mathsf{S}_b]\varphi \wedge [\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)\big)$ $\vee\big(\neg[\mathsf{S}_b]\varphi \wedge \neg[\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)\big)$ $\vee\big([\mathsf{S}_b]\varphi \wedge \neg[\mathsf{S}_c]\varphi\big)$	$(b, c \in H)$
$(RA5_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\neg[\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)$	$(c \in H, b \in G \setminus H)$
$(RA6_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$[\mathsf{S}_b]\varphi \vee \big(\neg[\mathsf{S}_b]\varphi \wedge (b \leqslant_a c)\big)$	$(b \in H, c \in G \setminus H)$
$(RA7_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi]\neg\psi$	$\leftrightarrow$	$\neg[H\Uparrow^a_\varphi]\psi$	
$(RA8_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[H\Uparrow^a_\varphi]\psi_1 \to [H\Uparrow^a_\varphi]\psi_2$	
$(RA9_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][1]\psi$	$\leftrightarrow$	$[1][H\Uparrow^a_\varphi]\psi$	
$(RA10_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][\mathsf{S_a}]\psi$	$\leftrightarrow$	$[\mathsf{S_a}][H\Uparrow^a_\varphi]\psi$	
$(RA11_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][\mathsf{B_a}]\psi$	$\leftrightarrow$	$[\mathsf{B_a}][H\Uparrow^a_\varphi]\psi$	
$(RA12_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][\pi\cup\pi']\psi$	$\leftrightarrow$	$[H\Uparrow^a_\varphi][\pi]\psi \wedge [H\Uparrow^a_\varphi][\pi']\psi$	
$(RA13_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][\pi;\pi']\psi$	$\leftrightarrow$	$[H\Uparrow^a_\varphi][\pi][\pi']\psi$	
$(RA14_{[H\Uparrow^a_\varphi]})$	$[H\Uparrow^a_\varphi][\varphi?]\psi$	$\leftrightarrow$	$[H\Uparrow^a_\varphi](\varphi \to \psi)$	
$(RA1_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi]p$	$\leftrightarrow$	p	
$(RA2_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$(RA3_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$(RA4_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\big([\mathsf{S}_b]\varphi \wedge [\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)\big)$ $\vee\big(\neg[\mathsf{S}_b]\varphi \wedge \neg[\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)\big)$ $\vee\big(\neg[\mathsf{S}_b]\varphi \wedge [\mathsf{S}_c]\varphi\big)$	$(b, c \in H)$
$(RA5_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$[\mathsf{S}_c]\varphi \vee \big(\neg[\mathsf{S}_c]\varphi \wedge (b \leqslant_a c)\big)$	$(c \in H, b \in G \setminus H)$
$(RA6_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](b \leqslant_a c)$	$\leftrightarrow$	$\neg[\mathsf{S}_b]\varphi \wedge (b \leqslant_a c)$	$(b \in H, c \in G \setminus H)$
$(RA7_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi]\neg\psi$	$\leftrightarrow$	$\neg[H\Downarrow^a_\varphi]\psi$	
$(RA8_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[H\Downarrow^a_\varphi]\psi_1 \to [H\Downarrow^a_\varphi]\psi_2$	
$(RA9_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][1]\psi$	$\leftrightarrow$	$[1][H\Downarrow^a_\varphi]\psi$	
$(RA10_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][\mathsf{S_a}]\psi$	$\leftrightarrow$	$[\mathsf{S_a}][H\Downarrow^a_\varphi]\psi$	
$(RA11_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][\mathsf{B_a}]\psi$	$\leftrightarrow$	$[\mathsf{B_a}][H\Downarrow^a_\varphi]\psi$	
$(RA12_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][\pi\cup\pi']\psi$	$\leftrightarrow$	$[H\Downarrow^a_\varphi][\pi]\psi \wedge [H\Downarrow^a_\varphi][\pi']\psi$	
$(RA13_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][\pi;\pi']\psi$	$\leftrightarrow$	$[H\Downarrow^a_\varphi][\pi][\pi']\psi$	
$(RA14_{[H\Downarrow^a_\varphi]})$	$[H\Downarrow^a_\varphi][\varphi?]\psi$	$\leftrightarrow$	$[H\Downarrow^a_\varphi](\varphi \to \psi)$	
$(RA1_{[H\downdownarrows^a]})$	$[H\downdownarrows^a]p$	$\leftrightarrow$	p	
$(RA2_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](b \leqslant_d c)$	$\leftrightarrow$	$b \leqslant_d c$	$(d \neq a)$
$(RA3_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$b \leqslant_a c$	$(b, c \in G \setminus H)$
$(RA4_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\top$	$(b, c \in H)$
$(RA5_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\top$	$(c \in H, b \in G \setminus H)$
$(RA6_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](b \leqslant_a c)$	$\leftrightarrow$	$\bot$	$(b \in H, c \in G \setminus H)$
$(RA7_{[H\downdownarrows^a]})$	$[H\downdownarrows^a]\neg\psi$	$\leftrightarrow$	$\neg[H\downdownarrows^a]\psi$	
$(RA8_{[H\downdownarrows^a]})$	$[H\downdownarrows^a](\psi_1 \to \psi_2)$	$\leftrightarrow$	$[H\downdownarrows^a]\psi_1 \to [H\downdownarrows^a]\psi_2$	
$(RA9_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][1]\psi$	$\leftrightarrow$	$[1][H\downdownarrows^a_\varphi]\psi$	
$(RA10_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][\mathsf{S_a}]\psi$	$\leftrightarrow$	$[\mathsf{S_a}][H\downdownarrows^a_\varphi]\psi$	
$(RA11_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][\mathsf{B_a}]\psi$	$\leftrightarrow$	$[\mathsf{B_a}][H\downdownarrows^a_\varphi]\psi$	
$(RA12_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][\pi\cup\pi']\psi$	$\leftrightarrow$	$[H\downdownarrows^a_\varphi][\pi]\psi \wedge [H\downdownarrows^a_\varphi][\pi']\psi$	
$(RA13_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][\pi;\pi']\psi$	$\leftrightarrow$	$[H\downdownarrows^a_\varphi][\pi][\pi']\psi$	
$(RA14_{[H\downdownarrows^a_\varphi]})$	$[H\downdownarrows^a_\varphi][\varphi?]\psi$	$\leftrightarrow$	$[H\downdownarrows^a_\varphi](\varphi \to \psi)$	
$(Nec_{[\mathbb{E},\Pi,e]})$ From ψ, infer $[\mathbb{E},\Pi,e]\psi$				
$(Nec_{[H\Uparrow^a_\varphi]})$ From ψ, infer $[H\Uparrow^a_\varphi]\psi$				
$(Nec_{[H\Downarrow^a_\varphi]})$ From ψ infer $[H\Downarrow^a_\varphi]\psi$				
$(Nec_{[H\downdownarrows^a]})$ From ψ infer $[H\downdownarrows^a]\psi$				

Table 4.2: Hilbert-style system **HBRRC** for $\mathcal{L}_{BRRC}$

Definition 58. *The translation* $t : \mathsf{Form}_{BRRC} \to \mathsf{Form}_{RC}$ *is defined as follows:*

$$
\begin{array}{lcll}
t(p) & = & p & \\
t(b \leqslant_d c) & = & b \leqslant_d c & \\
t(\neg\varphi) & = & \neg t(\varphi) & \\
t(\varphi \to \psi) & = & t(\varphi) \to t(\psi) & \\
t([1]\psi) & = & [1]t(\psi) & \\
t([\mathsf{S_a}]\psi) & = & [\mathsf{S_a}]t(\psi) & \\
t([\mathsf{B_a}]\psi) & = & [\mathsf{B_a}]t(\psi) & \\
t([\pi \cup \pi']\psi) & = & t([\pi]\psi) \wedge t([\pi']\psi) & \\
t([\pi;\pi']\psi) & = & t([\pi][\pi']\psi) & \\
t([\varphi?]\psi) & = & t(\varphi \to \psi) & \\
t([\mathbb{E},\Pi,e]p) & = & p & \\
t([\mathbb{E},\Pi,e]b \leqslant_d c) & = & b \leqslant_d c & \\
t([\mathbb{E},\Pi,e]\neg\psi) & = & \neg t([\mathbb{E},\Pi,e]\psi) & \\
t([\mathbb{E},\Pi,e](\psi_1 \to \psi_2)) & = & t([\mathbb{E},\Pi,e]\psi_1) \to t([\mathbb{E},\Pi,e]\psi_2) & \\
t([\mathbb{E},\Pi,e][1]\psi) & = & \bigwedge_{f\in E}[1]t([\mathbb{E},\Pi,f]\psi) & \\
t([\mathbb{E},\Pi,e][\mathsf{S_a}]\psi) & = & \bigwedge_{f\in U_a(e)}[\mathsf{S_a}]t([\mathbb{E},\Pi,f]\psi) & \\
t([\mathbb{E},\Pi,e][\mathsf{B_a}]\psi) & = & \bigwedge_{f\in D_a(e)} t([\pi_a(e,f)][\mathbb{E},\Pi,f]\psi) & \\
t([\mathbb{E},\Pi,e][\pi \cup \pi']\psi) & = & t([\mathbb{E},\Pi,e][\pi]\psi) \wedge t([\mathbb{E},\Pi,e][\pi']\psi) & \\
t([\mathbb{E},\Pi,e][\pi;\pi']\psi) & = & t([\mathbb{E},\Pi,e][\pi][\pi']\psi) & \\
t([\mathbb{E},\Pi,e][\varphi?]\psi) & = & t([\mathbb{E},\Pi,e]\varphi \to \psi) & \\
t([\mathbb{E}_1,\Pi,e_1][\mathbb{E}_2,\Pi,e_2]\psi) & = & t([\mathbb{E}_1,\Pi,e_1]t([\mathbb{E}_2,\Pi,e_2]\psi)) & \\
t([\mathbb{E},\Pi,e][H \Uparrow^a_\varphi]\psi) & = & t([\mathbb{E},\Pi,e]t([H \Uparrow^a_\varphi]\psi)) & \\
t([\mathbb{E},\Pi,e][H \Downarrow^a_\varphi]\psi) & = & t([\mathbb{E},\Pi,e]t([H \Downarrow^a_\varphi]\psi)) & \\
t([\mathbb{E},\Pi,e][H \downdownarrows^a_\varphi]\psi) & = & t([\mathbb{E},\Pi,e]t([H \downdownarrows^a_\varphi]\psi)) & \\
t([H \Uparrow^a_\varphi]p) & = & p & \\
t([H \Uparrow^a_\varphi]b \leqslant_d c) & = & b \leqslant_d c & \\
t([H \Uparrow^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (d \neq a) \\
t([H \Uparrow^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (b,c \in G \setminus H) \\
t([H \Uparrow^a_\varphi](b \leqslant_d c)) & = & \big([\mathsf{S}_b]t(\varphi) \wedge [\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c)\big) \vee & \\
 & & \big(\neg[\mathsf{S}_b]t(\varphi) \wedge \neg[\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c)\big) \vee & \\
 & & \big([\mathsf{S}_b]t(\varphi) \wedge \neg[\mathsf{S}_c]t(\varphi)\big) & (b,c \in H) \\
t([H \Uparrow^a_\varphi](b \leqslant_d c)) & = & \neg[\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c) & (c \in H, b \in G \setminus H) \\
t([H \Uparrow^a_\varphi](b \leqslant_d c)) & = & [\mathsf{S}_b]t(\varphi) \vee \big(\neg[\mathsf{S}_b]t(\varphi) \wedge (b \leqslant_a c)\big) & (b \in H, c \in G \setminus H) \\
t([H \Uparrow^a_\varphi]\neg\psi) & = & \neg t([H \Uparrow^a_\varphi]\psi) & \\
t([H \Uparrow^a_\varphi](\psi_1 \to \psi_2)) & = & t([H \Uparrow^a_\varphi]\psi_1) \to t([H \Uparrow^a_\varphi]\psi_2) & \\
t([H \Uparrow^a_\varphi][1]\psi) & = & [1]t([H \Uparrow^a_\varphi]\psi) & \\
t([H \Uparrow^a_\varphi][\mathsf{S_a}]\psi) & = & [\mathsf{S_a}]t([H \Uparrow^a_\varphi]\psi) & \\
t([H \Uparrow^a_\varphi][\mathsf{B_a}]\psi) & = & [\mathsf{B_a}]t([H \Uparrow^a_\varphi]\psi) & \\
t([H \Uparrow^a_\varphi][\pi \cup \pi']\psi) & = & t([H \Uparrow^a_\varphi][\pi]\psi) \wedge t([H \Uparrow^a_\varphi][\pi']\psi) & \\
t([H \Uparrow^a_\varphi][\pi;\pi']\psi) & = & t([H \Uparrow^a_\varphi][\pi][\pi']\psi) & \\
t([H \Uparrow^a_\varphi][\psi?]\theta) & = & t([H \Uparrow^a_\varphi](\psi \to \theta)) & \\
t([H \Uparrow^a_\varphi][\mathbb{E},\Pi,e]\psi) & = & t([H \Uparrow^a_\varphi]t([\mathbb{E},\Pi,e]\psi)) & \\
t([H \Uparrow^a_\varphi][H \Uparrow^a_\psi]\theta) & = & t([H \Uparrow^a_\varphi]t([H \Uparrow^a_\psi]\theta)) & \\
t([H \Uparrow^a_\varphi][H \Downarrow^a_\psi]\theta) & = & t([H \Uparrow^a_\varphi]t([H \Downarrow^a_\psi]\theta)) & \\
t([H \Uparrow^a_\varphi][H \downdownarrows^a_\psi]\theta) & = & t([H \Uparrow^a_\varphi]t([H \downdownarrows^a_\psi]\theta)) &
\end{array}
$$

$$
\begin{array}{lcll}
t([H \Downarrow^a_\varphi]p) & = & p & \\
t([H \Downarrow^a_\varphi]b \leqslant_d c) & = & b \leqslant_d c & \\
t([H \Downarrow^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (d \neq a) \\
t([H \Downarrow^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (b, c \in G \setminus H) \\
t([H \Downarrow^a_\varphi](b \leqslant_d c)) & = & \big([\mathsf{S}_b]t(\varphi) \wedge [\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c)\big) \vee & \\
 & & \big(\neg[\mathsf{S}_b]t(\varphi) \wedge \neg[\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c)\big) \vee & \\
 & & \big(\neg[\mathsf{S}_b]t(\varphi) \wedge [\mathsf{S}_c]t(\varphi)\big) & (b, c \in H) \\
t([H \Downarrow^a_\varphi](b \leqslant_d c)) & = & [\mathsf{S}_c]t(\varphi) \vee \big(\neg[\mathsf{S}_c]t(\varphi) \wedge (b \leqslant_a c)\big) & (c \in H, b \in G \setminus H) \\
t([H \Downarrow^a_\varphi](b \leqslant_d c)) & = & \neg\mathsf{Sign}(b, t(\varphi)) \wedge (b \leqslant_a c) & (b \in H, c \in G \setminus H) \\
t([H \Downarrow^a_\varphi]\neg\psi) & = & \neg t([H \Downarrow^a_\varphi]\psi) & \\
t([H \Downarrow^a_\varphi](\psi_1 \to \psi_2)) & = & t([H \Downarrow^a_\varphi]\psi_1) \to t([H \Downarrow^a_\varphi]\psi_2) & \\
t([H \Downarrow^a_\varphi][1]\psi) & = & [1]t([H \Downarrow^a_\varphi]\psi) & \\
t([H \Downarrow^a_\varphi][\mathsf{S_a}]\psi) & = & [\mathsf{S_a}]t([H \Downarrow^a_\varphi]\psi) & \\
t([H \Downarrow^a_\varphi][\mathsf{B_a}]\psi) & = & [\mathsf{B_a}]t([H \Downarrow^a_\varphi]\psi) & \\
t([H \Downarrow^a_\varphi][\pi \cup \pi']\psi) & = & t([H \Downarrow^a_\varphi][\pi]\psi) \wedge t([H \Downarrow^a_\varphi][\pi']\psi) & \\
t([H \Downarrow^a_\varphi][\pi; \pi']\psi) & = & t([H \Downarrow^a_\varphi][\pi][\pi']\psi) & \\
t([H \Downarrow^a_\varphi][\psi?]\theta) & = & t([H \Downarrow^a_\varphi](\psi \to \theta)) & \\
t([H \Downarrow^a_\varphi][\mathbb{E}, \Pi, e]\psi) & = & t([H \Downarrow^a_\varphi]t([\mathbb{E}, \Pi, e]\psi)) & \\
t([H \Downarrow^a_\varphi][H \Uparrow^a_\psi]\theta) & = & t([H \Downarrow^a_\varphi]t([H \Uparrow^a_\psi]\theta)) & \\
t([H \Downarrow^a_\varphi][H \Downarrow^a_\psi]\theta) & = & t([H \Downarrow^a_\varphi]t([H \Downarrow^a_\psi]\theta)) & \\
t([H \Downarrow^a_\varphi][H \downdownarrows^a_\psi]\theta) & = & t([H \Downarrow^a_\varphi]t([H \downdownarrows^a_\psi]\theta)) & \\
t([H \downdownarrows^a_\varphi]p) & = & p & \\
t([H \downdownarrows^a_\varphi]b \leqslant_d c) & = & b \leqslant_d c & \\
t([H \downdownarrows^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (d \neq a) \\
t([H \downdownarrows^a_\varphi](b \leqslant_d c)) & = & b \leqslant_d c & (b, c \in G \setminus H) \\
t([H \downdownarrows^a_\varphi](b \leqslant_d c)) & = & \top & (b, c \in H) \\
t([H \downdownarrows^a_\varphi](b \leqslant_d c)) & = & \top & (c \in H, b \in G \setminus H) \\
t([H \downdownarrows^a_\varphi](b \leqslant_d c)) & = & \bot & (b \in H, c \in G \setminus H) \\
t([H \downdownarrows^a_\varphi]\neg\psi) & = & \neg t([H \downdownarrows^a_\varphi]\psi) & \\
t([H \downdownarrows^a_\varphi](\psi_1 \to \psi_2)) & = & t([H \downdownarrows^a_\varphi]\psi_1) \to t([H \downdownarrows^a_\varphi]\psi_2) & \\
t([H \downdownarrows^a_\varphi][1]\psi) & = & [1]t([H \downdownarrows^a_\varphi]\psi) & \\
t([H \downdownarrows^a_\varphi][\mathsf{S_a}]\psi) & = & [\mathsf{S_a}]t([H \downdownarrows^a_\varphi]\psi) & \\
t([H \downdownarrows^a_\varphi][\mathsf{B_a}]\psi) & = & [\mathsf{B_a}]t([H \downdownarrows^a_\varphi]\psi) & \\
t([H \downdownarrows^a_\varphi][\pi \cup \pi']\psi) & = & t([H \downdownarrows^a_\varphi][\pi]\psi) \wedge t([H \downdownarrows^a_\varphi][\pi']\psi) & \\
t([H \downdownarrows^a_\varphi][\pi; \pi']\psi) & = & t([H \downdownarrows^a_\varphi][\pi][\pi']\psi) & \\
t([H \downdownarrows^a_\varphi][\psi?]\theta) & = & t([H \downdownarrows^a_\varphi](\psi \to \theta)) & \\
t([H \downdownarrows^a_\varphi][\mathbb{E}, \Pi, e]\psi) & = & t([H \downdownarrows^a_\varphi]t([\mathbb{E}, \Pi, e]\psi)) & \\
t([H \downdownarrows^a_\varphi][H \Uparrow^a_\psi]\theta) & = & t([H \downdownarrows^a_\varphi]t([H \Uparrow^a_\psi]\theta)) & \\
t([H \downdownarrows^a_\varphi][H \Downarrow^a_\psi]\theta) & = & t([H \downdownarrows^a_\varphi]t([H \Downarrow^a_\psi]\theta)) & \\
t([H \downdownarrows^a_\varphi][H \downdownarrows^a_\psi]\theta) & = & t([H \downdownarrows^a_\varphi]t([H \downdownarrows^a_\psi]\theta)) &
\end{array}
$$

Lemma 14. *For all formulas* $\varphi \in \mathsf{Form}_{BRRC}$*,*

$$\vdash_{\mathbf{HBRRC}} \varphi \leftrightarrow t(\varphi).$$

Theorem 15 (Soundness)**.** *Let* $\mathbb{M}$ *be the class of all models. For all* $\psi \in \mathsf{Form}_{BRRC}$*,*

$$\textit{if } \vdash_{\mathbf{HBRRC}} \psi, \textit{ then } \mathbb{M} \models \psi.$$

Proof. Suppose that $\vdash_{\mathbf{HBRRC}}$. Our goal is to show that $\mathbb{M} \models \psi$ for all ψ. It suffices to show that all axioms and all rules in **HBRRC** are valid on all models in a class $\mathbb{M}$ with respect to the semantics of $\mathcal{L}_{BRRC}$. This is straightforward. □

Theorem 16 (Completeness)**.** *Let* $\mathbb{M}$ *be the class of all models. For all* $\varphi \in \mathsf{Form}_{BRRC}$*,*

$$\textit{if } \mathbb{M} \models \varphi, \textit{ then } \vdash_{\mathbf{HBRRC}} \varphi.$$

Proof. Suppose that $\mathbb{M} \models \varphi$. Our goal is to show $\vdash_{\mathbf{HBRRC}} \varphi$ for all formulas φ. By the soundness theorem (Theorem 15) and Lemma 14, we obtain that $\mathbb{M} \models \varphi \leftrightarrow t(\varphi)$. By this and our supposition, we get $\mathbb{M} \models t(\varphi)$. Since the formula $t(\varphi)$ does not contain any dynamic operators, we can reduce the completeness of **HBRRC** to that of **HRC** in Theorem 14. Therefore, we obtain $\vdash_{\mathbf{HRC}} t(\varphi)$ by the completeness of **HRC** (Theorem 14). Since **HRC** is a sub system of **HBRRC**, we have that $\vdash_{\mathbf{HBRRC}} t(\varphi)$. By this and Lemma 14, we obtain $\vdash_{\mathbf{HBRRC}} \varphi$, as desired. □

Chapter 5

Dynamic Logical Analysis of Legal Cases

This chapter demonstrates how to analyze a legal case by our logical formalization. In Section 5.1, our proposed method for analyzing a judge's changing of belief and reliability in a judgment process is presented. With this method and our logical formalization, we develop an implementation for analyzing an agent's changing of belief and reliability (mentioned in Appendix B). By this implementation and our analysis method, six target legal cases are analyzed in Section 5.2.

5.1 Analysis Method

This section provides a description of our proposed method for analyzing a legal case based on our logical formalization consisting of the following steps.

(1) We will summarize the target legal case by extracting the facts and the decision (the more details of this step will be described in Section 5.2.1).[1] The result can be shown in Tables 5.2, 5.3 and 5.4.

(2) We will construct an initial Kripke model for the target legal case. This model is used for analyzing a judge's changing of belief and reliability. A construction of the initial model can be done by our implementation, as mentioned in the second feature of our implementation in Section B. This process can be summarized into the following steps:

(2.1) We will generate all possibilities which can be represented by possible belief states of an agent in a Kripke model. The number of all possible belief states can be calculated by 2^N where N is the number of signed agents or witnesses in the legal case. For example, the second legal case in Table 5.4 consists of a statement p and two witnesses b and f. That is, we obtain that N is equal to two, and the number of all possible belief states is four states as follows:

(w_1) Agent b gives statement p and agent f gives statement p.

(w_2) Agent b gives statement p and agent f gives statement $\neg p$.

[1]In this study, we did not apply legal text processing in the area of natural language processing (NLP) for summarizing legal cases and generating an initial Kripke model from a legal case.

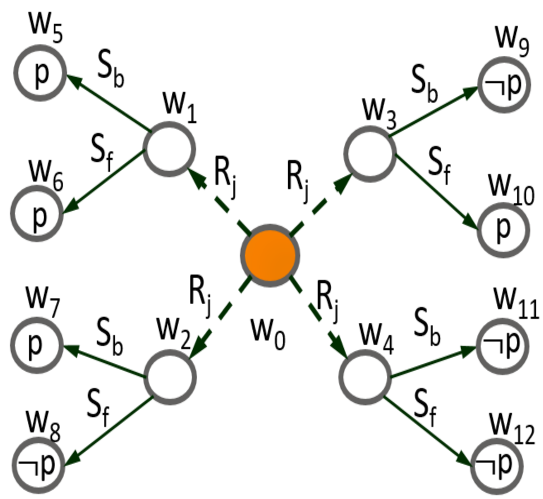

Figure 5.1: Initial Kripke model for the second legal case consisting of 13 states including four belief states of j (w_1, w_2,w_3, w_4), eight signed states including four signed states of b (w_5, w_7, w_9, w_{11}) and four signed states of f (w_6, w_8, w_{10}, w_{12}) and one current state (w_0) representing j's viewpoint

(w_3) Agent b gives statement $\neg p$ and agent f gives statement p.

(w_4) Agent b gives statement $\neg p$ and agent f gives statement $\neg p$.

From the above possible belief states, we can regard that each belief state consists of two signed states representing signed statements of b and f. If there is a lot of possible belief states, we may remove some states which are considered to be not important for analyzing the judge's changing of belief and reliability. That is, we may regard that such possibilities cannot occur in the legal judgment. This is for simplifying the initial model to be easy for analyzing the judge's changing of belief and reliability.

(2.2) Firstly, j representing a judge is defined as a belief agent who has an accessibility relation R_j. Two witnesses b and f are defined as signed agents who have accessibility relations S_b and S_f, respectively. A current state representing j's viewpoint is defined as w_0. Then, the links representing S_b and S_f will be added from each belief state to its signed states. For example, the possible belief state w_1 from Step (2.1) consists of two signed states w_5 and w_6 representing signed statements of b and f, respectively. Then, the links S_b and S_f will be added from belief state w_1 to signed states w_5 and w_6, respectively. Next, the links representing R_j will be added from the current state (w_0) to all possible belief states (w_1, w_2, w_3, w_4) because we assume that the judge should open to all possibilities at the initial stage. Finally, the initial model of our target legal case can be constructed as in Fig. 5.1 which is the same as the left-hand side of Fig. 5.2 outputting from our implementation. Note that Fig. 5.1 presents one way for generating an initial model from the second legal case. However, an initial model of the second legal case can be constructed in different ways, that is, identifying different key features of a legal case (*i.e.*, a statement, a belief agent and signed agents) or removing some possibilities. For example, if

Table 5.1: Summary of six dynamic logical operators

Type	Operator name	Logical formula	Goal
Formalizing belief change	Private announcement	$[\varphi \rightsquigarrow a]$	To restrict agent a's attention to the φ's states
	Careful policy	$[\mathsf{Careful}(a, \varphi)]$	To aggregate information about φ
	Private permission	$[\varphi \twoheadrightarrow a]$	To enlarge agent a's attention to cover all φ's states
Formalizing reliability change	Downgrade	$[H \Downarrow^a_\varphi]$	To make such agents who sign φ in H less reliable than all the other agents
	Upgrade	$[H \Uparrow^a_\varphi]$	To make such agents who sign φ in H more reliable than all the other agents
	Joint downgrade	$[H \downdownarrows^a]$	To make such agents in H equally reliable and less reliable than the agents in the other groups

we regard that giving statement p of agent f is not important, we can remove states w_1 and w_3. As a result, we obtain an initial Kripke model consisting of seven states (*i.e.*, two belief states (w_2, w_4), two signed states of b (w_7, w_{11}), two signed states of f (w_8, w_{12}) and one current state (w_0)) that is different from Fig. 5.1. With different initial models, we can obtain the same analysis result if such models have the essential information which is sufficient for a judgment.

(3) We can analyze the judge's changing of belief and reliability by inputting any dynamic logical operators including private announcement, careful policy, private permission, downgrade, upgrade and joint downgrade as mentioned in Chapters 3 and 4. These dynamic logical operators can be summarized in Table 5.1. Recall that our implementation aims at reducing the effort to decide which operators are to be applied for analyzing an agent's changing of belief and reliability. With this goal, we propose an application method of our dynamic operators as follows:

(3.1) We assume that the agent needs to apply two basic operations including private announcement and careful policy. When the agent receives a piece of information, he/she will apply the private announcement for admitting such information. The careful policy is used for deriving beliefs from signed information. Based on this idea and the above goal, there are the following options (OP1) and (OP2):

(OP1) The agent needs to apply only two kinds of operators, i.e., private announcement and careful policy. This means that whenever the agent receives a piece of information, he/she will accept the received information by applying the private announcement. If there is an inconsistency, the system will handle such inconsistency instead of the agent. That is, the inconsistency management policy will be automatically by our implementation by the following steps.

(i) The system will check if there is an inconsistency between the existing belief and new information or not. That is, there is an agent giving inconsistent statements or not.

(ii) If there is an inconsistency, the system applies the joint downgrade and the private permission operators by the following steps. First, the joint downgrade operator is employed for downgrading the agent who gives inconsistent statements less reliable than the other agents. Second, a process of belief restoration is performed by the private permission operators. For this process, the system will automatically restore all possibilities because it cannot determine which statements should be permitted to the agent's belief.

(iii) The system will check if there is the received information which is not inconsistent with the existing belief and is signed by the most reliable agent or not. If there is such information, the system will apply the private announcement operator for admitting such information.

An example of applying this policy will be described in Section 5.2.3. From the above policy, we can regard that the agent does not need to change his/her reliability or permit the possibility to his/her belief by him/herself.

(OP2) The agent needs to apply three kinds of operators, i.e., private announcement, downgrade/upgrade/joint downgrade and careful policy. That is, the agent needs to apply the operation for changing his/her reliability, i.e., downgrade, upgrade and joint downgrade. In this option, the agent needs to decide how to change his/her reliability ordering between other agents by him/herself.[2]

(3.2) In order to analyze the judge's changing of belief and reliability, we first use option (OP1) because it is an easy way that we do not need to take much effort to decide which operators are to be applied. If option (OP1) cannot work well, we will use option (OP2).

5.2 Analyzing Target Legal Cases

This section presents how we can analyze a legal case by our proposed method and implementation. First, six target legal cases are summarized in Section 5.2.1. Then, we analyze six target legal cases by our implementation as mentioned in Appendix B and the analysis results are stated in Section 5.2.2. Among six target legal cases, the second legal case is used to illustrate how to analyze belief change of a judge in Section 5.2.3.

[2]In this study, we will not analyze how an agent decides to change his/her reliability ordering between the other agents because this is a psychological issue and is out of our scope.

Table 5.2: Summary of the decisions of six target legal cases

Legal case	**Defendants**	**Decisions**	
		Inquiry	**Court**
1	d	d was charged with attempted murder	j decided that d was guilty
2	d	d was charged with murder	j acquitted d
3	d_1 and d_2	d_1 and d_2 were charged with first degree murder	j acquitted d_1 and d_2 of first degree murder
4	d	d was charged with manslaughter	j decided that d was not guilty
5	d	d was charged with attempted murder	j decided that d was not guilty
6	d_1 and d_2	d_1 and d_2 were charged with manslaughter	j decided that d_1 and d_2 were not guilty

5.2.1 Summary of Six Legal Cases

In this section, we present six target legal cases which have three characteristics. First, these legal cases are published judgments of the Supreme Court that can be retrieved from an on-line database. Second, we suppose that the judges in these legal cases need to change their belief and/or reliability ordering in order to derive their decision. This process can be demonstrated by our implementation. Third, these legal cases consist of three main components as follows:

- Facts provide the essential features of a legal case including all of the relevant people, actions, locations, evidences and so on.
- Decision explains how a judge decides a legal case in a court starting from the original trial decision to the final one.
- Reasoning provides an explanation of how a judge justified application of the law including the legal rules or precedents.

Among the above components, we consider the facts and the decision to be essential for us to analyze our target legal cases by our implementation. Based on this idea, six target legal cases can be summarized in Tables 5.2 and 5.3 (the more details of all target legal cases are presented in Appendix C). We can describe how to summarize our target legal cases as follows:

- Summarizing the decision of a legal case: this study focuses on only the inquiry stage and the court. Table 5.2 shows a summary of judgment in six target legal cases. In Table 5.2, j represents a judge in a legal case. In this study, we regard the judges in each legal case as a single agent j.
- Summarizing the facts of a legal case: we will extract only statements and witnesses that are most important to the judge for deriving his/her decision as shown in Table 5.3. Note that, in the first legal case, po gives statements $\mathsf{Sign}(v, p)$, $\mathsf{Sign}(f_1, p)$, $\mathsf{Sign}(f_2, \neg p)$ and $\mathsf{Sign}(mo, p)$ that can be denoted by $\mathsf{Sign}(po, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_1, p) \wedge$

Table 5.3: Summary of significant statements from witnesses in six target legal cases

Legal case	Witness	Statements		Note
		Inquiry	Court	
1	v	p	$\neg p$	p : d was the offender
	f_1	p	$\neg p$	
	f_2	$\neg p$	None	
	mo	p	None	
	po	None	$\mathsf{Sign}(v,p)$, $\mathsf{Sign}(f_1,p)$, $\mathsf{Sign}(f_2,\neg p)$, $\mathsf{Sign}(mo,p)$	
2	b	$\neg p$	p	p : d was the offender
	f_1	p	$\neg p$	
	f_2	p	$\neg p$	
	f_3	p	$\neg p$	
3	f_1	p	p	p : d_2 was the shooter
	f_2	$\neg p$	p	
4	f	$\neg p$	p	p : d was the offender
5	v	p	$\neg p$	p : d intended to kill v
	f_1	p	$\neg p$	
	f_2	p	p	
	b	p	p	
6	f_1	None	p, q, r	p : d_2 kicked v while v was on the ground,
	f_2	None	$\neg p$, $\neg q$	q : d_1 kicked v while v was on the ground,
	f_3	$\neg p$, $\neg q$	$\neg p$, $\neg q$	r : d_2 kicked v in the head
	f_4	p, $\neg r$	p, $\neg r$	
	f_5	$\neg r$	q, r	

$\mathsf{Sign}(f_2,\neg p) \wedge \mathsf{Sign}(mo,p))$. In this sense, we can regard that po first receives statements p, p, $\neg p$ and p from v, f_1, f_2 and mo, respectively, in the inquiry stage and then gives the received statements to the judge in the court.

- Simplifying a legal case: if there are some witnesses who give redundant statements, we can remove such witnesses. For example, in the second legal case, since three witnesses f_1, f_2 and f_3 give the same statements both in the inquiry stage and the court (see Table 5.3), we can merge them into one witness f as shown in Table 5.4.

5.2.2 Analysis Result

According to our analysis method in Section 5.1, six target legal cases are analyzed by our implementation and the results can be shown in Tables 5.5 and 5.6.

Table 5.5 shows a process for analyzing the judge's changing of belief and reliability in each legal judgment by our implementation including statements from Table 5.4, operations for changing an agent's belief and reliability, a goal of applying such operations and the final result. In Table 5.5, the operations which an agent employed to change his/her belief and reliability are shown. In addition, a goal of applying such operations will be presented in order to show why the agent decides to apply such operations. Table

Table 5.4: Simplified summary of significant statements from witnesses in six target legal cases

Legal case	**Witness**	**Statements**	
		Inquiry	**Court**
1	v (v, f_1)	p	$\neg p$
	f_2	$\neg p$	None
	mo	p	None
	po	None	$\mathsf{Sign}(v, p)$, $\mathsf{Sign}(f_2, \neg p)$, $\mathsf{Sign}(mo, p)$
2	b	$\neg p$	p
	f (f_1, f_2, f_3)	p	$\neg p$
3	f_1	p	p
	f_2	$\neg p$	p
4	f	$\neg p$	p
5	v (v, f_1)	p	$\neg p$
	b (b, f_2)	p	p
6	f_1	None	p, q, r
	f_3 (f_2, f_3)	$\neg p$, $\neg q$	$\neg p$, $\neg q$
	f_4	p, $\neg r$	p, $\neg r$
	f_5	$\neg r$	q, r

5.6 shows the results of analyzing six legal cases by our implementation including the following items:

- Number of statements are used to analyze the legal case (see Table 5.5).
- Number of steps are applied by an agent for changing his/her belief and reliability (see Table 5.5).
- Number of all operations are employed by both an agent and our implementation for changing such agent's belief and reliability.
- We will check if the inconsistency management policy is applied or not (the more details of this policy are described in Section B).
- We will check if the careful policy which is used to aggregate information can be applied or not.

From Table 5.5, we obtain the final result of all target legal cases, interpreted corresponding to the actual decision. From Table 5.6, we can interpret the results as follows:

- The number of statements can affect the number of steps and operations. Since this study assumes that an agent can consider only one information, our implementation will allow us to analyze only one statement at one time. However, we may need several statements for analyzing some legal case. For example, in the sixth legal case, we need three statements including p, q and r. In order to analyze such legal case, it is required to analyze each statement separately as in Table 5.5. From this table, three steps including three operations are performed for analyzing each statement. Thus, we need nine steps including nine operations for analyzing this legal case.

Table 5.5: Summary of analysis process of six target legal cases ('info.': information, 'Pri-Ann': private announcement and 'Agg' : careful policy for information aggregation)

Legal case	Statement	Operation	Goal	Final Result
1	p	(1) Pri-Ann	To admit statements of v and po in the court	j believes p
		(2) Upgrade	To upgrade agent po who signs $\mathsf{Sign}(v,p)$	
		(3) Agg	To aggregate statements of po	
		(4) Upgrade	To upgrade agents v and mo who sign p	
		(5) Agg	To aggregate info. about p	
2	p	(1) Pri-Ann	To admit statements of b and f in the court	j cannot determine on statements of both b and f
		(2) Pri-Ann	To admit statement of b in the inquiry stage	
		(3) Pri-Ann	To admit statement of f in the inquiry stage	
3	p	(1) Pri-Ann	To admit statements of f_1 and f_2 in the court	j believes p
		(2) Pri-Ann	To admit statement of f_2 in the inquiry stage	
		(3) Agg	To aggregate info. about p	
4	p	(1) Pri-Ann	To admit statement of f in the court	j cannot determine on statements of f
		(2) Pri-Ann	To admit statement of f in the inquiry stage	
5	p	(1) Pri-Ann	To admit statements of v and b in the court	j believes p
		(2) Pri-Ann	To admit statement of v in the inquiry stage	
		(3) Agg	To aggregate info. about p	
6	p	(1) Pri-Ann	To admit statements of f_1, f_3 and f_4 in the court	j believes $\neg p$
		(2) Downgrade	To downgrade agents f_1 and f_4 who sign p	
		(3) Agg	To aggregate info. about $\neg p$	
	q	(1) Pri-Ann	To admit statements of f_1, f_3 and f_5 in the court	j believes q
		(2) Downgrade	To downgrade agent f_3 who signs $\neg q$	
		(3) Agg	To aggregate info. about q	
	r	(1) Pri-Ann	To admit statements of f_1, f_4 and f_5 in the court	j believes $\neg r$
		(2) Downgrade	To downgrade agents f_1 and f_5 who sign r	
		(3) Agg	To aggregate info. about $\neg r$	

Table 5.6: Result of analyzing six target legal cases by the implementation

Legal case	Number of statements	Number of steps	Number of operations	Triggering inconsistency management	Capability for aggregation
1	1	5	5	No	Yes
2	1	3	10	Yes	No
3	1	3	7	Yes	Yes
4	1	2	5	Yes	No
5	1	3	7	Yes	Yes
6	3	9	9	No	Yes

- The number of steps depends on a way for applying our logical operations, as mentioned in the third step of Section 5.1. If we use option (OP1), the number of steps is less than the number of operations because some operations are performed by our implementation automatically. On the other hand, if we use option (OP2), the number of steps is equal to the number of operations. In this sense, our implementation cannot reduce the effort to apply logical operations.

- Triggering the inconsistency management policy can reduce the number of operations which are employed by the agent. That is, our implementation can help the agent to reduce the effort to apply logical operations. For example, in the second legal case, when the inconsistency management policy is applied, the system will automatically perform seven operations from the total of 10 (see Table 5.6). Thus, the agent needs to apply only three operations by him/herself.

- Although the careful policy cannot be applied for aggregating information in some legal cases, we can interpret the final result from our implementation corresponding to the actual decision. This can be illustrated by an example of analyzing the second legal case in Section 5.2.3.

5.2.3 Example of Analysis Process

This section aims to demonstrate how we can analyze a legal case based on our analysis method as mentioned in Section 5.1. In this section, our target legal case is the second legal case which is selected from our six legal cases in Section 5.2.1. Therefore, this section will describe only the analysis process of the second legal case. The analysis process of the other legal case is presented in Appendix D. Before describing a process of analysis, we will give a short description of our target legal case (cf. Appendix C) as follows:

> There was a fight between two groups of people, i.e., v's group (v and b) and d's group (d, f_1, f_2 and f_3). In the course of the fight, one of d's group pulled a knife and then stabbed v in the chest. Finally, v died.

Based on our analysis method (in Section 5.1), we first summarize this legal case as mentioned in the previous section (Section 5.2.1). With the summary of legal case (see Table 5.4), an initial Kripke model which is used to analyze belief change of a judge is constructed by our implementation (the more details of this process is described in Section

Table 5.7: Analysis process of the second legal case (first version)

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(b,p)\wedge \mathsf{Sign}(f,\neg p)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(f,\neg p)$	$\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge$ $\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(2)	$[\mathsf{Sign}(b,\neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(b,\neg p)$	None
(2.1)	$[\{b\} \downdownarrows^j]$	j downgrades agent b	$\mathsf{Bel}(j, f <_j b)$
(2.2)	$[\mathsf{Sign}(b,p) \twoheadrightarrow j]$ $[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(b,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(2.3)	$[\mathsf{Sign}(f,\neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge$ $\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(3)	$[\mathsf{Sign}(f,p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f,p)$	None
(3.1)	$[\{f\} \downdownarrows^j]$	j downgrades agent f	$\mathsf{Bel}(j, b \approx_j f)$
(3.2)	$[\mathsf{Sign}(f,\neg p) \twoheadrightarrow j]$ $[\mathsf{Sign}(f,p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(f,\neg p)$ and $\mathsf{Sign}(f,p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge$ $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$

5.1). The resultant initial Kripke model for the second legal case (see the left-hand side of Fig. 5.2) can be defined as follows:

$$
\begin{aligned}
&G = \{\, j, b, f \,\}\\
&W = \{\, w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12} \,\}\\
&R_j = \{\, (w_0, w_1), (w_0, w_2), (w_0, w_3), (w_0, w_4) \,\}\\
&S_b = \{\, (w_1, w_5), (w_2, w_7), (w_3, w_9), (w_4, w_{11}) \,\}\\
&S_f = \{\, (w_1, w_6), (w_2, w_8), (w_3, w_10), (w_4, w_{12}) \,\}\\
&V(p) = \{\, w_5, w_6, w_7, w_{10} \,\}\\
&@ := w_0
\end{aligned}
$$

Next, we will analyze the second legal case by using the first option (OP1) (mentioned in Section 5.1) as an application method of our dynamic operators. That is, only private announcement and careful policy are applied. Let us describe how to analyze the second legal case from a judge's viewpoint. At the initial stage, we assume that a judge j should open to all possibilities (see the left-hand side of Fig. 5.2). Firstly, we will focus on statements of witnesses b and f in the court. When j receives statements from b and f, the following steps are performed as in Table 5.7:

(1) j admits the statements of witnesses b and f in the court, i.e., $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(f,\neg p)$ by $[(\mathsf{Sign}(b,p)\wedge \mathsf{Sign}(f,\neg p)) \rightsquigarrow j]$. As a result, j believes $\mathsf{Sign}(b,p)$ and

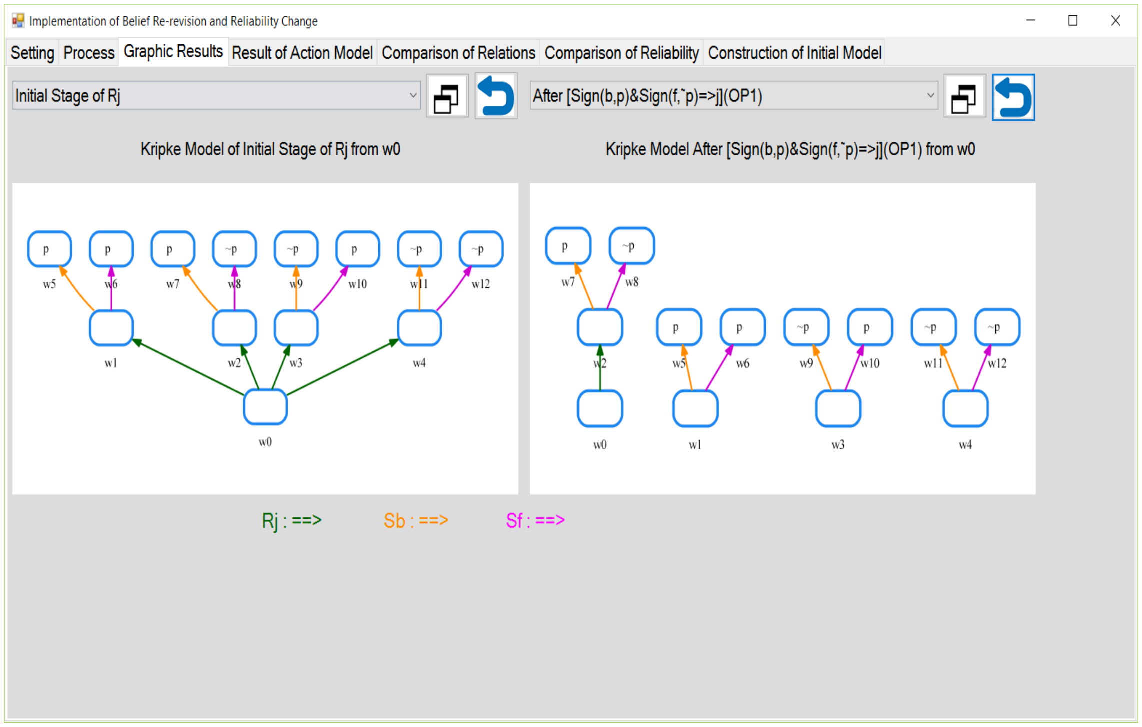

Figure 5.2: Left-hand side: Kripke model of the initial stage, Right-hand side: Kripke model after $[(\mathsf{Sign}(b,p) \wedge \mathsf{Sign}(f,\neg p)) \rightsquigarrow j]$

$\mathsf{Sign}(f,\neg p)$, i.e., $\mathsf{Bel}(j,\mathsf{Sign}(b,p))$ and $\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$ as shown in the right-hand side of Fig. 5.2.

(2) When j turns back to the inquiry stage, j commits him/herself to statement of b in the inquiry stage, i.e., $\mathsf{Sign}(b,\neg p)$ by $[\mathsf{Sign}(b,\neg p) \rightsquigarrow j]$, and the result can be shown in the left-hand side of Fig. 5.3. After that, the system can detect that the received statement $\mathsf{Sign}(b,\neg p)$ is inconsistent with j's belief. Thus, the system automatically performs the inconsistency management policy including four operations as the following sequences:

(2.1) When the system finds that b gives inconsistent statements, we can regard that b is unreliable. Therefore, the system will apply $[\{b\} \downdownarrows^j]$ in order to downgrade b to be less reliable than the other agents. The result of this downgrading is $\mathsf{Bel}(j, f <_j b)$ which means j believes that b becomes less reliable than f from j's perspective.

(2.2) By the update of $[\mathsf{Sign}(b,\neg p) \rightsquigarrow j]$ in Step (2), there is no link from state w_0 (see the left-hand side of Fig. 5.3). That is, there is no possibility in j's belief. Thus, we can regard that j needs to permit the possibility of both $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(b,\neg p)$ by $[\mathsf{Sign}(b,p) \twoheadrightarrow j]$ and $[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$, respectively. By these private permissions, j becomes undetermined on $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(b,\neg p)$, i.e., $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))$ and $\neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))$. The results of $[\mathsf{Sign}(b,p) \twoheadrightarrow j]$ and $[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$ are shown in the right-hand side of Fig. 5.3 and the left-hand side of Fig. 5.4, respectively.

(2.3) The system finds that there is the received statement of f, i.e., $\mathsf{Sign}(f,\neg p)$ which is not inconsistent with j's belief and is signed by f who is the most

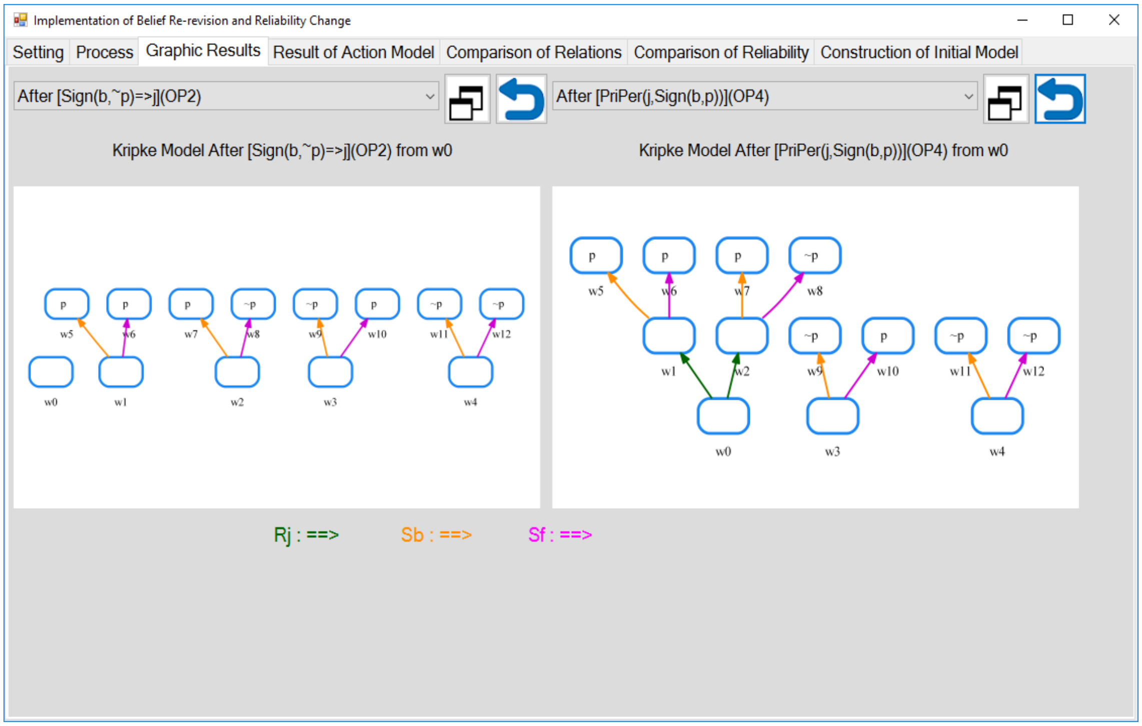

Figure 5.3: Left-hand side: Kripke model after $[\mathsf{Sign}(b, \neg p) \rightsquigarrow j]$, Right-hand side: Kripke model after $[\mathsf{Sign}(b, p) \twoheadrightarrow j]$

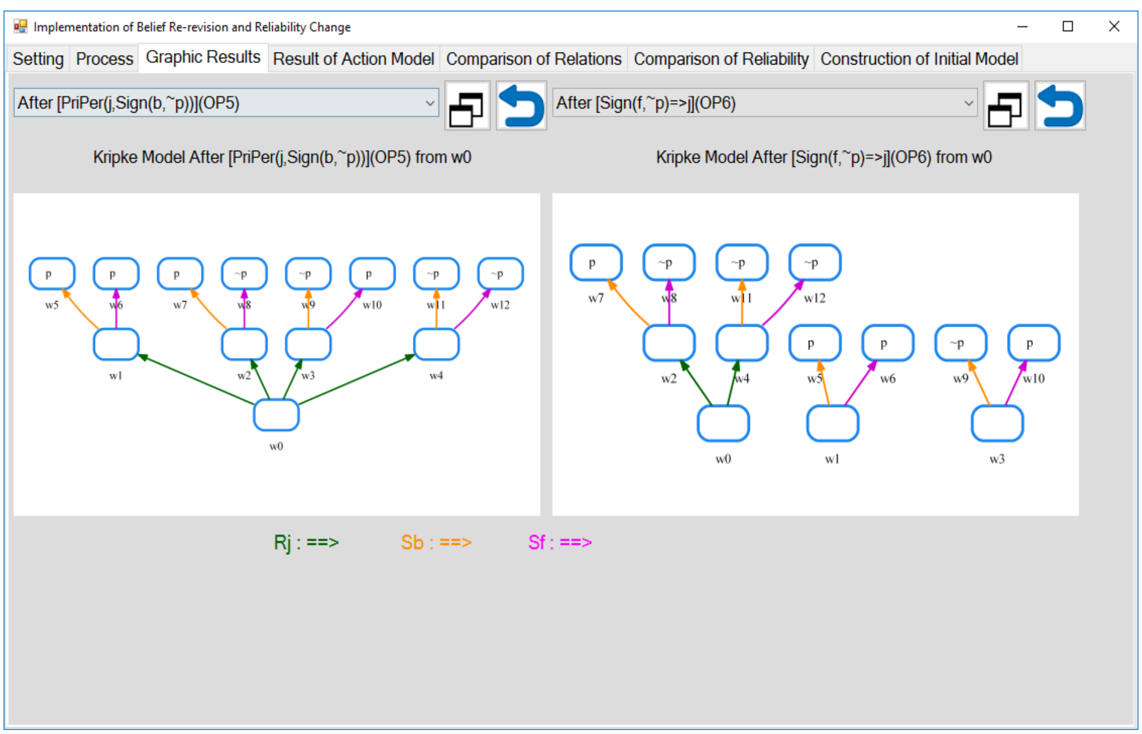

Figure 5.4: Left-hand side: Kripke model after $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow j]$, Right-hand side: Kripke model after $[\mathsf{Sign}(f, \neg p) \rightsquigarrow j]$

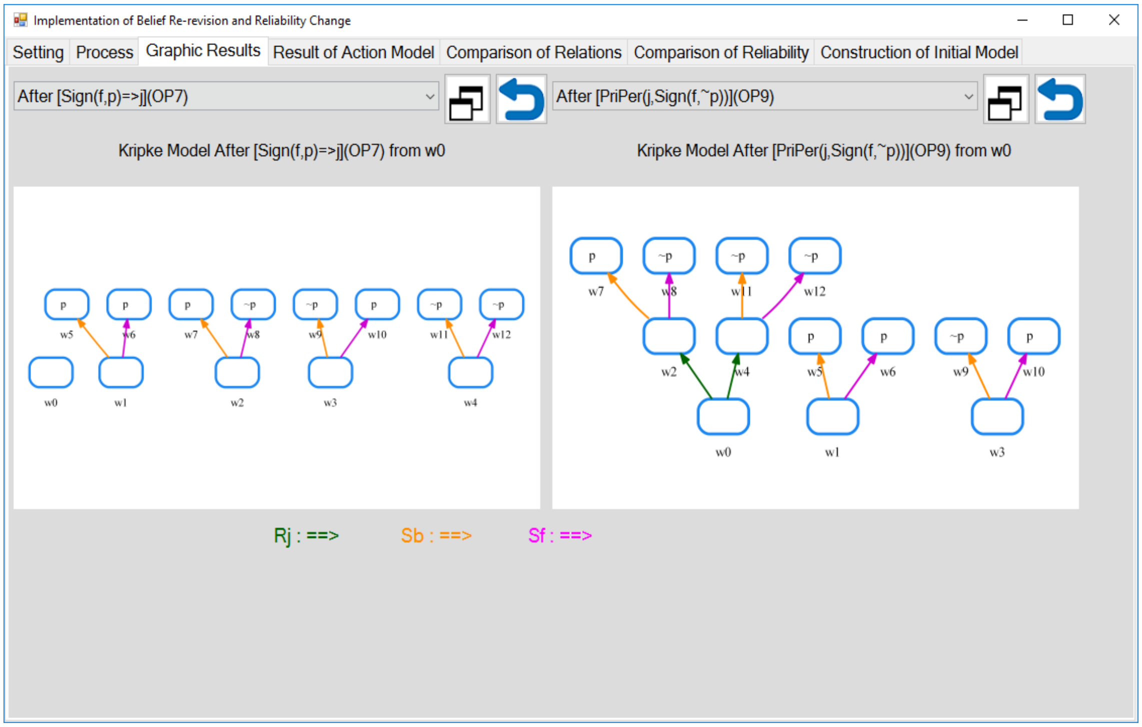

Figure 5.5: Left-hand side: Kripke model after $[\mathsf{Sign}(f,p) \rightsquigarrow j]$, Right-hand side: Kripke model after $[\mathsf{Sign}(f,\neg p) \twoheadrightarrow j]$

reliable agent. Thus, the system automatically employs $[\mathsf{Sign}(f,\neg p) \rightsquigarrow j]$ for admitting $\mathsf{Sign}(f,\neg p)$ to j. As a result, j will believe $\mathsf{Sign}(f,\neg p)$, i.e., $\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$ as shown in the right-hand side of Fig. 5.4.

(3) j also commits him/herself to statement of f in the inquiry stage, i.e., $\mathsf{Sign}(f,p)$ by $[\mathsf{Sign}(f,p) \rightsquigarrow j]$, and the result can be shown in the left-hand side of Fig. 5.5. After that, the system can detect that the received statement $\mathsf{Sign}(f,p)$ is inconsistent with j's belief. Thus, the system automatically performs the inconsistency management policy including three operations as the following sequences:

(3.1) Since the system can detect that f gives inconsistent statements, f is regraded to be unreliable. Thus, $[\{f\} \downdownarrows^{j}]$ is employed in order to downgrade f. After this downgrading, j believes that b and f become equally reliable and less reliable than the other agents from j's perspective, i.e., $\mathsf{Bel}(j, b \approx_j f)$.

(3.2) By the update of $[\mathsf{Sign}(f,p) \rightsquigarrow j]$ in Step (3), we can regard that there is no possibility in j's belief. For this reason, j needs to permit the possibility of both $\mathsf{Sign}(f,\neg p)$ and $\mathsf{Sign}(f,p)$ by $[\mathsf{Sign}(f,\neg p) \twoheadrightarrow j]$ and $[\mathsf{Sign}(f,p) \twoheadrightarrow j]$, respectively. These permissions will be performed by the system and the result is that j becomes undetermined on $\mathsf{Sign}(f,p)$ and $\mathsf{Sign}(f,\neg p)$, i.e., $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))$ and $\neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$ as shown in Fig. 5.6.

After the above process, if j employs the careful policy for information aggregation, we obtain that the careful policy cannot be done successfully because of the following reasons: By the careful policy, the system first finds a group of agents who are equally reliable. By Step (3.1), the system finds that agents b and f are equally reliable. Then,

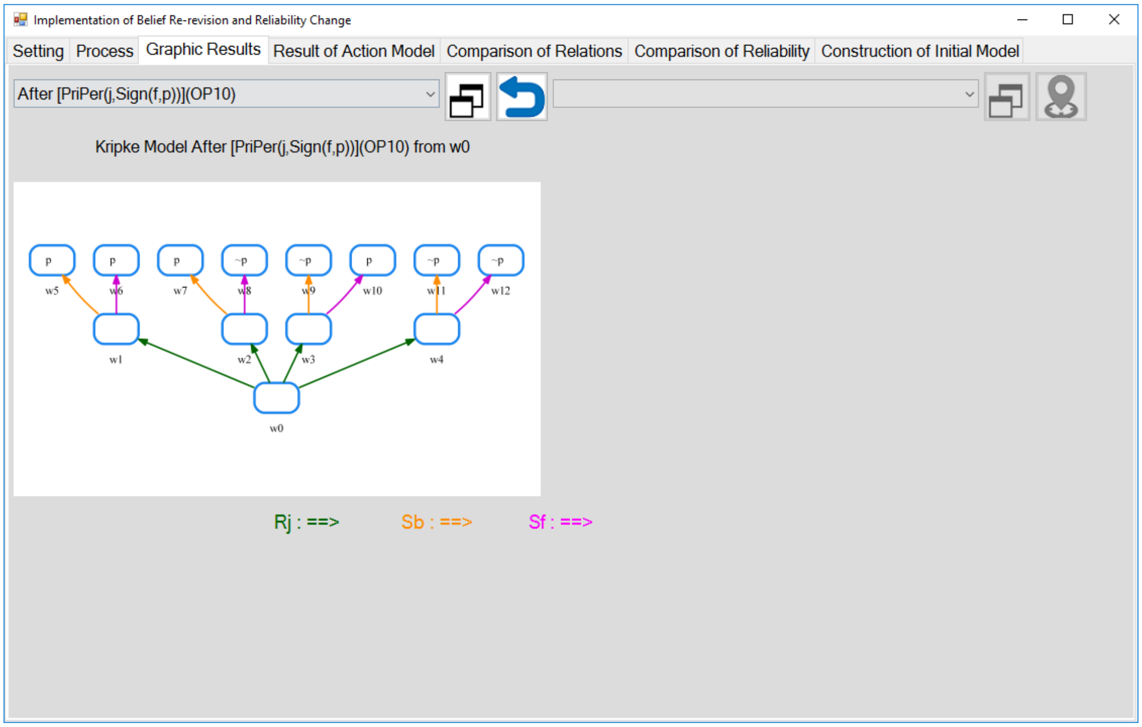

Figure 5.6: Kripke model after [$\mathsf{Sign}(f, p) \twoheadrightarrow j$]

the system will find statements which are universally signed by b and f. In this step, the system cannot find such statements because there is an inconsistency between statements of b and f. Thus, the careful policy cannot be employed.

Nevertheless, we can interpret the result from our implementation corresponding to the actual decision as follows: From Fig. 5.6, we can regard that j becomes undetermined on the statements of all witnesses. That is, j cannot decide which information he/she should believe. This can be interpreted as there is an absence of sufficient evidence. Consequently, we can regard that j acquits the defendant as in Table 5.2.

5.2.4 Discussion

According to analysis results in Section 5.2.2, we found two following problems.

(1) The aggregation policy cannot be applied in some legal cases.

As mentioned in the fifth feature of our implementation in Appendix B, since the system cannot decide which possibilities should be restored, it will automatically restore all possibilities to the agent's beliefs. However, this problem can be solved by two steps as follows:

- The system will allow us to select which statement should be permitted into an agent's belief by a consideration of the reliability of statements. That is, we will select the statement which is more reliable.
- The system will perform the private permission operator for restoring the possibility of the selected statement to an agent's belief.

Table 5.8: Analysis process of the second legal case (second version)

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(b,p)\wedge \mathsf{Sign}(f,\neg p)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(b,p)$ and $\mathsf{Sign}(f,\neg p)$	$\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(2)	$[\{f\} \downdownarrows^j]$	j downgrades agent f	$\mathsf{Bel}(j, b <_j f)$
(3)	$[\mathsf{Sign}(b,\neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(b,\neg p)$	None
(3.1)	$[\{b\} \downdownarrows^j]$	j downgrades agent b	$\mathsf{Bel}(j, b \approx_j f)$
(3.2)	$[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(b,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(4)	$[\mathsf{Sign}(f,\neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(b,p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(5)	$[\mathsf{Careful}(j,\neg p)]$	j aggregates information about $\neg p$	$\mathsf{Bel}(j,\neg p)$

Nevertheless, this solution has a limitation. That is, it requires that an agent can determine which statement to be more reliable. Otherwise, this solution cannot be employed. Let us illustrate how we can apply this solution to the second legal case. From Section 5.2.3, we found that the careful policy cannot be employed for information aggregation. In order to solve this problem, we will analyze the second legal case in a different way as shown in Table 5.8. Note that the analysis process in this table is called as the second version, while the first version refers to the one in Table 5.7 as mentioned in Section 5.2.3. We found that the second version differs from the first one in two aspects as follows:

- The second version applies the second option (OP2) as an application method of our dynamic operators (mentioned in Section 5.1) instead of the first option (OP1) which is used in the first version. For the second option (OP2), an agent needs to apply not only the private announcement and the careful policy but also dynamic operators for formalizing reliability change. That is, the joint downgrade operator is applied for downgrading agent f who is considered to be unreliable (see Step (2) in Table 5.8).

- For the inconsistency management policy, the system performs a process of belief restoration by applying only $[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$ for permitting the possibility of statement $\mathsf{Sign}(b,\neg p)$ (see Step (3.2) in Table 5.8). This is different from the first version that applies both $[\mathsf{Sign}(b,p) \twoheadrightarrow j]$ and $[\mathsf{Sign}(b,\neg p) \twoheadrightarrow j]$.

Let us explain why statement $\mathsf{Sign}(b,\neg p)$ is chosen. Since we regard that j considers

$\mathsf{Sign}(b, p)$ to be unreliable, j rejects $\mathsf{Sign}(b, p)$ but accepts $\mathsf{Sign}(b, \neg p)$ instead. By the analysis process in Table 5.8, the careful policy can be employed successfully. Finally, we obtain that j believes $\neg p$, i.e., j believes that the defendant was not the offender. Therefore, we can regard that j acquits the defendant that is the same as the actual decision in Table 5.2.

(2) We cannot interpret the result of an agent's reliability ordering in some legal cases.

Since this study focuses only on the result of an agent's belief, a question is how we can interpret the result of an agent's reliability ordering. In this study, the result of an agent's belief can be interpreted corresponding to the actual decision, but we cannot interpret the result of an agent's reliability ordering in some legal cases. Let us describe this problem by the first legal case. From Table 5.4, agent v gives both p and $\neg p$. Recall that this study has two hypotheses of reliability as mentioned in Section 1.3. With our hypotheses, there are two ways for interpreting the reliability of agent v as follows:

- Based on the first hypothesis (H1), we can interpret as: if statement p is considered to be reliable, agent v who gives statement p will be considered to be reliable.
- Based on the second hypothesis (H2), we can interpret as: if agent v gives inconsistent statements, agent v is considered to be unreliable.

Therefore, a question is how we can decide if agent v is reliable or not. Based on our analysis result from Table D.1, we obtain that agent v is reliable because of the following reason. From the analysis process in Table 5.5 in Section 5.2.2, since we regard that agent j believes that statement p of agent v in the inquiry stage is more reliable than statement $\neg p$ of agent v in the court, agent j upgrades the agents who sign statement p by $[\{v, f_2, mo\} \Uparrow^j_p]$ (see Step (4) in Table D.1). As a result, agent j believes that agents v and mo become more reliable. From this process, we can regard that the reliability of agent v is interpreted according to the first way as mentioned above. Nevertheless, since agent v signs inconsistent statements, he/she should be unreliable. For this reason, we cannot interpret the reliability of agent v in this case.

Chapter 6

Conclusion and Further directions

6.1 Conclusion

Our main goal of this study is to analyze a judge's belief change in a judgment process by a logical formalization. In order to achieve this goal, we proposed two components as follows:

Logical formalization for belief re-revision and reliability change

In order to analyze belief re-revision and reliability change of an agent, we proposed a logical formalization consisting of six dynamic operators, i.e., upgrade, downgrade, joint downgrade, private announcement, careful policy and private permission. Based on Lorini's framework [13] in Section 2.4, we first defined two notions including a signed statement and a reliability ordering. Information sources, i.e., agents who give information were represented by the signed statements and their reliability was represented by the reliability orderings. Nevertheless, our logical formalism was different from Lorini's framework [13] in two main aspects. First, we relativized the notion of the reliability orderings to a specific agent. Second, both the tell-action and the careful policy in [13] were captured by the private announcement. Then, we presented upgrade, downgrade and joint downgrade operators (mentioned in Chapter 3) which were used to allow an agent to change his/her reliability ordering between the other agents. The upgrade operator was applied for making some specific agents more reliable than the other agents, while downgrade operator aimed to downgrade all of them. In addition, an agent can make such agents in a specific group equally reliable and then downgrade them less reliable than agents in the other groups by employing the joint downgrade operator. For formalizing belief re-revision of an agent, we constructed private announcement, careful policy and private permission operators (mentioned in Chapter 4). An agent employed the private announcement for removing some beliefs, while the private permission was used to restore some possibilities to the agent's belief. When an agent received several statements, he/she had to derive his/her belief based on the received signed information by applying the careful policy. With a combination of our dynamic operators, we can analyze an agent's changing of belief and reliability.

Dynamic logical analysis of legal cases

In order to analyze a legal judgment from a logical point of view, we proposed our analysis method (mentioned in Chapter 5) including two main features. First, we stated how to construct an initial model for analyzing an agent's changing of belief. Second, we presented an approach for applying our dynamic operators. These features of our analysis method and our dynamic operators as mentioned above were implemented in a computer system. Our implementation consists of two main functions. First, the system can construct an initial model for formalizing a judge's changing of belief and reliability from a legal case. Second, the system can perform a policy for handling inconsistency by applying a combination of our dynamic operators. With the help of our implementation, we did not need to take much effort for analyzing a judge's changing of belief and reliability. That is, our implementation can automatically perform some operators instead. Therefore, this implementation can be regarded as a helpful tool for analyzing a legal judgment and can aid an understanding of a judge's reasoning in legal proceedings.

6.2 Further directions

Our future works consist of three goals. The first goal is to consider the reliability of statements. This study focuses only on the reliability of agents but does not consider the reliability of statements. This limitation leads to two problems as mentioned in Section 5.2.4. In order to solve these problems, we may formalize the reliability of statements by employing a preference modality based on [30] or the framework by [31].

The second goal is to consider more sophisticated ways to construct a restoration process of former beliefs by the private permission operator. Since this study supposed that an agent can consider only one information, we can analyze only one statement at one time. If we need to analyze several statements, it is required to analyze each statement separately. By this way, the private permission operator can work well. Nevertheless, if we consider multiple statements at the same time, it may cause a bad side-effect. For example, agent j first believes p and q ($\mathsf{Bel}(j,p) \wedge \mathsf{Bel}(j,q)$). Then, if j needs to permit the possibility of $\neg p$, $[\neg p \twoheadrightarrow j]$ is employed. By the update of $[\neg p \twoheadrightarrow j]$, j will not believe p ($\neg\mathsf{Bel}(j,p)$) and may not believe q ($\neg\mathsf{Bel}(j,q)$). In other words, $[\neg p \twoheadrightarrow j]$ affects not only $\mathsf{Bel}(j,p)$ but also $\mathsf{Bel}(j,q)$. In fact, $[\neg p \twoheadrightarrow j]$ should not affect the other propositions than p. Therefore, our goal is to avoid this problem.

For our last goal, the another interesting direction is to employ legal text processing in the area of natural language processing (NLP). Indeed, this study only focus on an application of logic to a legal case but does not consider an application of legal text processing. From our analysis method in Section 5.1, we manually summarize a legal case by extracting the facts and the decision. We may employ the automatic text summarization in our implementation for performing this process automatically.

Appendix A

Labelled Sequent Calculus for DEL with Relation Changers

First, we define a language $\mathcal{L}_{DELRC}$ for DEL with relation changers as follows:

Definition 59. *Let G be a fixed* finite *set of agents. The language $\mathcal{L}_{DELRC}$ consists of the following vocabulary: (i) a countably infinite set* $\mathsf{Prop} = \{ p, q, r, ... \}$ *of propositional letters, (ii) a countably infinite set* $\mathsf{AP} = \{ a, b, c, ... \}$ *of atomic programs, (iii) Boolean connectives: $\neg$, $\wedge$, (iv) program operators: $\cup$ (non-deterministic choice), ; (sequential composition), (v) mixed operators: ? (test), $[\cdot]$ (necessity), and (vi) a pointed action model $(\mathbb{E}, e)$ with $e \in E$. An* action model $\mathbb{E}$ *is a tuple $\mathbb{E} = (E, (Q_a)_{a\in\pi}, \mathrm{pre})$, where E is a non-empty set of action points, Q_a is a relation on E, and* pre *is a preconditions function. A set* Form_{RC} *of formulas φ of $\mathcal{L}_{DELRC}$ and a set* Prog *of programs π of $\mathcal{L}_{DELRC}$ are inductively defined as follows:*

$$\mathsf{Form}_{DELRC} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid [\pi]\varphi \mid [\mathbb{E}, e]\varphi$$

$$\mathsf{Prog} \ni \pi ::= a \mid \pi \cup \pi \mid \pi ; \pi \mid \varphi?$$

where $p \in \mathsf{Prop}$ and $a \in \mathsf{AP}$. We can regard $[\mathbb{E}, e]$ as $[\mathbb{E}, \Pi, e]$ in Section 4.1, and we can define Q_π for any $\pi \in \mathsf{Prog}$ as follows:

$$\begin{array}{lcl} Q_a & = & Q_a \\ Q_{\pi\cup\pi'} & = & Q_\pi \cup Q_{\pi'} \\ Q_{\pi;\pi'} & = & Q_\pi \circ Q_{\pi'} \\ Q_{\varphi?} & = & \{ (e_1, e_2) \mid e_1 = e_2 \} \end{array}$$

In order to define our labelled sequent calculus **GDELRC**, we first introduce the labelled formalism for **GDELRC** as follows. Let $\mathsf{Var} = \{ x, y, z, ... \}$ be a countably infinite set of variables. A set of labelled expressions (denoted by $A, B, C, ...$) is defined by:

$$A ::= x :^L \varphi \mid (x, L)R_\pi(y, L') \mid x = y,$$

where $\varphi \in \mathsf{Form}_{DELRC}$, $x, y \in \mathsf{Var}$, L and L' are the lists of pointed action models such that $L = ((\mathbb{E}_1, e_1), (\mathbb{E}_2, e_2), ..., (\mathbb{E}_n, e_n))$. For L, we use ε to define that L is empty. We say that $x :^L \varphi$ is a labelled formula, $(x, L)R_\pi(y, L')$ is a relational atom, and $x = y$ is an equality atom. Note that we can define $(x, L)R_\pi(y, L')$ for atomic program a and program π by: $(x, L)R_\pi(y, L') = (x, (\mathbb{E}_1, e_1), (\mathbb{E}_2, e_2), ..., (\mathbb{E}_n, e_n))R_\pi(y, (\mathbb{F}_1, f_1), (\mathbb{F}_2, f_2), ..., (\mathbb{F}_m, f_m))$, where $m = n$ and $\mathbb{E}_i = \mathbb{F}_i$ and $e_i(Q_i)_\pi f_i$ for all $1 \leqslant i \leqslant m$.

An underlying intuition for labelled expression is defined as follows. A labelled expression $x :^L \varphi$ is read as "after the successive updates of action models in L, a formula φ holds at state x." A labelled expression $(x, L)R_\pi(y, L')$ is read as "after the successive updates of action models in L and L', y is accessible from x by an execution of program π." We read $x = y$ as "state x is equal to state y." Note that our labelled sequent calculus **GDELRC** can be regarded as a formalized version of Kripke semantics. The length of these labelled expressions can be defined by the following definition.

Definition 60. *The length ℓ of formulas, programs, action models and labelled expressions are defined as follows:*

$$
\begin{array}{lcl}
\ell(p) & = & 1 \\
\ell(\neg\varphi) & = & \ell(\varphi) + 1 \\
\ell(\varphi \wedge \psi) & = & \ell(\varphi) + \ell(\psi) + 1 \\
\ell(\varphi \to \psi) & = & \ell(\varphi) + \ell(\psi) + 1 \\
\ell([\pi]\varphi) & = & \ell(\pi) + \ell(\varphi) + 1 \\
\ell(a) & = & 1 \\
\ell(\pi; \pi') & = & \ell(\pi) + \ell(\pi') + 1 \\
\ell(\pi \cup \pi') & = & \ell(\pi) + \ell(\pi') + 1 \\
\ell(\varphi?) & = & \ell(\varphi) + 1 \\
\ell(\varepsilon) & = & 0 \\
\ell(\mathbb{E}) & = & max\{\ell(\mathrm{pre}(e)) \mid e \in E\} \\
\ell([\mathbb{E}, e]\varphi) & = & \ell(\mathbb{E}) + \ell(\varphi) + 1 \\
\ell(L) & = & \ell((\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n)) = \ell(\mathbb{E}_1) + ... + \ell(\mathbb{E}_n) \\
\ell(x :^L \varphi) & = & \ell(L) + \ell(\varphi) \\
\ell((x, L)R_\pi(y, L')) & = & \ell(L) + \ell(\pi) \\
\ell(x = y) & = & 0
\end{array}
$$

In what follows, Γ and Δ are finite multiset of labelled expressions. From Table A.1, we define $\Gamma \Rightarrow \Delta$, where $\Gamma \Rightarrow \Delta$ is a sequent. The Hilbert-style system **HDELRC** for $\mathcal{L}_{DELRC}$ is presented in Table A.2.

Theorem 17 (Completeness)**.** *Let $\mathbb{M}$ be the class of models. For all $\psi \in \mathsf{Form}_{DELRC}$,*

$$\mathbb{M} \models \psi \text{ iff } \vdash_{\mathbf{HDELRC}} \psi.$$

From the labelled sequent calculus **GDELRC** in Table A.1, we can show that if a formula φ is provable in **HDELRC** , then $x :^\varepsilon \varphi$ is provable in **GDELRC** by Theorem 18. Before giving this theorem, we define the following derivable rules in **GDELRC** for the case of axiom $(RA8)$ in **HDELRC** as follows:

$$\frac{(x, \varepsilon)R_\pi(y, \varepsilon), \Gamma \Rightarrow \Delta, y :^\varepsilon \varphi}{\Gamma \Rightarrow \Delta, x :^\varepsilon [\pi]\varphi} \ (R[\pi]_1)^* \qquad \frac{\Gamma \Rightarrow \Delta, (x, \varepsilon)R_\pi(y, \varepsilon) \quad y :^\varepsilon \varphi, \Gamma \Rightarrow \Delta}{x :^\varepsilon [\pi]\varphi, \Gamma \Rightarrow \Delta} \ (L[\pi]_1)$$

where $*$: y does not appear in the lower sequent.

$$\frac{\Gamma \Rightarrow \Delta, x :^L \varphi}{\Gamma \Rightarrow \Delta, x :^L \bigwedge_{f \in Q_a(e)} \varphi} \ (R\bigwedge)^\dagger \qquad \frac{x :^L \varphi, \Gamma \Rightarrow \Delta}{\bigwedge_{f \in Q_a(e)} \varphi, \Gamma \Rightarrow \Delta} \ (L\bigwedge)^\ddagger$$

where $\dagger$: eQ_ad for some $d \in \mathbb{E}$ and d does not appear in the lower sequent, and $\ddagger$: $(e, d) \in Q_a$ for some $d \in \mathbb{E}$. Note that $\bigwedge$ is finite because Q_a is finite.

Table A.1: Labelled sequent calculus **GDELRC**

(Initial sequents)

$$x :^L \varphi \Rightarrow x :^L \varphi \qquad (x, L)R_\pi(y, L') \Rightarrow (x, L)R_\pi(y, L') \qquad x = y \Rightarrow x = y$$

(Structural rules)

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (Rw) \qquad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (Lw)$$

$$\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (Rc) \qquad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (Lc)$$

(Logical rules)

$$\frac{x :^L \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x :^L \neg\varphi}\ (R\neg) \qquad \frac{\Gamma \Rightarrow \Delta, x :^L \varphi}{x :^L \neg\varphi, \Gamma \Rightarrow \Delta}\ (L\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, x :^L \varphi_1 \quad \Gamma \Rightarrow \Delta, x :^L \varphi_2}{\Gamma \Rightarrow \Delta, x :^L \varphi_1 \wedge \varphi_2}\ (R\wedge) \qquad \frac{x :^L \varphi_i, \Gamma \Rightarrow \Delta}{x :^L \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L\wedge)_{i \in \{1,2\}}$$

$$\frac{x :^L \varphi_1, \Gamma \Rightarrow \Delta, x :^L \varphi_2}{\Gamma \Rightarrow \Delta, x :^L \varphi_1 \to \varphi_2}\ (R\to) \qquad \frac{\Gamma \Rightarrow \Delta, x :^L \varphi_1 \quad x :^L \varphi_2, \Gamma \Rightarrow \Delta}{x :^L \varphi_1 \to \varphi_2, \Gamma \Rightarrow \Delta}\ (L\to)$$

$$\frac{\left\{(x, (\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n))R_\pi(y, (\mathbb{E}_1, f_1), ..., (\mathbb{E}_n, f_n)), \Gamma \Rightarrow \Delta, y :^{(\mathbb{E}_1, f_1), ..., (\mathbb{E}_n, f_n)} \varphi \mid \Sigma\right\}}{\Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n)} [\pi]\varphi}\ (R[\pi])^\dagger$$

$$\frac{\Gamma \Rightarrow \Delta, (x, L)R_\pi(y, L') \quad y :^{L'} \varphi, \Gamma \Rightarrow \Delta}{x :^L [\pi]\varphi, \Gamma \Rightarrow \Delta}\ (L[\pi])$$

†: y does not appear in the lower sequent and $\Sigma = e_i(Q_i)_\pi f_i$ for all i.

(Action models rules)

$$\frac{\Gamma \Rightarrow \Delta, x :^L p}{\Gamma \Rightarrow \Delta, x :^{L, (\mathbb{E}, e)} p}\ (Rat) \qquad \frac{x :^L p, \Gamma \Rightarrow \Delta}{x :^{L, (\mathbb{E}, e)} p, \Gamma \Rightarrow \Delta}\ (Lat)$$

$$\frac{\Gamma \Rightarrow \Delta, x :^{L, (\mathbb{E}, e)} \varphi}{\Gamma \Rightarrow \Delta, x :^L [\mathbb{E}, e]\varphi}\ (R[\mathbb{E}, e]) \qquad \frac{x :^{L, (\mathbb{E}, e)} \varphi, \Gamma \Rightarrow \Delta}{x :^L [\mathbb{E}, e]\varphi, \Gamma \Rightarrow \Delta}\ (L[\mathbb{E}, e])$$

(Relational atom rules)

$$\frac{\Gamma \Rightarrow \Delta, (x, L)R_{\pi_a(e,f)}(y, L')}{\Gamma \Rightarrow \Delta, (x, L, (\mathbb{E}, e))R_a(y, L', (\mathbb{E}, f))} \ (Rrel)$$

$$\frac{(x, L)R_{\pi_a(e,f)}(y, L'), \Gamma \Rightarrow \Delta}{(x, L, (\mathbb{E}, e))R_a(y, L', (\mathbb{E}, f)), \Gamma \Rightarrow \Delta} \ (Lrel)$$

(Program rules)

$$\frac{\Gamma \Rightarrow \Delta, (x, L)R_{\pi_i}(y, L')}{\Gamma \Rightarrow \Delta, (x, L)R_{\pi_1 \cup \pi_2}(y, L')} \ (R\cup)_{i \in \{1,2\}}$$

$$\frac{(x, L)R_{\pi_1}(y, L'), \Gamma \Rightarrow \Delta \quad (x, L)R_{\pi_2}(y, L'), \Gamma \Rightarrow \Delta}{(x, L)R_{\pi_1 \cup \pi_2}(y, L'), \Gamma \Rightarrow \Delta} \ (L\cup)$$

$$\frac{\Gamma \Rightarrow \Delta, (x, L)R_{\pi_1}(z, L'') \quad \Gamma \Rightarrow \Delta, (z, L'')R_{\pi_2}(y, L')}{\Gamma \Rightarrow \Delta, (x, L)R_{\pi_1;\pi_2}(y, L')} \ (R;)$$

$$\frac{\{(x, (\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n))R_{\pi_1}(z, (\mathbb{E}_1, d_1), ..., (\mathbb{E}_n, d_n)), (z, (\mathbb{E}_1, d_1), ..., (\mathbb{E}_n, d_n))R_{\pi_2}(y, (\mathbb{E}_1, f_1), ..., (\mathbb{E}_n, f_n)), \Gamma \Rightarrow \Delta \mid \Sigma\}}{(x, (\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, f_1), ..., (\mathbb{E}_n, f_n)), \Gamma \Rightarrow \Delta} \ (L;)^{\ddagger}$$

‡: z does not appear in the lower sequent and $\Sigma = e_i(Q_i)_{\pi_1} d_i$ and $d_i(Q_i)_{\pi_2} f_i$ for all i.

$$\frac{\Gamma \Rightarrow \Delta, x = y \quad \Gamma \Rightarrow \Delta, x :^L \varphi}{\Gamma \Rightarrow \Delta, (x, L)R_{?\varphi}(y, L')} \ (R?)$$

$$\frac{x = y, \Gamma \Rightarrow \Delta}{(x, L)R_{?\varphi}(y, L'), \Gamma \Rightarrow \Delta} \ (L?_1) \qquad \frac{x :^L \varphi, \Gamma \Rightarrow \Delta}{(x, L)R_{?\varphi}(y, L'), \Gamma \Rightarrow \Delta} \ (L?_2)$$

(Equality rules)

$$\overline{\Rightarrow x = x} \ (R =)$$

$$\frac{x = y, \Gamma[x/w] \Rightarrow \Delta[x/w]}{x = y, \Gamma[y/w] \Rightarrow \Delta[y/w]} \ (L =_1) \qquad \frac{x = y, \Gamma[y/w] \Rightarrow \Delta[y/w]}{x = y, \Gamma[x/w] \Rightarrow \Delta[x/w]} \ (L =_2)$$

(Cut rule)

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ (Cut)$$

All instances of propositional tautologies			
$(K_{[\pi]})$	$[\pi](\varphi \to \psi) \to ([\pi]\varphi \to [\pi]\psi)$		
$(RA1)$	$[\pi \cup \pi']\varphi$	$\leftrightarrow$	$[\pi]\varphi \wedge [\pi']\varphi$
$(RA2)$	$[\pi; \pi']\varphi$	$\leftrightarrow$	$[\pi][\pi']\varphi$
$(RA3)$	$[\psi?]\varphi$	$\leftrightarrow$	$(\psi \to \varphi)$
$(RA4)$	$[\mathbb{E}, e]p$	$\leftrightarrow$	p
$(RA5)$	$[\mathbb{E}, e]\neg\varphi$	$\leftrightarrow$	$\neg[\mathbb{E}, e]\varphi$
$(RA6)$	$[\mathbb{E}, e](\varphi \wedge \psi)$	$\leftrightarrow$	$[\mathbb{E}, e]\varphi \wedge [\mathbb{E}, e]\psi$
$(RA7)$	$[\mathbb{E}, e](\varphi \to \psi)$	$\leftrightarrow$	$[\mathbb{E}, e]\varphi \to [\mathbb{E}, e]\psi$
$(RA8)$	$[\mathbb{E}, e][a]\varphi$	$\leftrightarrow$	$\bigwedge_{f \in Q_a(e)} [\pi_a(e, f)][\mathbb{E}, f]\varphi$
$(RA9)$	$[\mathbb{E}, e][\pi \cup \pi']\varphi$	$\leftrightarrow$	$[\mathbb{E}, e][\pi]\varphi \wedge [\mathbb{E}, e][\pi']\varphi$
$(RA10)$	$[\mathbb{E}, e][\pi; \pi']\varphi$	$\leftrightarrow$	$[\mathbb{E}, e][\pi][\pi']\varphi$
$(RA11)$	$[\mathbb{E}, e][\varphi?]\psi$	$\leftrightarrow$	$[\mathbb{E}, e](\varphi \to \psi)$
(MP)	From φ and $\varphi \to \psi$, infer ψ		
$(Nec_{[\pi]})$	From φ, infer $[\pi]\varphi$		
$(Nec_{[\mathbb{E},e]})$	From φ, infer $[\mathbb{E}, e]\varphi$		

Table A.2: Hilbert-style system **HDELRC** for $\mathcal{L}_{DELRC}$

Theorem 18. *For any formula $\varphi \in \mathsf{Form}_{DELRC}$ and any variable $x \in \mathsf{Var}$, if $\vdash_{\mathbf{HDELRC}} \varphi$, then $\vdash_{\mathbf{GDELRC}} \Rightarrow x :^{\varepsilon} \varphi$.*

Proof. Suppose that $\vdash_{\mathbf{HDELRC}} \varphi$. Our goal is to show that $\vdash_{\mathbf{GDELRC}} \Rightarrow x :^{\varepsilon} \varphi$. This proof can be conducted by induction on the height of derivation of **HDELRC**. Thus, we will show the following cases.

Case of axiom $(RA4)$: $[\mathbb{E}, e]p \leftrightarrow p$

First, we show the direction from left to right as follows:

$$\dfrac{\dfrac{\dfrac{x :^{\varepsilon} p \Rightarrow x :^{\varepsilon} p}{x :^{(\mathbb{E},e)} p \Rightarrow x :^{\varepsilon} p}\,(Lat)}{x :^{\varepsilon} [\mathbb{E}, e]p \Rightarrow x :^{\varepsilon} p}\,(L[\mathbb{E}, e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E}, e]p \to p}\,(R \to)$$

Second, we show the direction from right to left as follows:

$$\dfrac{\dfrac{\dfrac{x :^{\varepsilon} p \Rightarrow x :^{\varepsilon} p}{x :^{\varepsilon} p \Rightarrow x :^{(\mathbb{E},e)} p}\,(Rat)}{x :^{\varepsilon} p \Rightarrow x :^{\varepsilon} [\mathbb{E}, e]p}\,(R[\mathbb{E}, e])}{\Rightarrow x :^{\varepsilon} p \to [\mathbb{E}, e]p}\,(R \to)$$

Case of axiom $(RA5)$: $[\mathbb{E}, e]\neg\varphi \leftrightarrow \neg[\mathbb{E}, e]\varphi$

First, we show the direction from left to right as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{\varepsilon} [\mathbb{E},e]\varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}\,(L[\mathbb{E},e])}{\Rightarrow x :^{(\mathbb{E},e)} \varphi, x :^{\varepsilon} \neg[\mathbb{E},e]\varphi}\,(R\neg)}{x :^{(\mathbb{E},e)} \neg\varphi \Rightarrow x :^{\varepsilon} \neg[\mathbb{E},e]\varphi}\,(L\neg)}{x :^{\varepsilon} [\mathbb{E},e]\neg\varphi \Rightarrow x :^{\varepsilon} \neg[\mathbb{E},e]\varphi}\,(L[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e]\neg\varphi \to \neg[\mathbb{E},e]\varphi}\,(R\to)$$

Second, we show the direction from right to left as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \to x :^{\varepsilon} [\mathbb{E},e]\varphi}\,(R[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi, x :^{(\mathbb{E},e)} \neg\varphi}\,(R\neg)}{x :^{\varepsilon} \neg[\mathbb{E},e]\varphi \Rightarrow x :^{(\mathbb{E},e)} \neg\varphi}\,(L\neg)}{x :^{\varepsilon} \neg[\mathbb{E},e]\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\neg\varphi}\,(R[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} \neg[\mathbb{E},e]\varphi \to [\mathbb{E},e]\neg\varphi}\,(R\to)$$

Case of axiom ($RA6$**)**: $[\mathbb{E},e](\varphi \wedge \psi) \leftrightarrow [\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi$

First, we show the direction from left to right as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \wedge \psi \Rightarrow x :^{(\mathbb{E},e)} \varphi}\,(L\wedge)}{x :^{(\mathbb{E},e)} \varphi \wedge \psi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi}\,(R[\mathbb{E},e]) \qquad \dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{(\mathbb{E},e)} \varphi \wedge \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(L\wedge)}{x :^{(\mathbb{E},e)} \varphi \wedge \psi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\psi}\,(R[\mathbb{E},e])}{x :^{(\mathbb{E},e)} \varphi \wedge \psi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi}\,(R\wedge)}{x :^{\varepsilon} [\mathbb{E},e](\varphi \wedge \psi) \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi}\,(L[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e](\varphi \wedge \psi) \to ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi)}\,(R\to)$$

Second, we show the direction from right to left as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{\varepsilon} [\mathbb{E},e]\varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}\,(L[\mathbb{E},e])}{x :^{\varepsilon} ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi) \Rightarrow x :^{(\mathbb{E},e)} \varphi}\,(L\wedge) \qquad \dfrac{\dfrac{x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{\varepsilon} [\mathbb{E},e]\psi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(L[\mathbb{E},e])}{x :^{\varepsilon} ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi) \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(L\wedge)}{x :^{\varepsilon} ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi) \Rightarrow x :^{(\mathbb{E},e)} \varphi \wedge \psi}\,(R\wedge)}{x :^{\varepsilon} ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi) \Rightarrow x :^{\varepsilon} [\mathbb{E},e](\varphi \wedge \psi)}\,(R[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} ([\mathbb{E},e]\varphi \wedge [\mathbb{E},e]\psi) \to [\mathbb{E},e](\varphi \wedge \psi)}\,(R\to)$$

Case of axiom ($RA7$**)**: $[\mathbb{E},e](\varphi \to \psi) \leftrightarrow [\mathbb{E},e]\varphi \to [\mathbb{E},e]\psi$

First, we show the direction from left to right as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi}\,(Rw) \qquad \dfrac{x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(Lw)}{x :^{(\mathbb{E},e)} \varphi \to \psi, x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(L\to)}{x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi), x :^{\varepsilon} [\mathbb{E},e]\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\psi}\,([\mathbb{E},e])}{x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi) \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi \to [\mathbb{E},e]\psi}\,(R\to)}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi) \to ([\mathbb{E},e]\varphi \to [\mathbb{E},e]\psi)}\,(R\to)
$$

Second, we show the direction from right to left as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi}\,(Rw) \qquad \dfrac{x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(Lw)}{x :^{(\mathbb{E},e)} \varphi \to \psi, x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(L\to)}{x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi), x :^{\varepsilon} [\mathbb{E},e]\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\psi}\,([\mathbb{E},e])}{x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi) \Rightarrow x :^{\varepsilon} [\mathbb{E},e]\varphi \to [\mathbb{E},e]\psi}\,(R\to)}{\Rightarrow x :^{\varepsilon} ([\mathbb{E},e]\varphi \to [\mathbb{E},e]\psi) \to [\mathbb{E},e](\varphi \to \psi)}\,(R\to)
$$

Case of axiom ($RA8$): $[\mathbb{E},e][a]\varphi \leftrightarrow \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi$

First, we show the direction from left to right as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon) \Rightarrow (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon)}{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon)}\,(Rw)}{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_a\,(y,(\mathbb{E},f))}\,(Rrel) \qquad \dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon), x :^{(\mathbb{E},e)} [a]\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[a])}{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon), x :^{(\mathbb{E},e)} [a]\varphi \Rightarrow y :^{\varepsilon} [\mathbb{E},f]\varphi}\,(R[\mathbb{E},f])}{x :^{(\mathbb{E},e)} [a]\varphi \Rightarrow x :^{\varepsilon} [\pi_a(e,f)][\mathbb{E},f]\varphi}\,(R[\pi_a(e,f)])}{x :^{(\mathbb{E},e)} [a]\varphi \Rightarrow x :^{\varepsilon} \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi}\,(R\textstyle\bigwedge)}{x :^{\varepsilon} [\mathbb{E},e][a]\varphi \Rightarrow x :^{\varepsilon} \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi}\,(L[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][a]\varphi \to \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi}\,(R\to)
$$

Second, we show the direction from right to left as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon) \Rightarrow (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon)}{(x,(\mathbb{E},e))\,\mathsf{R}_a\,(y,(\mathbb{E},f)) \Rightarrow (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon)}\,(Lrel)}{(x,(\mathbb{E},e))\,\mathsf{R}_a\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,\varepsilon)\,\mathsf{R}_{\pi_a(e,f)}\,(y,\varepsilon)}\,(Rw) \qquad \dfrac{\dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{\varepsilon} [\mathbb{E},f]\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[\mathbb{E},f])}{y :^{\varepsilon} [\mathbb{E},f]\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_a\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{\left\{(x,(\mathbb{E},e))\,\mathsf{R}_a\,(y,(\mathbb{E},f)), x :^{\varepsilon} [\pi_a(e,f)][\mathbb{E},f]\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_af\right\}}\,(L[\pi_a(e,f)]_1)}{x :^{\varepsilon} [\pi_a(e,f)][\mathbb{E},f]\varphi \Rightarrow x :^{(\mathbb{E},e)} [a]\varphi}\,(R[a])}{x :^{\varepsilon} [\pi_a(e,f)][\mathbb{E},f]\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][a]\varphi}\,(L[\mathbb{E},e])}{x :^{\varepsilon} \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][a]\varphi}\,(L\textstyle\bigwedge)}{\Rightarrow x :^{\varepsilon} \bigwedge_{f \in Q_a(e)} [\pi_a(e,f)][\mathbb{E},f]\varphi \to [\mathbb{E},e][a]\varphi}\,(R\to)
$$

Case of axiom ($RA9$): $[\mathbb{E},e][\pi \cup \pi']\varphi \leftrightarrow [\mathbb{E},e][\pi]\varphi \land [\mathbb{E},e][\pi']\varphi$

First, we show the direction from left to right as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_1\\ x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi\end{array} \quad \begin{array}{c}\vdots\,\mathfrak{D}_2\\ x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi']\varphi\end{array}}{x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi}\,(R\wedge)}{x :^{\varepsilon} [\mathbb{E},e][\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi}\,(L[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi \cup \pi']\varphi \to [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi}\,(R\to)$$

where $\mathfrak{D}_1$ is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)}, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}\,(Rw)}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi\cup\pi'}\,(y,(\mathbb{E},f))}\,(R\cup) \quad \dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{\left\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)), x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_{\pi}f\right\}}\,(L[\pi\cup\pi'])}{x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{(\mathbb{E},e)} [\pi]\varphi}\,(R[\pi])}{x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi}\,(R[\mathbb{E},e])$$

Similarly, $\mathfrak{D}_2$ is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)}, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}\,(Rw)}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi\cup\pi'}\,(y,(\mathbb{E},f))}\,(R\cup) \quad \dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{\left\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)), x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_{\pi'}f\right\}}\,(L[\pi\cup\pi'])}{x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{(\mathbb{E},e)} [\pi']\varphi}\,(R[\pi'])}{x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi']\varphi}\,(R[\mathbb{E},e])$$

Second, we show the direction from right to left as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_1\\ x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi\end{array} \quad \begin{array}{c}\vdots\,\mathfrak{D}_2\\ x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi\end{array}}{\left\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi\cup\pi'}\,(y,(\mathbb{E},f)), x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_{\pi\cup\pi'}f\right\}}\,(L\cup)}{x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi \Rightarrow x :^{(\mathbb{E},e)} [\pi \cup \pi']\varphi}\,(R[\pi\cup\pi'])}{x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi \cup \pi']\varphi}\,(R[\mathbb{E},e])}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi \to [\mathbb{E},e][\pi \cup \pi']\varphi}\,(R\to)$$

where $\mathfrak{D}_1$ is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}\,(Rw) \quad \dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{x :^{(\mathbb{E},e)} [\pi]\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[\pi])}{x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[\mathbb{E},e])}{x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L\wedge)$$

Similarly, $\mathfrak{D}_2$ is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}\,(Rw) \quad \dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)}{x :^{(\mathbb{E},e)} [\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[\pi'])}{x :^{\varepsilon} [\mathbb{E},e][\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L[\mathbb{E},e])}{x :^{\varepsilon} [\mathbb{E},e][\pi]\varphi \wedge [\mathbb{E},e][\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(L\wedge)$$

Case of axiom ($RA10$): $[\mathbb{E},e][\pi;\pi']\varphi \leftrightarrow [\mathbb{E},e][\pi][\pi']\varphi$

First, we show the direction from left to right as follows:

$$
\dfrac{
\dfrac{
\dfrac{
\begin{pmatrix}(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)),\\ (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))\end{pmatrix} \Rightarrow \begin{pmatrix} z :^{(\mathbb{E},d)} \varphi,\\ (x,(\mathbb{E},e))\,\mathsf{R}_{\pi;\pi'}\,(z,(\mathbb{E},d))\end{pmatrix}\ (\mathfrak{D}_1)
\qquad
\dfrac{z :^{(\mathbb{E},d)} \varphi \Rightarrow z :^{(\mathbb{E},d)} \varphi}{\begin{pmatrix} z :^{(\mathbb{E},d)} \varphi,\\ (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)),\\ (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))\end{pmatrix} \Rightarrow z :^{(\mathbb{E},d)} \varphi}\,(Lw)
}{
\big\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)), (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d)), x :^{(\mathbb{E},e)} [\pi;\pi']\varphi \Rightarrow z :^{(\mathbb{E},d)} \varphi \mid fQ_{\pi'}d\big\}
}\,(L[\pi;\pi'])
}{
\big\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)), x :^{(\mathbb{E},e)} [\pi;\pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} [\pi']\varphi \mid eQ_{\pi}f\big\}
}\,(R[\pi'])
}{
\dfrac{
\dfrac{x :^{(\mathbb{E},e)} [\pi;\pi']\varphi \Rightarrow x :^{(\mathbb{E},e)} [\pi][\pi']\varphi}{x :^{\varepsilon} [\mathbb{E},e][\pi;\pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi][\pi']\varphi}\,([\mathbb{E},e])
}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi;\pi']\varphi \to [\mathbb{E},e][\pi][\pi']\varphi}\,(R\to)
}\,(R[\pi])
$$

where $\mathfrak{D}_1$ is constructed as follows:

$$
\dfrac{
\dfrac{
\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}{\begin{pmatrix}(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)),\\ (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))\end{pmatrix} \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f))}\,(Lw)
\quad
\dfrac{(y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d)) \Rightarrow (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))}{\begin{pmatrix}(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)),\\ (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))\end{pmatrix} \Rightarrow (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d))}\,(Lw)
}{
(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)), (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi;\pi'}\,(z,(\mathbb{E},d))
}\,(R;)
}{
(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(y,(\mathbb{E},f)), (y,(\mathbb{E},f))\,\mathsf{R}_{\pi'}\,(z,(\mathbb{E},d)) \Rightarrow z :^{(\mathbb{E},d)} \varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi;\pi'}\,(z,(\mathbb{E},d))
}\,(Rw)
$$

Second, we show the direction from right to left as follows:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d)) \Rightarrow (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d))}{\begin{pmatrix}(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d)),\\ (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))\end{pmatrix} \Rightarrow \begin{pmatrix} y :^{(\mathbb{E},f)} \varphi,\\ (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d))\end{pmatrix}}\,(w)
\quad
\begin{pmatrix} z :^{(\mathbb{E},d)} [\pi']\varphi,\\ (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d)),\\ (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))\end{pmatrix} \Rightarrow y :^{(\mathbb{E},f)} \varphi\ (\mathfrak{D}_1)
}{
\big\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d)), (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)), x :^{(\mathbb{E},e)} [\pi][\pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_{\pi}d, dQ_{\pi'}f\big\}
}\,L[\pi]
}{
\big\{(x,(\mathbb{E},e))\,\mathsf{R}_{\pi;\pi'}\,(y,(\mathbb{E},f)), x :^{(\mathbb{E},e)} [\pi][\pi']\varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi \mid eQ_{\pi;\pi'}f\big\}
}\,(L;)
}{
\dfrac{
\dfrac{x :^{(\mathbb{E},e)} [\pi][\pi']\varphi \Rightarrow x :^{(\mathbb{E},e)} [\pi;\pi']\varphi}{x :^{\varepsilon} [\mathbb{E},e][\pi][\pi']\varphi \Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi;\pi']\varphi}\,([\mathbb{E},e])
}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][\pi][\pi']\varphi \to [\mathbb{E},e][\pi;\pi']\varphi}\,(R\to)
}\,(R[\pi;\pi'])
$$

where $\mathfrak{D}_1$ is constructed as follows:

$$
\dfrac{
\dfrac{
\dfrac{(z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}{(z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi, (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f))}\,(Rw)
\quad
\dfrac{y :^{(\mathbb{E},f)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \varphi}{y :^{(\mathbb{E},f)} \varphi, (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi}\,(Lw)
}{
z :^{(\mathbb{E},d)} [\pi']\varphi, (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi
}\,(L[\pi'])
}{
z :^{(\mathbb{E},d)} [\pi']\varphi, (x,(\mathbb{E},e))\,\mathsf{R}_{\pi}\,(z,(\mathbb{E},d)), (z,(\mathbb{E},d))\,\mathsf{R}_{\pi'}\,(y,(\mathbb{E},f)) \Rightarrow y :^{(\mathbb{E},f)} \varphi
}\,(Lw)
$$

Case of axiom ($RA11$): $[\mathbb{E},e][\varphi?]\psi \leftrightarrow [\mathbb{E},e](\varphi \to \psi)$

First, we show the direction from left to right as follows:

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Rightarrow x = x}\,(R=)}{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi, x = x}\,(w)
\quad
\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi}\,(Rw)
}{
x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi, (x,(\mathbb{E},e))\,\mathsf{R}_{\varphi?}\,(x,(\mathbb{E},e))
}\,(R?)
\quad
\dfrac{x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{(\mathbb{E},e)} \psi, x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \psi}\,(Lw)
}{
\dfrac{
\dfrac{
\dfrac{x :^{(\mathbb{E},e)} \varphi, x :^{(\mathbb{E},e)} [\varphi?]\psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x :^{(\mathbb{E},e)} [\varphi?]\psi \Rightarrow x :^{(\mathbb{E},e)} \varphi \to \psi}\,(R\to)
}{x :^{\varepsilon} [\mathbb{E},e][\varphi?]\psi \Rightarrow x :^{\varepsilon} [\mathbb{E},e](\varphi \to \psi)}\,([\mathbb{E},e])
}{\Rightarrow x :^{\varepsilon} [\mathbb{E},e][\varphi?]\psi \to [\mathbb{E},e](\varphi \to \psi)}\,(R\to)
}\,(L[\varphi?])
$$

Second, we show the direction from right to left as follows:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{x :^{(\mathbb{E},e)} \varphi \Rightarrow x :^{(\mathbb{E},e)} \varphi}{x :^{(\mathbb{E},e)} \varphi \Rightarrow y :^{(\mathbb{E},f)} \psi, x :^{(\mathbb{E},e)} \varphi}\,(Rw)
}{(x, (\mathbb{E}, e))\, \mathsf{R}_{\varphi?}\, (y, (\mathbb{E}, f)) \Rightarrow y :^{(\mathbb{E},f)} \psi, x :^{(\mathbb{E},e)} \varphi}\,(L?_2)
\quad
\dfrac{
\dfrac{
\dfrac{x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},e)} \psi}{x = y, x :^{(\mathbb{E},e)} \psi \Rightarrow x :^{(\mathbb{E},f)} \psi}\,(Lw)
}{x = y, x :^{(\mathbb{E},e)} \psi \Rightarrow y :^{(\mathbb{E},f)} \psi}\,(L =_1)
}{x :^{(\mathbb{E},e)} \psi, (x, (\mathbb{E}, e))\, \mathsf{R}_{\varphi?}\, (y, (\mathbb{E}, f)) \Rightarrow y :^{(\mathbb{E},f)} \psi}\,(L?_1)
}{\big\{(x, (\mathbb{E}, e))\, \mathsf{R}_{\varphi?}\, (y, (\mathbb{E}, f)), x :^{(\mathbb{E},e)} \varphi \to \psi \Rightarrow y :^{(\mathbb{E},f)} \psi \mid eQ_{\varphi?} f\big\}}\,(L \to)
}{x :^{(\mathbb{E},e)} \varphi \to \psi \Rightarrow x :^{(\mathbb{E},e)} [\varphi?]\psi}\,(R[\varphi?])
}{
\dfrac{x :^{\varepsilon} [\mathbb{E}, e](\varphi \to \psi) \Rightarrow x :^{\varepsilon} [\mathbb{E}, e][\varphi?]\psi}{\Rightarrow x :^{\varepsilon} [\mathbb{E}, e](\varphi \to \psi) \to [\mathbb{E}, e][\varphi?]\psi}\,(R \to)
}\,([\mathbb{E}, e])
$$

□

Next, we will provide a proof of cut elimination of **GDELRC**. Before giving the details of the proof, let us define the substitution by the following definition.

Definition 61. *We define a substitution $A[y/x]$ (whose reading is the result of substituting x in A with y) as:*

$$
\begin{array}{rcl}
z[y/x] & \equiv & z \text{ if } x \neq z \\
z[y/x] & \equiv & y \text{ if } x = z \\
(z, L)R_\pi(w, L')[y/x] & \equiv & (z[y/x], L)R_\pi(w[y/x], L') \\
(z :^L \varphi)[y/x] & \equiv & z[y/x] :^L \varphi \\
(w = z)[y/x] & \equiv & w[y/x] = z[y/x]
\end{array}
$$

where $w, x, y, z \in \mathsf{Var}$, $\varphi \in \mathsf{Form}_{DELRC}$, $\pi \in \mathsf{PR}$, L and L' are the lists of pointed action models, and A be a labelled expression. Given a multiset Γ of labelled expressions, we define a substitution $\Gamma[y/x]$ as:

$$\Gamma[y/x] \;\; := \;\; \{\, A[y/x] \mid A \in \Gamma \,\}$$

By the above definition, we give the following lemma.

Lemma 15. *If $\Gamma \Rightarrow \Delta$ is derivable in* **GDELRC** *by a derivation $\mathfrak{D}$, then $\Gamma[y/x] \Rightarrow \Delta[y/x]$ is also derivable by a derivation $\mathfrak{D}'$ which has the same height as $\mathfrak{D}$.*

Then, we define the following rule of the extended cut.

Definition 62. *Let A be a labelled expression.*

$$\dfrac{\Gamma \Rightarrow \Delta, A^m \quad A^n, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

where $m, n \geq 0$ and A is called an Ecut labelled expression*.*

Let us define **GDELRC**$^-$ as **GDELRC** without the cut rule. Next, we give the definition of an Ecut-bottom form as follows:

Definition 63. *A derivation $\mathfrak{D}$ is an Ecut-bottom form if*

$$\dfrac{\begin{array}{c}\vdots\, \mathfrak{D}_L \\ \Gamma \Rightarrow \Delta, A^m\end{array} \quad \begin{array}{c}\vdots\, \mathfrak{D}_R \\ A^n, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

where we have no applications of $(Ecut)$ in $\mathfrak{D}_L$ nor $\mathfrak{D}_R$.

Now, we provide the following lemma which is used for the proof of cut elimination.

Lemma 16. *If there is an Ecut-bottom form derivation of* $\Gamma \Rightarrow \Delta$*, then* $\mathbf{GDELRC}^{-} \vdash \Gamma \Rightarrow \Delta$.

Proof. We can show this proof by the method of Ono and Komori [32]. First, we give an Ecut-bottom form derivation $\mathfrak{D}$ as follows:

$$\dfrac{\begin{array}{c}\vdots\, \mathfrak{D}_L \\ \Gamma \Rightarrow \Delta, A^m\end{array} \quad \begin{array}{c}\vdots\, \mathfrak{D}_R \\ A^n, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (Ecut)$$

Then, the proof is conducted by double induction on the complexity $c(\mathfrak{D})$ and the weight $w(\mathfrak{D})$ of $\mathfrak{D}$, where $c(\mathfrak{D})$ is the length of the Ecut labelled expression in the last application of $(Ecut)$, and $w(\mathfrak{D})$ is the number of the total sequents in $\mathfrak{D}_L$ and $\mathfrak{D}_R$ (note that $c(\mathfrak{D}) \geqslant 0$ and $w(\mathfrak{D}) \geqslant 2$). In addition, we can define the lexicographic order $(c, w) \leqslant (c', w')$ as $(c, w) < (c', w') := c < c'$ or ($c = c'$ and $w \leqslant w'$), where $c = c(\mathfrak{D})$ and $w = w(\mathfrak{D})$. From the above Ecut-bottom form, we suppose that $m = 0$ and $n = 0$. Let us use $\texttt{rule}(L)$ and $\texttt{rule}(R)$ to mean the last rules of $\mathfrak{D}_L$ and $\mathfrak{D}_R$, respectively. The proof can be divided into the following five cases.

(1) At least one of $\texttt{rule}(L)$ and $\texttt{rule}(R)$ is an initial sequent or $(R =)$.

(2) At least one of $\texttt{rule}(L)$ and $\texttt{rule}(R)$ is a structural rule.

(3) At least one of $\texttt{rule}(L)$ and $\texttt{rule}(R)$ is a logical rule, an action model rule, a relational rule, or a program rule where the cut labelled expression is not introduced by the rule.

(4) Both $\texttt{rule}(L)$ and $\texttt{rule}(R)$ are logical rules, action model rules, relational rules, or program rules where the cut labelled expression is introduced by the rules.

(5) At least one of $\texttt{rule}(L)$ and $\texttt{rule}(R)$ is equality rules $(L =_1)$ or $(L =_2)$.

From all the above cases, we will focus only on cases (1), (3), (4) and (5).

Case (1): we will show only the case for $(R =)$.

Case (1.1): $\texttt{rule}(L)$ is $(R =)$. We can construct the derivation $\mathfrak{D}$ as follows:

$$\dfrac{\dfrac{}{\Rightarrow x = x}\ (R =) \quad \begin{array}{c}\vdots\, \mathfrak{D}_R \\ (x = x)^n, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma' \Rightarrow \Delta'}\ (Ecut)$$

This case is reduced to check what is the last rule of $\texttt{rule}(R)$.

Case (3): we will show only the cases for (Rat), $(R[\mathbb{E}, e])$ and $(Rrel)$.

Case (3.1): `rule`(L) is (Rat).

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L} p, A^m\end{array}}{\Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} p, A^m}\,(Rat) \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L,(\mathbb{E},e)} p}\,(Ecut)$$

This is transformed into the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L} p, A^m\end{array} \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L} p}\,(Ecut)}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L,(\mathbb{E},e)} p}\,(Rat)$$

Case (3.2): `rule`(L) is $(R[\mathbb{E},e])$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} \varphi, A^m\end{array}}{\Gamma \Rightarrow \Delta, x :^{L} [\mathbb{E},e]\varphi, A^m}\,(R[\mathbb{E},e]) \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L} [\mathbb{E},e]\varphi}\,(Ecut)$$

This is transformed into the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} \varphi, A^m\end{array} \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L,(\mathbb{E},e)} \varphi}\,(Ecut)}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', x :^{L} [\mathbb{E},e]\varphi}\,(R[\mathbb{E},e])$$

Case (3.3): `rule`(L) is $(Rrel)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, (x,L)\,\mathsf{R}_{[\pi_a(e,f)]}\,(y,L'), A^m\end{array}}{\Gamma \Rightarrow \Delta, (x,L,(\mathbb{E},e))\,\mathsf{R}_a\,(y,L',(\mathbb{E},f))}\,(Rrel) \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', (x,L,(\mathbb{E},e))\,\mathsf{R}_a\,(y,L',(\mathbb{E},f))}\,(Ecut)$$

This is transformed into the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, (x,L)\,\mathsf{R}_{[\pi_a(e,f)]}\,(y,L'), A^m\end{array} \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ A^n,\Gamma' \Rightarrow \Delta'\end{array}}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', (x,L)\,\mathsf{R}_{[\pi_a(e,f)]}\,(y,L')}\,(Ecut)}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta', (x,L,(\mathbb{E},e))\,\mathsf{R}_a\,(y,L',(\mathbb{E},f))}\,(Rrel)$$

Case (4): we will show only the cases for (Rat), $(R[\mathbb{E},e])$ and $(Rrel)$.

Case (4.1): `rule`(L) is (Rat).

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L} p, (x :^{L,(\mathbb{E},e)} p)^{m-1}\end{array}}{\Gamma \Rightarrow \Delta, (x :^{L,(\mathbb{E},e)} p)^{m}}\,(Rat) \quad \dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L,(\mathbb{E},e)} p)^{n-1}, x :^{L} p, \Gamma' \Rightarrow \Delta'\end{array}}{(x :^{L,(\mathbb{E},e)} p)^{n}, \Gamma' \Rightarrow \Delta'}\,(Lat)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

From the above derivation, we can construct the derivation $\mathfrak{D}_1$ as follows:

$$\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L} p, (x :^{L,(\mathbb{E},e)} p)^{m-1}\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L,(\mathbb{E},e)} p)^{n-1}, x :^{L} p, \Gamma' \Rightarrow \Delta'\end{array}}{(x :^{L,(\mathbb{E},e)} p)^{n}, \Gamma' \Rightarrow \Delta'}\,(Lat)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x :^{L} p}\,(Ecut)$$

In addition, we can construct the derivation $\mathfrak{D}_2$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L} p, (x :^{L,(\mathbb{E},e)} p)^{m-1}\end{array}}{\Gamma \Rightarrow \Delta, (x :^{L,(\mathbb{E},e)} p)^{m}}\,(Rat) \quad \begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L,(\mathbb{E},e)} p)^{n-1}, x :^{L} p, \Gamma' \Rightarrow \Delta'\end{array}}{x :^{L} p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

By $\mathfrak{D}_1$ and $\mathfrak{D}_2$, we can transform the original derivation into the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{1}\\ \Gamma, \Gamma' \Rightarrow \Delta, \Delta', x :^{L} p\end{array} \quad \begin{array}{c}\vdots\,\mathfrak{D}_{2}\\ x :^{L} p, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\end{array}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(c)$$

Case (4.2): `rule`(L) is ($R[\mathbb{E},e]$).

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} \varphi, (x :^{L} [\mathbb{E},e]\varphi)^{m-1}v\end{array}}{\Gamma \Rightarrow \Delta, (x :^{L} [\mathbb{E},e]\varphi)^{m}}\,(R[\mathbb{E},e]) \quad \dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L} [\mathbb{E},e]\varphi)^{n-1}, x :^{L,(\mathbb{E},e)} \varphi, \Gamma' \Rightarrow \Delta'\end{array}}{(x :^{L} [\mathbb{E},e]\varphi)^{n}, \Gamma' \Rightarrow \Delta'}\,(L[\mathbb{E},e])}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

From the above derivation, we can construct the derivation $\mathfrak{D}_1$ as follows:

$$\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} \varphi, (x :^{L} [\mathbb{E},e]\varphi)^{m-1}\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L} [\mathbb{E},e]\varphi)^{n-1}, x :^{L,(\mathbb{E},e)} \varphi, \Gamma' \Rightarrow \Delta'\end{array}}{(x :^{L} [\mathbb{E},e]\varphi)^{n}, \Gamma' \Rightarrow \Delta'}\,(L[\mathbb{E},e])}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x :^{L,(\mathbb{E},e)} \varphi}\,(Ecut)$$

In addition, we can construct the derivation $\mathfrak{D}_2$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathfrak{D}_{L'}\\ \Gamma \Rightarrow \Delta, x :^{L,(\mathbb{E},e)} \varphi, (x :^{L} [\mathbb{E},e]\varphi)^{m-1}\end{array}}{\Gamma \Rightarrow \Delta, (x :^{L} [\mathbb{E},e]\varphi)^{m}}\,(R[\mathbb{E},e]) \quad \begin{array}{c}\vdots\,\mathfrak{D}_{R'}\\ (x :^{L} [\mathbb{E},e]\varphi)^{n-1}, x :^{L,(\mathbb{E},e)} \varphi, \Gamma' \Rightarrow \Delta'\end{array}}{x :^{L,(\mathbb{E},e)} \varphi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

By $\mathfrak{D}_1$ and $\mathfrak{D}_2$, we can transform the original derivation into the following derivation.

$$\dfrac{\dfrac{\overset{\vdots\ \mathfrak{D}_1}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x :^{L,(\mathbb{E},e)} \varphi} \quad \overset{\vdots\ \mathfrak{D}_2}{x :^{L,(\mathbb{E},e)} \varphi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(c)$$

Case (4.3): $\texttt{rule}(L)$ is $(Rrel)$. Let $A = (x, L, (\mathbb{E}, e))\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L', (\mathbb{E}, f))$.

$$\dfrac{\dfrac{\overset{\vdots\ \mathfrak{D}_{L'}}{\Gamma \Rightarrow \Delta, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), A^{m-1}}}{\Gamma \Rightarrow \Delta, A^m}\,(Rrel) \quad \dfrac{\overset{\vdots\ \mathfrak{D}_{R'}}{A^{n-1}, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), \Gamma' \Rightarrow \Delta'}}{A^n, \Gamma' \Rightarrow \Delta'}\,(Lrel)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

From the above derivation, we can construct the derivation $\mathfrak{D}_1$ as follows:

$$\dfrac{\overset{\vdots\ \mathfrak{D}_{L'}}{\Gamma \Rightarrow \Delta, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), A^{m-1}} \quad \dfrac{\overset{\vdots\ \mathfrak{D}_{R'}}{A^{n-1}, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), \Gamma' \Rightarrow \Delta'}}{A^n, \Gamma' \Rightarrow \Delta'}\,(Lrel)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L')}\,(Ecut)$$

In addition, we can construct the derivation $\mathfrak{D}_2$ as follows:

$$\dfrac{\dfrac{\overset{\vdots\ \mathfrak{D}_{L'}}{\Gamma \Rightarrow \Delta, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), A^{m-1}}}{\Gamma \Rightarrow \Delta, A^m}\,(Rrel) \quad \overset{\vdots\ \mathfrak{D}_{R'}}{A^{n-1}, (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), \Gamma' \Rightarrow \Delta'}}{(x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(Ecut)$$

By $\mathfrak{D}_1$ and $\mathfrak{D}_2$, we can transform the original derivation into the following derivation.

$$\dfrac{\dfrac{\overset{\vdots\ \mathfrak{D}_1}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', (x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L')} \quad \overset{\vdots\ \mathfrak{D}_2}{(x, L)\, \mathsf{R}_{[\pi_a(e,f)]}\, (y, L'), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,(Ecut)}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,(c)$$

Case (5): we will show two examples as follows:

Case (5.1): $\texttt{rule}(R)$ is $(L =_1)$. Let $A = A'[x/w]$. Suppose that $\texttt{rule}(L)$ is $(R =)$. We can construct the derivation $\mathfrak{D}$ as follows:

$$\dfrac{\dfrac{}{\Rightarrow x = x}\,(R =) \quad \dfrac{\overset{\vdots\ \mathfrak{D}_{R'}}{x = x, (A'[x/w])^n, \Gamma'[x/w] \Rightarrow \Delta'[x/w]}}{x = x, (A'[x/w])^n, \Gamma'[x/w] \Rightarrow \Delta'[x/w]}\,(L =_1)}{\Gamma'[x/w] \Rightarrow \Delta'[x/w]}\,(Ecut)$$

where $A'[x/w]$ is $x = x$. Thus, we can regard $x = x, (A'[x/w])^n$ as $(x = x)^{n+1}$. Then, the above derivation is transformed into the following form.

$$\dfrac{\dfrac{}{\Rightarrow x = x}\,(R =) \quad \overset{\vdots\ \mathfrak{D}_R}{x = x, (A'[x/w])^n, \Gamma'[x/w] \Rightarrow \Delta'[x/w]}}{\Gamma'[x/w] \Rightarrow \Delta'[x/w]}\,(Ecut)$$

Case (5.2): `rule`(L) is $(L =_1)$. Let $A = A'[y/w]$. In this case, we assume that there are no occurrences of a substitution of w in A'.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}\\ x = y, \Gamma[x/w] \Rightarrow \Delta[x/w], (A'[x/w])^m\end{array}}{x = y, \Gamma[y/w] \Rightarrow \Delta[y/w], (A'[y/w])^m}\,(L =_1) \quad \begin{array}{c}\vdots\ \mathfrak{D}_R\\ (A'[y/w])^n, \Gamma' \Rightarrow \Delta'\end{array}}{x = y, \Gamma[y/w], \Gamma' \Rightarrow \Delta[y/w], \Delta'}\,(Ecut)$$

This is transformed into the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathfrak{D}_{L'}[x/y]\\ x = x, \Gamma[x/w][x/y] \Rightarrow \Delta[x/w][x/y], (A'[x/w][x/y])^m\end{array} \quad \begin{array}{c}\vdots\ \mathfrak{D}_R[x/y]\\ (A'[y/w][x/y])^n, \Gamma'[x/y] \Rightarrow \Delta'[x/y]\end{array}}{x = x, \Gamma[x/w][x/y], \Gamma'[x/y] \Rightarrow \Delta[x/w][x/y], \Delta'[x/y]}\,(Ecut)}{x = y, x = x, \Gamma[x/w][x/y], \Gamma'[x/y] \Rightarrow \Delta[x/w][x/y], \Delta'[x/y]}\,(Lw)}{x = y, x = y, \Gamma[y/w], \Gamma' \Rightarrow \Delta[y/w], \Delta'}\,(L =_1)}{x = y, \Gamma[y/w], \Gamma' \Rightarrow \Delta[y/w], \Delta'}\,(Lc)$$

□

Theorem 19 (Cut Elimination)**.** *For any* Γ *and* Δ*, if* $\vdash_{\mathbf{GDELRC}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{GDELRC}^-} \Gamma \Rightarrow \Delta$*.*

Proof. Suppose $\vdash_{\mathbf{GDELRC}} \Gamma \Rightarrow \Delta$. Our goal is to show that $\vdash_{\mathbf{GDELRC}^-} \Gamma \Rightarrow \Delta$. The proof can be established by Lemma 16. □

Next, we will show the soundness of labelled sequent calculus **GDELRC** for Kripke semantics.

Definition 64. *Let* $\mathfrak{M} = (W, (R_a)_{a \in \mathsf{Prog}}, V)$ *be a model and* $f : \mathsf{Var} \to W$ *be an* assignment *on* $\mathfrak{M}$*. Given any model* $\mathfrak{M}$*, any assignment* f *on* $\mathfrak{M}$ *and any labelled expression* A*, we can define* $\mathfrak{M}, f \models A$ *as follows:*

$\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n)} \varphi$	iff	$\mathfrak{M}^{\otimes \mathbb{E}_1 \otimes ... \otimes \mathbb{E}_n}, (f(x), e_1, ..., e_n) \models \varphi$
$\mathfrak{M}, f \models x = y$	iff	$f(x) = f(y)$
$\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1), ..., (\mathbb{E}_n, e_n)) R_\pi (y, (\mathbb{E}_1, d_1), ..., (\mathbb{E}_n, d_n))$	iff	$(f(x), e_1, ..., e_n) R_\pi^{\otimes \mathbb{E}_1 \otimes ... \otimes \mathbb{E}_n} (f(y), d_1, ..., d_n))$

Lemma 17. *For any* Γ *and* Δ*,*

$$\text{if } \vdash_{\mathbf{GDELRC}} \Gamma \Rightarrow \Delta, \text{ then } \mathfrak{M}, f \models \Gamma \Rightarrow \Delta.$$

Proof. We show the proof by induction on the height h of a derivation of $\Gamma \Rightarrow \Delta$ in **GDELRC**.

Basis (initial sequents):
Our goal is to show that

$$\mathfrak{M}, f \models A \Rightarrow A \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any $\mathfrak{M}$ and f. Suppose that $\mathfrak{M}, f \models A$. It suffices to show that $\mathfrak{M}, f \models A$. This is trivial by our assumption.

Case where the last applied rule of our derivation is ($R[\pi]$):
First, we can write the rule ($R[\pi]$) by a simple case as follows:

$$\frac{\big\{(x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)),\Gamma \Rightarrow \Delta, y :^{(\mathbb{E}_1,d_1)} \varphi \mid e_1 Q_\pi d_1\big\}}{\Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi}\ (R[\pi])$$

Then, we suppose that $\mathfrak{M}, f \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)),\Gamma \Rightarrow \Delta, y :^{(\mathbb{E}_1,d_1)} \varphi$ such that $e_1 Q_\pi d_1$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi \text{ for all } \mathfrak{M} \text{ and all } f,$$

where y of our assumption does not appear in Γ, Δ and $x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi$. Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\begin{aligned}&\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi\\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi.\end{aligned}$$

Suppose that $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$. We show that

$$\begin{aligned}\mathfrak{M}, f \models x :^{(\mathbb{E}_1,e_1)} [\pi]\varphi &\text{ iff } \mathfrak{M}^{\otimes\mathbb{E}_1}, (f(x), e_1) \models [\pi]\varphi\\ &\text{ iff } \forall_{(v,d_1)\in W^{\otimes\mathbb{E}_1}}((f(x),e_1)R_\pi^{\otimes\mathbb{E}_1}(v,d_1) \Rightarrow \mathfrak{M}^{\otimes\mathbb{E}_1}, (v,d_1) \models \varphi).\end{aligned}$$

Fix any $(v, d_1) \in W^{\otimes\mathbb{E}_1}$ such that $(f(x), e_1)R_\pi^{\otimes\mathbb{E}_1}(v, d_1)$. Let us show that

$$\mathfrak{M}^{\otimes\mathbb{E}_1}, (v, d_1) \models \varphi.$$

Define new assignment function $g : \mathsf{Var} \to W$ by

$$g(z) = \begin{cases} v & \text{if } z = y,\\ f(z) & \text{if } z \neq y.\end{cases}$$

By our first assumption, $\mathfrak{M}, g \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)),\Gamma \Rightarrow \Delta, y :^{(\mathbb{E}_1,d_1)} \varphi$, i.e.,

$$\begin{aligned}&\text{if } \mathfrak{M}, g \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)) \text{ and } \mathfrak{M}, g \models C \text{ for all } C \in \Gamma,\\ &\text{then } \mathfrak{M}, g \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, g \models y :^{(\mathbb{E}_1,d_1)} \varphi.\end{aligned}$$

Let us show

$$\mathfrak{M}, g \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)) \text{ and } \mathfrak{M}, g \models C \text{ for all } C \in \Gamma.$$

By our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$ and y is fresh, we can get $\mathfrak{M}, g \models C$ for all $C \in \Gamma$. Next, we will show

$$\mathfrak{M}, g \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1)) \text{ iff } (g(x), e_1)R_\pi^{\otimes\mathbb{E}_1}(g(y), d_1).$$

By definition of g, we get $(f(x), e_1)R_\pi^{\otimes\mathbb{E}_1}(v, d_1)$ which is our assumption. By $\mathfrak{M}, g \models (x,(\mathbb{E}_1,e_1))R_\pi(y,(\mathbb{E}_1,d_1))$ and $\mathfrak{M}, g \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned}&\mathfrak{M}, g \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, g \models y :^{(\mathbb{E}_1,d_1)} \varphi\\ &\text{iff if } \mathfrak{M}, g \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, g \models y :^{(\mathbb{E}_1,d_1)} \varphi.\end{aligned}$$

By our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$ and the freshness of y in Δ, we get $\mathfrak{M}, g \not\models D$ for all $D \in \Delta$. It follows that

$$\mathfrak{M}, g \models y :^{(\mathbb{E}_1, d_1)} \varphi \text{ iff } \mathfrak{M}^{\otimes \mathbb{E}_1}, (g(y), d_1) \models \varphi.$$

By definition of g, we get $\mathfrak{M}^{\otimes \mathbb{E}_1}, (v, d_1) \models \varphi$, which is our goal.

Case where the last applied rule of our derivation is ($L[\pi]$):
First, we can write the rule ($L[\pi]$) as follows:

$$\frac{\Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1)) \quad y :^{(\mathbb{E}_1, d_1)} \varphi, \Gamma \Rightarrow \Delta}{x :^{(\mathbb{E}_1, e_1)} [\pi]\varphi, \Gamma \Rightarrow \Delta} \ (L[\pi])$$

Then, we suppose that $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1))$ and $\mathfrak{M}, f \models y :^{(\mathbb{E}_1, d_1)} \varphi, \Gamma \Rightarrow \Delta$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} [\pi]\varphi, \Gamma \Rightarrow \Delta \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} [\pi]\varphi$ and $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By our assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1))$, i.e., if $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, then $\mathfrak{M}, f \models D$ for some $D \in \Delta$ or $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1))$. By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1)).$$

From $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_\pi (y, (\mathbb{E}_1, d_1))$, we get $(f(x), e_1) R_\pi^{\otimes \mathbb{E}_1} (f(y), d_1)$. From our assumption of $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} [\pi]\varphi$,

$$\begin{aligned} \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} [\pi]\varphi &\text{ iff } \mathfrak{M}^{\otimes \mathbb{E}_1}, (f(x), e_1) \models [\pi]\varphi \\ &\text{ iff } \forall_{(v, d_1) \in W^{\otimes \mathbb{E}_1}} ((f(x), e_1) R_\pi^{\otimes \mathbb{E}_1} (v, d_1) \Rightarrow \mathfrak{M}^{\otimes \mathbb{E}_1}, (v, d_1) \models \varphi). \end{aligned}$$

By this implication and $(f(x), e_1) R_\pi^{\otimes \mathbb{E}_1} (f(y), d_1)$, we get

$$\mathfrak{M}^{\otimes \mathbb{E}_1}, (f(y), d_1) \models \varphi \text{ iff } \mathfrak{M}, f \models y :^{(\mathbb{E}_1, d_1)} \varphi.$$

From our assumption, $\mathfrak{M}, f \models y :^{(\mathbb{E}_1, d_1)} \varphi, \Gamma \Rightarrow \Delta$, i.e.,

$$\text{if } \mathfrak{M}, f \models y :^{(\mathbb{E}_1, d_1)} \varphi \text{ and } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By this implication, $\mathfrak{M}, f \models y :^{(\mathbb{E}_1, d_1)} \varphi$ and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we obtain $\mathfrak{M}, f \models D$ for some $D \in \Delta$, which is our goal.

Case where the last applied rule of our derivation is (Rat):
First, we can write the rule (Rat) as follows:

$$\frac{\Gamma \Rightarrow \Delta, x :^{\varepsilon} p}{\Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1)} p} \ (Rat)$$

Suppose that $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{\varepsilon} p$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1)} p \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\begin{aligned} &\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p \\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p. \end{aligned}$$

Suppose that $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$. Our goal is to show that

$$\begin{aligned} \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p &\text{ iff } \mathfrak{M}^{\otimes \mathbb{E}_1}, (f(x), e_1) \models p \\ &\text{ iff } (f(x), e_1) \in V^{\otimes \mathbb{E}_1}(p) \\ &\text{ iff } f(x) \in V(p) \\ &\text{ iff } \mathfrak{M}, f(x) \models p \\ &\text{ iff } \mathfrak{M}, f \models x :^{\varepsilon} p. \end{aligned}$$

By our first assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{\varepsilon} p$, i.e.,

$$\text{if } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{\varepsilon} p.$$

By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned} &\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{\varepsilon} p \\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models x :^{\varepsilon} p. \end{aligned}$$

By our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$, we obtain $\mathfrak{M}, f \models x :^{\varepsilon} p$, which is our goal.

Case where the last applied rule of our derivation is (*Lat*):
First, we can write the rule (*Lat*) as follows:

$$\frac{x :^{\varepsilon} p, \Gamma \Rightarrow \Delta}{x :^{(\mathbb{E}_1, e_1)} p, \Gamma \Rightarrow \Delta} \ (Lat)$$

Suppose that $\mathfrak{M}, f \models x :^{\varepsilon} p, \Gamma \Rightarrow \Delta$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p, \Gamma \Rightarrow \Delta \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p$ and $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

From our assumption of $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p$,

$$\begin{aligned} \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} p &\text{ iff } \mathfrak{M}^{\otimes \mathbb{E}_1}, (f(x), e_1) \models p \\ &\text{ iff } (f(x), e_1) \in V^{\otimes \mathbb{E}_1}(p) \\ &\text{ iff } f(x) \in V(p) \\ &\text{ iff } \mathfrak{M}, f(x) \models p \\ &\text{ iff } \mathfrak{M}, f \models x :^{\varepsilon} p. \end{aligned}$$

By our first assumption, $\mathfrak{M}, f \models x :^{\varepsilon} p, \Gamma \Rightarrow \Delta$, i.e.,

$$\text{if } \mathfrak{M}, f \models x :^{\varepsilon} p \text{ and } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By this implication, $\mathfrak{M}, f \models x :^{\varepsilon} p$ and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we obtain $\mathfrak{M}, f \models D$ for some $D \in \Delta$, which is our goal.

Case where the last applied rule of our derivation is ($R[\mathbb{E}, e]$):
First, we can write the rule ($R[\mathbb{E}, e]$) as follows:

$$\frac{\Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1)} \varphi}{\Gamma \Rightarrow \Delta, x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi} \ (R[\mathbb{E}, e])$$

Suppose that $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1)} \varphi$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\begin{aligned} &\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi \\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi. \end{aligned}$$

Suppose that $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$. Our goal is to show that

$$\begin{aligned} \mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi \ &\text{iff } \mathfrak{M}, f(x) \models [\mathbb{E}_1, e_1]\varphi \\ &\text{iff } \mathfrak{M}^{\otimes \mathbb{E}_1}, (f(x), e_1) \models \varphi \\ &\text{iff } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi. \end{aligned}$$

By our first assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, x :^{(\mathbb{E}_1, e_1)} \varphi$, i.e.,

$$\text{if } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi.$$

By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned} &\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi \\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi. \end{aligned}$$

By our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$, we obtain $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi$, which is our goal.

Case where the last applied rule of our derivation is ($L[\mathbb{E}, e]$):
First, we can write the rule ($L[\mathbb{E}, e]$) as follows:

$$\frac{x :^{(\mathbb{E}_1, e_1)} \varphi, \Gamma \Rightarrow \Delta}{x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi, \Gamma \Rightarrow \Delta} \ (L[\mathbb{E}, e])$$

Suppose that $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi, \Gamma \Rightarrow \Delta$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi, \Gamma \Rightarrow \Delta \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi$ and $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By our assumption of $\mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi$,

$$\begin{aligned} \mathfrak{M}, f \models x :^{\varepsilon} [\mathbb{E}_1, e_1]\varphi \text{ iff } & \mathfrak{M}, f(x) \models [\mathbb{E}_1, e_1]\varphi \\ \text{iff } & \mathfrak{M}^{\otimes \mathbb{E}_1}, (f(x), e_1) \models \varphi \\ \text{iff } & \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi. \end{aligned}$$

From our first assumption, $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi, \Gamma \Rightarrow \Delta$, i.e.,

$$\text{if } \mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi \text{ and } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By this implication, $\mathfrak{M}, f \models x :^{(\mathbb{E}_1, e_1)} \varphi$ and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we obtain $\mathfrak{M}, f \models D$ for some $D \in \Delta$, which is our goal.

Case where the last applied rule of our derivation is ($Rrel$):
First, we can write the rule ($Rrel$) as follows:

$$\frac{\Gamma \Rightarrow \Delta, (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon)}{\Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1)) R_a (y, (\mathbb{E}_1, d_1))} \ (Rrel)$$

Suppose that $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon)$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1)) R_a (y, (\mathbb{E}_1, d_1)) \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\begin{aligned} & \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_a (y, (\mathbb{E}_1, d_1)) \\ & \text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_a (y, (\mathbb{E}_1, d_1)). \end{aligned}$$

Suppose that $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$. Our goal is to show that

$$\begin{aligned} \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1)) R_a (y, (\mathbb{E}_1, d_1)) \text{ iff } & (f(x), e_1) R_a^{\otimes \mathbb{E}_1} (f(y), d_1) \\ \text{iff } & f(x) R_{\pi_a(e_1, d_1)} f(y) \\ \text{iff } & \mathfrak{M}, f \models (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon). \end{aligned}$$

By our first assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon)$, i.e.,

$$\text{if } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon).$$

By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned} & \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon) \\ & \text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models (x, \varepsilon) R_{\pi_a(e_1, d_1)} (y, \varepsilon). \end{aligned}$$

By this implication and our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$, we obtain $\mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon)$, which is our goal.

Case where the last applied rule of our derivation is ($Lrel$):
First, we can write the rule ($Lrel$) as follows:

$$\frac{(x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon), \Gamma \Rightarrow \Delta}{(x, (\mathbb{E}_1, e_1))R_a(y, (\mathbb{E}_1, d_1)), \Gamma \Rightarrow \Delta}\ (Lrel)$$

Suppose that $\mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon), \Gamma \Rightarrow \Delta$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_a(y, (\mathbb{E}_1, d_1)), \Gamma \Rightarrow \Delta \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_a(y, (\mathbb{E}_1, d_1))$ and $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

From our assumption of $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_a(y, (\mathbb{E}_1, d_1))$,

$$\begin{aligned}\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_a(y, (\mathbb{E}_1, d_1)) &\text{ iff } (f(x), e_1)R_a^{\otimes \mathbb{E}_1}(f(y), d_1)\\ &\text{ iff } f(x)R_{\pi_a(e_1,d_1)}f(y)\\ &\text{ iff } \mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon).\end{aligned}$$

By our first assumption, $\mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon), \Gamma \Rightarrow \Delta$, i.e.,

$$\text{if } \mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon) \text{ and } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By this implication, $\mathfrak{M}, f \models (x, \varepsilon)R_{\pi_a(e_1,d_1)}(y, \varepsilon)$ and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we obtain $\mathfrak{M}, f \models D$ for some $D \in \Delta$, which is our goal.

Case where the last applied rule of our derivation is ($R;$):
First, we can write the rule ($R;$) as follows:

$$\frac{\Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)) \quad \Gamma \Rightarrow \Delta, (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))}{\Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2))}\ (R;)$$

Then, we suppose that $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3))$ and $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)) \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\begin{aligned}&\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2))\\ &\text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)).\end{aligned}$$

Suppose that $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$. Our goal is to show

$$\begin{aligned}\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)) &\text{ iff } (f(x), e_1)R^{\otimes \mathbb{E}_1}_{\pi_1;\pi_2}(f(y), e_2)\\ &\text{ iff } (f(x), e_1)R^{\otimes \mathbb{E}_1}_{\pi_1}(v, e_3) \text{ and } (v, e_3)R^{\otimes \mathbb{E}_1}_{\pi_2}(f(y), e_2)\\ &\quad \text{for some } (v, e_3) \in W^{\otimes \mathbb{E}_1}.\end{aligned}$$

From our assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta, (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3))$, i.e.,

$$\begin{aligned}&\text{if } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma,\\ &\quad \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)).\end{aligned}$$

By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned}&\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3))\\ &\quad \text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)).\end{aligned}$$

By this implication and our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$, we get

$$\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)) \text{ iff } (f(x), e_1)R^{\otimes \mathbb{E}_1}_{\pi_1}(f(z), e_3).$$

From our assumption, $\mathfrak{M}, f \models \Gamma \Rightarrow \Delta$, $(z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$, i.e.,

$$\begin{aligned}&\text{if } \mathfrak{M}, f \models C \text{ for all } C \in \Gamma,\\ &\quad \text{then } \mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)).\end{aligned}$$

By this implication and our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$, we get

$$\begin{aligned}&\mathfrak{M}, f \models D \text{ for some } D \in \Delta \text{ or } \mathfrak{M}, f \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))\\ &\quad \text{iff if } \mathfrak{M}, f \not\models D \text{ for all } D \in \Delta, \text{then } \mathfrak{M}, f \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)).\end{aligned}$$

By this implication and our assumption of $\mathfrak{M}, f \not\models D$ for all $D \in \Delta$, we get

$$\mathfrak{M}, f \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)) \text{ iff } (f(z), e_3)R^{\otimes \mathbb{E}_1}_{\pi_2}(f(y), e_2).$$

By $(f(x), e_1)R^{\otimes \mathbb{E}_1}_{\pi_1}(f(z), e_3)$ and $(f(z), e_3)R^{\otimes \mathbb{E}_1}_{\pi_2}(f(y), e_2)$, we obtain $(f(x), e_1)R^{\otimes \mathbb{E}_1}_{\pi_1}(v, e_3)$ and $(v, e_3)R^{\otimes \mathbb{E}_1}_{\pi_2}(f(y), e_2)$ for some $(v, e_3) \in W^{\otimes \mathbb{E}_1}$, which is our goal.

Case where the last applied rule of our derivation is ($L;$):
First, we can write the rule $(L;)$ by a simple case as follows:

$$\frac{\{(x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)), (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)), \Gamma \Rightarrow \Delta \mid e_1 Q_{\pi_1} e_3, e_3 Q_{\pi_2} e_2\}}{(x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)), \Gamma \Rightarrow \Delta}\ (L;)$$

Then, we suppose that $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)), (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$, $\Gamma \Rightarrow \Delta$ such that $e_1 Q_{\pi_1} e_3$ and $e_3 Q_{\pi_2} e_2$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)), \Gamma \Rightarrow \Delta \text{ for all } \mathfrak{M} \text{ and all } f,$$

where z of our assumption does not appear in Γ, Δ and $(x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2))$. Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2))$ and $\mathfrak{M}, f \models C$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D \text{ for some } D \in \Delta.$$

By our assumption of $\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2))$,

$$\begin{aligned}\mathfrak{M}, f \models (x, (\mathbb{E}_1, e_1))R_{\pi_1;\pi_2}(y, (\mathbb{E}_1, e_2)) &\text{ iff } (f(x), e_1)R^{\otimes\mathbb{E}_1}_{\pi_1;\pi_2}(f(y), e_2)\\ &\text{ iff } (f(x), e_1)R^{\otimes\mathbb{E}_1}_{\pi_1}(v, e_3) \text{ and } (v, e_3)R^{\otimes\mathbb{E}_1}_{\pi_2}(f(y), e_2)\\ &\quad \text{for some } (v, e_3) \in W^{\otimes\mathbb{E}_1}.\end{aligned}$$

Define new assignment function $g : \mathsf{Var} \to W$ by

$$g(u) = \begin{cases} v & \text{if } u = z, \\ f(u) & \text{if } u \neq z. \end{cases}$$

By our first assumption, $\mathfrak{M}, g \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)), (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$, $\Gamma \Rightarrow \Delta$, i.e.,

$$\begin{aligned}&\text{if } \mathfrak{M}, g \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)) \text{ and } \mathfrak{M}, g \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)) \text{ and}\\ &\quad \mathfrak{M}, g \models C \text{ for all } C \in \Gamma, \text{then } \mathfrak{M}, g \models D \text{ for some } D \in \Delta.\end{aligned}$$

Let us show $\mathfrak{M}, g \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3))$ and $\mathfrak{M}, g \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$ and $\mathfrak{M}, g \models C$ for all $C \in \Gamma$. By our assumption of $\mathfrak{M}, f \models C$ for all $C \in \Gamma$ and z is fresh, we can get $\mathfrak{M}, g \models C$ for all $C \in \Gamma$. Next, we will show

$$\mathfrak{M}, g \models (x, (\mathbb{E}_1, e_1))R_{\pi_1}(z, (\mathbb{E}_1, e_3)) \text{ iff } (g(x), e_1)R^{\otimes\mathbb{E}_1}_{\pi_1}(g(z), e_3).$$

By definition of g, we get $(f(x), e_1)R^{\otimes\mathbb{E}_1}_{\pi_1}(v, e_3)$, which is our assumption. From our assumption of $(z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2))$,

$$\mathfrak{M}, g \models (z, (\mathbb{E}_1, e_3))R_{\pi_2}(y, (\mathbb{E}_1, e_2)) \text{ iff } (g(z), e_3)R^{\otimes\mathbb{E}_1}_{\pi_2}(g(y), e_2).$$

By definition of g, we get $(v, e_3)R^{\otimes\mathbb{E}_1}_{\pi_1}(f(y), e_2)$, which is our assumption. Thus, by our assumptions and the freshness of z in Δ, we obtain $\mathfrak{M}, f \models D$ for some $D \in \Delta$, which is our goal.

Case where the last applied rule of our derivation is $(L =_1)$:
First, we can write the rule $(L =_1)$ as follows:

$$\frac{x = y, \Gamma[x/w] \Rightarrow \Delta[x/w]}{x = y, \Gamma[y/w] \Rightarrow \Delta[y/w]} \; (L =_1)$$

Then, we suppose that $\mathfrak{M}, f \models x = y, \Gamma[x/w] \Rightarrow \Delta[x/w]$ for all models $\mathfrak{M}$ and all assignments f. Our goal is to show that

$$\mathfrak{M}, f \models x = y, \Gamma[y/w] \Rightarrow \Delta[y/w] \text{ for all } \mathfrak{M} \text{ and all } f.$$

Fix any model $\mathfrak{M}$ and any assignment f. Suppose that $\mathfrak{M}, f \models x = y$ and $\mathfrak{M}, f \models C[y/w]$ for all $C \in \Gamma$. It suffices to show that

$$\mathfrak{M}, f \models D[y/w] \text{ for some } D \in \Delta.$$

From our assumption of $\mathfrak{M}, f \models x = y$,

$$\mathfrak{M}, f \models x = y \text{ iff } f(x) = f(y).$$

Define new assignment function $g : \mathsf{Var} \to W$ by

$$g(z) = \begin{cases} f(y) & \text{if } z = x, \\ f(z) & \text{if } z \neq x. \end{cases}$$

Based on definition of g and our assumption of $f(x) = f(y)$, we have our claim as follows:

$$\mathfrak{M}, f \models C[y/w] \text{ iff } \mathfrak{M}, g \models C[x/w] \text{ for all } C.$$

By our first assumption, $\mathfrak{M}, g \models x = y, \Gamma[x/w] \Rightarrow \Delta[x/w]$, i.e.,

$$\text{if } \mathfrak{M}, g \models x = y \text{ and } \mathfrak{M}, g \models C[x/w] \text{ for all } C \in \Gamma,$$
$$\text{then } \mathfrak{M}, g \models D[x/w] \text{ for some } D \in \Delta.$$

By this implication and our claim, we obtain $\mathfrak{M}, f \models D[y/w]$ for some $D \in \Delta$, which is our goal. □

Theorem 20 (Soundness). *If* $\vdash_{\mathbf{GDELRC}^-} \Rightarrow x :^{\varepsilon} \varphi$ *for all* $x \in \mathsf{Var}$*, then* φ *is valid on all Kripke models.*

Proof. Suppose $\vdash_{\mathbf{GDELRC}^-} \Rightarrow x :^{\varepsilon} \varphi$ for all $x \in \mathsf{Var}$. Our goal is to show that φ is valid on all models. The proof can be shown by Lemma 17. □

Corollary 1. *Given any formula* φ*, the following are equivalent:*

(*i*) φ *is valid on all models*

(*ii*) $\vdash_{\mathbf{HDELRC}} \varphi$

(*iii*) $\vdash_{\mathbf{GDELRC}} \Rightarrow x :^{\varepsilon} \varphi$ *for all* $x \in \mathsf{Var}$

(*iv*) $\vdash_{\mathbf{GDELRC}^-} \Rightarrow x :^{\varepsilon} \varphi$ *for all* $x \in \mathsf{Var}$

Proof. First, the direction from (*i*) to (*ii*) can be established by Theorem 17 (completeness of **HDELRC**). Then, the direction from (*ii*) to (*iii*) is shown by Theorem 18 (all formulas φ in **HDELRC** are derivable in **GDELRC**). Next, the direction from (*iii*) to (*iv*) can be shown by Theorem 19 (cut-elimination of **GDELRC**). Finally, the direction from (*iv*) to (*i*) can be established by Theorem 20 (soundness of **GDELRC** for Kripke semantics). □

Appendix B

Implementation for Realizing Changing of Belief and Reliability

This chapter describes an implementation for realizing an agent's changing of belief and reliability based on our analysis method (mentioned in Section 5.1) and our logical formalization (mentioned in Chapters 3 and 4). We have developed a Windows-Based Application with Visual C# ^{TM}. This program outputs the truth value of propositions, together with world accessibility relations in `dot` format. Thus, we can visualize the `dot` file by Graphviz^{TM}.

The main features of our implementation are summarized as follows:

(1) We can input and edit the definition of an initial Kripke model corresponding to Definition 30 in Section 3.1 that consists of the following six items:

- G : a finite set of agents
- W : a finite non-empty set of states
- $(R_a)_{a \in G}$: an accessibility relation representing beliefs of agent a
- $(S_a)_{a \in G}$: an accessibility relation representing signatures of agent a
- $(\preccurlyeq_a)_{a \in G}$: a reliability ordering between agents by agent a
- V : a valuation

In addition, we have to input Prop, which is a set of propositions, and @ which is a current state representing agent a's viewpoint. For the reliability ordering, the system will automatically define that all agents are equally reliable, i.e., their rank is 1, at the initial stage. Note that this study fixes the rank of reliability to be 0, 1 and 2. That is, we may regard that 0 represents unreliable, 1 represents neutral, and 2 represents reliable.[1] However, the system allows us to edit the reliability ordering of all agents manually. Furthermore, we can export the definition of an initial model into the text file and import such model into the system. An example of inputting an initial model of the second legal case can be shown by Fig. B.1. This initial model can be visualized as in Fig. B.2.

[1] In the real world, a judge cannot categorize the reliability of witnesses to be several groups as [13]. Thus, this study proposes to simplify the rank of reliability in a real situation by fixing to be 0, 1 and 2.

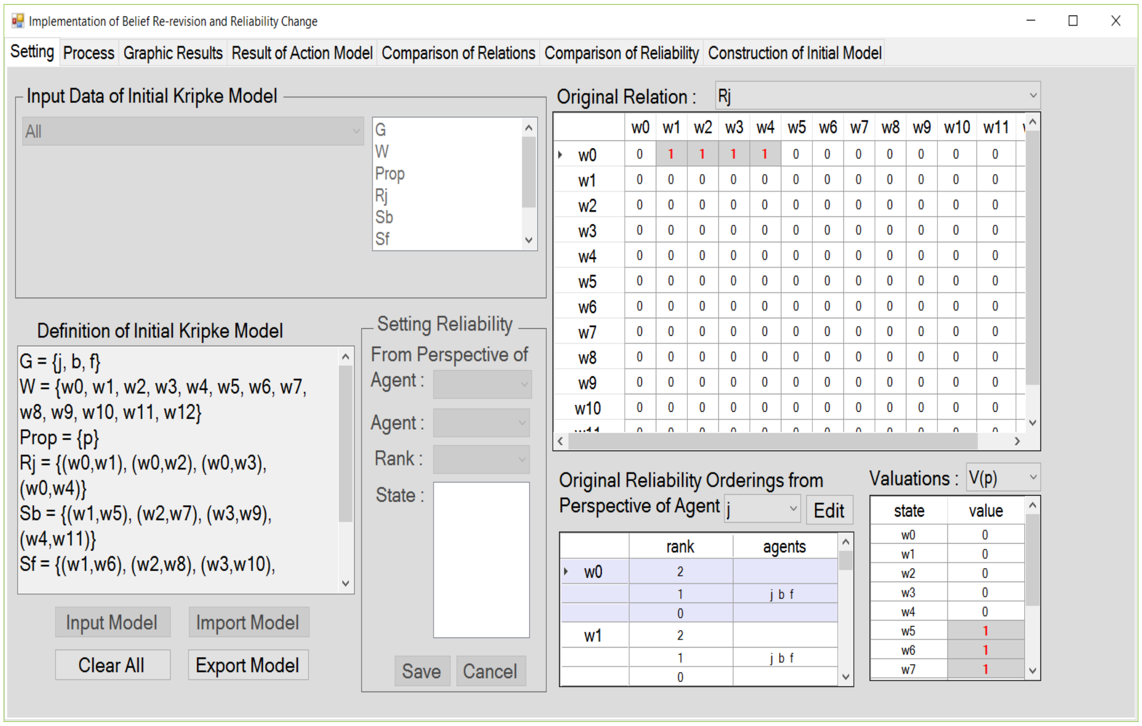

Figure B.1: Inputting an initial model of the second legal case

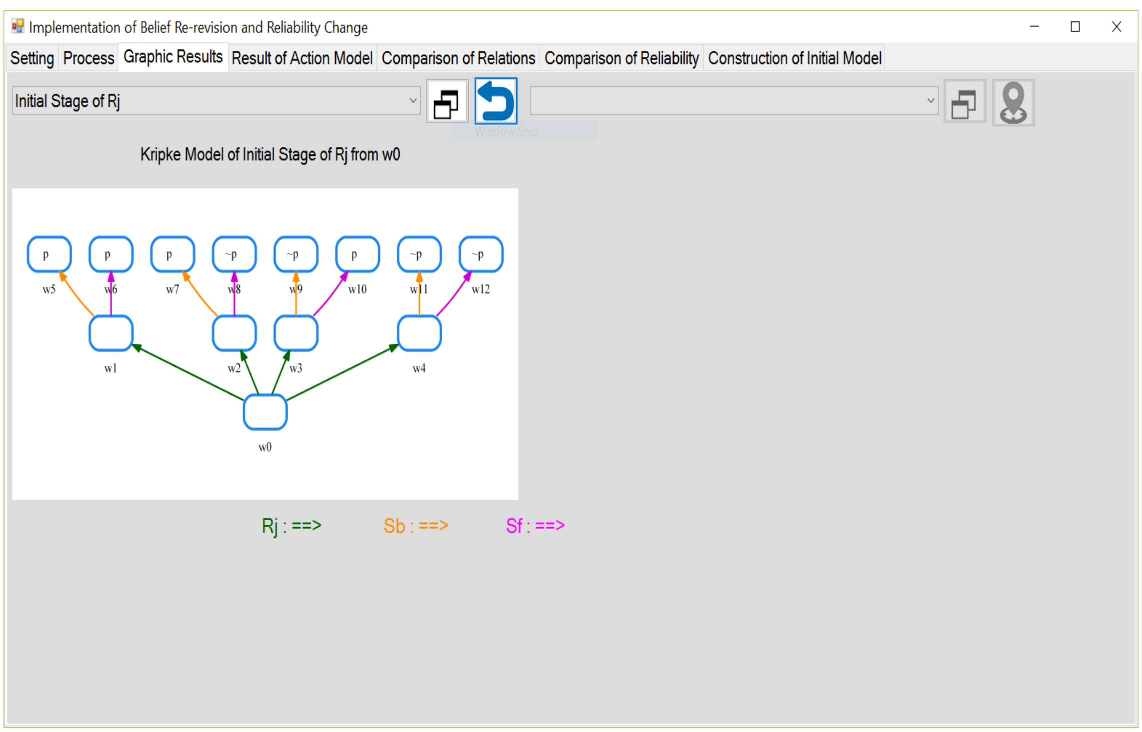

Figure B.2: Initial Kripke model of the second legal case

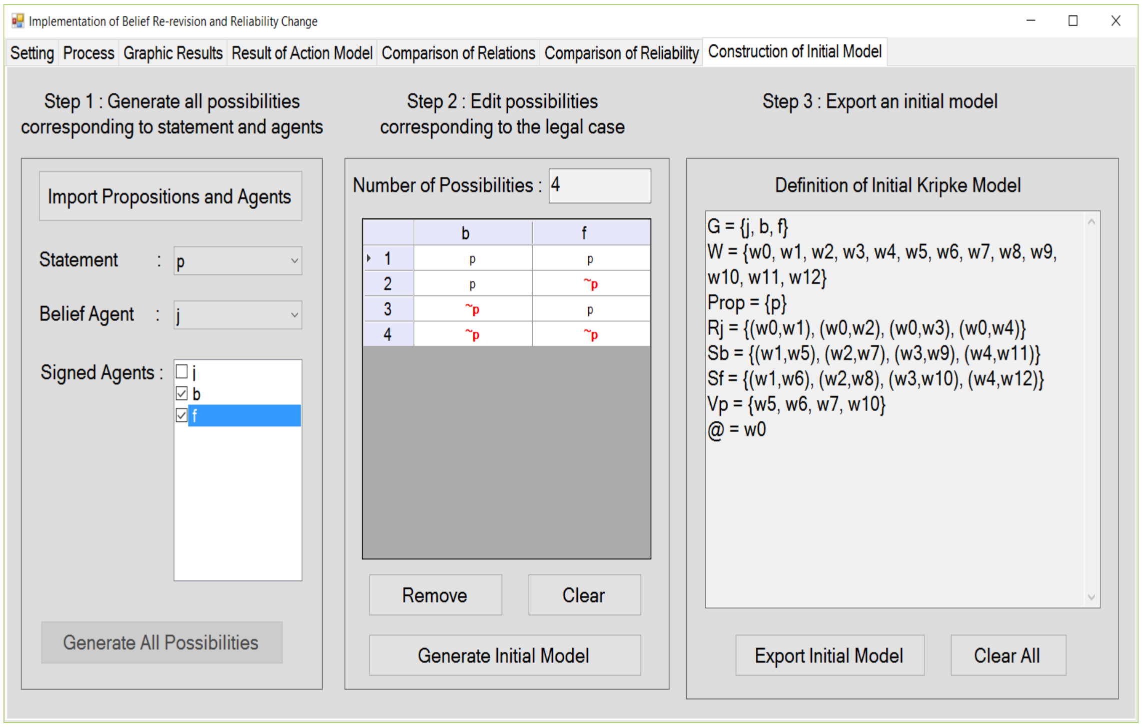

Figure B.3: Generating an initial model for the second legal case

Table B.1: Format of dynamic logical operators for inputting into the implementation

Operator name	Logical formula	Format in the system
Private announcement	$[p \rightsquigarrow a]$	`[p=>a]`
Private permission	$[p \twoheadrightarrow a]$	`[PriPer(a,p)]`
Careful policy	$[\mathsf{Careful}(a, p)]$	`[Agg(a,p)]`
Downgrade	$[H \Downarrow^a_p]$	`[downgrade(a,H,p)]`
Upgrade	$[H \Uparrow^a_p]$	`[upgrade(a,H,p)]`
Joint downgrade	$[H \downdownarrows^a]$	`[jdowngrade(a,H)]`

(2) The system can generate an initial Kripke model based on our proposed method described in Section 5.1 by the following steps:

- We first import a set of agents (G) and a set of propositions (Prop). Then, the system allows us to input three key features of a legal case including a statement, a belief agent and signed agents. Note that a statement represents a proposition that needs to be analyzed, a belief agent is an agent that needs to be analyzed his/her changing of belief and reliability, and signed agents are agents who give such statement in the legal case. After that, the system will generate all possibilities according to the input features as shown in Fig. B.3.
- The system allows us to edit such possibilities obtained from the previous step by removing some of them. Then, the system will generate a definition of an initial Kripke model.
- We can export the definition of the initial model into the text file that can be imported into the system as described in the previous feature.

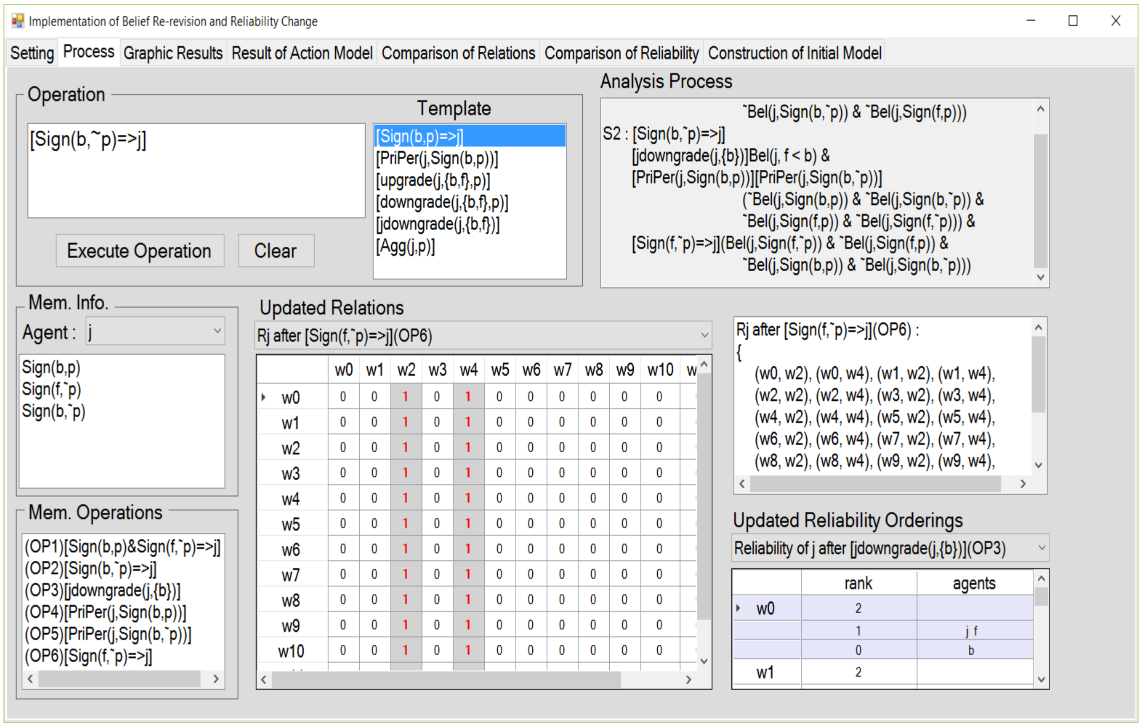

Figure B.4: Result after inputting $[\mathsf{Sign}(b, \neg p) \rightsquigarrow j]$

(3) We can input any dynamic logical operators consisting of six operators as shown in Table B.1. Then, the system automatically calculates such operator according to Chapters 3 and 4, and outputs the result on the screen such as in Fig. B.4 which consists of the following parts:

- `Operation` is an input of any dynamic operators.
- `Template` is a guideline for inputting any dynamic operators. The system will automatically generate this template corresponding to an input definition of an initial Kripke model.
- `Mem.Info.` is a memory of received information which is used to check if there is an inconsistency or not.
- `Mem.Operations` is a memory of operations presenting all operations are employed by both an agent and our system.
- `Analysis Process` demonstrates how dynamic operators are applied for formalizing an agent's changing of belief and reliability step by step. An output is shown in terms of formulas.
- `Updated Relations` represent an updated relation R_a which is the result after calculating three kinds of dynamic operators including private announcement, careful policy and private permission for formalizing belief re-revision.
- `Updated Reliability Orderings` represent an updated reliability ordering $\preccurlyeq_a$ resulting from three dynamic operators including downgrade, upgrade and joint downgrade.

In addition to the result in Fig. B.4, the system can automatically visualize the resultant states such as in Fig. B.5 which represents a result after calculating

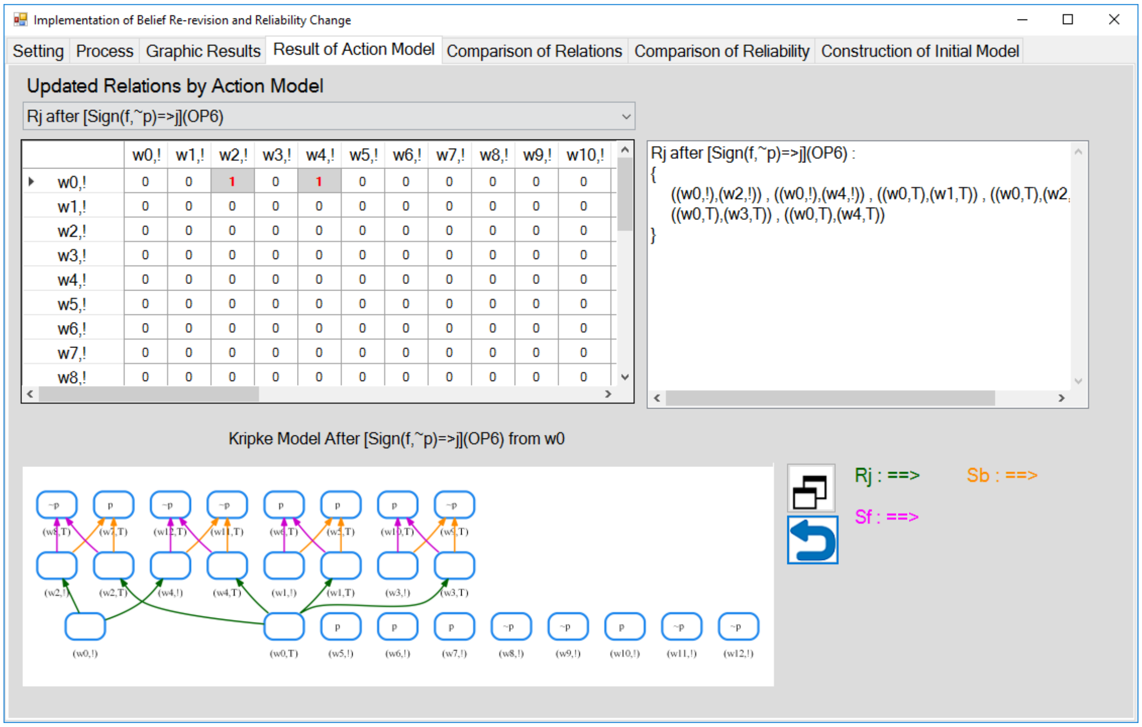

Figure B.5: Result after calculating [Sign$(f, \neg p) \rightsquigarrow j$] by a product update operation

[Sign$(f, \neg p) \rightsquigarrow j$] by a product update operation. Since the result in Fig. B.5 is difficult to understand, the system automatically converts such result into a simple one (which focuses only an announcing action '!') such as in Fig. B.6.

(4) The system can verify if an input of dynamic logical operator is correct syntax or not by a logical formula parser. Let us describe how to construct the logical formula parser. There are three following steps.

(4.1) The context-free grammar is defined corresponding to the syntax of our logical formalization (mentioned in Chapters 3 and 4) as follows:

S	$\rightarrow$	[actions]
actions	$\rightarrow$	brvFunc(agent,infos) |
		jdowngrade(agent, agentSet) |
		grading(agent,agentSet,infos) |
		infos => agent
brvFunc	$\rightarrow$	PriPer | Agg
grading	$\rightarrow$	downgrade | upgrade
agent	$\rightarrow$	a | b | c
agentSet	$\rightarrow$	{agents}
agents	$\rightarrow$	agents | agent, agent | agents, agent
infos	$\rightarrow$	info
info	$\rightarrow$	prop | ~ info | (info) | info & info | info # info |
		info -> info | info <-> info | Bel(agent,infos) |
		Sign(agent,infos)
prop	$\rightarrow$	p

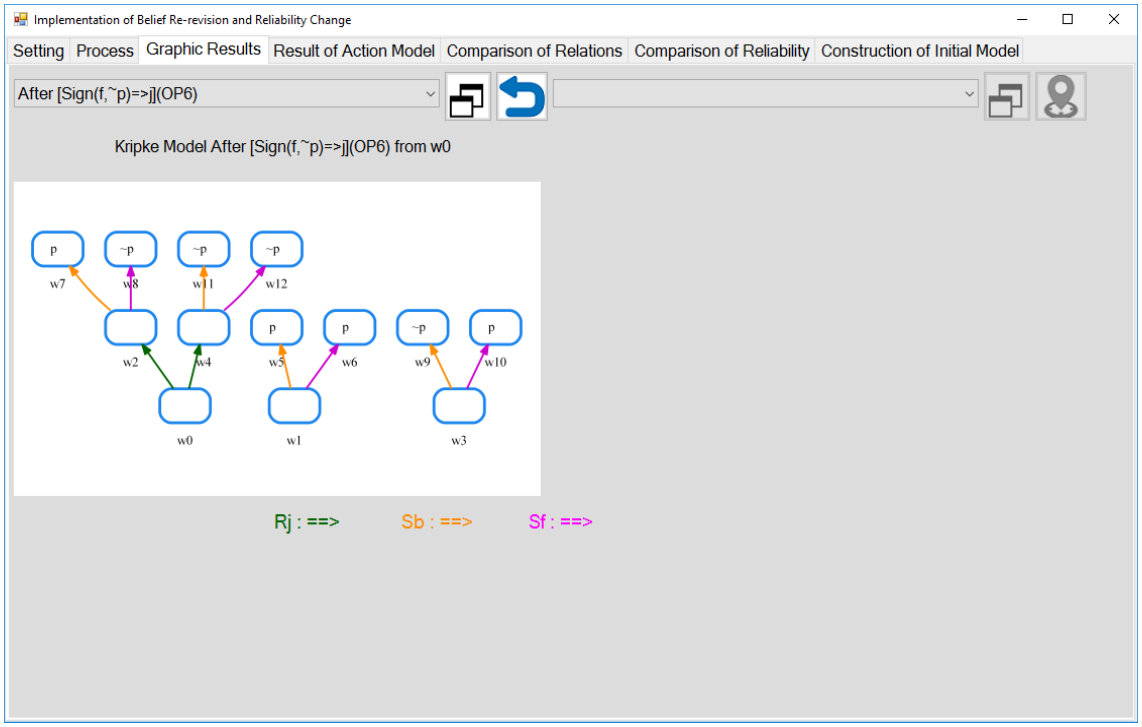

Figure B.6: Kripke model after calculating $[\mathsf{Sign}(f, \neg p) \rightsquigarrow j]$

Note that `infos` represents a set of formulas. The boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are represented by `~`, `&`, `#`, `->` and `<->`, respectively. In Table B.1, a set of agents `H`, which is in `[downgrade(a,H,p)]`, `[upgrade(a,H,p)]` and `[jdowngrade(a,H)]`, can be represented by `agentSet` as defined above.

(4.2) A process of parsing is implemented based on the Earley algorithm [33] which is a well-known top-down parsing algorithm for parsing strings that belong to a given context-free grammar. The process of parsing consists of three steps as follows: First, the context-free grammar from (4.1) is entered into the parser. Second, the process of parsing is performed based on the Earley algorithm. Finally, the parse tree will be retrieved from such chart entries.

(4.3) If the input has syntax errors, the parser can detect where an error occurred and may correct such error by a process of error recovery automatically. However, some errors cannot be automatically corrected by the system. That is, it requires the user to correct them by him/herself. In this case, the system will indicate a position where the error occurred and generate an error message including information about such error and a suggestion for error recovery such as in Fig. B.7. This figure shows an example of an error message indicating information of the error consisting of type, position and solution.

(5) The system can perform an inconsistency management policy as follows:

(5.1) The system will check if there is an inconsistency between the existing belief and new information or not. That is, our goal is to check if there is an agent giving inconsistent statements or not.

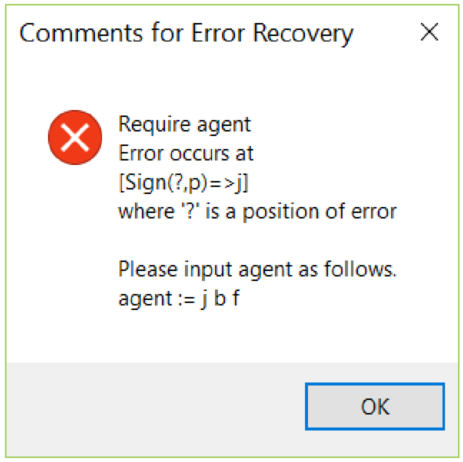

Figure B.7: Error message after inputting [`Sign`(, `p`) => `j`]

(5.2) If there is an inconsistency, the system applies joint downgrade and private permission operators by the following steps. First, the joint downgrade operator is employed for downgrading the agent who gives inconsistent statements less reliable. Second, a process of belief restoration is performed by the private permission operators. For this process, the system will automatically restore all possibilities because it cannot determine which statements should be permitted to the agent's belief.

(5.3) The system will check if there is the received information which is not inconsistent with the existing belief and is signed by the most reliable agent or not. If there is such information, the system will apply the private announcement operator for admitting such information.

The above process can be illustrated by Fig. B.4. When $[\mathsf{Sign}(b, \neg p) \rightsquigarrow j]$ is calculated and an inconsistency is detected, the system will perform four operations corresponding to the above process as follows: $[\{b\} \downdownarrows^j]$, $[\mathsf{Sign}(b, p) \twoheadrightarrow j]$, $[\mathsf{Sign}(b, \neg p) \twoheadrightarrow j]$ and $[\mathsf{Sign}(f, \neg p) \rightsquigarrow j]$. For a process of belief restoration, the system applies two private permission operators for permitting both $\mathsf{Sign}(b, p)$ and $\mathsf{Sign}(b, \neg p)$. The more details of this process are described in Section 5.2.3.

(6) The system can keep track of all the changes of an agent's belief and reliability by showing a comparison of relations and a comparison of reliability orderings from a specific agent's perspective. Fig. B.8 illustrates changing of j's belief as follows: First, j does not believe $\mathsf{Sign}(b, p)$ and $\mathsf{Sign}(f, \neg p)$, i.e., $\neg\mathsf{Bel}(j, \mathsf{Sign}(b, p))$ and $\neg\mathsf{Bel}(j, \mathsf{Sign}(f, \neg p))$ (see the left-hand side of Fig. B.8). Then, when j admits statements $\mathsf{Sign}(b, p)$ and $\mathsf{Sign}(f, \neg p)$ by $[(\mathsf{Sign}(b, p) \wedge \mathsf{Sign}(f, \neg p)) \rightsquigarrow j]$, j believes $\mathsf{Sign}(b, p)$ and $\mathsf{Sign}(f, \neg p)$, i.e., $\mathsf{Bel}(j, \mathsf{Sign}(b, p))$ and $\mathsf{Bel}(j, \mathsf{Sign}(f, \neg p))$ (see the right-hand side of Fig. B.8). Reliability change of agent j can be demonstrated by Fig. B.9. First, j believes that agents b and f are equally reliable, i.e., $\mathsf{Bel}(j, b \approx_j f)$ (see the left-hand side of Fig. B.9). Then, when j considers agent b to be unreliable, j downgrades agent b by $[\{b\} \downdownarrows^j]$. As a result, j believes that agent b is less reliable than agent f, i.e., $\mathsf{Bel}(j, f <_j b)$ (see the right-hand side of Fig. B.9).

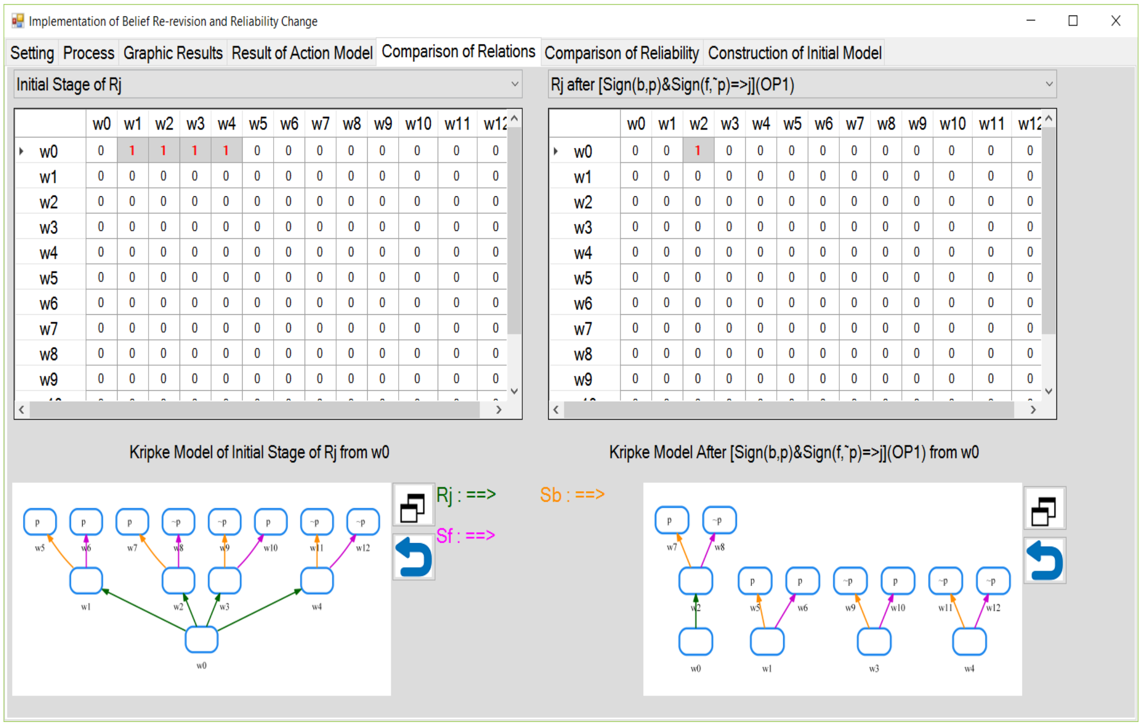

Figure B.8: Comparison of relations R_j: the left-hand side is R_j and Kripke model of the initial stage, and the right-hand side is R_j and Kripke model after $[(\mathsf{Sign}(b,p) \wedge \mathsf{Sign}(f,\neg p)) \rightsquigarrow j]$.

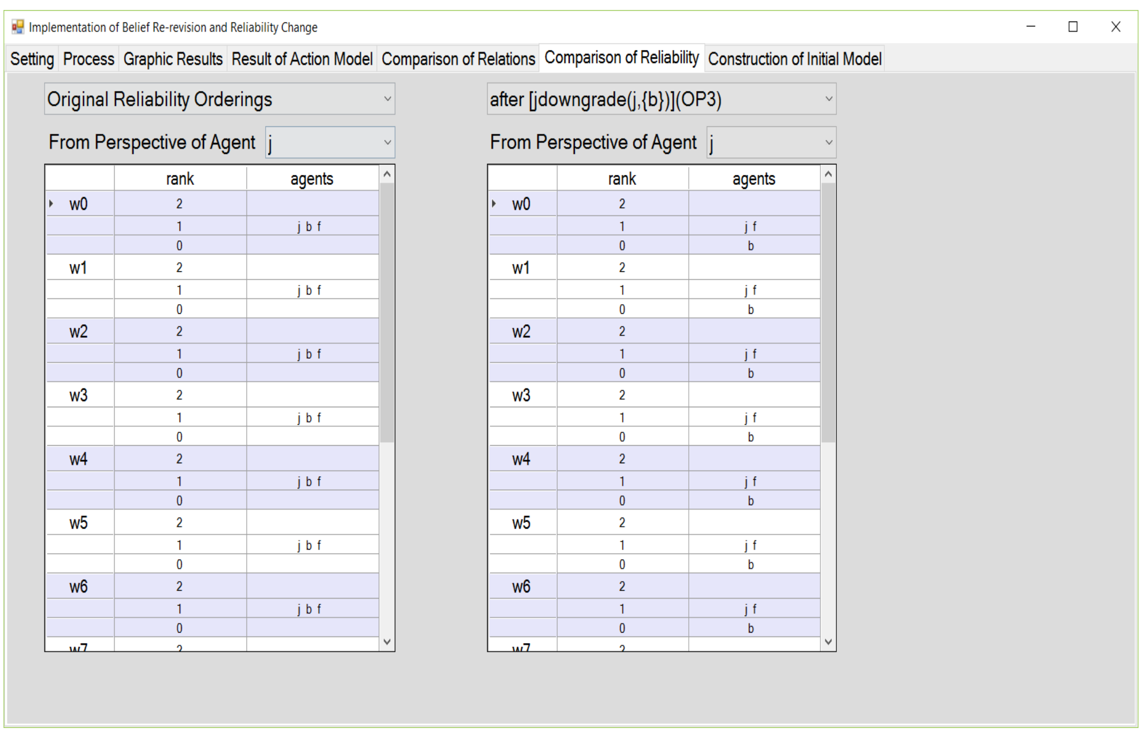

Figure B.9: Comparison of reliability orderings $\preccurlyeq_j$: the left-hand side is $\preccurlyeq_j$ in the initial stage, and the right-hand side is $\preccurlyeq_j$ after $[\{b\} \downdownarrows^j]$.

Appendix C

Details of Six Target Legal Cases

The story and the judgment of six target legal cases can be summarized as follows:

1) Legal case from Thailand (in [34]) occurred on January 26, 2003 in Trang province, Thailand.[1] The story can be summarized as follows:

   One day, Choochart (v) had a drink with his friends including Saichol (f_1), Ekachai (f_2) and Sommai (d) at f_2's house. After that, v was punched and stabbed with a hand scraper in the back by an offender, and as a result, v had bleeding in the lung. However, v was still alive.

   The details of judgment can be summarized as follows:

   In the inquiry stage, four witnesses v, f_1, f_2 and mo (mother of v) were interviewed by a police po, who is an official inquiry, as follows: v, f_1 and mo told that d was the offender, while f_2 told that d was not the offender. After the interview, d was charged with attempted murder. In the Civil Court, v and f_1 changed their statements, i.e., both of them told that d was not the offender. po was called to be a witness for testifying all statements in the inquiry stage. From these testimonies, the judge believed that the statements of v and f_1 in the Civil Court are less reliable than that in the inquiry stage. Thus, the judge believed that d was the offender and decided that d was guilty.

2) Legal case from Canada (in [35]) occurred on April 24, 1988 in Ontario, Canada.[2] The story can be summarized as follows:

   One day, Joseph (v) and his brother, Steven (b), got off a bus at an intersection. At the same time, the respondent, K.G.B. (d) and three other men including P.L. (f_1), P.M. (f_2) and M.T. (f_3) were driving past the same intersection. An argument started among them and shortly thereafter a fight occurred. v and b were unarmed. In the course of the fight, one of the four men from the car pulled a knife and then stabbed v in the chest. Finally, v died.

[1] This legal case can be referred from `http://deka2007.supremecourt.or.th/deka/web/search.jsp` (in Thai).

[2] This legal case can be referred from `http://www.canlii.org/en/`.

The details of judgment can be summarized as follows:

> In the inquiry stage, four witnesses b, f_1, f_2 and f_3 were interviewed as follows: b told that d was not the offender, while f_1, f_2 and f_3 told that d was the offender. After the interview, d was charged with murder. In the Youth Court, since all witnesses recanted their statements, the judge could not consider the prior statements of all witnesses as evidence. Thus, the judge acquitted d.

3) Legal case from British Columbia (in [36]) occurred on August 31, 1996 in Surrey, British Columbia.[2] The story can be summarized as follows:

> One day, while Basant Singh (v) with new friends including Sher (f_1), Jarnail (f_2), and the others gathered for social purposes, a van consisting of two respondents Sukhminder (d_1), Ajmer (d_2) and the others slowly approached the group of v. A burst of gun fire swept the group of v, shooting on a low trajectory into the ground. As a result, v died and three others including f_2 were wounded.

The details of judgment can be summarized as follows:

> In the inquiry stage, two witnesses f_1 and f_2 were interviewed as follows: f_1 told that d_1 was a driver of the van and d_2 was the shooter, while f_2 told that d_1 was a driver of the van and d_2 was not the shooter. After the interview, both d_1 and d_2 were charged with the first degree murder, the attempted murder of three other persons and aggravated assault on the same three persons. In the Crown Court, f_2 changed his statement, i.e., he told that d_2 was the shooter. Since there is an inconsistency in the statements of f_2, the judge considered only f_1's statement for identifying the shooter to be truthful. In addition, the judge believed that both d_1 and d_2 did not intend to kill v. For this reason, the judge acquitted both d_1 and d_2 of first degree murder but convicted them of manslaughter. The judge also convicted them of aggravated assault instead of attempted murder of the other three victims.

4) Legal case from Nova Scotia occurred on December 31, 2009 in Halifax, Nova Scotia.[2] The story can be summarized as follows:

> One day, Welsh (v) went to a New Year's Eve Party with his girlfriend Gautreau (f). While f was drinking in the party, v went outside the party to have a cigarette. Later on, f went outside and found v was punched then fell backward and struck his head on the pavement. Finally, v died.

The details of judgment can be summarized as follows:

> In the inquiry stage, only one witness f was interviewed as follows: f told that Leeds (d) was the offender. After the interview, d was charged with manslaughter. In the Crown Court, the judge found that f's recollection of the event was affected by her alcohol assumption, and there

were many inconsistencies in f's evidence such as the identification of d as the offender. Thus, the judge believed that f was not a reliable witness. Accordingly, the judge decided that d was not guilty.

5) Legal case from Nova Scotia occurred on August 6, 2011 in Halifax, Nova Scotia.[2] The story can be summarized as follows:

    One day, Barry (v) and his friends including Fisher (f_1), Marsh (f_2) and Slaunwhite (f_3) were drinking alcohol and smoking marijuana at v's home. Then, v together with his friends f_1, f_2 and f_3 drove to the house of Neil (d). While v was driving the vehicle at d, d was scared and fired the shot that injured v. However, v was still alive.

    The details of judgment can be summarized as follows:

    In the inquiry stage, four witnesses v, f_1, f_2 and Beaupre (b) who was d's neighbor were interviewed as follows: v and f_1 told that d intended to kill v, while f_2 and b told that d did not intend to kill v. After the interview, d was charged with the following offences: attempted murder, aggravated assault, using of a weapon in committing an assault, discharging a firearm with intent to endanger the life, intentionally discharging a firearm into a place, using of a firearm in a careless manner and possessing a weapon for a purpose dangerous to the public peace. In the Crown Court, the judge considered v and f_1 to be unreliable because v could not recall the events because of a combination of his intoxication by both drugs and alcohol on the evening in the events, and f_1's evidence was inconsistent within itself. Thus, the judge only accepted the evidence from f_2 and b that d did not intend to kill v; in fact, d just defended himself against v's attack. That is, d's actions were justified to be self-defense. Therefore, the judge decided that d was not guilty of all counts in the indictment.

6) Legal case from Nova Scotia occurred on July 17, 2004 in Bedford, Nova Scotia.[2] The story can be summarized as follows:

    One day, Bobby (v) was intoxicated at Busters Bar and having been denied further drinks from the bar. While Comer (d_1) and his friends including Warner (f_1), Maes (f_2), Southwell (f_3) and Morrison (f_4) were drinking, v approached d_1's table and asked for some beer but his request was refused. Then, v attempted to take d_1's beer but his attempt was prevented from f_1. After that, Smith (d_2) and his friend, Hodgson (f_5), arrived at the bar and joined the group at d_1's table. v left the bar first, then d_1, d_2 and f_5 left the bar. When f_1, f_2, f_3 and f_4 exited the bar, they came upon a verbal exchange between v, d_1, d_2 and f_5. Then, v kicked d_2 first, then all three including v, d_1 and d_2 were punching each other. The fight was of short duration. After v fell to the ground, d_1, d_2 and f_5 ran off.

    The details of judgment can be summarized as follows:

In the inquiry stage, five witnesses f_1, f_2, f_3, f_4 and f_5 were interviewed as follows: f_1 and f_2 told that they could not see what happened when v was on the ground, but f_2 stated that he saw d_2 kicked v once above the belt. f_3 told that he did not see anyone kick v while v was on the ground. f_4 told that v kicked d_2 first, then d_2 kicked v while v was on the ground. However, f_4 was not sure if d_2's kick was to v's head or not. f_5 told that he could not say where d_1's kick landed on v. After the interview, both d_1 and d_2 were charged with manslaughter in the death of v. In the Crown Court, three witnesses f_1, f_2 and f_5 changed their statements as follows: f_1 testified that d_1 and d_2 kicked v while v was on the ground, and all the kicks he saw landed on v's upper body between the belt and the head. f_2 told that he could not say if anyone kicked v while v was on the ground because people were in front of him and blocking his view. f_5 told that v kicked d_2 first, then d_1 kicked v in the head while v was on the ground. The judge found that the reliability of evidence of all witnesses was questionable because of the following reasons: f_1 and f_5 gave inconsistent statements, f_2's view of the events was affected by the fact that he was not wearing his eyeglasses, and f_3 and f_4 turned away from the fight. Based on these reasons, the judge was not satisfied on the evidence that d_2 kicked v while v was on the ground. Thus, the judge believed that d_2's act was in self-defense and was not excessive. Accordingly, d_2 was found not guilty. On the other hand, the judge believed that the kicking of d_1 was not in self-defense and was excessive because of the evidence that d_1 kicked v while v was on the ground. However, the judge cannot conclude that the kicking of d_1 was the cause of v's death because there is no evidence to support a finding that d_1 kicked v in the head. Accordingly, d_1 was found not guilty.

Appendix D

Analysis Process of Five Target Legal Cases

Section 5.2.2 presents only the summary of analysis process of six target legal cases in Table 5.5. Then, only the details of analysis process of the second legal case are shown in Table 5.7 (mentioned in in Section 5.2.3) for describing how to analyze a judge's changing of belief and reliability. Therefore, this chapter demonstrates the details of analysis process of five target legal cases in the following tables.

Table D.1: Analysis process of the first legal case

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(v, \neg p) \wedge \mathsf{Sign}(po, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_2, \neg p) \wedge \mathsf{Sign}(mo, p))) \rightsquigarrow j]$	j admits $\mathsf{Sign}(v, \neg p)$ and $\mathsf{Sign}(po, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_2, \neg p) \wedge \mathsf{Sign}(mo, p))$	$\mathsf{Bel}(j, \mathsf{Sign}(v, \neg p) \wedge \mathsf{Sign}(po, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_2, \neg p) \wedge \mathsf{Sign}(mo, p)))$
(2)	$[\{v, po\} \Uparrow^j_{\mathsf{Sign}(v,p)}]$	j upgrades agent po who signs $\mathsf{Sign}(v, p)$	$\mathsf{Bel}(j, po <_j v \approx_j f_2 \approx_j mo)$
(3)	$[\mathsf{Careful}(j, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_2, \neg p) \wedge \mathsf{Sign}(mo, p))]$	j aggregates statements of po	$\mathsf{Bel}(j, \mathsf{Sign}(v, p) \wedge \mathsf{Sign}(f_2, \neg p) \wedge \mathsf{Sign}(mo, p))$
(4)	$[\{v, f_2, mo\} \Uparrow^j_p]$	j upgrades agents v and mo who sign p	$\mathsf{Bel}(j, po \approx_j v \approx_j mo <_j f_2)$
(5)	$[\mathsf{Careful}(j, p)]$	j aggregates information about p	$\mathsf{Bel}(j, p)$

Table D.2: Analysis process of the third legal case

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(f_1, p) \wedge \mathsf{Sign}(f_2, p)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_1, p)$ and $\mathsf{Sign}(f_2, p)$	$\mathsf{Bel}(j, \mathsf{Sign}(f_1, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_1, \neg p)) \wedge \mathsf{Bel}(j, \mathsf{Sign}(f_2, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_2, \neg p))$
(2)	$[\mathsf{Sign}(f_2, \neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_2, \neg p)$	None
(2.1)	$[\{f_2\} \Downarrow^j]$	j downgrades agent f_2	$\mathsf{Bel}(j, f_1 <_j f_2)$
(2.2)	$[\mathsf{Sign}(f_2, p) \twoheadrightarrow j]$ $[\mathsf{Sign}(f_2, \neg p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(f_2, p)$ and $\mathsf{Sign}(f_2, \neg p)$	$\neg\mathsf{Bel}(j, \mathsf{Sign}(f_1, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_1, \neg p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_2, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_2, \neg p))$
(2.3)	$[\mathsf{Sign}(f_1, p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_1, p)$	$\mathsf{Bel}(j, \mathsf{Sign}(f_1, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_1, \neg p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_2, p)) \wedge \neg\mathsf{Bel}(j, \mathsf{Sign}(f_2, \neg p))$
(3)	$[\mathsf{Careful}(j, p)]$	j aggregates information about p	$\mathsf{Bel}(j, p)$

Table D.3: Analysis process of the fourth legal case

Step	Operation	Meaning	Result
(1)	$[\mathsf{Sign}(f,p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f,p)$	$\mathsf{Bel}(j,\mathsf{Sign}(f,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$
(2)	$[\mathsf{Sign}(f,\neg p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f,\neg p)$	None
(2.1)	$[\{f\} \downdownarrows^j]$	j downgrades agent f	$\mathsf{Bel}(j, j <_j f)$
(2.2)	$[\mathsf{Sign}(f,p) \twoheadrightarrow j]$ $[\mathsf{Sign}(f,\neg p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(f,p)$ and $\mathsf{Sign}(f,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(f,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f,\neg p))$

Table D.4: Analysis process of the fifth legal case

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(v,\neg p) \wedge \mathsf{Sign}(b,p)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(v,\neg p)$ and $\mathsf{Sign}(b,p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(v,p)) \wedge \mathsf{Bel}(j,\mathsf{Sign}(v,\neg p)) \wedge \mathsf{Bel}(j,\mathsf{Sign}(b,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))$
(2)	$[\mathsf{Sign}(v,p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(v,p)$	None
(2.1)	$[\{v\} \downdownarrows^j]$	j downgrades agent v	$\mathsf{Bel}(j, b <_j v)$
(2.2)	$[\mathsf{Sign}(v,p) \twoheadrightarrow j]$ $[\mathsf{Sign}(v,\neg p) \twoheadrightarrow j]$	j permits $\mathsf{Sign}(v,p)$ and $\mathsf{Sign}(v,\neg p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(v,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(v,\neg p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(b,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))$
(2.3)	$[\mathsf{Sign}(b,p) \rightsquigarrow j]$	j admits $\mathsf{Sign}(b,p)$	$\neg\mathsf{Bel}(j,\mathsf{Sign}(v,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(v,\neg p)) \wedge \mathsf{Bel}(j,\mathsf{Sign}(b,p)) \wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(b,\neg p))$
(3)	$[\mathsf{Careful}(j,p)]$	j aggregates information about p	$\mathsf{Bel}(j,p)$

Table D.5: Analysis process of the sixth legal case

Step	Operation	Meaning	Result
(1)	$[(\mathsf{Sign}(f_1,p)\wedge \mathsf{Sign}(f_3,\neg p)\wedge \mathsf{Sign}(f_4,p)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_1,p)$, $\mathsf{Sign}(f_3,\neg p)$ and $\mathsf{Sign}(f_4,p)$	$\mathsf{Bel}(j,\mathsf{Sign}(f_1,p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_1,\neg p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_3,p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_3,\neg p))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_4,p))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_4,\neg p))$
(2)	$[\{f_1,f_3,f_4\}\Downarrow^j_p]$	j downgrades agents f_1 and f_4 who sign p	$\mathsf{Bel}(j,f_3<_j f_1\approx_j f_4)$
(3)	$[\mathsf{Careful}(j,\neg p)]$	j aggregates information about $\neg p$	$\mathsf{Bel}(j,\neg p)$
(4)	$[(\mathsf{Sign}(f_1,q)\wedge \mathsf{Sign}(f_3,\neg q)\wedge \mathsf{Sign}(f_5,q)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_1,q)$, $\mathsf{Sign}(f_3,\neg q)$ and $\mathsf{Sign}(f_5,q)$	$\mathsf{Bel}(j,\mathsf{Sign}(f_1,q))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_1,\neg q))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_3,q)\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_3,\neg q))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_5,q))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_5,\neg q))$
(5)	$[\{f_1,f_3,f_5\}\Downarrow^j_{\neg q}]$	j downgrades agent f_3 who signs $\neg q$	$\mathsf{Bel}(j,f_1\approx_j f_5<_j f_3)$
(6)	$[\mathsf{Careful}(j,q)]$	j aggregates information about q	$\mathsf{Bel}(j,q)$
(7)	$[(\mathsf{Sign}(f_1,r)\wedge \mathsf{Sign}(f_4,\neg r)\wedge \mathsf{Sign}(f_5,r)) \rightsquigarrow j]$	j admits $\mathsf{Sign}(f_1,r)$, $\mathsf{Sign}(f_4,\neg r)$ and $\mathsf{Sign}(f_5,r)$	$\mathsf{Bel}(j,\mathsf{Sign}(f_1,r))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_1,\neg r))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_4,r)\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_4,\neg r))\wedge \mathsf{Bel}(j,\mathsf{Sign}(f_5,r))\wedge \neg\mathsf{Bel}(j,\mathsf{Sign}(f_5,\neg r))$
(8)	$[\{f_1,f_4,f_5\}\Downarrow^j_r]$	j downgrades agents f_1 and f_5 who sign r	$\mathsf{Bel}(j,f_4<_j f_1\approx_j f_5)$
(9)	$[\mathsf{Careful}(j,\neg r)]$	j aggregates information about $\neg r$	$\mathsf{Bel}(j,\neg r)$

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