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A survey and recent results about common developments of two or more boxes

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1. Introduction

Polygons that can fold into a convex polyhedron have been investigated since Lubiw and O'Rourke posed the problem in 1996 [**8**]. Demaine and O'Rourke published a great book about geometric folding algorithms that includes many results about the topic [**6**, Chapter 25]. In this context, there are few general results for the relationship between polygons and polyhedra folded from the polygons. Almost only one nice result is the following characterization of the polygons that fold into a tetramonohedron[1], which is characterized by a tiling as follows (see [**2, 3**] for the details): A polygon P is a development of a tetramonohedron if and only if (1) P has a p2 tiling, (2) four of the rotation centers consist in the triangular lattice formed by the triangular faces of the tetramonohedron, (3) the four rotation centers are the lattice points, and (4) no two of the four rotation centers belong to the same equivalent class on the tiling.

In this article, we concentrate on polygons that consist of unit squares, and orthogonal convex polyhedra, i.e., *boxes*, folded from them. Biedl et al. first find two polygons that fold into two incongruent orthogonal boxes [**5**] (see also [**6**, Figure 25.53]). The first one folds into two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$, and the second one folds into two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$. Are these two polygons exceptional? The answer is "no." You can see another example in Figure 1.

We survey the series of our research on this topic. Especially, we give an affirmative answer to the natural question that asks whether there exists a polygon that folds into three different boxes: Yes, there exist infinitely many polygons that fold into three different boxes. So far, it is still open whether if there exists a polygon folding to four or more distinct boxes.

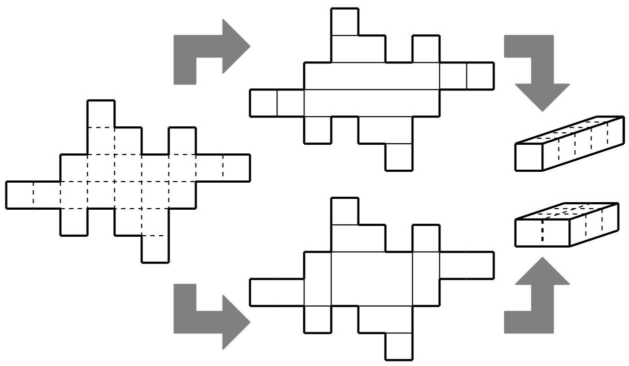

FIGURE 1. A polygon that folds into two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$.

[1]A *tetramonohedron* is a tetrahedron that consists of four congruent triangular faces.

2. Preliminaries

We concentrate on orthogonal polygons that consist of unit squares. A convex orthogonal polyhedron of six rectangular faces is called a *box*. For a positive integer S, we denote by $P(S)$ the set of three integers a, b, c with $0 < a \leq b \leq c$ and $ab + bc + ca = S$, i.e., $P(S) = \{(a, b, c) \mid ab + bc + ca = S\}$. Intuitively, $2S = 2(ab+bc+ca)$ indicates the surface area of the box of size $a \times b \times c$. Therefore, it is necessary to satisfy $|P(S)| \geq k$ to have a polygon of size $2S$ that can fold into k incongruent orthogonal boxes. For example, the two known polygons in [**5**] correspond to $P(11) = \{(1, 1, 5), (1, 2, 3)\}$ and $P(17) = \{(1, 1, 8), (1, 2, 5)\}$. Using a simple algorithm that computes $ab+bc+ca$ for all possible combinations of a, b, c with $1 \leq a \leq b \leq c \leq 50$, we have

$$P(11) =\{(1,1,5),(1,2,3)\}, P(15) = \{(1,1,7),(1,3,3)\}, P(17) = \{(1,1,8),(1,2,5)\},$$
$$P(19) =\{(1,1,9),(1,3,4)\}, P(23) = \{(1,1,11),(1,2,7),(1,3,5)\}, P(27) = \{(1,1,13),(1,3,6),(3,3,3)\},$$
$$P(29) =\{(1,1,14),(1,2,9),(1,4,5)\}, P(31) = \{(1,1,15),(1,3,7),(2,3,5)\},$$
$$P(32) =\{(1,2,10),(2,2,7),(2,4,4)\}, P(35) = \{(1,1,17),(1,2,11),(1,3,8),(1,5,5)\},$$
$$P(44) =\{(1,2,14),(1,4,8),(2,2,10),(2,4,6)\}, P(45) = \{(1,1,22),(2,5,5),(3,3,6)\},$$
$$P(47) =\{(1,1,23),(1,2,15),(1,3,11),(1,5,7),(3,4,5)\},$$
$$P(56) =\{(1,2,18),(2,2,13),(2,3,10),(2,4,8),(4,4,5)\},$$
$$P(59) =\{(1,1,29),(1,2,19),(1,3,14),(1,4,11),(1,5,9),(2,5,7)\},$$
$$P(68) =\{(1,2,22),(2,2,16),(2,4,10),(2,6,7),(3,4,8)\},$$
$$P(75) =\{(1,1,37),(1,3,18),(3,3,11),(3,4,9),(5,5,5)\},$$

and so on. That is, there is no polygon that folds into two different boxes if its surface area is less than $22 = 2 \times 11$ since $P(i) < 2$ for all $0 < i < 11$. On the other hand, if we try to find a polygon that folds into three different boxes, its surface should be at least $2 \times 23 = 46$, and in this case, three possible combinations of height, width, and depth are $1 \times 1 \times 11$, $1 \times 2 \times 7$, and $1 \times 3 \times 5$.

3. Polygons folding into Two Boxes

Even for small surface area, it is not easy to check all common developments of some boxes since they are too huge. In 2008, we first developed some randomized algorithms that check a part of common developments [**9**]. By computational experiments, we obtain over 25000 common developments of two different boxes (including one in Figure 1, which is my most favorite one). Thousands of them can be found at `http://www.jaist.ac.jp/~uehara/etc/origami/nets/index-e.html`. We give here some interesting ones among them.

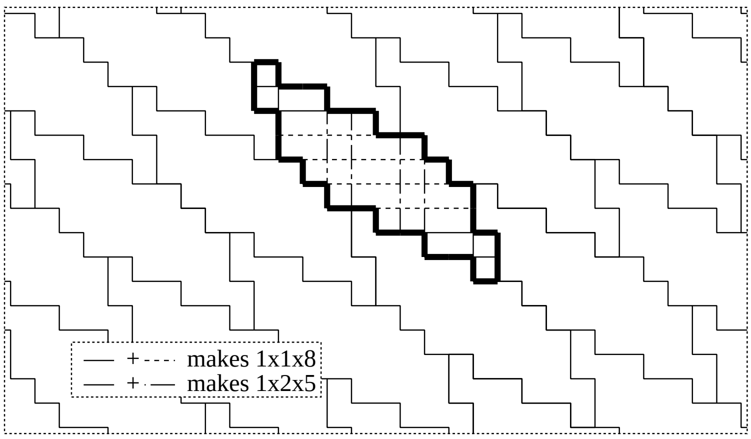

FIGURE 2. Polygon folding into two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$, and tiling the plane.

Tiling pattern: The discovered polygonal patterns reminded us of *tiling*. Indeed, there exists a simple polygon that can fold into two boxes and it forms a tiling. The polygon in Figure 2 can fold into two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$, and it tiles the plane.

A polygon is called a *double packable solid* if it tiles the plane and a polyhedron folded from the polygon fills the space [**7**, Section 3.5.2]. It is easy to see that every orthogonal box fills the space. Therefore, the polygon in Figure 2 forms two double packable solids.

As shown in Introduction, any development of a tetra(mono)hedron is characterized by the notion of p2 tiling [**2, 3**]. We have not yet checked if the developments of two boxes can fold into a tetra(mono)hedron.

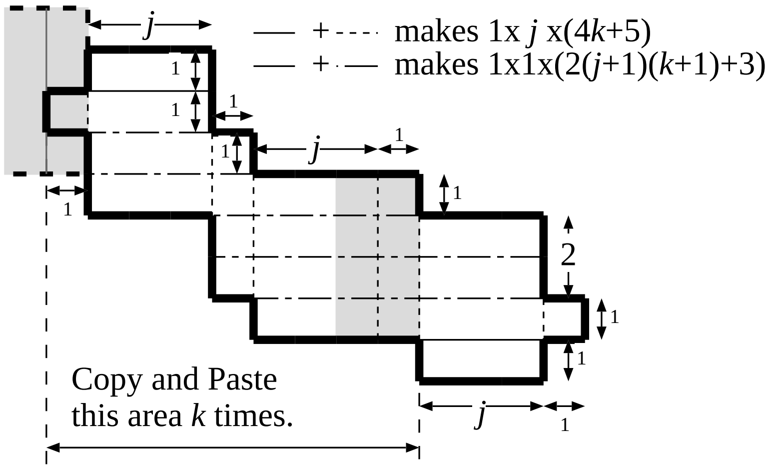

FIGURE 3. Polygon folding into two boxes of size $1 \times 1 \times (2(j+1)(k+1)+3)$ and $1 \times j \times (4k+5)$.

Infinitely many polygons: A natural question is whether or not there are infinite distinct[2] polygons that can fold into two boxes? The answer is "yes." Some polygons in our catalogue can be generalized. From one of them, we find a polygon that can fold into two boxes of size $1 \times 1 \times (2(j+1)(k+1)+3)$ and $1 \times j \times (4k+5)$ for any positive integers j and k (Figure 3).

The first parameter j just stretches each rectangle in Figure 3 in the same rate, which has no effect to construct two distinct boxes; two ways of folding are similar to the polygon in Figure 2. For the second parameter k, we copy in the leftside polygon in Figure 3 and glue it to the leftmost square (with overlapping at gray areas) and repeat it k times. Then, the way of folding of the box of size $1 \times 1 \times (2(j+1)(k+1)+3)$ is essentially the same for every k; just we roll up four unit squares vertically. The way of folding of the box of size $1 \times j \times (4k+5)$ depends on k. We spiral up the polygon k times, and obtain vertically long rectangles. By these ways of folding, we have two distinct boxes of different sizes from a polygon. Therefore, there exist an infinite of distinct polygons that can fold into two boxes.

4. Polygons folding to Three Boxes

In 2011, we succeeded to enumerate all common developments of surface area 22, which is the smallest one admitting two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$. By an exhaustive search, we found that the number of common developments of two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$ is 2263 ([**1**]). Among resulting common developments, there is only one exceptional development which folds into not only two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$, but also of size $0 \times 1 \times 11$ (Figure 4; it is also a tiling pattern). Each column of the development has height 2 except both endpoints, which admits to fold the third box of volume 0. But this is a kind of cheating: if you admit to have volume 0, a long ribbon can wrap doubly covered rectangles in many ways (see [**1**] for further details).

In 2013, we finally found a development that folds into three different boxes of size $2 \times 13 \times 58$, $7 \times 14 \times 38$, and $7 \times 8 \times 56$ (Figure 5). The basic idea is simple; first we start a common development of size $1 \times 1 \times 8$

[2]Precisely, *distinct* means $gcd(a, b, c, a', b', c') = 1$ for two boxes of size $a \times b \times c$ and $a' \times b' \times c'$.

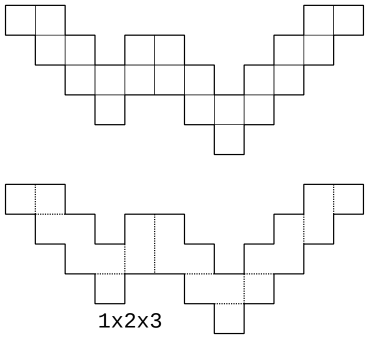

FIGURE 4. Polygon folding into two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$, and a box(?) of size $0 \times 1 \times 11$.

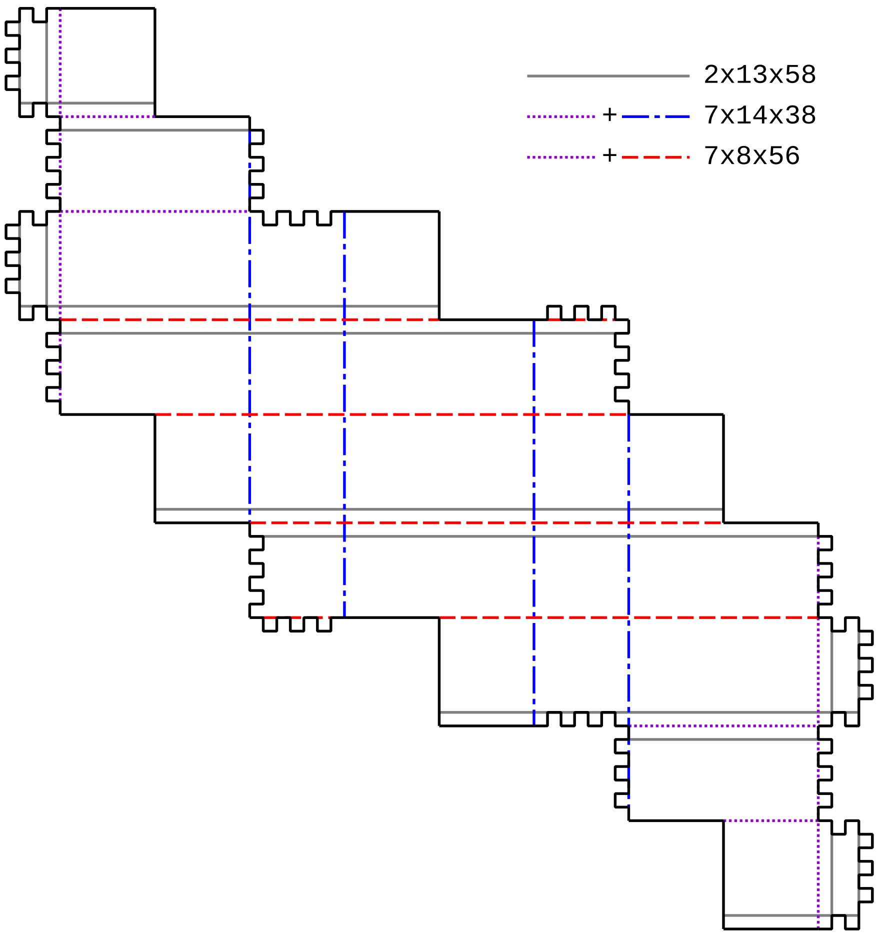

FIGURE 5. Polygon folding into three boxes of size $2 \times 13 \times 58$, $7 \times 14 \times 38$, and $7 \times 8 \times 56$.

and $1 \times 2 \times 5$. The third one is obtained by "squashing" the box of size $1 \times 1 \times 8$ into half height of size, roughly, $1/2 \times 2 \times 8$. But this intuitive idea does not work in a straightforward way; a square of size 1×1 has perimeter 4, which is not equal to the perimeter 5 of the rectangle of size $1/2 \times 2$. So we use a trick to move some area from two lid squares of size 1×1 to four side rectangles of size 1×8 in a nontrivial way

(Figure 6). Intuitively, the zig-zag pattern can be generalized as shown in Figure 7, and we finally obtain an infinitely many polygons that fold into three different boxes of positive volumes. See [**10**] for further details.

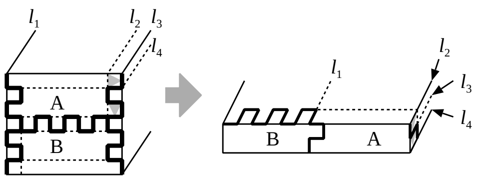

FIGURE 6. Squash the box with moving a part of lid to four sides.

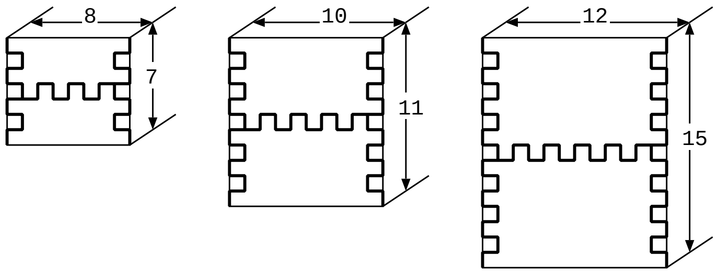

FIGURE 7. General pattern for squashing.

In 2012, Toshihiro Shirakawa accidentally found two polygons that can fold into two boxes of size $1 \times 1 \times 7$ and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. These polygons have surface area 30, which may admit to fold into another box of size $1 \times 3 \times 3$. We have examined to enumerate all common developments of two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$, whose surface area is 22, but it takes 10 hours in 2011, and 5 hours in 2014. We use supercomputer (Cray XC30) few months with nontrivial hybrid search of the breadth-first and depth-first searches (see [**11**] for further details). As a result, we succeed to enumerate all common developments of two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$, and the number is 1076. For these common developments, we design a new algorithm that checks if an orthogonal polygon of area 30 can fold into a box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. The details of the algorithm is on preparation [**12**]. Among 1076 common developments of two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$, 9 polygons can fold into the other box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. Surprisingly, among these 9 developments, one polygon can fold into the box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$ in two different ways. This amazing polygon can be found in Figure 8.

Concluding remarks

So far, the smallest polygon folding into three boxes based on the same idea in Figure 5 requires more than 500 unit squares. On the other hand, we have enumerated all common developments of surface area 30 which can fold into two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$. Therefore, enumeration of all common developments of the smallest surface area 46 which may fold into three different boxes of size $1 \times 1 \times 11$, $1 \times 2 \times 7$, and $1 \times 3 \times 5$ is the next challenging problem.

The main motivation of this research is investigation of relationship between a polygon and polyhedra which can be folded from the polygon and vice versa. From this viewpoint, the extensions to nonorthogonal and/or nonconvex ones are also interesting future work. For example, Araki, Horiyama, and Uehara have investigated the set of polygons obtained from Johnson-Zalgaller solids by edge cutting [**4**]. From the set, they extract all polygons that can fold into regular tetrahedra. However, general characterization of the relationship between a polygon and a polyhedra folded from it is still widely open.

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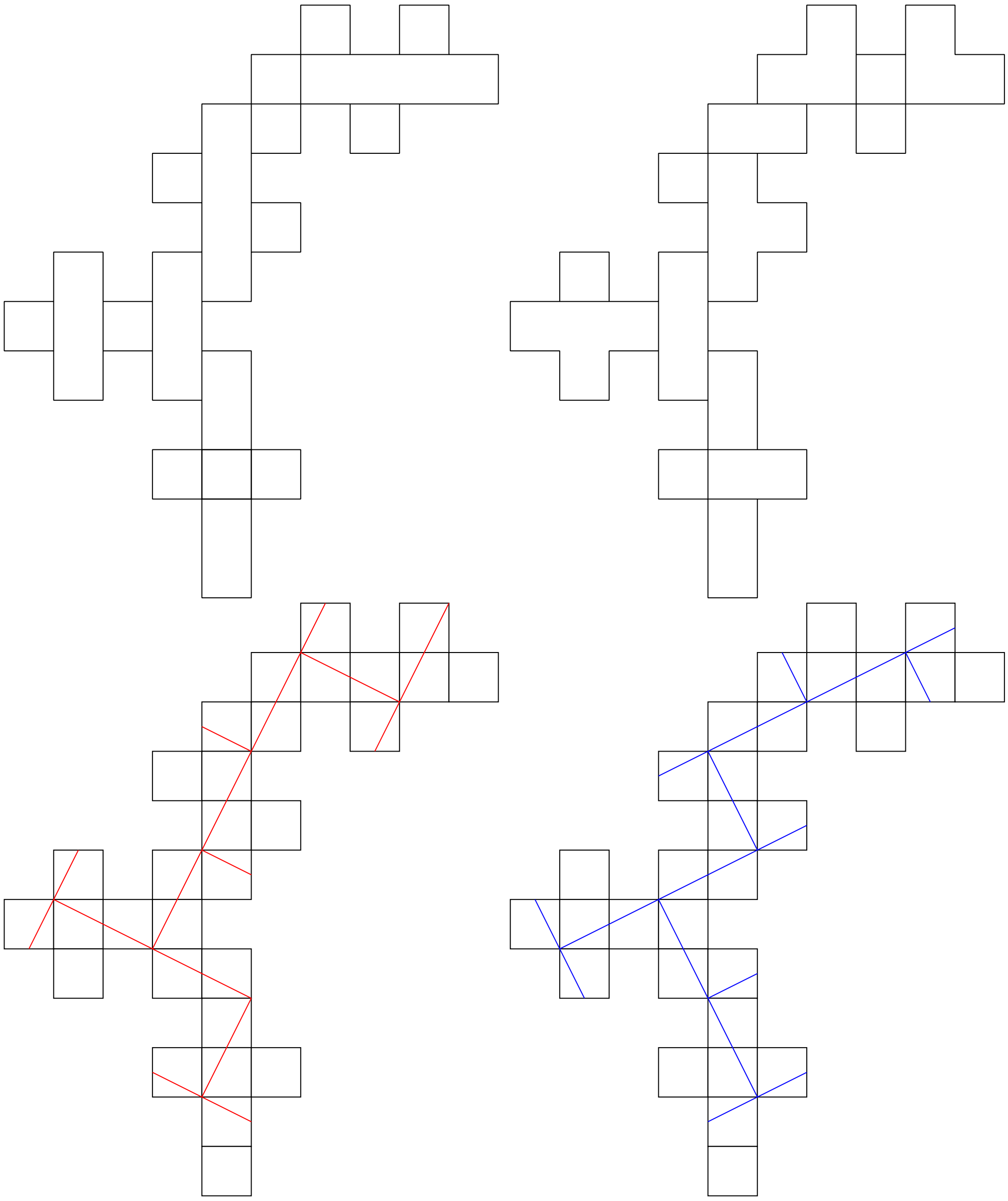

FIGURE 8. Polygon folding into three boxes of size $1 \times 1 \times 7$, $1 \times 3 \times 3$, and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. The last box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$ can be folded in two different ways.

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