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Single-Player and Two-Player Buttons & Scissors Games

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Abstract

The Buttons & Scissors puzzle was recently shown to be NP-hard. In this paper we continue studying the complexity of various versions of the puzzle. For example, we show that it is NP-hard when the puzzle consists of $C = 2$ colors, and polytime solvable for $C = 1$. Similarly, it is NP-hard when each color is used by at most $F = 4$ buttons, and polytime solvable for $F = 3$. We also consider restrictions on the board size, cut directions, and cut lengths. Finally, we introduce new two-player games and show that they are PSPACE-complete.

1 Introduction

Buttons & Scissors is a single-player puzzle by KyWorks that was recently studied by Gregg et al. [2]. A *level* is an $n \times n$ grid of buttons of different colors; each position is either empty or has a button of a single color sewn to it. The goal is to remove all buttons using horizontal, vertical and diagonal scissor cuts. A cut is *feasible* if it removes at least two buttons of the same color and no buttons of any other color. Figures 1(a)–(b) show a sample level and solution. See [2] for further clarification of the rules and terminology.

Deciding whether a level is solvable is NP-complete [2]. We show that several restricted versions of the puzzle remain hard, and provide polytime algorithms for a number of easier versions. We also introduce two-player Buttons & Scissors games and show that they are PSPACE-complete. Due to space restrictions, most proofs are sketched or omitted.

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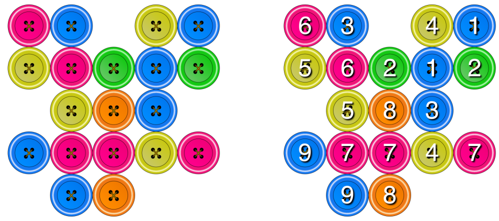

(a) (b)

Figure 1: (a) Level 6 in the Buttons & Scissors app is on a 5×5 grid with $C = 5$ colors, each used at most $F = 6$ times; (b) a solution using nine cuts of minimum length $\ell = 2$ and all four directions ($d = \ast$).

2 Notation

Each Buttons & Scissors level can be parameterized as follows (see Figure 1 for an example):

1. The *board size* $m \times n$.
2. The *number of colors* C.
3. The *maximum frequency* F of an individual color.
4. The *cut directions* d can be limited from the original four directions, which we denote by $\ast$. We only consider $d \in \{\ast, \times, +, \nearrow, -\}$ because an $n \times m$ board can be rotated 90° to an equivalent $m \times n$ board, or 45° to an equivalent $k \times k$ board for $k = n + m - 1$ by adding blank squares.
5. The *cut length* ℓ is the minimum number of buttons required to be removed by a feasible cut.

These parameters give the following decision problem (the original problem is $B\&S[n \times n, \infty, \infty, \ast, 2](B)$):
Decision Problem: $B\&S[m \times n, C, F, d, \ell](B)$.
Input: Given an $m \times n$ board B with buttons of C colors, where each color is used at most F times.
Output: True if B has a solution with minimum cut length ℓ using d directions. Otherwise, False.

3 Single-Player Puzzle

We now present our results on the single-player puzzle.

3.1 Board Size

Remark 1 *If a Buttons & Scissors board can be solved, then it can be solved using only cuts of* 2 *or* 3 *buttons.*

Theorem 1 *Buttons & Scissors is polytime solvable on $1 \times m$ boards. That is, $B\&S[1 \times n, \infty, \infty, -, 2](B) \in P$.*

Proof. Consider the following context-free grammar,

$$\begin{array}{llll} S \to \epsilon & S \to S\square & S \to xSx & S \to xSxx \\ S \to SS & S \to \square S & & S \to xxSx \end{array}$$

where $\square$ is an empty square and $x \in \{1, 2, \ldots, C\}$. By Remark 1, the solvable $1 \times m$ boards are in one-to-one correspondence with the strings in this language. $\square$

Theorem 2 *Given a full $2 \times m$ Buttons & Scissors board with $C = 2$ and a constant s, there exists a polynomial time algorithm that removes all but s buttons from the board with feasible cuts.*

3.2 Number of Colors

Theorem 3 *Buttons & Scissors is polytime solvable for 1-color, and is NP-complete for 2-colors. That is, $B\&S[n \times n, 1, \infty, *, 2](B) \in P$ and $B\&S[n \times n, 2, \infty, *, 2](B)$ is NP-complete.*

Proof Sketch: In a graph G, the maximum number of vertices that can be covered by edge-disjoint K_2 and K_3 subgraphs is polytime computable in the size of G (see Cornuéjols et al. [1]). We convert each 1-color board B into a graph whose vertices can be perfectly covered if and only if B is solvable. The transformation uses Remark 1 and is non-trivial since the order of cuts is important. The 2-color reduction is in the full paper. $\square$

3.3 Frequency of Colors

Remark 2 *If board B' is obtained from board B by removing every button of a single color, then $B\&S[m \times n, C, F, d, \ell](B) \implies B\&S[m \times n, C, F, d, \ell](B')$ (i.e., it is impossible that Buttons & Scissors is solvable on B, but not solvable on B' with the same parameters).*

Theorem 4 *Buttons & Scissors is polytime solvable for maximum color frequency $F = 3$, and is NP-complete for $F = 4$. That is, $B\&S[n \times n, \infty, 3, *, 2](B) \in P$ and $B\&S[n \times n, \infty, 4, *, 2](B)$ is NP-complete.*

Proof Sketch: If $F = 3$, then each color is removed by a single cut in any solution. By Remark 2, these cuts cannot make a solvable board unsolvable. Thus, there is a simple greedy algorithm for deciding solvability.

It was proven that $B\&S[n \times n, \infty, 7, *, 2](B)$ is NP-complete via 3-SAT [2]. We use the same reduction for $F = 4$, with Figure 2 replacing each OR gadget. $\square$

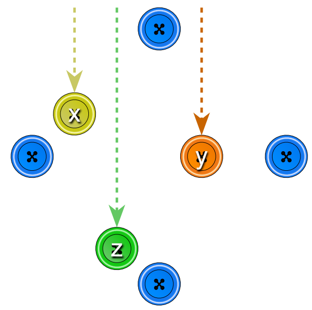

Figure 2: The left blue button can be removed if and only if "x is removed $\vee$ y is removed $\vee$ z is removed".

3.4 Cut Directions

NP-completeness for horizontal and vertical cuts and $F = 7$ was proven in [2]. We improve this to $F = 6$.

Theorem 5 *$B\&S[n \times n, \infty, 6, +, 2](B)$ is NP-complete.*

3.5 Cut Lengths

In Section 3.2 we saw that the 1-color version has a polytime algorithm. However, if the minimum cut length is set to $\ell = 3$ (instead of $\ell = 2$) then it is NP-complete.

Theorem 6 *$B\&S[n \times n, 1, \infty, *, 3](B)$ is NP-complete.*

4 Two-Player Games

In our two-player games, the players take turns making feasible cuts on a common board B. In a *partisan game*, some cuts are available to one player, but not the other. In an *impartial game* all feasible cuts can be made by both players. We consider two losing conditions:

1. The player cannot execute a feasible cut (LAST).
2. The player removed fewer buttons (MAX).

Theorem 7 *The partisan LAST two-player game is PSPACE-complete, where player 1 cuts blue buttons, player 2 cuts red, and both cut green (if any).*

Theorem 8 *The impartial MAX and LAST two-player Buttons & Scissors games are PSPACE-complete.*

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