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Title	A proportional linguistic distribution based model for multiple attribute decision making under linguistic uncertainty
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Citation	Annals of Operations Research, 256(2): 305-328
Issue Date	2016-12-09
Type	Journal Article
Text version	author
URL	http://hdl.handle.net/10119/15296
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Description	

JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY

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Noname manuscript No.
(will be inserted by the editor)

A Proportional Linguistic Distribution Based Model for Multiple Attribute Decision Making under Linguistic Uncertainty

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Songsak Sriboonchitta**

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Abstract This paper aims at developing a proportional fuzzy linguistic distribution model for multiple attribute decision making problems, which is based on the nature of symbolic linguistic model combined with distributed assessments. Particularly, in this model the evaluation on attributes of alternatives is represented by distributions on the linguistic term set used as an instrument for assessment. In addition, this new model is also able to deal with incomplete linguistic assessments so that it allows evaluators to avoid the dilemma of having to supply complete assessments when not available. As for aggregation and ranking problems of proportional fuzzy linguistic distributions, the extension of conventional aggregation operators as well as the expected utility in this proportional fuzzy linguistic distribution model are also examined. Finally, the proposed model will be illustrated with an application in product evaluation.

Keywords: Computing with words, decision making, incomplete assessments, linguistic modeling, multiple attribute.

1 Introduction

In practice most of the multiple attribute decision making (MADM) problems involve both types of qualitative and quantitative attributes, which are often organized in a hierarchical structure [1, 2, 17]. While quantitative attributes

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This paper is a significantly revised and extended version of [6].

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can be measured by means of numeric scales in the form of numbers, interval or fuzzy numbers, qualitative attributes can be only assessed by subjective judgements with uncertainty. In such situations of decision making, the problem of how to represent and aggregate qualitative attributes essentially plays an important role in decision analysis. So far, there are several approaches proposed to deal with this problem. A reasonable way is to make use of a fuzzy linguistic approach to represent qualitative attributes by means of linguistic values of linguistic variables [32], [33], [34], and a mechanism for computing with words (CW) [35] for fusing linguistic information so as to provide an evaluation for decision making.

Basically, most early work on decision making with linguistic information made use of fuzzy sets as a tool for modeling linguistic information and aggregation methods were then developed based on Zadeh's extension principle, e.g., [4]. Notably also, another approach also aimed to develop the linguistic symbolic computational model based on ordinal scales [25]. Because of the inherent operation mechanism of these two linguistic computational models, the results of a computational process usually do not exactly match any of the initial linguistic terms and, hence, a process of linguistic approximation must be applied to convert the computational results into linguistic terms of the initial linguistic domain. This linguistic approximation process causes a loss of information and consequently leads to the lack of precision in the final results [3]. In order to overcome this limitation of information loss in the computational stage for CW, Herrera and Martínez [8] developed the so-called 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation so as to improve the precision of the final results. This 2-tuple linguistic representation model has been widely applied to a range of applications [5, 9, 10, 16, 20, 23]. However, as pointed out by its authors, the 2-tuple fuzzy linguistic representation model was only suitable for handling linguistic variables with equidistant labels. In addition, as argued by Lawry [13], although Herrera and Martínez's symbolic approach offered a computationally much more feasible method than those approaches using the extension principle in CW, it did not directly take into account the underlying vagueness of linguistic terms.

In an attempt to improve Herrera and Martínez's 2-tuple fuzzy linguistic representation model so as to be able to deal with unbalanced linguistic term sets while simultaneously taking the underlying semantics of terms into account, Wang and Hao [21] proposed a so-called proportional 2-tuple fuzzy linguistic representation model for CW making use of the canonical characteristic values (CCVs) of linguistic terms determined by their corresponding semantics. Wang and Hao's proportional 2-tuple fuzzy linguistic representation model interestingly provides a suitable and more flexible space in a computation stage for CW, which could allow evaluators in various decision models flexibly evaluate performances of alternatives by not just one label but with proportional 2-tuples of the form $(\alpha A, \beta B)$, where A and B are two consecutive linguistic terms, and $\alpha, \beta \in [0, 1], \alpha + \beta = 1$. However, as we have seen, by definition, this model cannot deal with decision situations where alternative performances are generally assessed by means of uncertain linguistic judgments as studied in, e.g., [11, 24, 26, 29–31].

Recently Zhang et al. [36] also proposed a concept of distribution assessment overcoming the limitations of Wang and Hao's model [21], however, due to the premise that the summation of symbolic proportions must equal to 1, Zhang et al.'s model, and Wang and Hao's proportional 2-tuple model as well, cannot handle incomplete and ignoring information. In order words, they are only applicable under the context that all the linguistic assessments are complete. As a matter of fact, incomplete assessments emerge commonly when evaluators are lack of confidence, especially in the case of facing with uncertain and incomplete information. As such, it would be desirable that a new linguistic representation model could be developed so as to be able to deal with linguistic assessments with incomplete information. It is of interest to note that the precision and the reasonability of the final result will be obviously improved if such a model appropriately touches upon the incomplete assessments or ignoring information. In fact, in the evidential reasoning (ER) approach to MADM developed in [29, 30], uncertain assessments of attributes are modeled by means of mass functions in Dempster-Shafer theory of evidence [18] and then it can also handle incomplete and ignoring information. Also, the attribute aggregation process taking the relative weights of attributes into account is performed by making use of the so-called discounting operation and Dempster's rule of combination, as discussed in [11]. Therefore, the ER approach could not be able to deal with decision situations where the attribute weight information is expressed linguistically and, in addition, many aggregation operators in the numerical setting often used in decision analysis could not be extendable for use in the ER approach.

In this paper, we are dealing with MADM problems with uncertain linguistic information given not only in evaluation on attributes of alternatives but also in attribute weights. We will propose a proportional fuzzy linguistic distribution model that not only inherits advantages of 2-tuple and proportional 2-tuple fuzzy linguistic representation models, but also overcomes the above-mentioned limitations of these models. The rest of this paper is organized as follows. Section 2 presents some preliminaries about 2-tuple fuzzy linguistic representation model and proportional 2-tuple fuzzy linguistic representation model. Then, Section 3 will explore the proportional fuzzy linguistic distribution model and its computational operators as well as expected utility in proportional fuzzy linguistic distribution. In Section 4, an extension of conventional aggregation operators to deal with the aggregation problem of proportional fuzzy linguistic distributions will be examined. In Section 5, an example taken from [30] will be used for illustration of the proposed model. We first use the original numerical weights for the purpose of comparing the results and illuminating the applicability of the proposed model. Then, we replace the numerical weights with linguistic weights in order to further explain the capability of this model for handling uncertain linguistic information. Finally, Section 6 presents some concluding remarks.

2 Preliminaries

In this section, we will briefly review basics of Herrera and Martínez's 2-tuple fuzzy linguistic representation model [8] and Wang and Hao's proportional 2-tuple fuzzy linguistic representation model [21] developed for CW. For a comprehensive and detailed exposition of theory and applications of the 2-tuple linguistic model, the interested reader might be referred to the survey paper [14] and the recent book [15].

2.1 The 2-Tuple Fuzzy Linguistic Representation Model

Let $S = \{s_0, s_1, \ldots, s_n\}$ be a linguistic term set, and the term s_i with $i = 0, \ldots, n$, represents a possible value for a linguistic variable. The total order on S is defined as: $s_i \leq s_j \Leftrightarrow i \leq j$. There is a negation operator: $\mathrm{Neg}(s_i) = s_j$ such that $j = n - i$, where $n + 1$ is the cardinality of S. In general, using a symbolic method to aggregate linguistic information [7], we often get a value $\beta \in [0, n]$, and $\beta \notin \{0, \ldots, n\}$. Therefore, an approximation function must be used in order to conveniently express the index of the result in S.

To avoid any approximation process which consequently causes a loss of information [8], 2-tuple (s_i, α) that expresses the equivalent information to β is defined by the following function:

$$\begin{aligned} &\triangle : [0, n] \to S \times [-0.5, 0.5) \\ &\triangle\ (\beta) = (s_i, \alpha), \text{with} \begin{cases} s_i, & i = \mathrm{round}(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \end{aligned}$$

where round $(\cdot)$ is the usual round operation, s_i has the closest index label to β, and α is the value of the symbolic translation.

Inversely, a 2-tuple $(s_i, \alpha) \in S \times [-0.5, 0.5)$ can also be equivalently represented by a numerical value in $[0, n]$ by means of the following transformation:

$$\begin{aligned} &\triangle^{-1} : S \times [-0.5, 0.5) \to [0, n] \\ &\triangle^{-1}\ (s_i, \alpha) = i + \alpha = \beta. \end{aligned}$$

The negation operator over 2-tuples is defined by

$$\mathrm{Neg}((s_i, \alpha)) = \triangle(n - (\triangle^{-1}(s_i, \alpha)))$$

where $n + 1$ is the cardinality of $S = \{s_0, s_1, \ldots, s_n\}$.

Because 2-tuples can be transformed into numerical values without loss of information, naturally, conventional aggregation operators can be extended for 2 tuples. Therefore, many 2-tuple aggregation operators based on conventional aggregation operators have been proposed in the literature, such as

2-tuple arithmetic mean, 2-tuple weighted average operator, and 2-tuple ordered weighted average operator. See [8] for more details about 2-tuple fuzzy linguistic representation model.

2.2 The Proportional 2-Tuple Fuzzy Linguistic Representation Model

Let $S = \{s_0, s_1, \ldots, s_n\}$ be an ordinal term set with $s_0 < s_1 < \cdots < s_n$, $I = [0, 1]$ and

$$IS \equiv I \times S = \{(\alpha, s_i) | \alpha \in [0, 1] \text{ and } s_i \in S\}.$$

Given a pair (s_i, s_{i+1}) of two successive ordinal terms of S, any two elements (α, s_i), (β, s_{i+1}) of IS are called a symbolic proportion pair and α, β are called a pair of symbolic proportions of the pair (s_i, s_{i+1}) if $\alpha + \beta = 1$. A symbolic proportion pair (α, s_i), $(1-\alpha, s_{i+1})$ is denoted by $(\alpha s_i, (1-\alpha)s_{i+1})$ and the set of all the symbolic proportion pairs is denoted by S^*, i.e., $S^* = \{(\alpha s_i, (1-\alpha)s_{i+1}) | \alpha \in [0, 1] \text{ and } s_i, s_{i+1} \in S\}$. The set S^* is called the ordinal proportional 2-tuple set generated by S and the members of S^* are called ordinal proportional 2-tuples [21].

More particularly, the authors in [21] also introduced the so-called canonical characteristic values (CCV, for short) as an equivalent representation for fuzzy number based semantics of linguistic terms and then developed an efficient method for computing with words based on the proportional 2-tuple fuzzy linguistic model. The CCV of a fuzzy number can be defined by making use of its expected value, center of gravity, or mean of maxima. For example, if the semantics of a linguistic term is simply defined by a symmetrical triangular fuzzy number in [0, 1], denoted by $T = [c-\delta, c, c+\delta]$, then the expected value (EV) of this fuzzy number can be used as CCV, i.e., $EV(T) = c$.

With the notions of proportional 2-tuple and CCV, the computation operator used for transforming a proportional 2-tuple into a numerical value belonging to [0, 1] is defined as follows.

Let $c_i \in [0, 1]$ with $c_0 < c_1 < \cdots < c_n$ be the canonical characteristic values of s_i, i.e., $CCV(s_i) = c_i$ for $i = 0, 1, \ldots, n$. Then, the function CCV on S^* is defined by

$$\begin{aligned} CCV : S^* &\to [0, 1] \\ CCV((\alpha s_i, (1-\alpha)s_{i+1})) &= \alpha CCV(s_i) + (1-\alpha) CCV(s_{i+1}) \\ &= \alpha c_i + (1-\alpha) c_{i+1} \\ &= z \in [0, 1] \end{aligned}$$

and we call it the corresponding canonical characteristic value function on S^* generated by CCV on S. It has been proved by Wang and Hao [21] that the CCV is a bijection from S^* to $[c_0, c_n]$. Specifically, let us define $f : [0, n] \to [c_0, c_n]$ by

$$f(x) = c_i + \beta(c_{i+1} - c_i)$$

where $i = E(x)$, E is the integral part function and $\beta = x - i$. Then f is a bijection. Since,

$$\begin{aligned} CCV(((1-\beta)s_i, \beta s_{i+1})) &= (1-\beta)c_i + \beta c_{i+1} \\ &= c_i + \beta(c_{i+1} - c_i) \\ &= f(i+\beta) \\ &= f(\pi((1-\beta)s_i, \beta s_{i+1})) \end{aligned}$$

for $i = 0, 1, \ldots, n-1, \beta \in [0,1]$, thus $CCV = f \circ \pi$. Here, π is the position index function of ordinal 2-tuples, i.e.,

$$\begin{aligned} \pi : S^* &\to [0,n] \\ \pi((\alpha s_i, (1-\alpha)s_{i+1})) &= i + (1-\alpha) \end{aligned}$$

and its inverse $\pi^{-1} : [0,n] \to S^*$ is defined by

$$\pi^{-1}(x) = ((1-\beta)s_i, \beta s_{i+1}))$$

where $i = E(x)$, E is the integral part function and $\beta = x - i$. So, CCV is a bijection, and its inverse will be denoted by CCV^{-1}.

It is not difficult to see that the role of $\triangle$ with CCV^{-1} and the role of $\triangle^{-1}$ with CCV are the same if we consider it from the perspective of transformation between 2-tuples and numerical values. Therefore, conventional average operators can also be extended easily for proportional 2-tuples. (For more details, see e.g. [21].)

3 Proportional Fuzzy Linguistic Distribution Model

In this section, we will introduce a proportional fuzzy linguistic distribution model for MADM problems. In this model, we use proportions as evaluators' confidence levels indicating their belief degrees that each linguistic term fits an evaluation. Further, since the uncertainty may be assigned not only to any single evaluation grade but also to their rational combinations [28], each attribute can be directly evaluated using subjective judgments with the uncertainty being assigned to any number of adjacent single evaluation grades. Moreover, with introducing a variable representing the extent of ignoring information, the proportional fuzzy linguistic distribution model is capable of dealing with incomplete assessments. Thus, it is not necessary for evaluators having to supply complete assessments when not available. Particularly, in the proportional fuzzy linguistic distribution model, evaluation on an attribute of alternative can be represented by means of proportional fuzzy linguistic distributions defined as follows.

3.1 Proportional Fuzzy Linguistic Distribution

Again, let $S = \{s_0, s_1, \ldots, s_n\}$ be an ordinal term set with $s_0 < s_1 < \cdots < s_n$, $I = [0, 1]$ and

$$IS \equiv I \times S = \{(\alpha, s_i) | \alpha \in [0, 1] \text{ and } s_i \in S\}.$$

Given a sequence $(s_i, s_{i+1}, \ldots, s_{i+m})$ of $(m+1)$ successive ordinal terms of S, any $(m+1)$ elements (α_i, s_i), $(\alpha_{i+1}, s_{i+1}), \ldots, (\alpha_{i+m}, s_{i+m})$ of IS are called a symbolic proportion sequence, and it will be denoted by

$$\begin{cases} (\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, 0) \text{ if } \sum_{j=i}^{i+m} \alpha_j = 1 \\ (\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, \varepsilon) \text{ if } \sum_{j=i}^{i+m} \alpha_j < 1 \end{cases} \tag{1}$$

where ε represents the extent of ignoring information. The set of all the symbolic proportion sequences is denoted by S^*, i.e.,

$$S^* = \left\{ (\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, \varepsilon) \;\middle|\; \begin{array}{l} \alpha_i \in (0, 1], \alpha_{i+m} \in (0, 1], \\ 0 < \sum_{j=i}^{i+m} \alpha_j \leq 1, \\ \varepsilon = 1 - \sum_{j=i}^{i+m} \alpha_j, \\ 0 \leq i, \text{ and } i + m \leq n \end{array} \right\}$$

The set S^* is called proportional fuzzy linguistic distribution set generated by S and the members of S^* are called proportional fuzzy linguistic distributions.

In the sequel, a proportional fuzzy linguistic distribution

$$(\alpha_i s_i,\ \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m},\ \varepsilon)$$

will be used to represent an evaluator's subjective judgment. Here, i is called the starting label; s_i is the No. i linguistic term; α_j is the proportional coefficient in front of the related linguistic term. It represents the confidence levels that to which degree the evaluator believes a linguistic term fits an evaluation. Similarly, $i + m$ is called the ending label.

An assessment $(\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m},\ \varepsilon)$ is called complete (respectively, incomplete) if $\sum_{j=i}^{i+m} \alpha_j = 1$ (respectively, $\sum_{j=i}^{i+m} \alpha_j < 1$). For example, in evaluation of the performance of four types of motorcycle [30] as discussed in Section 5, the following types of uncertain subjective judgments of a motorcycle, say "Honda", are frequently used.

1) The *responsiveness* of engine is evaluated to be *good* with a confidence degree of 1.

2) The *fuel economy* of engine is evaluated to be *indifferent* with a confidence degree of 0.5, and to be *average* with a confidence degree of 0.5.
3) The *quietness* of engine is evaluated to be *good* with a confidence degree of 0.5 and to be *excellent* with a confidence degree of 0.3.
4) The *stopping power* of brake is *good* with a confidence degree of 0.6.

The four assessments 1)–4) given in the above can be represented in the form of proportional fuzzy linguistic distributions defined by (1) as

$S^*(responsiveness) = (1s_3, 0)$
$S^*(fuel\ economy) = (0.5s_1, 0.5s_2, 0)$
$S^*(quietness) = (0.5s_3, 0.3s_4, 0.2)$
$S^*(stopping\ power) = (0.6s_3, 0.4)$

where s_i with i = 1, 2, 3 and 4 are the linguistic terms of the term set S_1 as shown in (14).

It is worth mentioning that we quoted original data from [30] in order to compare the result. Hence, the above four statements used two linguistic terms at most to describe a linguistic variable (e.g. statements 2) and 3)). Actually, evaluators can use the combination of any number of linguistic terms to describe a linguistic variable if they believe it is reasonable in some special situations. For example, assessing the fuel economy of "Honda" based on mountain road, ordinary road and highway, the following uncertain statement perhaps could be used.

The *fuel economy* of Honda is *indifferent* on mountain roads, *average* on ordinary roads, *excellent* on highways, and unclear on the other kinds of roads. Then the uncertain statement can be represented in the form of proportional fuzzy linguistic distribution as:

$S^*(fuel\ economy) = (0.25s_1, 0.25s_2, 0s_3, 0.25s_4, 0.25)$.

3.2 Comparison of Proportional Fuzzy Linguistic Distributions

Let $S = \{s_0, s_1, \ldots, s_n\}$ be an linguistic term set and S^* be the proportional fuzzy linguistic distribution set generated by S. For any two proportional fuzzy linguistic distributions $\Gamma, \Lambda \in S^*$, where

$$\Gamma = (\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, \varepsilon)$$
$$\Lambda = (\beta_g s_g, \beta_{g+1} s_{g+1}, \ldots, \beta_{g+f} s_{g+f}, \varepsilon)$$

their comparison can be described as follows.

1) If ε_1 =0 and $\varepsilon_2 = 0$, then we define

$$\begin{aligned}&(\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, 0) < (\beta_g s_g, \beta_{g+1} s_{g+1}, \ldots, \beta_{g+f} s_{g+f}, 0)\\ &\Leftrightarrow \alpha_i \cdot i + \alpha_{i+1} \cdot (i+1) + \cdots + \alpha_{i+m} \cdot (i+m)\\ &< \beta_g \cdot g + \beta_{g+1} \cdot (g+1) + \cdots + \beta_{g+f} \cdot (g+f)\\ &\Leftrightarrow \sum_{j=i}^{i+m} (\alpha_j \cdot j) < \sum_{k=g}^{g+f} (\beta_k \cdot k). \end{aligned} \tag{2}$$

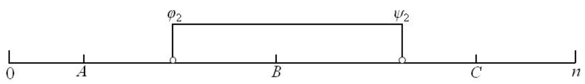

Fig. 1 The relationship between a complete and an incomplete proportional fuzzy linguistic distribution

2) If $\varepsilon_1 = 0$ and $\varepsilon_2 \neq 0$, the latter will generate an interval value $[\varphi, \ \psi]$ because it includes ignoring information. Thus, we need to allocate ε_2 in order to obtain the minimum value φ and maximum value ψ.
   For the minimum value φ, we can consider an extreme situation that ε_2 is allocated to s_0 completely, i.e.,

$$\begin{aligned}\varphi &= \beta_g \cdot g + \ \beta_{g+1} \cdot (g+1) + \cdots + \ \beta_{g+f} \cdot (g+f) + \varepsilon_2 \cdot 0 \\ &= \sum_{k=g}^{g+f} (\beta_k \cdot k). \end{aligned} \tag{3}$$

   For the maximum value ψ, we can consider an extreme situation that ε_2 is allocated to s_n completely, i.e.,

$$\begin{aligned}\psi &= \beta_g \cdot g + \ \beta_{g+1} \cdot (g+1) + \cdots + \ \beta_{g+f} \cdot (g+f) + \varepsilon_2 \cdot n \\ &= \sum_{k=g}^{g+f} (\beta_k \cdot k) + \varepsilon_2 \cdot n. \end{aligned} \tag{4}$$

   Then, the interval value can be represented as

$$\left[\sum_{k=g}^{g+f} (\beta_k \cdot k), \ \sum_{k=g}^{g+f} (\beta_k \cdot k) + \varepsilon_2 \cdot n \right]$$

   and the relationship between the two proportional fuzzy linguistic distributions can be graphically described in Figure 1. A, B and C represent the possible relative locations of the former proportional fuzzy linguistic distribution. $[\varphi_2, \psi_2]$ is the interval value generated by the latter proportional fuzzy linguistic distribution.
3) If $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$, the two proportional fuzzy linguistic distributions will respectively generate interval values $[\varphi_1, \psi_1]$, $[\varphi_2, \psi_2]$. Similarly, the relationships between the two proportional fuzzy linguistic distributions can be graphically described in Figure 2.

3.3 Computational Operators of Proportional Fuzzy Linguistic Distribution

As a 2-tuple based linguistic computational model, the proportional fuzzy linguistic distribution model is a kind of symbolic model and the related calculations can be carried out directly on the labels and proportional coefficients.

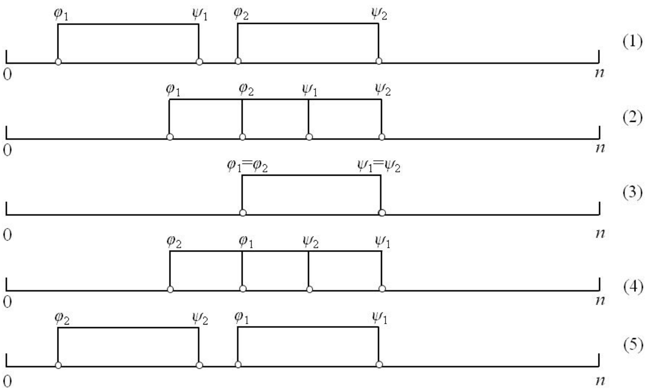

Fig. 2 The relationship between two incomplete proportional fuzzy linguistic distributions

In addition, in this model we can also extend weighted aggregation operators so as to be able to deal with linguistic decision situations where attribute weights are represented not by numerical values but by complete proportional fuzzy linguistic distributions. This can be done by making use of the CCV of complete proportional fuzzy linguistic distributions defined as follows.

Formally, let $S = \{s_0, s_1, \ldots, s_n\}$ be an ordinal term set with $s_0 < s_1 < \cdots < s_n$, and S^* is the proportional fuzzy linguistic distribution set generated by S. Then the CCV of a complete proportional fuzzy linguistic distribution

$$(\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, 0)$$

is defined as follows:

$$\begin{aligned}
&CCV(\alpha_i s_i, \alpha_{i+1} s_{i+1}, \ldots, \alpha_{i+m} s_{i+m}, 0) = \\
&= (\alpha_i CCV(s_i), \alpha_{i+1} CCV(s_{i+1}), \ldots, \alpha_{i+m} CCV(s_{i+m}), 0) \\
&= (\alpha_i c_i, \alpha_{i+1} c_{i+1}, \ldots \alpha_{i+m} c_{i+m}, 0) \\
&= z_i + z_{i+1} + \cdots + z_{i+m} \\
&= \sum_{j=i}^{i+m} z_j \qquad (5)
\end{aligned}$$

with $j = i, i+1, \ldots, i+m$. We call it the corresponding canonical characteristic value function on S^* generated by CCV on S, where $c_i < c_{i+1} < \cdots < c_{i+m} \in [0, 1]$ are the CCV of $s_i, s_{i+1}, \ldots, s_{i+m}$ respectively.

3.4 Expected Utility in Proportional Fuzzy Linguistic Distribution

The concept of expected utility has been widely used in decision making under uncertainty. Basically, given a set of alternatives X, the preference relation

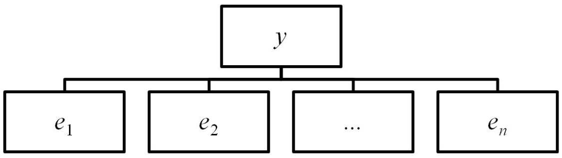

Fig. 3 Two level hierarchy

on X can be characterized by a single-valued function $u(x)$, called expected utility, such that for any $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Then, maximization of $u(x)$ will yield the solution to the problem of selecting the best alternative from X.

When the proportional fuzzy linguistic distribution model is used for MADM with uncertain linguistic information, the aggregated assessment for each alternative is represented by a proportional fuzzy linguistic distribution. In order to define a ranking order among alternatives based on their aggregated assessments, we can also use the concept of expected utility in a similar way as discussed in the ER approach for MADM [30].

Particularly, in the proportional fuzzy linguistic distribution model we assume a utility function

$$u' : S \to [0, 1]$$

satisfying

$$u'(s_{i+1}) > u'(s_i), \text{ if } s_{i+1} \text{ is preferred to } s_i.$$

This utility function $u'(x)$ can be estimated using the probability assignment method [12,22] or other methods as discussed in [29,30].

For simplicity, suppose that the hierarchy of attributes has two levels with only an attribute y on the top level, and a finite set $E = \{e_1, e_2, \ldots, e_n\}$ of its basic attributes at the bottom level, as shown in Figure 3. If all assessments for basic attributes are complete, i.e., $\sum_{j=i}^{i+m} \alpha_j = 1$, or $\varepsilon = 0$, then, the expected utility of an alternative on the only attribute y is defined by

$$u(y) = \sum_{j=i}^{i+m} \alpha_j u'(s_j). \tag{6}$$

Then, for any alternatives a and b, we say that a is strictly preferred to b if and only if $u(y(a)) > u(y(b))$.

If any assessment for the basic attribute is incomplete, then the assessment for y is also incomplete, i.e., $\sum_{j=i}^{i+m} \alpha_j < 1$, or $\varepsilon > 0$. In such case, the confidence interval $[\alpha_j, (\alpha_j + \varepsilon)]$ provides the range of the likelihood to which y may be assessed to the evaluation grades [30]. Without loss of generality, assume that s_0 is the least preferred grade having the lowest utility and s_n is the most preferred grade having the highest utility. Then, the maximum, minimum and average expected utilities on y of an alternative in proportional fuzzy linguistic distribution model are given by

$$u_{\max}(y) = \sum_{j=0}^{n-1} \alpha_j u'(s_j) + (\alpha_n + \varepsilon) u'(s_n) \tag{7}$$

$$u_{\min}(y) = (\alpha_0 + \varepsilon) u'(s_0) + \sum_{j=1}^{n} \alpha_j u'(s_j) \tag{8}$$

$$u_{\text{avg}}(y) = \frac{u_{\max}(y) + u_{\min}(y)}{2}. \tag{9}$$

If all original assessments are complete, then $\varepsilon = 0$, and $u(y) = u_{\max}(y) = u_{\min}(y) = u_{\text{avg}}(y)$. If the original assessments include incomplete information, then the ranking of two alternatives a and b on y is based on their utility intervals by

– $a \succ_y b$ if and only if $u_{\min}(y(a)) > u_{\max}(y(b))$
– $a \sim_y b$ if and only if $u_{\min}(y(a)) = u_{\min}(y(b))$ and $u_{\max}(y(a)) = u_{\max}(y(b))$

Otherwise, the average expected utility can be used to generate a ranking, i.e.,

– $a \succ_y b$ on an average basis, if $u_{\text{avg}}(y(a)) > u_{\text{avg}}(y(b))$.

Note that the ranking order based on the average expected utility may be questionable in some situation. For example, it may happen that $u_{\text{avg}}(y(a)) > u_{\text{avg}}(y(b))$, but $u_{\max}(y(b)) > u_{\min}(y(b)) > u_{\min}(y(a))$.

Alternatively, in a similar way of applying the so-called pignistic transformation [19] to induce a probability function from an incomplete aggregated assessment as done in [11], if the overall aggregated assessment of an alternative is incomplete, we will first transform it to a complete proportional fuzzy linguistic distribution by uniformly distributing the degree of ignoring to any individual evaluation grades, and then compute the expected utility of the obtained complete proportional fuzzy linguistic distribution for decision making. Specifically, we have the expected utility of alternative on the only attribute y defined as follows

$$u(y) = \sum_{j=i}^{i+m} \left(\alpha_j + \frac{\varepsilon}{m+1} \right) u'(s_j) \tag{10}$$

4 Proportional Fuzzy Linguistic Distribution Aggregation Operators

In MADM problems one usually needs to aggregate different information of attributes so as to obtain an integrated value that represents an overall evaluation of alternative. In the proportional fuzzy linguistic distribution model, attribute aggregation of proportional fuzzy linguistic distributions will result in a proportional fuzzy linguistic distribution as well. In this section, we are going to introduce several proportional fuzzy linguistic distribution aggregation operators based on conventional aggregation operators and CCV function defined above.

4.1 Weighted average operator

Typically, attributes in a MADM problem may have different weights that represent their relative important contributing to overall evaluation of alternatives. Also, the weighted average operator is widely used in many situations of MADM. In this section we will introduce an extended version of the weighted average operator in the context of MADM with uncertain linguistic information based on the proportional fuzzy linguistic distribution model.

Let

$$\mathbf{X} = \left\{ \begin{array}{l} (\alpha^1_{i_1} s_{i_1}, \alpha^1_{i_1+1} s_{i_1+1}, \ldots, \alpha^1_{i_1+m_1} s_{i_1+m_1}, \varepsilon_1), \\ (\alpha^2_{i_2} s_{i_2}, \alpha^2_{i_2+1} s_{i_2+1}, \ldots, \alpha^2_{i_2+m_2} s_{i_2+m_2}, \varepsilon_2), \\ \cdots, \\ (\alpha^p_{i_p} s_{i_p}, \alpha^p_{i_p+1} s_{i_p+1}, \ldots, \alpha^p_{i_p+m_p} s_{i_p+m_p}, \varepsilon_p) \end{array} \right\}$$

be a set of proportional fuzzy linguistic distributions, and $\mathbf{W} = \{\omega_1, \omega_2, \ldots, \omega_p\}$ be their associated weights, respectively. Then, the weighted average of proportional fuzzy linguistic distributions in $\mathbf{X}$, which is also a proportional fuzzy linguistic distribution denoted by $(\delta_i s_i, \delta_{i+1} s_{i+1}, \ldots, \delta_{i+m} s_{i+m}, \bar{\varepsilon})$, is defined as follows.

1) Let i be the minimum index of the starting labels of proportional fuzzy linguistic distributions in $\mathbf{X}$, i.e., $i = \min\{i_1, i_2, \ldots, i_p\}$.
2) Let $(i+m)$ be the maximum index of the ending labels of proportional fuzzy linguistic distributions in $\mathbf{X}$, i.e., $i+m = \max\{i_1 + m_1, i_2 + m_2, \ldots, i_p + m_p\}$.
3) Without loss of generality, proportional fuzzy linguistic distributions in $\mathbf{X}$ can be represented such that they have the same starting labels and ending labels as follows

$$\mathbf{X} = \left\{ \begin{array}{l} (\alpha^1_{i} s_{i}, \alpha^1_{i+1} s_{i+1}, \ldots, \alpha^1_{i+m} s_{i+m}, \varepsilon_1), \\ (\alpha^2_{i} s_{i}, \alpha^2_{i+1} s_{i+1}, \ldots, \alpha^2_{i+m} s_{i+m}, \varepsilon_2), \\ \cdots, \\ (\alpha^p_{i} s_{i}, \alpha^p_{i+1} s_{i+1}, \ldots, \alpha^p_{i+m} s_{i+m}, \varepsilon_p) \end{array} \right\}$$

   where

$$\begin{array}{l} \alpha^1_k = 0, \text{ for } k < i_1; k > (i_1 + m_1) \\ \alpha^2_k = 0, \text{ for } k < i_2; k > (i_2 + m_2) \\ \ldots, \\ \alpha^p_k = 0, \text{ for } k < i_p; k > (i_p + m_p) \end{array}$$

4) Then, the weighted average $(\delta_i s_i, \delta_{i+1} s_{i+1}, \ldots, \delta_{i+m} s_{i+m}, \bar{\varepsilon})$ is given by

$$\left\{ \begin{array}{rl} \delta_i s_i &= \left(\dfrac{\sum_{k=1}^{p} \alpha^k_i \cdot \omega_k}{\sum_{k=1}^{p} \omega_k} \right) s_i \\ &\vdots \\ \delta_{i+m} s_{i+m} &= \left(\dfrac{\sum_{k=1}^{p} \alpha^k_{i+m} \cdot \omega_k}{\sum_{k=1}^{p} \omega_k} \right) s_{i+m} \\ \bar{\varepsilon} &= \dfrac{\sum_{k=1}^{p} \varepsilon_k \cdot \omega_k}{\sum_{k=1}^{p} \omega_k} \end{array} \right. \tag{11}$$

4.2 Linguistic weighted average operator

In addition, the weighted average operator for proportional fuzzy linguistic distributions defined above can be also extended for the case where the weights are expressed by means of uncertain linguistic weights instead of numerical values. In particular, let

$$\mathbf{X} = \left\{ \begin{array}{l} (\alpha^1_{i_1} s_{i_1}, \alpha^1_{i_1+1} s_{i_1+1}, \ldots, \alpha^1_{i_1+m_1} s_{i_1+m_1}, \varepsilon_1), \\ (\alpha^2_{i_2} s_{i_2}, \alpha^2_{i_2+1} s_{i_2+1}, \ldots, \alpha^2_{i_2+m_2} s_{i_2+m_2}, \varepsilon_2), \\ \cdots, \\ (\alpha^p_{i_p} s_{i_p}, \alpha^p_{i_p+1} s_{i_p+1}, \ldots, \alpha^p_{i_p+m_p} s_{i_p+m_p}, \varepsilon_p) \end{array} \right\}$$

be a set of proportional fuzzy linguistic distributions, and

$$\mathbf{W} = \left\{ \begin{array}{l} W_1 = (\beta^1_{j_1} \omega_{j_1}, \beta^1_{j_1+1} \omega_{j_1+1}, \ldots, \beta^1_{j_1+n_1} \omega_{j_1+n_1}, 0), \\ W_2 = (\beta^2_{j_2} \omega_{j_2}, \beta^2_{j_2+1} \omega_{j_2+1}, \ldots, \beta^2_{j_2+n_2} \omega_{j_2+n_2}, 0), \\ \cdots, \\ W_p = (\beta^p_{j_p} \omega_{j_p}, \beta^p_{j_p+1} \omega_{j_p+2}, \ldots, \beta^p_{j_p+n_p} \omega_{j_p+n_p}, 0) \end{array} \right\}$$

be the set of their associated linguistic weights which are also represented in the form of complete proportional fuzzy linguistic distributions on the term set of linguistic weights. Then, by a similar procedure as for weighted average operator above, the linguistic weighted average of proportional fuzzy linguistic distributions in $\mathbf{X}$ is defined as follows.

1) Let i be the minimum index of the starting labels of proportional fuzzy linguistic distributions in $\mathbf{X}$, i.e., $i = \min\{i_1, i_2, \ldots, i_p\}$.
2) Let $(i+m)$ be the maximum index of the ending labels of proportional fuzzy linguistic distributions in $\mathbf{X}$, i.e., $i+m = \max\{i_1 + m_1, i_2 + m_2, \ldots, i_p + m_p\}$.
3) Representing proportional fuzzy linguistic distributions in $\mathbf{X}$ such that they have the same starting labels and ending labels as follows

$$\mathbf{X} = \left\{ \begin{array}{l} (\alpha^1_i s_i, \alpha^1_{i+1} s_{i+1}, \ldots, \alpha^1_{i+m} s_{i+m}, \varepsilon_1), \\ (\alpha^2_i s_i, \alpha^2_{i+1} s_{i+1}, \ldots, \alpha^2_{i+m} s_{i+m}, \varepsilon_2), \\ \cdots, \\ (\alpha^p_i s_i, \alpha^p_{i+1} s_{i+1}, \ldots, \alpha^p_{i+m} s_{i+m}, \varepsilon_p) \end{array} \right\}$$

   where

$$\begin{array}{l} \alpha^1_k = 0, \text{ for } k < i_1; k > (i_1 + m_1) \\ \alpha^2_k = 0, \text{ for } k < i_2; k > (i_2 + m_2) \\ \ldots, \\ \alpha^p_k = 0, \text{ for } k < i_p; k > (i_p + m_p) \end{array}$$

4) Transform linguistic weights represented by complete proportional fuzzy linguistic distributions into numerical weights by means of the CCV oper-

ator as defined by (5), i.e.,

$$\begin{cases} CCV(\beta^1_{j_1}\omega_{j_1}, \beta^1_{j_1+1}\omega_{j_1+1}, \ldots, \beta^1_{j_1+m_1}\omega_{j_1+m_1}, 0) = \sum_{j=j_1}^{j_1+m_1} \beta^1_j \cdot CCV(\omega_j) = Z_1 \\ \vdots \\ CCV(\beta^p_{j_p}\omega_{j_p}, \beta^p_{j_p+1}\omega_{j_p+2}, \ldots, \beta^p_{j_p+m_p}\omega_{j_p+m_p}, 0) = \sum_{j=j_p}^{j_p+m_p} \beta^p_j \cdot CCV(\omega_j) = Z_p \end{cases} \tag{12}$$

where Z_k, $k = 1, \ldots, p$, is the numerical value transformed by CCV over linguistic weight W_k, $k = 1, \ldots, p$.

5) Then, the linguistic weighted average, denoted by $(\delta_i s_i, \delta_{i+1} s_{i+1}, \ldots, \delta_{i+m} s_{i+m}, \bar{\varepsilon})$, of proportional fuzzy linguistic distributions in $\mathbf{X}$ is given by

$$\begin{cases} \delta_i s_i = \left(\frac{\sum_{k=1}^p \alpha_i^k \cdot Z_k}{\sum_{k=1}^p Z_k} \right) s_i \\ \vdots \\ \delta_{i+m} s_{i+m} = \left(\frac{\sum_{k=1}^p (\alpha_{i+m}^k \cdot Z_k}{\sum_{k=1}^p Z_k} \right) s_{i+m} \\ \bar{\varepsilon} = \frac{\sum_{k=1}^p \varepsilon_k \cdot Z_k}{\sum_{k=1}^p Z_k} \end{cases} \tag{13}$$

In the following we will illustrate how these aggregation operators can be used for attribute aggregation in a MADM problem with uncertain linguistic information.

5 Example: Motorcycle Assessment Problem

In this section, we apply the proportional fuzzy linguistic distribution model to deal with a MADM problem taken from [30]. We first use the original data including distinct evaluation grades and weights for a comparative study. Then, instead of using the numerical weights we suppose a set of linguistic weights represented by complete proportional fuzzy linguistic distributions in order to further explain the capability of handling uncertain weighting information of the proposed model.

5.1 Motorcycle Assessment Problem

The problem is to evaluate the performances of four types of motorcycle, namely, Kawasaki, Yamaha, Honda, and BMW. Therefore, we have to know the overall performance of each motorcycle. The overall performance of each motorcycle is based on evaluating three major qualitative attributes, which

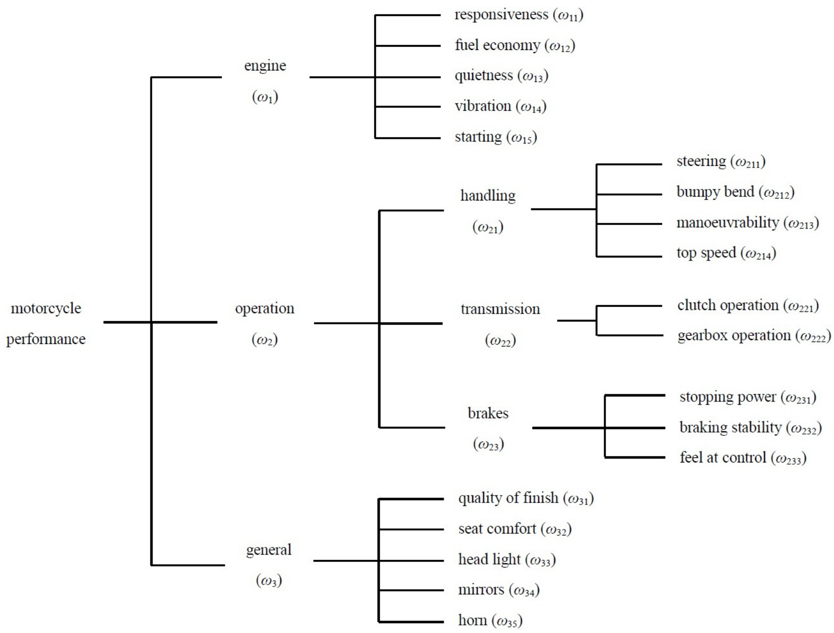

Fig. 4 Evaluation hierarchy for motorcycle performance assessment [30]

are quality of engine, operation, and general finish, although quantitative attributes may also be included [27], [29]. Because these attributes are general and difficult to assess directly, they need to be decomposed into more detailed sub-attributes to facilitate the assessment. As a result of decomposition, an attribute hierarchy for evaluation of motorcycles is graphically depicted in Figure 4, where ω_i, ω_{ij} and ω_{ijk} are the weights of corresponding attributes at level 1, level 2, and level 3 respectively.

It is essential to define linguistic term set and associated semantics to supply evaluator with an instrument to assess the attributes of the operation of a motorcycle naturally. In this paper, we take the same linguistic term set of distinct evaluation grades as in [30], which is defined as

$$S_1 = \left\{s_0^1(\text{Poor}), s_1^1(\text{Indifferent}), s_2^1(\text{Average}), s_3^1(\text{Good}), s_4^1(\text{Excellent})\right\} \quad (14)$$

The subjective judgments on attributes of motorcycle are summarized in Table 1, where P, I, A, G, E are the abbreviations of the evaluation grades of *Poor*, *Indifferent*, *Average*, *Good*, and *Excellent*, respectively, and a number in a bracket denotes a degree of belief to which an attribute is assessed to an evaluation grade.

For the purpose of comparing the final result, all relevant attributes are assumed to be equal relative importance as done in [30], i.e.

$$\begin{gathered}\omega_1 = \omega_2 = \omega_3 = 0.333 \\ \omega_{11} = \omega_{12} = \omega_{13} = \omega_{14} = \omega_{15} = 0.2 \\ \omega_{21} = \omega_{22} = \omega_{23} = 0.333 \\ \omega_{211} = \omega_{212} = \omega_{213} = \omega_{214} = 0.25 \\ \omega_{221} = \omega_{222} = \ 0.5 \\ \omega_{231} = \omega_{232} = \omega_{233} = 0.333 \\ \omega_{31} = \omega_{32} = \omega_{33} = \omega_{34} = \omega_{35} = 0.2.\end{gathered}$$

5.2 Aggregating Assessments via Proportional Fuzzy Linguistic Distribution Model

After the evaluator supplies all the subjective judgments for qualitative attributes, the evaluation procedure based on proportional fuzzy linguistic distribution model will be carried out as described in the following.

1) Proportional fuzzy linguistic distributions transformation: According to the linguistic term set of distinct evaluation grades, the original linguistic assessments shown in Table 1 should be converted into corresponding proportional fuzzy linguistic distributions by using symbolic translation value of $s_i, i = 0, 1, \ldots, 4$ and the statements with the associated representation method such as 1)-4) discussed in Section 3. The general decision matrix for motorcycle assessment represented by proportional fuzzy linguistic distributions is shown in Table 2, where s_0, s_1, s_2, s_3, and s_4 are the expressions of *Poor*, *Indifferent*, *Average*, *Good*, and *Excellent*, respectively, and the numerical coefficients in front of s_0, s_1, s_2, s_3, and s_4 denote the confidence levels to which degree an attribute is assessed to a grade.

Table 1 Generalized decision matrix for motorcycle assessment [30]

General attributes		Basic attributes	Type of motorcycle (alternatives) *Kawasaki* (a_1)	*Yamaha* (a_2)	*Honda* (a_3)	*BMW* (a_4)
engine ω_1		responsiveness ω_{11}	E (0.8)	G (0.3), E (0.6)	G (1.0)	I (1.0)
		fuel economy ω_{12}	A (1.0)	I (1.0)	I (0.5), A (0.5)	E (1.0)
		quietness ω_{13}	I (0.5), A (0.5)	A (1.0)	G (0.5), E (0.3)	E (1.0)
		vibration ω_{14}	G (1.0)	I (1.0)	G (0.5), E (0.5)	P (1.0)
		starting ω_{15}	G (1.0)	A (0.6), G (0.3)	G (1.0)	A (1.0)
operation ω_2	handling ω_{21}	steering ω_{211}	E (0.9)	G (1.0)	A (1.0)	A (0.6)
		bumpy bends ω_{212}	A (0.5), G (0.5)	G (1.0)	G (0.8), E (0.1)	P (0.5), I (0.5)
		maneuverability ω_{213}	A (1.0)	E (0.9)	I (1.0)	P (1.0)
		top speed stability ω_{214}	E (1.0)	G (1.0)	G (1.0)	G (0.6), E (0.4)
	transmission ω_{22}	clutch operation ω_{221}	A (0.8)	G (1.0)	E (0.85)	I (0.2), A (0.8)
		gearbox operation ω_{222}	A (0.5), G (0.5)	I (0.5), A (0.5)	E (1.0)	P (1.0)
	brakes ω_{23}	stopping power ω_{231}	G (1.0)	A (0.3), G (0.6)	G (0.6)	E (1.0)
		braking stability ω_{232}	G (0.5), E (0.5)	G (1.0)	A (0.5), G (0.5)	E (1.0)
		feel at control ω_{233}	P (1.0)	G (0.5), E (0.5)	G (1.0)	G (0.5), E (0.5)
general ω_3		quality of finish ω_{31}	P (0.5), I (0.5)	G (1.0)	E (1.0)	G (0.5), E (0.5)
		seat comfort ω_{32}	G (1.0)	G (0.5), E (0.5)	G (0.6)	E (1.0)
		headlight ω_{33}	G (1.0)	A (1.0)	E (1.0)	G (0.5), E (0.5)
		mirrors ω_{34}	A (0.5), G (0.5)	G (0.5), E (0.5)	E (1.0)	G (1.0)
		horn ω_{35}	A (1.0)	G (1.0)	G (0.5), E (0.5)	E (1.0)

Table 2 Generalized decision matrix for motorcycle assessment represented by proportional fuzzy linguistic distributions

General attributes		Basic attributes	Type of motorcycle (alternatives)			
			Kawasaki (a_1)	*Yamaha* (a_2)	*Honda* (a_3)	*BMW* (a_4)
		responsiveness ω_{11}	$(0.8s_4, 0.2)$	$(0.3s_3, 0.6s_4, 0.1)$	$(1s_3, 0)$	$(1s_1, 0)$
		fuel economy ω_{12}	$(1s_2, 0)$	$(1s_1, 0)$	$(0.5s_1, 0.5s2, 0)$	$(1s_4, 0)$
engine ω_1		quietness ω_{13}	$(0.5s_1, 0.5s_2, 0)$	$(1s_2, 0)$	$(0.5s_3, 0.3s_4, 0.2)$	$(1s_4, 0)$
		vibration ω_{14}	$(1s_3, 0)$	$(1s_1, 0)$	$(0.5s_3, 0.5s_4, 0)$	$(1s_0, 0)$
		starting ω_{15}	$(1s_3, 0)$	$(0.6s_2, 0.3s_3, 0.1)$	$(1s_3, 0)$	$(1s_2, 0)$
operation ω_2	handling ω_{21}	steering ω_{211}	$(0.9s_4, 0.1)$	$(1s_3, 0)$	$(1s_2, 0)$	$(0.6s_2, 0.4)$
		bumpy bends ω_{212}	$(0.5s_2, 0.5s_3, 0)$	$(1s_3, 0)$	$(0.8s_3, 0.1s_4, 0.1)$	$(0.5s_0, 0.5s_1, 0)$
		maneuverability ω_{213}	$(1s_2, 0)$	$(0.9s_4, 0.1)$	$(1s_1, 0)$	$(1s_0, 0)$
		top speed stability ω_{214}	$(1s_4, 0)$	$(1s_3, 0)$	$(1s_3, 0)$	$(0.6s_3, 0.4s_4, 0)$
	transmission ω_{22}	clutch operation ω_{221}	$(0.8s_2, 0.2)$	$(1s_3, 0)$	$(0.85s_4, 0.15)$	$(0.2s_1, 0.8s_2, 0)$
		gearbox operation ω_{222}	$(0.5s_2, 0.5s_3, 0)$	$(0.5s_1, 0.5s_2, 0)$	$(1s_4, 0)$	$(1s_0, 0)$
	brakes ω_{23}	stopping power ω_{231}	$(1s_3, 0)$	$(0.3s_2, 0.6s_3, 0.1)$	$(0.6s_3, 0.4)$	$(1s_4, 0)$
		braking stability ω_{232}	$(0.5s_3, 0.5s_4, 0)$	$(1s_3, 0)$	$(0.5s_2, 0.5s_3, 0)$	$(1s_4, 0)$
		feel at control ω_{233}	$(1s_0, 0)$	$(0.5s_3, 0.5s_4, 0)$	$(1s_3, 0)$	$(0.5s_3, 0.5s_4, 0)$
		quality of finish ω_{31}	$(0.5s_0, 0.5s_1, 0)$	$(1s_3, 0)$	$(1s_4, 0)$	$(0.5s_3, 0.5s_4, 0)$
		seat comfort ω_{32}	$(1s_3, 0)$	$(0.5s_3, 0.5s_4, 0)$	$(0.6s_3, 0.4)$	$(1s_4, 0)$
general ω_3		headlight ω_{33}	$(1s_3, 0)$	$(1s_2, 0)$	$(1s_4, 0)$	$(0.5s_3, 0.5s_4, 0)$
		mirrors ω_{34}	$(0.5s_2, 0.5s_3, 0)$	$(0.5s_3, 0.5s_4, 0)$	$(1s_4, 0)$	$(1s_3, 0)$
		horn ω_{35}	$(1s_2, 0)$	$(1s_3, 0)$	$(0.5s_3, 0.5s4, 0)$	$(1s_4, 0)$

2) Proportional fuzzy linguistic distributions computation and aggregation: The attribute aggregation process will then be carried out in a bottom-up fashion. That is, for each alternative we first aggregate the third level attributes via (11), and then the aggregated results obtained for the second level attributes are aggregated in the same way. Finally, we aggregate the results for the first level attributes in order to obtain an overall proportional fuzzy linguistic distribution for the alternative. By this way, we can get the final result of the overall performances of four types of motorcycle represented by proportional fuzzy linguistic distributions as shown in Table 3.

Table 3 The overall performances represented by proportional fuzzy linguistic distributions

	The overall performances
Kawasaki	$(0.07s_0, 0.066s_1, 0.314s_2, 0.398s_3, 0.125s_4, 0.027)$
Yamaha	$(0s_0, 0.16s_1, 0.213s_2, 0.457s_3, 0.151s_4, 0.019)$
Honda	$(0s_0, 0.061s_1, 0.079s_2, 0.401s_3, 0.393s_4, 0.066)$
BMW	$(0.164s_0, 0.092s_1, 0.128s_2, 0.168s_3, 0.437s_4, 0.011)$

3) Proportional fuzzy linguistic distributions conversion: Convert the overall value of performances of four types of motorcycle represented by proportional fuzzy linguistic distributions into the corresponding linguistic terms of distinct evaluation grades, which are shown in Table 4. The distributed assessments on the four types of motorcycle can be shown graphically as in Figure 5.

Table 4 Distributed assessments on four types of motorcycle

	Poor (P)	Indifferent (I)	Average (A)	Good (G)	Excellent (E)	ε
Kawasaki	0.07	0.066	0.314	0.398	0.125	0.027
Yamaha	0	0.16	0.213	0.457	0.151	0.019
Honda	0	0.061	0.079	0.401	0.393	0.066
BMW	0.164	0.092	0.128	0.168	0.437	0.011

5.3 Computing the Expected Utilities of Four Types of Motorcycle

Now, in order for selecting the best motorcycle, the expected utilities of four types of motorcycle should be calculated. Taking the utility function $u' : S_1 \rightarrow [0,1]$ as in [30] that is defined by

$$u'(P) = 0, u'(I) = 0.35, u'(A) = 0.55,$$
$$u'(G) = 0.85, u'(E) = 1$$

then, using (7), (8) and (9), we easily obtain the expected utilities of four types of motorcycle as shown in Table 5. In Table 5, we can find that the minimum

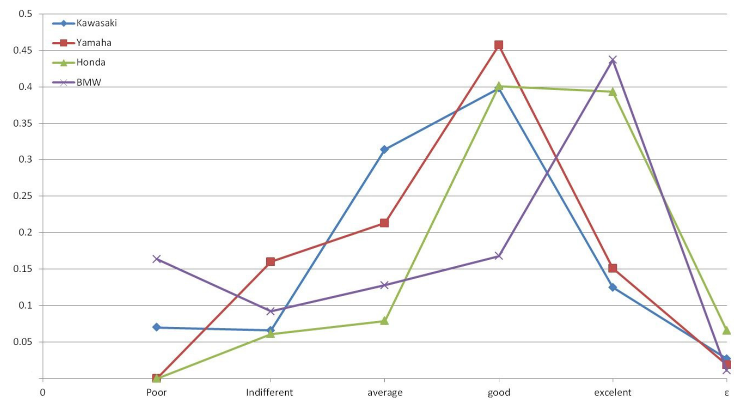

Fig. 5 The distributed assessments on the four types of motorcycle

utility of Honda is larger than the maximum utilities of the other three types of motorcycle. Hence, according to the ranking principle of expected utility, Honda is ranked first among the four types of motorcycle. Similarly, Yamaha is ranked second as its minimum utility is larger than the maximum utilities of the remaining two. Finally, as the average utility of BMW is larger than that of Kawasaki, it is ranked third and Kawasaki is the last. Eventually, the ranking of the four types of motorcycle is given by

$$\text{Honda} \succ \text{Yamaha} \succ \text{BMW} \succ \text{Kawasaki}$$

Table 5 The expected utilities of four types of motorcycle

	Maximum utility	Minimum utility	Average utility
Kawasaki	0.6861	0.6591	0.6726
Yamaha	0.7316	0.7126	0.7221
Honda	0.86465	0.79865	0.83165
BMW	0.6934	0.6824	0.6879

It is clear there is not so much difference between the results obtained by the proportional fuzzy linguistic distribution model and those obtained by the modified ER algorithm [30]. The ranking order of the four types of motorcycle is the same. However, it would be worth noticing that, because the proportional fuzzy linguistic distributions employ weighted average operator to aggregate multiple attribute, it clearly has a linear behavior. The modified ER method exhibits a quasi-linear behavior with equal weights and strongly nonlinear behavior with unequal weights [11].

5.4 Motorcycle Assessment Problem with Linguistic Weights

In order to further explain the capability of handling uncertain weighting information of the proposed model, we now assume that the weights of attributes are given linguistically in form of complete proportional fuzzy linguistic distributions. To this end, a distinctive evaluation set providing standards for assessing the relative importance of attributes needs to be defined first.

Specifically, let S_2 be a linguistic term set which is used to linguistically evaluate the relative importance of different attributes,

$$\begin{aligned} S_2 = \{ & s_0^2 \text{ (Very Low)}, s_1^2\text{(Low)}, s_2^2\text{(Fairly Low)}, \\ & s_3^2 \text{ (Fairly High)}, s_4^2\text{(High)}, s_5^2\text{(Very High)}\} \end{aligned} \tag{15}$$

and the associated fuzzy set semantics is shown in Figure 6.

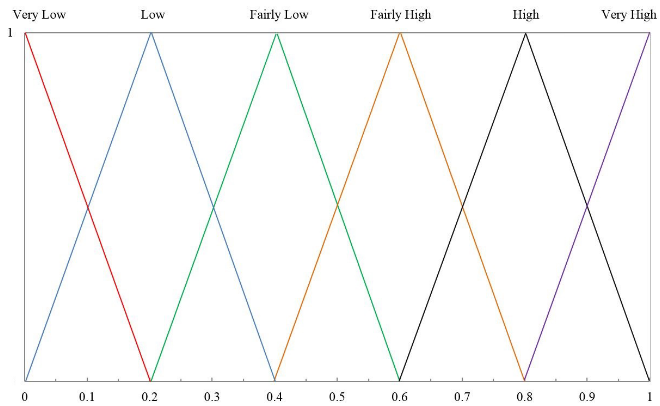

Fig. 6 Linguistic weights and associated fuzzy number semantics

According to the linguistic weight term set and associated fuzzy number semantics, the evaluator gives the relative importance of different attributes of each motorcycle by means of proportional fuzzy linguistic distributions as shown in Table 6, where s_0, s_1, s_2, s_3, s_4 and s_5 are the expressions of *Very Low*, *Low*, *Fairly Low*, *Fairly High*, *High* and *Very High*, respectively.

Followed by the aggregation procedure of proportional fuzzy linguistic distribution model, the final results of the overall performances of four types of motorcycle represented by proportional fuzzy linguistic distributions can be obtained via (12) and (13), as shown in Table 7, which is then converted into the corresponding linguistic terms of distinct evaluation grades, as shown in Table 8 and graphically depicted in Figure 7. Then, using the same utilities of five individual evaluation grades mentioned above, the expected utilities of four types of motorcycle can be obtained via (7), (8) and (9), which are shown in Table 9. Similarly, according to the ranking principle of expected utility, Honda is still the most preferred among the four types of motorcycle, and the ranking of the four types of motorcycle is given by

Table 6 Linguistic weights represented by proportional fuzzy linguistic distributions

Attributes	The third level	Linguistic weights
handling ω_{21}	steering ω_{211}	$(0.4s_3, 0.6s_4, 0)$
	bumpy bends ω_{212}	$(1s_4, 0)$
	maneuverability ω_{213}	$(0.5s_4, 0.5s_5, 0)$
	top speed stability ω_{214}	$(1s_3, 0)$
transmission ω_{22}	clutch operation ω_{221}	$(0.5s_3, 0.5s_4, 0)$
	gearbox operation ω_{222}	$(0.5s_3, 0.5s_4, 0)$
brakes ω_{23}	stopping power ω_{231}	$(1s_5, 0)$
	braking stability ω_{232}	$(1s_5, 0)$
	feel at control ω_{233}	$(0.5s_4, 0.5s_5, 0)$
Attributes	The second level	Linguistic weights
engine ω_1	responsiveness ω_{11}	$(0.4s_4, 0.6s_5, 0)$
	fuel economy ω_{12}	$(0.3s_3, 0.3s_4, 0.4s_5, 0)$
	quietness ω_{13}	$(1s_3, 0)$
	vibration ω_{14}	$(1s_3, 0)$
	starting ω_{15}	$(0.5s_3, 0.5s_4, 0)$
operation ω_2	handling ω_{21}	$(0.4s_4, 0.6s_5, 0)$
	transmission ω_{22}	$(1s_5, 0)$
	brakes ω_{23}	$(1s_5, 0)$
general ω_3	quality of finish ω_{31}	$(1s_3, 0)$
	seat comfort ω_{32}	$(0.6s_1, 0.4s_2, 0)$
	headlight ω_{33}	$(0.2s_3, 0.8s_4, 0)$
	mirrorsω_{34}	$(1s_2, 0)$
	horn ω_{35}	$(1s_2, 0)$
Attributes	The first level	Linguistic weights
overall performance	engine ω_1	$(0.4s_4, 0.6s_5, 0)$
	operation ω_2	$(0.4s_4, 0.6s_5, 0)$
	general ω_3	$(1s_4, 0)$

Table 7 The overall performances represented by proportional fuzzy linguistic distributions by using linguistic weights

	The overall performances
Kawasaki	$(0.074s_0, 0.066s_1, 0.307s_2, 0.384s_3, 0.136s_4, 0.033)$
Yamaha	$(0s_0, 0.166s_1, 0.234s_2, 0.434s_3, 0.143s_4, 0.023)$
Honda	$(0s_0, 0.072s_1, 0.086s_2, 0.386s_3, 0.402s_4, 0.054)$
BMW	$(0.164s_0, 0.115s_1, 0.13s_2, 0.166s_3, 0.414s_4, 0.011)$

Table 8 Distributed assessments on four types of motorcycle by using linguistic weights

	Poor (P)	Indifferent (I)	Average (A)	Good (G)	Excellent (E)	ε
Kawasaki	0.074	0.066	0.307	0.384	0.136	0.033
Yamaha	0	0.166	0.234	0.434	0.143	0.023
Honda	0	0.072	0.086	0.386	0.402	0.054
BMW	0.164	0.115	0.13	0.166	0.414	0.011

Table 9 The expected utilities of four types of motorcycle by using linguistic weights

	Maximum utility	Minimum utility	Average utility
Kawasaki	0.68735	0.65435	0.67085
Yamaha	0.7217	0.6987	0.7102
Honda	0.8566	0.8026	0.8296
BMW	0.67785	0.66685	0.67235

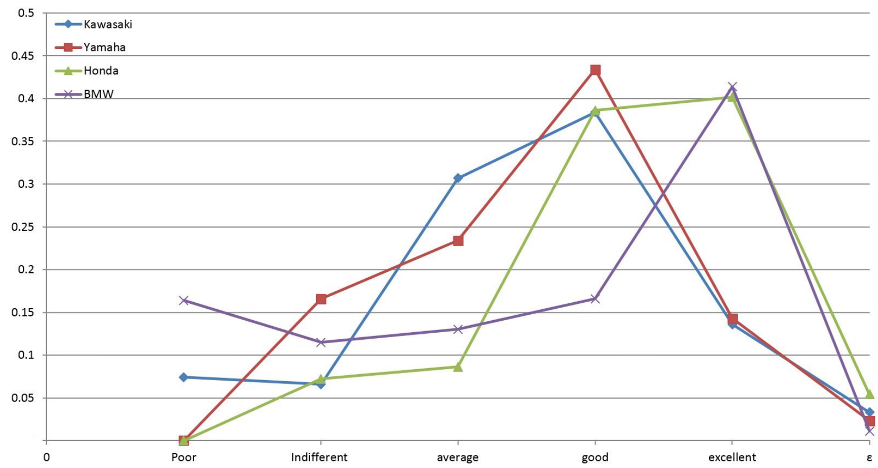

Fig. 7 The distributed assessments on the four types of motorcycle by using linguistic weights

$$\text{Honda} \succ \text{Yamaha} \succ \text{BMW} \succ \text{Kawasaki}.$$

As we can observe, the overall assessments of four types of motorcycle obtained by the linguistic weighted average operator as shown in Fig. 7 have almost the same shape as those obtained by the weighted average operator depicted in Fig. 5. Note that linguistic weights are generated in such a way that their CCVs (5) after normalization are close to numerical weights of equally importance. Consequently, the three kinds of expected utilities of each motorcycle obtained by linguistic weighted aggregation operator are also in similar shape to those obtained by weighted average operator. As such the same ranking result has been obtained. It is worth emphasizing here that the ER approach could not be able to deal with MADM situations where attribute weights are given linguistically as discussed above.

6 Concluding Remarks

So far, linguistic computational models based on 2-tuple representation have been widely researched and applied to many areas. However, due to MADM problems often involving uncertainty, some limitations gradually emerge, leading to a conclusion that the applicability of these models has been affected. After carefully analyzing the causes of these limitations, we proposed a proportional fuzzy linguistic distribution model aiming at supplying a new approach for dealing with MADM under uncertain linguistic information, and meanwhile, overcoming these limitations. By relaxing the restrictions, the rational combinations of any number of linguistic terms associated with corresponding proportions can be used as evaluator's subjective judgments. Moreover, with introducing a new variable representing the extent of ignoring information, incomplete linguistic assessment as a response to the uncertainty could directly be used during the evaluation process. The applicability of the proposed model as well as it advantages have been illustrated by a tutorial example in product evaluation.

In summary, on the one hand, the proposed model of proportional fuzzy linguistic distributions could overcome the limitation imposed on previously developed symbolic models [21,31] in dealing with incomplete linguistic information; while on the other hand it could also provide a more efficient method for attribute aggregation as well as capability in dealing with uncertain linguistic weights in comparison to the ER approach [29,30]. In addition, many conventional aggregation operators could be also extendable within the proposed proportional fuzzy linguistic distribution model for use in decision analysis with uncertain linguistic information.

The proposed model in this paper could be extended to apply to many areas where linguistic assessments are employed. For future work, there are still some issues which would be worth addressed. One aspect is that, in some special situations, proportion alone perhaps is not sufficient to capture the vagueness and uncertainty, more complicated combinations could be used, such as interval. The other aspect is related to aggregation operators. We developed two linear aggregation operators in this paper. As we know, adopting linear additive method to aggregate assessment information requires all the attributes to be additively independent. However, linear additive independence assumption may not always be acceptable in reality. Therefore, some non-linear aggregation operators may be considered for the evaluation model proposed in this paper.

Acknowledgment

This work has been partly supported by KAKENHI Grant-in-Aid for Scientific Research A–25240049.

References

1. Adil Baykasoğlu, İlker Gölcük, and Derya Eren Akyol. A fuzzy multiple-attribute decision making model to evaluate new product pricing strategies. *Annals of Operations Research*, pages 1–38, 2015.
2. I. Bogardi and A. Baedossy. Application of madm to geological exploration. In P. Hansen, editor, *Essays and Surveys on Multiple Criteria Decision Making.* New York: Springer-Verlag, 1983.
3. C. Carlsson and R. Fullér. Benchmarking and linguistic importance weighted aggregations. *Fuzzy Sets Systems*, 114(1):35–42, 2000.
4. R. Degani and G. Bortolan. The problem of linguistic approximation in clinical decision making. *International Journal of Approximate Reasoning*, 2(2):143–162, 1988.
5. M. Delgado, F. Herrera, E. Herrera-Viedma, J. M. Martin-Bautista, L. Martínez, and A. M. Vila. A communication model based on the 2-tuple fuzzy linguistic representation for a distributed intelligent agent system on internet. *Soft Computing*, 6(5):320–328, 2002.

6. W.-T. Guo and V.-N. Huynh. A fuzzy linguistic representation model for decision making under uncertainty. In *2014 IEEE International Conference on Industrial Engineering and Engineering Management, IEEM 2014, Selangor Darul Ehsan, Malaysia, December 9-12, 2014*, pages 29–33. IEEE, 2014.
7. F. Herrera, E. Herrera-Viedma, and J. L. Verdegay. Direct approach processes in group decision making using linguistic OWA operators. *Fuzzy Sets Systems*, 79(2):175–190, 1996.
8. F. Herrera and L. Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8(6):746–752, 2000.
9. E. Herrera-Viedma, A. G. LÓPez-Herrera, M. Luque, and C. Porcel. A fuzzy linguistic irs model based on a 2-tuple fuzzy linguistic approach. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15(02):225–250, 2007.
10. V.-N. Huynh and Y. Nakamori. A linguistic screening evaluation model in new product development. *IEEE Trans. Engineering Management*, 58(1):165–175, 2011.
11. V.-N. Huynh, Y. Nakamori, T. B. Ho, and T. Murai. Multiple-attribute decision making under uncertainty: the evidential reasoning approach revisited. *IEEE Transactions on Systems, Man and Cybernetics—Part A: Systems and Humans*, 36(4):804–822, 2006.
12. R. L. Keeney and H. Raiffa. *Decisions With Multiple Objectives.* 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1993.
13. J. Lawry. An alternative approach to computing with words. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9:3–16, 2001.
14. L. Martínez and F. Herrera. An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, 207:1 – 18, 2012.
15. L. Martínez, R. M. Rodriguez, and F. Herrera. *The 2-tuple Linguistic Model: Computing with Words in Decision Making.* Springer, 2015.
16. J. Qin and X. Liu. 2-tuple linguistic muirhead mean operators for multiple attribute group decision making and its application to supplier selection. *Kybernetes*, 45(1):2–29, 2016.
17. T. L. Saaty. *The Analytic Hierarchy Process.* Pittsburgh, PA: Univ. Pittsburgh, 1998.
18. G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, 1976.
19. P. Smets. Decision making in the tbm: the necessity of the pignistic transformation. *International Journal of Approximate Reasoning*, 38(2):133 – 147, 2005.
20. C.-F. Tsai and Z.-Y. Chen. Crossing the fuzzy front end chasm: Effective product project concept selection using a 2-tuple fuzzy linguistic approach. *J. Intell. Fuzzy Syst.*, 25(3):755–770, May 2013.
21. J. H. Wang and J. Y. Hao. A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 14(3):435–445, 2006.
22. W. L. Winston. *Operations Research—Applications and Algorithms.* Belmont, CA: Duxbury Press, 1994.
23. Z. Wu, J. Xu, and Z. Xu. A multiple attribute group decision making framework for the evaluation of lean practices at logistics distribution centers. *Annals of Operations Research*, pages 1–23, 2015.
24. D.-L. Xu. An introduction and survey of the evidential reasoning approach for multiple criteria decision analysis. *Annals of Operations Research*, 195(1):163–187, 2012.
25. R. R. Yager. A new methodology for ordinal multiobjective decisions based on fuzzy sets. *Decision Sciences*, 12(4):589–600, 1981.
26. H.B. Yan, V. N. Huynh, and Y. Nakamori. A probabilistic model for linguistic multi-expert decision making involving semantic overlapping. *Expert Systems with Applications*, 38(7):8901–8912, 2011.
27. J. B. Yang. Rule and utility based evidential reasoning approach for multiple attribute decision analysis under uncertainty. *European Journal of Operational Research*, 131(1):31–61, 2001.
28. J. B. Yang and P. Sen. A general multi-level evaluation process for hybrid madm with uncertainty. *IEEE Transactions on Systems, Man and Cybernetics*, 24(10):1458–1473, 1994.
29. J. B. Yang and M. G. Singh. An evidential reasoning approach for multiple attribute decision making with uncertainty. *IEEE Transactions on Systems, Man and Cybernetics*, 24(1):1–18, 1994.

30. J. B. Yang and D. L. Xu. On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty. *IEEE Transactions on Systems, Man and Cybernetics—Part A: Systems and Humans*, 32(3):289–304, 2002.
31. X. You, J. You, H. Liu, and L. Zhen. Group multi-criteria supplier selection using an extended vikor method with interval 2-tuple linguistic information. *Expert Systems with Applications*, 42(4):1906–1916, 2015.
32. L. A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning pt. I. *Information Sciences*, 8:199–249, 1975.
33. L. A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning pt. II. *Information Sciences*, 8:301–357, 1975.
34. L. A. Zadeh. The concept of a linguistic variable and its applications to approximate reasoning pt. III. *Information Sciences*, 9:43–80, 1975.
35. L. A. Zadeh. Fuzzy logic = computing with words. *IEEE Transactions on Fuzzy Systems*, 4(2):103–111, 1996.
36. G.Q. Zhang, Y. C. Dong, and Y.F. Xu. Consistency and consensus measures for linguistic preference relations based on distribution assessments. *Information Fusion*, 17:46–55, 2014.