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A study on type assignment systems and their models.

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Master Thesis

A study on type assignment systems and their models.

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Abstract

In this paper, we document three type assignment systems and prove their completeness through filter models. We clarify several ambiguity in the proof, and reconstruct the cut-elimination proof in the union type theory. Our main focus is on the union type assignment system in which several definitions and proofs are inconsistent in the original paper. We also construct a sequent calculus system for the type theory of the intersection type assignment system, and find out that quasi-cut rule is necessary to prove that the cut elimination holds in that system.

Keywords: Type assignment system; Lambda calculus; Cut-elimination

Contents

1 Introduction

1.1 Research background

As we all know, λ-calculus is a Turing-complete computational model. In particular, typed λ-calculus has wide application not only in programming languages, but also in some semantics of natural language and proof assistant. For example, type inference has become one of the theoretical foundations of type checking in compiling process.

A type assignment system (TA system) is a set of rules in order to assign type properties to λ-terms. In order to prove its completeness to semantic world, Barendregt et al. [1] created a filter λ-model, which was mainly based on several axioms and rules of a partial order relation $\leq$ between types. Due to its similarity to the derivability $\vdash$ of a logical system, a new idea has emerged in which we can find some properties of logical systems to define the relation $\leq$.

1.2 Previous research

In previous work [2], Ishihara and Kurata defined the relation $\leq$ by a LK system $\Rightarrow$ as follows.

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta} \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \quad \cdots$$

The advantage is that it makes later proof easier, because the well-known cut-elimination property can be applied to this LK system. In particular, following theorem can be proved.

Theorem 1.1. *$L_T \vdash \Gamma \Rightarrow \theta$ if and only if $\bigwedge \Gamma \leq \theta$.*

Because the cut-rule is admissible in the type theory, important lemmas concerning the relation $\leq$ can be easier proved by induction without discussion about the cut-rule.

1.3 Research purpose

The objective of this research is divided into three parts. The first is to establish a solid relation between the partial order relations $\leq$ of filter λ-models and the derivabilities $\vdash$ of logical systems. We shall build a new logical system $L_{T\wedge}$ for the relation $\leq$ in the intersection system, and prove the equivalence between the relation $\leq$ and the derivability $\vdash$ of $L_{T\wedge}$. The second is to solve the proof problem occurred in the cut-elimination proof of L_T. The third is to extend this solution to $L_{T\wedge}$ and prove the important lemma by $L_{T\wedge}$.

2 Preliminaries

For abbreviations, we use **I.H** for **Induction Hypothesis** and **def** for **definition** in this paper.

2.1 Untyped lambda calculus

Definition 2.1. *The set of **untyped λ-terms** Λ is defined as follows.*

- *The variables $x_0, x_1, x_2 \cdots$ are untyped λ-terms.*
- *For each variable x and λ-term M, $\lambda x.M$ is also a untyped λ-term, denoted as **abstraction form**.*
- *For two λ-terms M and N, MN is also a untyped λ-term, denoted as **application form**.*

We use the following abbreviations:

$$MN_1 \cdots N_k \equiv (..(MN_1) \cdots N_k)$$
$$\lambda x_1 \cdots x_n.M \equiv (\lambda x_1(..(\lambda x_n.M)..))$$

Definition 2.2. *The set of **free variables** of M, denoted as FV(M), is defined as follows.*

- *If M is a variable, then itself is the only element in FV(M).*
- *If M is a abstraction form as $\lambda x.N$, then $FV(M) = FV(N) \setminus \{x\}$.*
- *If M is a application form as M_1M_2, then FV(M) is simply the union of the two sets as $FV(M_1) \cup FV(M_2)$.*

Note:
*If FV(M) $= \emptyset$, then we say M is **closed**.*

Definition 2.3. *For each M, $N \in \Lambda$ and each variable x, **M[x:=N]** is defined inductively as follows.*

M	$M[x{:=}N]$
x	N
$y \not\equiv x$	y
M_1M_2	$M_1[x := N]M_2[x := N]$
$\lambda x.M_1$	$\lambda x.M_1$
$\lambda y.M_1, y \not\equiv x$	$\lambda z.M_1[y := z][x := N]$

where $z \equiv y$ *if* $x \notin FV(M_1)$ *or* $y \notin FV(N)$*, else z is the first variable in the sequence* $x_0, x_1, x_2, \cdots$ *not in* M_1 *or N.*

Definition 2.4.

- *For a binary relation R on* Λ*, and each* $(M, N) \in \boldsymbol{R}$*, M is called (R-)****redex*** *and N is called (R-)****contractum*** *of M.*
- *A binary relation* $\boldsymbol{R}$ *on* Λ *is a* ***compatible*** *(with the operations) if for every* $(M, M') \in \boldsymbol{R}$ *and x as a random variable with* $Z \in \Lambda$*,* (ZM, ZM')*,* $(MZ, M'Z)$ *and* $(\lambda x.M, \lambda x.M')$ *are also in* $\boldsymbol{R}$*.*
- *A R-****equality*** $=_R$ *(or* ***congruence****) on* Λ *is a compatible, reflexive, symmetric and transitive relation.*
- *A R-****reduction*** $\twoheadrightarrow_R$ *on* Λ *is one which is only compatible, reflexive, and transitive without symmetric property.*
- *A one step* $\boldsymbol{R}$*-reduction* $\rightarrow_R$ *is simply a R-****reduction*** *without transitive property.*
- *If* R_1, R_2 *are reductions, then their union relation* R_1R_2 *is defined as* $R_1 \cup R_2$*.*

Definition 2.5.

- $\lambda x.M \twoheadrightarrow_\alpha \lambda y.M[x := y]$ *(α-reduction)(*$y \notin FV(M) \cup BV(M)$*)*
- $(\lambda x.M)N \twoheadrightarrow_\beta M[x := N]$ *(β-reduction)*
- $\lambda x.Mx \twoheadrightarrow_\eta M$ *(η-reduction)(*$x \notin FV(M)$*)*

Note:

The equality and the one step reduction corresponding to α, β, η *can be defined as* ***Definition 2.4****.*

The λ-terms are often considered equal on $=_\beta$ *or* $=_{\beta\eta}$*.*

For further reference or detail, one should read [3].

2.2 Lambda model

Definition 2.6. *(Variable interpretation)*

- *Let D be a set holding all interpretations for variables of untyped lambda calculus.*
  *A **(term) environment** ρ in D is simply a total map between all variables of untyped lambda calculus and D as follows.*

  $$\rho : V \to D$$

  Env_D is defined as the set holding all environments in D.

- *If $\rho \in Env_D$, $d \in D$, then $\rho[x := d]$ is defined by $\rho' \in Env_D$ as follows.*

  $$\rho'(y) = \begin{cases} d & \text{if } y{=}x \\ \rho(y) & \text{otherwise} \end{cases}$$

Definition 2.7. *(Lambda model)*

- *We define an **applicative structure** as a pair $\langle D, \cdot \rangle$ consisting of a set D with a binary operation $\cdot : D \times D \to D$ on it.*

- *A **lambda model** $\mathscr{M}$ is defined as follows.*

  $$\mathscr{M} = \left\langle D, \ \cdot \ , [\![\ ]\!]^{\mathscr{M}} \right\rangle$$

  in which $\langle D, \cdot \rangle$ is an applicative structure and $[\![\ ]\!]^{\mathscr{M}} : \Lambda \times Env_D \to D$ satisfies following equations.

  1. $[\![x]\!]^{\mathscr{M}}_{\rho} = \rho(x)$
  2. $[\![MN]\!]^{\mathscr{M}}_{\rho} = [\![M]\!]^{\mathscr{M}}_{\rho} \cdot [\![N]\!]^{\mathscr{M}}_{\rho}$
  3. $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} = [\![\lambda y.M[x := y]]\!]^{\mathscr{M}}_{\rho}$
     where $y \notin FV(M)$
  4. $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} \cdot d = [\![M]\!]^{\mathscr{M}}_{\rho[x:=d]}$
  5. *If $\forall d \in D[[\![M]\!]^{\mathscr{M}}_{\rho[x:=d]} = [\![N]\!]^{\mathscr{M}}_{\rho[x:=d]}]$ then $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} = [\![\lambda x.N]\!]^{\mathscr{M}}_{\rho}$*
  6. *If $\rho \restriction_{FV(M)} = \rho' \restriction_{FV(M)}$ then $[\![M]\!]^{\mathscr{M}}_{\rho} = [\![M]\!]^{\mathscr{M}}_{\rho'}$*

2.3 Call-by-value lambda model

Definition 2.8. *(Call-by-value lambda model)*
*A **call-by-value lambda model** $\mathscr{M}$ is defined as follows.*

$$\mathscr{M} = \left\langle D, K, \ \cdot \ , [\![\ ]\!]^{\mathscr{M}} \right\rangle$$

where $K \subseteq D$, $\langle D, \cdot \rangle$ is an applicative structure and $[\![\ ]\!]^{\mathscr{M}} : \Lambda \times Env_K \to D$ satisfies following equations.

1. $[\![x]\!]^{\mathscr{M}}_{\rho} = \rho(x)$
2. $[\![MN]\!]^{\mathscr{M}}_{\rho} = [\![M]\!]^{\mathscr{M}}_{\rho} \cdot [\![N]\!]^{\mathscr{M}}_{\rho}$
3. $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} = [\![\lambda y.M[x := y]]\!]^{\mathscr{M}}_{\rho}$
   where $y \notin FV(M)$
4. $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} \cdot k = [\![M]\!]^{\mathscr{M}}_{\rho[x:=k]}$, *where $k \in K$*
5. *If $\forall k \in K[[\![M]\!]^{\mathscr{M}}_{\rho[x:=k]} = [\![N]\!]^{\mathscr{M}}_{\rho[x:=k]}]$ then $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} = [\![\lambda x.N]\!]^{\mathscr{M}}_{\rho}$*
6. *If $\rho \restriction_{FV(M)} = \rho' \restriction_{FV(M)}$ then $[\![M]\!]^{\mathscr{M}}_{\rho} = [\![M]\!]^{\mathscr{M}}_{\rho'}$*
7. *If $M \in Val$, then $[\![M]\!]^{\mathscr{M}}_{\rho} \in K$*

3 Type assignment systems

Definition 3.1.

- *We define the (type assignment) **statement** as follows.*

  $$M : \alpha \ (M \in \Lambda \text{ with } \alpha \in \mathbb{T})$$

  *In this statement, M is called the **subject** and α is called the **predicate**.*

- *We define a **basis** as a set of statements with **different** variables as subjects.*
  Note: One may notice that we define the basis differently comparing to the original paper, here we follow the new definition in [4].

- *We define that a statement $M : \alpha$ is derivable from a basis Γ written as following.*

$$\Gamma \vdash M : \alpha$$

- *A rule R is said to be **admissible**, if for all instances $S_0, \cdots, S_{n-1}, S$ of R it is the case that*

$$\text{if for all } i \le n \vdash^i S_i, \text{ then} \vdash S.$$

where $S_0, \cdots, S_{n-1}$ are the deduction elements assigned to the immediate successors of node v and S is assigned to the node v. $\vdash^i$ represents the derivability in the formal system to which S_i belongs.

Definition 3.2. *R-reduction or R-expansion holds in a type assignment system means that following two rules are admissible in the system respectively.*

$$\frac{M \twoheadrightarrow_R N \quad M : \alpha}{N : \alpha}\ (R - reduction) \quad \frac{M \twoheadleftarrow_R N \quad M : \alpha}{N : \alpha}\ (R - expansion)$$

It is clear that when both rules above are admissible in the type assignment system, the following rule is also admissible in the system.

$$\frac{M =_R N \quad M : \alpha}{N : \alpha}\ (R - equality)$$

And the equality also finds its position in semantic world as we want, it is in the following form for β-equality.

$$M =_\beta N \Leftrightarrow [\![M]\!]^{\mathscr{M}}_\rho = [\![N]\!]^{\mathscr{M}}_\rho.$$

3.1 The simple type assignment system

The main idea of designing the simple typed system is to build an abstraction for function spaces. As if M gets type $A \to B$ and N gets type A, then N applied to M can be viewed as valid and MN gets type B. In this way types help determine what terms fit together.

In the meantime, requiring terms to have simple types implies that they are strongly normalizing so that equality of terms of a certain type can be reduced to equality of terms in a fixed type.

Definition 3.3. *The set of simple type can be defined as follows.*

$$\mathbb{A} := \varphi_0, \varphi_1, \varphi_2 \cdots$$
$$\mathbb{T} := \mathbb{A} \mid \mathbb{T} \to \mathbb{T}$$

Definition 3.4. *The simple type assignment system is defined in the natural deduction manner as follows.*

$$\dfrac{\begin{array}{c}[x:\sigma]\\ \vdots\\ M:\tau\end{array}}{\lambda x.M:\sigma\to\tau}\ (\to I) \qquad \dfrac{M:\sigma\to\tau \quad N:\sigma}{MN:\tau}\ (\to E)$$

Lemma 3.5. *(The free variable lemma)*

$\Gamma \vdash M : \alpha \Rightarrow \Gamma \restriction_{FV(M)} \vdash M : \alpha$ *, where* $\Gamma \restriction_{FV(M)} = \{x : \alpha \in \Gamma \mid x \in FV(M)\}$.

Proof. Induction on the derivation of $\Gamma \vdash M : \alpha$. □

Lemma 3.6. *(The generation lemma)*

1. $\Gamma \vdash x : \alpha \Rightarrow \{x : \alpha\} \in \Gamma$.
2. $\Gamma \vdash MN : \alpha \Rightarrow \exists \beta \in \mathbb{T}[\Gamma \vdash M : \beta \to \alpha$ *and* $\Gamma \vdash N : \beta]$.
3. $\Gamma \vdash \lambda x.M : \alpha \Rightarrow \exists \sigma, \tau \in \mathbb{T}[\alpha \equiv \sigma \to \tau$ *and* $\Gamma, x : \sigma \vdash M : \tau]$.

Proof. Induction on the derivation of LHS. These three cases can be easily proved as the only non-trivial case is (axiom), (→ E),(→ I), respectively. □

Lemma 3.7. *(The substitution lemma)*

$$\Gamma, x : \alpha \vdash M : \beta \text{ and } \Gamma \vdash N : \alpha \Rightarrow \Gamma \vdash M[x := N] : \beta.$$

Proof. Induction on the derivation of $\Gamma, x : \alpha \vdash M : \beta$.

Basis: $M \equiv y$.
By the generation lemma, $\{y : \beta\} \in \Gamma \cup \{x : \alpha\}$

- $\{y : \beta\} \in \Gamma$. Then $y \not\equiv x$, so $\Gamma \vdash y[x := N] : \beta \equiv y : \beta$.
- $y : \beta \equiv x : \alpha$. Then $\Gamma \vdash y[x : N] : \beta \equiv N : \alpha$.

Induction Steps:

- The last rule applied is (→ I).

$$\dfrac{\begin{array}{c}x:\alpha,[y:\sigma]\\ \vdots\\ M':\tau\end{array}}{(M\equiv)\lambda y.M':\sigma\to\tau(\equiv\beta)}\ (\to I)$$

($y \equiv x$**)** Reduces to <u>Basis</u>.

($y \not\equiv x$**)** By the I.H, we have $\Gamma, y : \sigma \vdash M'[x := N] : \tau$, then by (→ I), we have $\Gamma \vdash (M[x := N] \equiv)\lambda y.M'[x := N] : \sigma \to \tau(\equiv \beta)$.

- The last rule applied is (→ E).

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ M_1 : \sigma \to \beta & N_1 : \sigma \end{array}}{(M \equiv) M_1 N_1 : \beta} \; (\to E)$$

By the I.H, we have $\Gamma \vdash M_1[x := N] : \sigma \to \beta$ and $\Gamma \vdash N_1[x := N] : \sigma$, then by (→ E), we have $\Gamma \vdash (M[x := N] \equiv) M_1[x := N] N_1[x := N] : \beta$.

□

Lemma 3.8. *(The βη-reduction property)*

The following rule is admissible in this system.

$$\dfrac{M \twoheadrightarrow_{\beta\eta} N \quad M : \alpha}{N : \alpha} \; (\beta\eta - reduction)$$

Proof. By the definition of reduction, it suffices to show only one-step reduction cases.
Induction on the derivation of $\Gamma \vdash M : \alpha$.

Basis: $M \equiv x$. By the definition of reduction, there is no contractum of variables, so this case is vacuous true.

Induction Steps:

- The last rule applied is (→ I).

$$\dfrac{\begin{array}{c} [x : \sigma] \\ \vdots \\ M_1 : \tau \end{array}}{(M \equiv) \lambda x.M_1 : \sigma \to \tau (\equiv \alpha)} \; (\to I)$$

($N \equiv \lambda x.M_2$ **with** $M_1 \twoheadrightarrow_{\beta\eta} M_2$**)** By the I.H, we have $\Gamma, x : \sigma \vdash M_1 : \tau$, then by (→ I), we have $\Gamma \vdash (N \equiv) \lambda x.M_2 : \sigma \to \tau (\equiv \alpha)$.

($M_1 \equiv Nx$ **with** $x \notin FV(N)$**)** By the generation lemma, we have $\Gamma, x : \sigma \vdash N : \sigma' \to \tau$ and $\Gamma, x : \sigma \vdash x : \sigma'$ for some σ'. Apply the generation lemma again, we have $\sigma \equiv \sigma'$. By the free variable lemma, we have $\Gamma \vdash N : \sigma \to \tau (\equiv \alpha)$.

- The last rule applied is (→ E).

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ M_1 : \sigma \to \alpha & M_2 : \sigma \end{array}}{(M \equiv) M_1 M_2 : \alpha} \; (\to E)$$

($N \equiv N_1M_2$ **with** $M_1 \twoheadrightarrow_{\beta\eta} N_1$)

($N \equiv M_1N_2$ **with** $M_2 \twoheadrightarrow_{\beta\eta} N_2$) These two cases can be treated similarly. Simply by applying the I.H twice with ($\rightarrow$ E), we have $\Gamma \vdash N : \alpha$.

($M_1 \equiv \lambda x.M'$ **with** $N \equiv M'[x := M_2]$) By the generation lemma, we have $\Gamma, x : \sigma \vdash M' : \alpha$. By the substitution lemma, we have $\Gamma \vdash M'[x := M_2](\equiv N)$.

□

To see why we failed to prove that expansion rule is also admissible in the simple type assignment system, we here show one simple example.

Example 3.9. *Suppose we have assigned a type σ for yy, now we want to assign the same type to $(\lambda x.xx)y$. By applying the generation lemma, we can construct following deduction.*

$$\dfrac{\dfrac{\dfrac{x : \alpha \rightarrow \sigma \quad x : \alpha}{xx : \sigma}}{\lambda x.xx : \beta \rightarrow \sigma}\,(\rightarrow I) \qquad y : \beta}{(\lambda x.xx)y : \sigma}$$

The problem is at $(\rightarrow I)$ on the left side. Because the subject x has two different types, $(\rightarrow I)$ can not be applied due to our restriction on the basis(different subject). Even if we give up this restriction on basis which means that we must put a restriction on the $(\rightarrow I)$ which will also leads to this circumstance.

3.2 The intersection type assignment system

The intersection type assignment system is an extended system of simple type assignment system by adding intersection type. The intersection type intends to be assigned to the λ-term which is holding two or more types.

The motivation for creating such system lies in the requirement that not only subject reduction but also subject expansion holds. Suppose we have $\vdash M[x := N] : \alpha$, then in order to assign the same type to $(\lambda x.M)N$, it is natural to think of application of ($\rightarrow$ E) as follows.

$$\dfrac{\lambda x.M : \sigma \rightarrow \alpha \quad N : \sigma}{(\lambda x.M)N : \alpha}\,(\rightarrow E)$$

The problem is that there may be several occurrences of x in M, so we need a type holding other types.

Another problem appears when there is no occurrence of x in M, so that N may be not typable at all. To solve this problem, a universal type ω is needed to hold all λ-terms which is the motivation for building the $\leq$ relation on types.

Definition 3.10. *The set of intersection type can be defined as follows.*

$$\mathbb{A} := \omega \mid \varphi_0, \varphi_1, \varphi_2 \cdots$$
$$\mathbb{T} := \mathbb{A} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \wedge \mathbb{T}$$

Definition 3.11. *The intersection type assignment system is defined in the natural deduction manner as follows.*

$$\dfrac{\begin{matrix}[x:\sigma]\\ \vdots \\ M:\tau\end{matrix}}{\lambda x.M:\sigma\to\tau}\ (\to I) \qquad \dfrac{M:\sigma\to\tau \quad N:\sigma}{MN:\tau}\ (\to E)$$

$$\dfrac{M:\sigma \quad M:\tau}{M:\sigma\wedge\tau}\ (\wedge I) \qquad \dfrac{M:\sigma\wedge\tau}{M:\tau(\sigma)}\ (\wedge E)$$

$$\dfrac{}{M:\omega}\ (\omega) \qquad \dfrac{M:\sigma \quad \sigma\leq\tau}{M:\tau}\ (\leq)$$

Definition 3.12. *The relation $\leq$ is inductively defined as follows.*

$\alpha \leq \alpha$ *(ref)*	$\alpha \leq \beta \leq \gamma \Rightarrow \alpha \leq \gamma$ *(trans)*
$\alpha \leq \omega$ *(ω-top)*	$\omega \leq \omega \to \omega$ *(ω-arrow)*
$\alpha \leq \alpha \wedge \alpha$	$\alpha \wedge \beta \leq \alpha \quad \alpha \wedge \beta \leq \beta$
$(\alpha \to \beta) \wedge (\alpha \to \gamma) \leq \alpha \to (\beta \wedge \gamma)$	
$\dfrac{\alpha \leq \alpha' \quad \beta \leq \beta'}{\alpha \wedge \beta \leq \alpha' \wedge \beta'}\ (\wedge - mono)$	$\dfrac{\alpha' \leq \alpha \quad \beta \leq \beta'}{\alpha \to \beta \leq \alpha' \to \beta'}\ (\to -mono)$

Note: One can prove easily that ($\wedge$ E) is derivable due to ($\leq$).

We use the notation $\sigma \sim \tau$ for $\sigma \leq \tau \leq \sigma$.

Lemma 3.13. $(\alpha \to \beta) \sim \omega \Leftrightarrow \beta \sim \omega$

Proof. We define $\Omega \subseteq \mathbb{T}$ as follows.

$$\Omega := \omega \mid \mathbb{T} \to \Omega \mid \Omega \wedge \Omega$$

Then we will prove that $\sigma \in \Omega \Rightarrow \sigma \sim \omega$ by induction on the complexity of $\sigma \in \Omega$.

Base case: This case is straightforward as $\omega \sim \omega$.

Induction Steps:

$(\sigma \to \tau)$ $\tau \in \Omega$, so we can prove this case as follows.

$$\dfrac{\dfrac{\sigma \leq \omega \quad \dfrac{\dfrac{\tau \in \Omega}{\tau \sim \omega}\,(I.H)}{\omega \leq \tau}}{\omega \leq \omega \to \omega \leq \sigma \to \tau}\,(\to -mono)}{\omega \leq \sigma \to \tau}\,(trans)$$

$(\sigma \wedge \tau)$ $\sigma, \tau \in \Omega$, so we can prove this case as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\sigma \in \Omega}{\omega \leq \sigma}\,(I.H) \quad \dfrac{\tau \in \Omega}{\omega \leq \tau}\,(I.H)}{\omega \wedge \omega \leq \sigma \wedge \tau}}{\omega \leq \omega \wedge \omega \leq \sigma \wedge \tau}}{\omega \leq \sigma \wedge \tau}$$

Then we can easily prove that $\sigma \in \Omega, \sigma \leq \tau \Rightarrow \tau \in \Omega$ by induction on the definition of $\leq$. We omit the proof here because of its triviality.
Finally we can prove this lemma as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{(\alpha \to \beta) \sim \omega}{\omega \leq \alpha \to \beta}}{\alpha \to \beta \in \Omega}}{\beta \in \Omega}}{\beta \sim \omega} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \sim \omega}{\omega \leq \beta}}{\beta \in \Omega}}{\alpha \to \beta \in \Omega}}{(\alpha \to \beta) \sim \omega}$$

□

Definition 3.14. *The type theory $L_{T\wedge}$ is a sequent calculus system defined as follows.*

Axiom:

$\Gamma \Rightarrow \omega$
$\Gamma, a, \Delta \Rightarrow a \ (a \in \mathbb{A})$

Inference Rules:

$$\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta}\,(\wedge \Rightarrow) \qquad \dfrac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\,(\Rightarrow \wedge)$$

$$\dfrac{\omega \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\,(\Rightarrow\to) \qquad \dfrac{\alpha' \Rightarrow \alpha \quad \beta \Rightarrow \beta'}{\Gamma, \alpha \to \beta, \Delta \Rightarrow \alpha' \to \beta'}\,(\to\Rightarrow\to)$$

$$\dfrac{\Gamma \Rightarrow \alpha \to \beta \quad \beta \Rightarrow \gamma}{\Gamma \Rightarrow \alpha \to \gamma}\,(\Rightarrow\to\Rightarrow)(\star) \qquad \dfrac{\Gamma \Rightarrow \beta \to \gamma \quad \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \gamma}\,(\Rightarrow\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \Gamma \Rightarrow \alpha \to \gamma \quad \beta \wedge \gamma \Rightarrow \sigma}{\Gamma \Rightarrow \alpha \to \sigma} \ (\Rightarrow\to \wedge)$$

($\star$) *This rule can be derived by* $(\Rightarrow\to \wedge)$ *as Lemma 3.35.*

Note: In the rules above, Γ *and* Δ *are called the* ***context****. In the conclusion of each rule, the types other than* θ *which are not in the context is called the* ***principal*** *types.*

It is easy to see that this system is a subset of L_T which will be defined later in the union type assignment system with quasi-cut rules added. Because we have no rules concerning $\vee$ anymore, $(\Rightarrow\Rightarrow\to)$ rule can not be derived. We have to add it to $L_{T\wedge}$ to make cut-elimination work.

Definition 3.15. *We define* $T \vdash_n \Gamma \Rightarrow \theta$ *as that* $\Gamma \Rightarrow \theta$ *has a proof of depth at most* n *in the sequent calculus system* T*.*

Lemma 3.16. *We can prove the following structure properties under* $L_{T\wedge}$*.*

1. *If* $L_{T\wedge} \vdash_n \Gamma, \Delta \Rightarrow \theta$*, then* $L_{T\wedge} \vdash_n \Gamma, \alpha, \Delta \Rightarrow \theta$*.(Weakening-L)*
2. *If* $L_{T\wedge} \vdash_n \Gamma, \alpha, \beta, \Delta \Rightarrow \theta$*, then* $L_{T\wedge} \vdash_n \Gamma, \beta, \alpha, \Delta \Rightarrow \theta$*.(Exchange-L)*
3. *If* $L_{T\wedge} \vdash_n \Gamma, \alpha, \alpha, \Delta \Rightarrow \theta$*, then* $L_{T\wedge} \vdash_n \Gamma, \alpha, \Delta \Rightarrow \theta$*.(Contraction-L)*

Proof. Because $L_{T\wedge}$ includes a subset of axioms and rules of L_T. So every property inside L_T holds in $L_{T\wedge}$ too. For the detailed proof, one can check the proof under Lemma 3.37. As to the new added $(\Rightarrow\Rightarrow\to)$ rule, it is easy to prove the properties above also hold. □

Proposition 3.17. $L_{T\wedge} + Cut \vdash \Gamma \Rightarrow \theta$ *if and only if* $\bigwedge \Gamma \leq \theta$*.*

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \Sigma \Rightarrow \theta} \ (Cut)$$

Proof. This proof is part of the proof of Proposition 3.39, so we omit the detail here. As to the new added $(\Rightarrow\Rightarrow\to)$ rule, it can be proved as follows.

$$\frac{\dfrac{\Gamma \Rightarrow \beta \to \gamma}{\bigwedge \Gamma \leq \beta \to \gamma} \ I.H \qquad \dfrac{\dfrac{\alpha \Rightarrow \beta}{\alpha \leq \beta} \ I.H \qquad \gamma \leq \gamma}{\beta \to \gamma \leq \alpha \to \gamma} \ (\to -mono)}{\bigwedge \Gamma \leq \alpha \to \gamma}$$

□

Theorem 3.18. *Cut elimination holds for* $L_{T\wedge} + Cut$*.*

Proof. This proof is part of the proof of Theorem 3.40, so we omit the detail here. The new case need to be discussed is in Subcase 3c with the left premise being (⇒⇒→), because (⇒⇒→) is no longer derivable. This case can be proved as follows.

Subcase 3c (⇒⇒→) In this case, the proof is as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow \gamma \to \beta \quad \alpha \Rightarrow \gamma}{\Gamma \Rightarrow \alpha \to \beta} \quad \dfrac{\alpha' \Rightarrow \alpha \quad \beta \Rightarrow \beta'}{\Delta, \alpha \to \beta, \Sigma \Rightarrow \alpha' \to \beta'}}{\Delta, \Gamma, \Sigma \Rightarrow \alpha' \to \beta'}\ (Cut)$$

It can be transformed into the following proof.

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \gamma \to \beta \quad \beta \Rightarrow \beta'}{\Gamma \Rightarrow \gamma \to \beta'} \quad \dfrac{\alpha' \Rightarrow \alpha \quad \alpha \Rightarrow \gamma}{\alpha' \Rightarrow \gamma}\ (Cut)}{\Gamma \Rightarrow \alpha' \to \beta'}}{\Delta, \Gamma, \Sigma \Rightarrow \alpha' \to \beta'}\ (\mathrm{Weakening} - \mathrm{L})$$

□

Theorem 3.19. $L_{T\wedge} \vdash \Gamma \Rightarrow \theta$ *if and only if* $\bigwedge \Gamma \leq \theta$.

Proof. This theorem can be derived from Proposition 3.17 and Theorem 3.18. □

Lemma 3.20. $(\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_m \to \beta_m) \leq \alpha \to \beta$ *and* $\beta \not\sim \omega$, *then there are* $i_1, \cdots, i_l \in \{1, \cdots, m\}$ *such that* $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_l} \geq \alpha$ *and* $\beta_{i_1} \wedge \cdots \wedge \beta_{i_l} \leq \beta$.

Proof. We will prove this lemma by $L_{T\wedge}$. Because cut-elimination holds in this system, we do not need to discuss about the cut-rule which makes the proof easier than the original one. The original proof is in appendix.

By Theorem 3.19, it suffices to prove the following statement implies the same conclusion.

$$L_{T\wedge} \vdash \alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m \Rightarrow \alpha \to \beta \text{ and } \beta \not\sim \omega$$

We prove this by induction on the derivation, then the only cases need to be treated are (⇒→), (→⇒→), (⇒⇒→) and (⇒→ ∧).

(⇒→) Because we can derive $\beta \sim \omega$ from the assumption of this rule, this case is trivial.

(→⇒→)

$$\dfrac{\alpha \Rightarrow \alpha_k \quad \beta_k \Rightarrow \beta}{\alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m \Rightarrow \alpha \to \beta}\ (\star)(\to\Rightarrow\to)$$

($\star$) $1 \leq k \leq m$
By Theorem 3.41, we have $\alpha \leq \alpha_k$ and $\beta_k \leq \beta$ from the assumptions.
We simply set $l = 1$ and $i_1 = k$.

($\Rightarrow\rightarrow \wedge$)

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \alpha \rightarrow \gamma_1\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \alpha \rightarrow \gamma_2\end{array} \quad \begin{array}{c}\vdots\\ \gamma_1 \wedge \gamma_2 \Rightarrow \beta\end{array}}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\star)(\Rightarrow\rightarrow \wedge)$$

($\star$) $\Gamma = \alpha_1 \rightarrow \beta_1, \cdots, \alpha_m \rightarrow \beta_m$
From I.H, there exists $i_1, \cdots, i_j \in \{1, \cdots, m\}$ and $i'_1, \cdots, i'_j \in \{1, \cdots, m\}$, such that

$$\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_j} \geq \alpha \text{ and } \beta_{i_1} \wedge \cdots \wedge \beta_{i_j} \leq \gamma_1,$$
$$\alpha_{i'_1} \wedge \cdots \wedge \alpha_{i'_j} \geq \alpha \text{ and } \beta_{i'_1} \wedge \cdots \wedge \beta_{i'_j} \leq \gamma_2.$$

From ($\wedge - mono$), we have

$$(\beta_{i_1} \wedge \cdots \wedge \beta_{i_j}) \wedge (\beta_{i'_1} \wedge \cdots \wedge \beta_{i'_j}) \leq \gamma_1 \wedge \gamma_2 \leq \beta$$

and

$$\alpha \sim \alpha \wedge \alpha \leq (\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_j}) \wedge (\alpha_{i'_1} \wedge \cdots \wedge \alpha_{i'_j}).$$

In this case, $\{1, \cdots, l\} = \{i_1, \cdots, i_j\} \cup \{i'_1, \cdots, i'_j\}$.

($\Rightarrow\Rightarrow\rightarrow$)

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \gamma \rightarrow \beta\end{array} \quad \begin{array}{c}\vdots\\ \alpha \Rightarrow \gamma\end{array}}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\Rightarrow\Rightarrow\rightarrow)(\star)$$

($\star$) $\Gamma = \alpha_1 \rightarrow \beta_1, \cdots, \alpha_m \rightarrow \beta_m$
From I.H, there exists $i_1, \cdots, i_j \in \{1, \cdots, m\}$ such that

$$\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_j} \geq \gamma (\geq \alpha) \text{ and } \beta_{i_1} \wedge \cdots \wedge \beta_{i_j} \leq \beta.$$

In this case, $\{1, \cdots, l\} = \{i_1, \cdots, i_j\}$.

□

The reason why we need to restrain β is that if $\beta \sim \omega$ then by Lemma 3.13, we have $(\alpha \rightarrow \beta) \sim \omega$, which means that the assumption is true for all $(\alpha_1 \rightarrow \beta_1) \wedge \cdots \wedge (\alpha_n \rightarrow \beta_n)$.

Definition 3.21. *A **filter** is a non-empty subset $d \subseteq \mathbb{T}$ satisfies following conditions:*

- $\alpha, \beta \in d \Rightarrow \alpha \wedge \beta \in d$;
- $\beta \in d$ *and* $\alpha \geq \beta \Rightarrow \alpha \in d$.

Lemma 3.22. *Let T be a non-empty set of types, then $\uparrow T$ defined as follows is called the filter generated by T.*

$$\uparrow T = \{\alpha \in \mathbb{T} \mid \exists n \geq 1, \exists \beta_1, \cdots, \beta_n \in T \cup \{\omega\}[\beta_1 \wedge \cdots \wedge \beta_n \leq \alpha]\}.$$

Proof. Firstly, we shall prove $\uparrow T$ is a filter by induction on the definition of filter.

- $\alpha, \beta \in d \Rightarrow \alpha \wedge \beta \in d$.
  By the definition, we have $\exists \alpha_1, \cdots, \alpha_n \in T \cup \{\omega\}[\alpha_1 \wedge \cdots \wedge \alpha_n \leq \alpha]$ and $\exists \beta_1, \cdots, \beta_m \in t \cup \{\omega\}[\beta_1 \wedge \cdots \wedge \beta_m \leq \beta]$. By $(\wedge - mono)$, we have $\exists \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_m \in T \cup \{\omega\}[\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \beta_1 \wedge \cdots \wedge \beta_m \leq \alpha \wedge \beta$, so by the definition of filter, we have $\alpha \wedge \beta \in d$.

- $\beta \in d$ and $\alpha \geq \beta \Rightarrow \alpha \in d$.
  By the definition, we have $\exists \beta_1, \cdots, \beta_n \in T \cup \{\omega\}[\beta_1 \wedge \cdots \wedge \beta_n \leq \beta]$. By (trans) we have $\exists \beta_1, \cdots, \beta_n \in T \cup \{\omega\}[\beta_1 \wedge \cdots \wedge \beta_n \leq \beta \leq \alpha]$, so by the definition of filter, we have $\alpha \in d$.

Secondly, we need to prove $\uparrow T$ is the smallest set satisfying the definition of filter. Suppose we have another filter $F \subseteq \uparrow T$, meaning $\exists \alpha \in \uparrow T[\alpha \notin F]$. By the definition above, we have $\exists \alpha_1, \cdots, \alpha_n \in T \cup \{\omega\}[\alpha_1 \wedge \cdots \wedge \alpha_n \leq \alpha]$. Because $T \subseteq F$, we have $\alpha_1, \cdots, \alpha_n \in F$ also. So by the definition of filter, $\alpha_1 \wedge \cdots \wedge \alpha_n \in F$, this leads to $\alpha \in F$ which is a contradiction. □

Lemma 3.23.

1. $\{\alpha \mid \Gamma \vdash_\wedge M : \alpha\}$ *is a filter.*

2. $\Gamma \vdash_\wedge x : \alpha$ *if and only if* α *is in the filter generated by* $\{\beta \mid x : \beta \in \Gamma\}$.

Proof.

1. This lemma can be proved by rules $(\omega), (\leq)$, and $(\wedge I)$.

2. $(\Rightarrow)$ By induction on the derivation of $\Gamma \vdash_\wedge x : \alpha$. Because the subject is a variable, the only cases are $(\omega), (\leq)$, and $(\wedge I)$ which can be straightforward proved from the definition of filter.
   $(\Leftarrow)$ By Lemma 3.22, we have

$$\alpha \in \{\gamma \in \mathbb{T} \mid \exists n \geq 1, \exists \beta_1, \cdots, \beta_n \in \{\beta \mid x : \beta \in \Gamma\} \cup \{\omega\} \\ [\beta_1 \wedge \cdots \wedge \beta_n \leq \gamma]\}.$$

   Then we can have

$$\dfrac{\beta_1 \wedge \cdots \wedge \beta_n \leq \alpha \quad \dfrac{x : \beta_1, \cdots, x : \beta_n}{x : \beta_1 \wedge \cdots \wedge \beta_n} (\wedge I)}{x : \alpha} (\leq)$$

□

Proposition 3.24. $\Gamma \vdash_{\wedge n} \lambda x.M : \gamma \Rightarrow$ $\exists \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m \in \mathbb{T}[\forall i \in \{1, 2, \cdots, m\}[\Gamma \vdash_{\wedge n-1} \lambda x.M : \alpha_i \to \beta_i], (\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_m \to \beta_m) \leq \gamma]$ *or* $\Gamma, x : \sigma \vdash_{\wedge n-1} M : \tau$ *such that* $\gamma \equiv \sigma \to \tau$*.*

where $\Gamma \vdash_{\wedge n} M : \alpha$ *means that* $M : \alpha$ *can be derived by a proof of at most n depth under the intersection system.*

Proof. By induction on the derivation of $\Gamma \vdash_{\wedge n} \lambda x.M : \gamma$, and because the subject is in abstraction form, the only cases are $(\omega), (\to I), (\leq)$, and $(\wedge I)$.

(ω) This case can be proved by $(\omega \leq \omega \to \omega)$.

$(\to I)$ This case naturally stands.

$(\leq)$ From the first part of the I.H, it naturally stands. As for the second part, we have $\Gamma, x : \sigma \vdash_{\wedge n-2} M : \tau$ such that $\gamma' \equiv \sigma \to \tau$, then by $(\to I)$ we have $\Gamma \vdash_{\wedge n-1} \lambda x.M : \sigma \to \tau$ with $\sigma \to \tau \leq \gamma$.

$(\wedge I)$ This case can be proved from $(\wedge - mono)$ and I.H.

□

Lemma 3.25. *(The generation lemma)*

1. $\Gamma \vdash_{\wedge} MN : \alpha \Rightarrow \exists \beta \in \mathbb{T}[\Gamma \vdash_{\wedge} M : \beta \to \alpha$ *and* $\Gamma \vdash_{\wedge} N : \beta]$.
2. $\forall \alpha, \beta \in \mathbb{T}[\Gamma, x : \alpha \vdash_{\wedge} M : \beta \Rightarrow \Gamma, x : \alpha \vdash_{\wedge} N : \beta]$*, then* $\forall \gamma \in \mathbb{T}[\Gamma \vdash_{\wedge} \lambda x.M : \gamma \Rightarrow \Gamma \vdash_{\wedge} \lambda x.N : \gamma]$.
3. $\Gamma \vdash_{\wedge} \lambda x.M : \alpha \Rightarrow \exists \sigma, \tau \in \mathbb{T}[\alpha \equiv \sigma \to \tau$ *and* $\Gamma, x : \sigma \vdash_{\wedge} M : \tau]$.

Proof.

1. By induction on the derivation of $\Gamma \vdash_{\wedge} MN : \alpha$. Because the subject is in application form, the only cases are $(\omega), (\to E), (\leq)$, and $(\wedge I)$.

   $(\to E)$ This case naturally stands.

   (ω) This case can be proved from $\omega \leq \omega \to \omega$.

($\leq$) This case can be proved from the I.H.

($\wedge I$) $\alpha \equiv \alpha_1 \wedge \alpha_2$, then

$$\frac{\begin{array}{cc} \vdots & \vdots \\ MN : \alpha_1 & MN : \alpha_2 \end{array}}{MN : \alpha_1 \wedge \alpha_2}$$

By the I.H, we have $\exists \beta_1, \beta_2 \in \mathbb{T}$ such that

$$\begin{array}{l} \Gamma \vdash_\wedge M : \beta_1 \to \alpha_1 \text{ and } \Gamma \vdash_\wedge N : \beta_1, \\ \Gamma \vdash_\wedge M : \beta_2 \to \alpha_2 \text{ and } \Gamma \vdash_\wedge N : \beta_2. \end{array}$$

Then we have $\Gamma \vdash_\wedge N : \beta_1 \wedge \beta_2$ and $\Gamma \vdash_\wedge M : (\beta_1 \to \alpha_1) \wedge (\beta_2 \to \alpha_2)$ by ($\wedge I$). By the definition of $\leq$, we have $(\beta_1 \to \alpha_1) \wedge (\beta_2 \to \alpha_2) \leq (\beta_1 \wedge \beta_2 \to \alpha_1) \wedge (\beta_1 \wedge \beta_2 \to \alpha_2) \leq (\beta_1 \wedge \beta_2) \to (\alpha_1 \wedge \alpha_2)$. Then by ($trans$) and ($\leq$), we have $\Gamma \vdash_\wedge M : (\beta_1 \wedge \beta_2) \to (\alpha_1 \wedge \alpha_2)$.

2. By induction on the derivation of $\Gamma \vdash_\wedge \lambda x.M : \gamma$. Because the subject is in abstraction form, the only cases are $(\omega), (\to I), (\leq)$, and $(\wedge I)$.

   (ω) This case naturally stands.

   ($\leq$) This case can be proved from the I.H and ($\leq$).

   ($\wedge I$) This case can be proved from the I.H and ($\wedge I$).

   ($\to I$) This case can be proved directly from the assumption.

3. By induction on the derivation of $\Gamma \vdash_\wedge \lambda x.M : \alpha$. Suppose $\tau \sim \omega$, then by (ω) and ($\leq$), we have $\Gamma, x : \sigma \vdash_\wedge M : \tau$. So we may suppose $\tau \not\sim \omega$, and because the subject is in abstraction form, the only cases are $(\omega), (\to I), (\leq)$, and $(\wedge I)$.

   (ω) This case is reduced to $\tau \sim \omega$.

   ($\wedge I$) $\alpha \not\equiv \sigma \to \tau$, so this case vacuously stands.

   ($\to I$) This case naturally stands.

   ($\leq$) From I.H, we have $\beta \leq \alpha$, such that $\beta \equiv \sigma' \to \tau'$ and $\Gamma, x : \sigma' \vdash_\wedge M : \tau'$. From the second part of conclusion of Proposition 3.24, it naturally stands. From the first part, we have $\exists \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m \in \mathbb{T}[\forall i \in \{1, 2, \cdots, m\}[\Gamma \vdash_{\wedge n-1} \lambda x.M : \alpha_i \to \beta_i], (\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_m \to \beta_m) \leq \sigma \to \tau]$. From Lemma 3.20, we have $i_1, \cdots, i_k \in \{1, \cdots, m\}$ such that $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \geq \sigma$ and $\beta_{i_1} \wedge \cdots \wedge \beta_{i_k} \leq \tau$. Then we can build $\Gamma, x : \sigma \vdash_\wedge M : \tau$ as follows.

$$
\cfrac{\cfrac{x : \sigma}{x : \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}}\,(\leq)}{x : \alpha_{i_p}(1 \leq p \leq k)}\,(\leq)
$$

$$
\vdots\ (\star)
$$

$$
\cfrac{\cfrac{M : \beta_{i_p}(1 \leq p \leq k)}{M : \beta_{i_1} \wedge \cdots \wedge \beta_{i_k}}\,(\wedge I)}{M : \tau}\,(\leq)
$$

$(\star)$: It is easy to see that $\forall p[x : \alpha_{i_p} \vdash_\wedge \beta_{i_p}]$, because if $x : \alpha_{i_p} \not\vdash_\wedge \beta_{i_p}$, we can apply Proposition 3.24 again until we get to the second part of the conclusion. This procedure is like we climb up the derivation to collect all $(\to I)$ applications on which $\lambda x.M : \alpha$ depends and take their conjunction type.

□

Note: The free variable lemma as follows also holds in this system.

$$
\Gamma \vdash_\wedge M : \alpha \Rightarrow \Gamma \restriction_{FV(M)} \vdash_\wedge M : \alpha,
$$
where $\Gamma \restriction_{FV(M)} = \{\text{x} : \alpha \in \Gamma \mid x \in FV(M)\}$.

3.3 The union type assignment system

After extending to intersection types turns out to be a success, we consider further adding union types to the system. But several difficulties arise when we try to prove β-reduction holds under the new system [5].

However, if we restrain the argument to the set Val defined as follows,

$$
\text{Val} := V \mid \lambda V.\Lambda
$$

we can prove the terms in the new system are invariant under the so-called call-by-value evaluation($\to_v$), which is a weaker version of β-reduction.

$$
(\lambda x.M)N \twoheadrightarrow_v M[x := N]\ (N \in Val)
$$

Definition 3.26. *The set of union type can be defined as follows.*

$$
\begin{aligned}
\mathbb{A} &:= \omega \mid \varphi_0, \varphi_1, \varphi_2 \cdots \\
\mathbb{T} &:= \mathbb{A} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \wedge \mathbb{T} \mid \mathbb{T} \vee \mathbb{T}
\end{aligned}
$$

Definition 3.27. *The union type assignment system TA is defined in the natural deduction manner as follows.*

$$\frac{\begin{array}{c}[x:\sigma]\\ \vdots\\ M:\tau\end{array}}{\lambda x.M:\sigma\to\tau}\ (\to I) \qquad \frac{M:\sigma\to\tau \quad N:\sigma}{MN:\tau}\ (\to E)$$

$$\frac{M:\sigma \quad M:\tau}{M:\sigma\wedge\tau}\ (\wedge I) \qquad \frac{M:\sigma \quad \sigma\le\tau}{M:\tau}\ (\le)$$

$$\frac{}{M:\omega}\ (\omega) \qquad \frac{N:\sigma\vee\tau \quad \begin{array}{c}[x:\sigma]\\ \vdots\\ M:\theta\end{array} \quad \begin{array}{c}[x:\tau]\\ \vdots\\ M:\theta\end{array}}{M[x:=N]:\theta}\ (\vee E)(\star)$$

($\star$) $N \in Val$.

To see why we need the restriction on N, we will show that by a simple example as follows.

Example 3.28. *We consider the following reduction sequence.*

$$\lambda xyz.x((\lambda t.t)yz)((\lambda t.t)yz) \to_v \lambda xyz.x(yz)((\lambda t.t)yz) \to_v \lambda xyz.x(yz)(yz)$$

Now we suppose there is no restriction on N, then we can assign a type to terms on both sides of the sequence as follows.

$x : (\alpha \to (\alpha \to \gamma)) \wedge (\beta \to (\beta \to \gamma))$

$y : \delta \to (\alpha \vee \beta)$

$z : \delta$

$t : \alpha \vee \beta$

Let Γ include all four statement above, then for $\lambda xyz.x(yz)(yz)$, the crucial part of the deduction is as follows.

$$\cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ yz:\alpha\vee\beta\end{array} \quad \cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ x:\alpha\to(\alpha\to\gamma)\end{array} \quad [w:\alpha]}{xw:\alpha\to\gamma} \quad [w:\alpha]}{xww:\gamma} \quad \cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ x:\beta\to(\beta\to\gamma)\end{array} \quad [w:\beta]}{xw:\beta\to\gamma} \quad [w:\beta]}{xww:\gamma}}{xww[w:=yz]:\gamma}}{\begin{array}{c}x(yz)(yz):\gamma\\ \vdots\end{array}}$$

One can easily prove $(\lambda t.t)yz : \alpha \vee \beta$, so we can replace yz for $(\lambda t.t)yz$ in the deduction above and get the same type for $\lambda xyz.x((\lambda t.t)yz)((\lambda t.t)yz)$. But we can not assign the same type for the intermediate one because the substitution applies to all occurrences.

Definition 3.29. *We define another weaker system TA^- by replacing the $(\vee E)$ with the following rule.*

$$\dfrac{x:\sigma\vee\tau \quad \begin{matrix}[x:\sigma]\\ \vdots \\ M:\theta\end{matrix} \quad \begin{matrix}[x:\tau]\\ \vdots \\ M:\theta\end{matrix}}{M:\theta}\ (\vee E)^-$$

Definition 3.30. *The relation $\leq$ is inductively defined as an extension of the same relation in the intersection type assignment system with following rules concerning $\vee$ added.*

$$\alpha \leq \alpha\vee\beta,\ \ \beta\leq\alpha\vee\beta,\ \ \alpha\vee\alpha\leq\alpha \qquad \alpha\wedge(\beta\vee\gamma)\leq(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\ \textit{(dis)}$$

$$(\alpha\to\gamma)\wedge(\beta\to\gamma)\leq\alpha\vee\beta\to\gamma \qquad \dfrac{\alpha\leq\alpha' \quad \beta\leq\beta'}{\alpha\vee\beta\leq\alpha'\vee\beta'}\ (\vee-mono)$$

Proposition 3.31. *$\wedge$ and $\vee$ are associative and commutative modulo $\sim$.*

Proof. This proposition can be proved from the monotonicity of $\wedge$ and $\vee$. □

Lemma 3.32. *We can prove that following equivalences are derivable in this extended type theory.*

$$\alpha\wedge\alpha\sim\alpha \qquad \alpha\vee\alpha\sim\alpha$$
$$\alpha\wedge(\beta\vee\gamma)\sim(\alpha\wedge\beta)\vee(\alpha\wedge\gamma) \qquad \alpha\vee(\beta\wedge\gamma)\sim(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$$

Proof. The first two equivalences can be easily derived from the definition, so we only prove the later ones.

$\alpha\wedge(\beta\vee\gamma)\leq(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$ This case can be derived from the definition.

$\alpha\wedge(\beta\vee\gamma)\geq(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$ From $(\vee-mono)$, we have $(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\leq(\alpha\wedge(\beta\vee\gamma))\vee(\alpha\wedge(\beta\vee\gamma))\leq\alpha\wedge(\beta\vee\gamma)$.

$\alpha\vee(\beta\wedge\gamma)\geq(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$ From the distributive property and the monotonicity, we have $(\alpha\vee\beta)\wedge(\alpha\vee\gamma)\leq((\alpha\vee\beta)\wedge\alpha)\vee((\alpha\vee\beta)\wedge\gamma)\leq(\alpha\wedge\alpha)\vee(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\vee(\beta\wedge\gamma)\leq\alpha\vee\alpha\vee\alpha\vee(\beta\wedge\gamma)\leq\alpha\vee(\beta\wedge\gamma)$.

$\alpha\vee(\beta\wedge\gamma)\leq(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$ From $(\wedge-mono)$, we have $(\alpha\vee\beta)\wedge(\alpha\vee\gamma)\geq(\alpha\vee(\beta\wedge\gamma))\wedge(\alpha\vee(\beta\wedge\gamma))\geq\alpha\vee(\beta\wedge\gamma)$.

□

Lemma 3.33. *Let $\{\alpha_i \mid i\in I\}$ and $\{\beta_j \mid j\in J\}$ be two non-empty finite sets of types, the following two equivalences can be derived from this extended type theory.*

- $\bigvee_{i\in I, j\in J}(\alpha_i \wedge \beta_j) \sim (\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j)$
- $\bigwedge_{i\in I, j\in J}(\alpha_i \vee \beta_j) \sim (\bigwedge_{i\in I}\alpha_i) \vee (\bigwedge_{j\in J}\beta_j)$

Proof. These two are symmetric equivalences, so we only show the first one.

($\leq$) Since for every $i \in I, j \in J$, we have $\alpha_i \wedge \beta_j \leq (\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j)$ from the definition. So from $(\vee - mono)$, we have $\bigvee_{i\in I, j\in J}(\alpha_i \wedge \beta_j) \leq ((\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j)) \vee \cdots \vee ((\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j)) \sim (\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j)$.

($\geq$) By the distributive property and the monotonicity, we have $(\bigvee_{i\in I}\alpha_i) \wedge (\bigvee_{j\in J}\beta_j) \leq ((\bigvee_{i\in I}\alpha_i) \wedge \beta_1) \vee \cdots \vee ((\bigvee_{i\in I}\alpha_i) \wedge \beta_j) \leq (\bigvee_{i\in I}(\alpha_i \wedge \beta_1)) \vee \cdots \vee (\bigvee_{i\in I}(\alpha_i \wedge \beta_j)) \sim \bigvee_{i\in I, j\in J}(\alpha_i \wedge \beta_j)$.

□

Definition 3.34. *The type theory L_T is a sequent calculus system defined as follows.*

Axiom:

$\Gamma \Rightarrow \omega$

$\Gamma, a, \Delta \Rightarrow a \ (a \in \mathbb{A})$

Inference Rules:

$$\frac{\omega \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \theta \quad \Gamma, \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \theta}\ (\vee \Rightarrow) \qquad \frac{\alpha' \Rightarrow \alpha \quad \beta \Rightarrow \beta'}{\Gamma, \alpha \to \beta, \Delta \Rightarrow \alpha' \to \beta'}\ (\to\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \Gamma \Rightarrow \alpha \to \gamma \quad \beta \wedge \gamma \Rightarrow \sigma}{\Gamma \Rightarrow \alpha \to \sigma}\ (\Rightarrow\to \wedge)$$

$$\frac{\Gamma \Rightarrow \alpha \to \gamma \quad \Gamma \Rightarrow \beta \to \gamma \quad \sigma \Rightarrow \alpha \vee \beta}{\Gamma \Rightarrow \sigma \to \gamma}\ (\Rightarrow \vee \to)$$

Note: In the rules above, Γ and Δ are called the ***context****. In the conclusion of each rule, the types other than θ which are not in the context is called the* ***principal*** *types.*

Lemma 3.35. *The following quasi-cut rules are derivable under L_T.*

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \beta \Rightarrow \gamma}{\Gamma \Rightarrow \alpha \to \gamma} (\Rightarrow\to\Rightarrow) \qquad \frac{\Gamma \Rightarrow \beta \to \gamma \quad \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \gamma} (\Rightarrow\Rightarrow\to)$$

Proof.

- The first one can be derived as a special case of ($\Rightarrow\to \wedge$) with the structure property (Weakening-L) which will be treated later.

$$\frac{\Gamma \Rightarrow \alpha \to \beta \quad \Gamma \Rightarrow \alpha \to \beta \quad \dfrac{\dfrac{\beta \Rightarrow \gamma}{\beta, \beta \Rightarrow \gamma}\,(\text{Weakening} - \text{L})}{\beta \wedge \beta \Rightarrow \gamma}\,(\wedge \Rightarrow)}{\Gamma \Rightarrow \alpha \to \gamma}\,(\Rightarrow\to \wedge)$$

- The second one can be derived as a special case of ($\Rightarrow \vee \to$).

$$\frac{\Gamma \Rightarrow \beta \to \gamma \quad \Gamma \Rightarrow \beta \to \gamma \quad \dfrac{\alpha \Rightarrow \beta}{\sigma \Rightarrow \beta \vee \beta}\,(\Rightarrow \vee 1)}{\Gamma \Rightarrow \alpha \to \gamma}\,(\Rightarrow \vee \to)$$

□

Proposition 3.36. *The following statements are true under L_T.*

1. *If $L_T \vdash_n \Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta$, then $L_T \vdash_n \Gamma, \alpha, \beta, \Delta \Rightarrow \theta$.*
2. *If $L_T \vdash_n \Gamma, \alpha \vee \beta, \Delta \Rightarrow \theta$, then $L_T \vdash_n \Gamma, \alpha, \Delta \Rightarrow \theta$ and $L_T \vdash_n \Gamma, \beta, \Delta \Rightarrow \theta$.*

Proof.

1. For the cases with no principal type or the cases with principal type but $\alpha \wedge \beta$ is in the context, they can be proved by the rule applied after I.H. For example, ($\Rightarrow \wedge$) case can be proved as follows.

$$\frac{\dfrac{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \sigma}{\Gamma, \alpha, \beta, \Delta \Rightarrow \sigma}\,(I.H) \quad \dfrac{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \tau}{\Gamma, \alpha, \beta, \Delta \Rightarrow \tau}\,(I.H)}{\Gamma, \alpha, \beta, \Delta \Rightarrow \sigma \wedge \tau}\,(\Rightarrow \wedge)$$

($\to\Rightarrow\to$) case is a little special, but can be proved by its inner weakening property.

$$\frac{\sigma' \Rightarrow \sigma \quad \tau \Rightarrow \tau'}{\Gamma, \alpha, \beta, \sigma \to \tau, \Delta \Rightarrow \sigma' \to \tau'} \ (\to\Rightarrow\to)$$

For the only case with the principal type $\alpha \wedge \beta$, it naturally stands.

2. This inverse property can be proved similarly as above.

□

Lemma 3.37. *We can prove the following structure properties under L_T.*

1. *If $L_T \vdash_n \Gamma, \Delta \Rightarrow \theta$, then $L_T \vdash_n \Gamma, \alpha, \Delta \Rightarrow \theta$.(Weakening-L)*
2. *If $L_T \vdash_n \Gamma, \alpha, \beta, \Delta \Rightarrow \theta$, then $L_T \vdash_n \Gamma, \beta, \alpha, \Delta \Rightarrow \theta$.(Exchange-L)*
3. *If $L_T \vdash_n \Gamma, \alpha, \alpha, \Delta \Rightarrow \theta$, then $L_T \vdash_n \Gamma, \alpha, \Delta \Rightarrow \theta$.(Contraction-L)*

Proof.

1. All cases can be proved by putting the rule applied below the I.H. For example, $(\Rightarrow \wedge)$ case can be proved as follows.

$$\frac{\dfrac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, \alpha, \Delta \Rightarrow \sigma} \ (I.H) \quad \dfrac{\Gamma, \Delta \Rightarrow \tau}{\Gamma, \alpha, \Delta \Rightarrow \tau} \ (I.H)}{\Gamma, \alpha, \Delta \Rightarrow \sigma \wedge \tau} \ (\Rightarrow \wedge)$$

2. For the cases with no principal type, they can be proved by putting the rule applied below the I.H. For example, $(\Rightarrow \wedge)$ case can be proved as follows.

$$\frac{\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \sigma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \sigma} \ (I.H) \quad \dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \tau}{\Gamma, \beta, \alpha, \Delta \Rightarrow \tau} \ (I.H)}{\Gamma, \beta, \alpha, \Delta \Rightarrow \sigma \wedge \tau} \ (\Rightarrow \wedge)$$

$(\to\Rightarrow\to)$ case can be proved by its inner weakening property.
For other cases with principal types, they can be proved by the rule applied after the I.H twice. For example, $(\wedge \Rightarrow)$ can be proved as follows.

$$\frac{\dfrac{\Gamma, \sigma, \tau, \beta, \Delta \Rightarrow \theta}{\Gamma, \beta, \sigma, \tau, \Delta \Rightarrow \theta} \ (I.H) * 2}{\Gamma, \beta, \sigma \wedge \tau, \Delta \Rightarrow \theta} \ (\wedge \Rightarrow)$$

3. For the cases with no principal type, they can be proved by the rule applied after the I.H. For example, $(\Rightarrow \wedge)$ case can be proved as follows.

$$\dfrac{\dfrac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \sigma}{\Gamma, \alpha, \Delta \Rightarrow \sigma}(I.H) \quad \dfrac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \tau}{\Gamma, \alpha, \Delta \Rightarrow \tau}(I.H)}{\Gamma, \alpha, \Delta \Rightarrow \sigma \wedge \tau}(\Rightarrow \wedge)$$

($\rightarrow\Rightarrow\rightarrow$) case can be proved by its inner weakening property.

($\wedge \Rightarrow$) and ($\vee \Rightarrow$) can be proved similarly, here we only show the ($\wedge \Rightarrow$) case.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \alpha, \beta, \alpha \wedge \beta, \Delta \Rightarrow \theta}{\Gamma, \alpha, \beta, \alpha, \beta, \Delta \Rightarrow \theta}(3.36)}{\Gamma, \alpha, \alpha, \beta, \beta, \Delta \Rightarrow \theta}(\mathrm{Exchange}-\mathrm{L})}{\Gamma, \alpha, \beta, \Delta \Rightarrow \theta}(I.H) * 2}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \theta}(\wedge \Rightarrow)$$

□

Lemma 3.38. *The same equivalence as Lemma 3.13 is also true under this extended type theory.*

$$(\alpha \rightarrow \beta) \sim \omega \Leftrightarrow \beta \sim \omega$$

Proof. We can prove this lemma as Lemma 3.13 by some change on the definition of Ω as follows.

$$\Omega := \omega \mid \mathbb{T} \rightarrow \Omega \mid \Omega \wedge \Omega \mid \mathbb{T} \vee \Omega$$

□

Proposition 3.39. *$L_T + Cut \vdash \Gamma \Rightarrow \theta$ if and only if $\bigwedge \Gamma \leq \theta$.*

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \Sigma \Rightarrow \theta}(Cut)$$

Proof.

($\Rightarrow$) We prove this proposition by induction on the derivation of $\Gamma \Rightarrow \theta$. The only non-trivial cases are ($\vee \Rightarrow$), ($\rightarrow\Rightarrow\rightarrow$), ($\Rightarrow\rightarrow \wedge$), ($\Rightarrow\rightarrow$) and ($\Rightarrow \vee \rightarrow$).

($\vee \Rightarrow$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \alpha, \Delta \Rightarrow \theta}{\bigwedge\{\Gamma, \Delta\} \wedge \alpha \leq \theta}(I.H) \quad \dfrac{\Gamma, \beta, \Delta \Rightarrow \theta}{\bigwedge\{\Gamma, \Delta\} \wedge \beta \leq \theta}(I.H)}{(\bigwedge\{\Gamma, \Delta\} \wedge \alpha) \vee (\bigwedge\{\Gamma, \Delta\} \wedge \beta) \leq \theta \vee \theta \sim \theta}(\vee - mono)}{(\star)(\alpha \vee \beta) \wedge (\bigwedge\{\gamma_1 \vee \gamma_2\}) \wedge (\bigwedge\{\gamma_3 \vee \alpha\}) \wedge (\bigwedge\{\gamma_4 \vee \beta\}) \leq \theta}(3.33)}{\bigwedge\{\Gamma, \Delta\} \wedge \cdots \wedge \bigwedge\{\Gamma, \Delta\} \wedge (\alpha \vee \beta) \leq \theta}(\wedge - mono)}{\bigwedge\{\Gamma, \Delta\} \wedge (\alpha \vee \beta) \leq \theta}(3.32)$$

($\star$) $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \{\Gamma, \Delta\}$

$(\to\Rightarrow\to)$

$$\dfrac{\dfrac{\alpha' \Rightarrow \alpha}{\alpha' \leq \alpha}\,(I.H) \quad \dfrac{\beta \Rightarrow \beta'}{\beta \leq \beta'}\,(I.H)}{\bigwedge\Gamma \wedge (\alpha \to \beta) \wedge \bigwedge\Delta \leq \alpha \to \beta \leq \alpha' \to \beta'}\,(\to -mono)$$

$(\Rightarrow\to\wedge)$

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \to \beta \quad \Gamma \Rightarrow \alpha \to \gamma}{\bigwedge\Gamma \leq (\alpha \to \beta) \wedge (\alpha \to \gamma) \leq \alpha \to \beta \wedge \gamma}\,(I.H) \quad \dfrac{\beta \wedge \gamma \Rightarrow \sigma}{\beta \wedge \gamma \leq \sigma}\,(I.H)}{\bigwedge\Gamma \leq \alpha \to \sigma}\,(\to -mono)$$

$(\Rightarrow\to)$ This case can be proved by Lemma 3.38.

$(\Rightarrow\vee\to)$ This case can be proved as $(\Rightarrow\to\wedge)$ similarly.

$(\Leftarrow)$ By induction on the definition of $\leq$. The only non-trivial cases are (ref) and (dis).

(ref) It suffices to show $L_T \vdash \alpha \Rightarrow \alpha$ by induction on the complexity of α. $\alpha \in \mathbb{A}$ case comes directly from the axiom. $\alpha \equiv \sigma \wedge \tau$, $\alpha \equiv \sigma \vee \tau$ and $\alpha \equiv \sigma \to \tau$ can be proved similarly, so here we only show $\alpha \equiv \sigma \wedge \tau$ case.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\sigma \Rightarrow \sigma}}{\sigma, \tau \Rightarrow \sigma}\,(\mathrm{Weakening-L}) \quad \dfrac{\dfrac{}{\tau \Rightarrow \tau}}{\sigma, \tau \Rightarrow \tau}\,(\mathrm{Weakening-L})}{\sigma, \tau \Rightarrow \sigma \wedge \tau}\,(\Rightarrow\wedge)}{\sigma \wedge \tau \Rightarrow \sigma \wedge \tau}\,(\wedge\Rightarrow))$$

(dis) We need to show that $\alpha, \beta\vee\gamma \Rightarrow (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$ can be derived.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}\,(ref)}{\alpha, \beta \Rightarrow \alpha} \quad \dfrac{\overline{\beta \Rightarrow \beta}\,(ref)}{\alpha, \beta \Rightarrow \beta}}{\alpha, \beta \Rightarrow \alpha \wedge \beta}\,(\Rightarrow\wedge)}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\Rightarrow\vee 1) \quad \dfrac{\dfrac{\dfrac{\overline{\alpha \Rightarrow \alpha}\,(ref)}{\alpha, \beta \Rightarrow \alpha} \quad \dfrac{\overline{\beta \Rightarrow \beta}\,(ref)}{\alpha, \beta \Rightarrow \beta}}{\alpha, \beta \Rightarrow \alpha \wedge \beta}\,(\Rightarrow\wedge)}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\Rightarrow\vee 2)}{\alpha, \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}$$

□

Theorem 3.40. *Cut elimination holds for $L_T + Cut$.*

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \Sigma \Rightarrow \theta}\,(Cut)$$

Note: α is defined as the **cut-type**.

Proof. It suffices to show that we can remove an **innermost** cut in a proof tree. When we say an innermost cut, we mean that it is applied above all other cut rule applications. We define the **level** of a cut and the **rank** of a cut as the sum of the depths of the premises and the number of occurrences of type constructors in the cut-type, respectively. For intuition thinking, you can take the **level** of a cut as its depth in the proof tree and the **rank** of a cut as the complexity of the cut-type.

In order to prove this theorem, we proceed by induction on the rank, with a subinduction on the level, and under this method, we can divide the proof into following three cases.

- At least one of the premises is an axiom.
- None of the two premises is an axiom, and the cut-type is not principal in at least one of the premises.
- The cut-type is principal in both premises.

Case 1 At least one of the premises is an axiom.

Subcase 1a The left premise is an axiom. ($\alpha \in \mathbb{A}$)

- The cut-type is principal in the right premise, which means that the right premise is also the axiom.

$$\frac{\Gamma', \alpha, \Delta' \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \alpha}{\Delta, \Gamma', \alpha, \Delta', \Sigma \Rightarrow \alpha}\ (Cut) \qquad \frac{\Gamma \Rightarrow \omega \quad \Delta, \omega, \Sigma \Rightarrow \omega}{\Delta, \Gamma, \Sigma \Rightarrow \omega}\ (Cut)$$

  The conclusions of the two are axioms, so they can be directly derived without the cut rule.

- The cut-type is not principal in the right premise.

$$\frac{\Gamma', \alpha, \Delta' \Rightarrow \alpha \quad \begin{matrix}\vdots \\ \Delta, \alpha, \Sigma \Rightarrow \theta\end{matrix}}{\Delta, \Gamma', \alpha, \Delta', \Sigma \Rightarrow \theta}\ (Cut) \qquad \frac{\Gamma \Rightarrow \omega \quad \begin{matrix}\vdots \\ \Delta, \omega, \Sigma \Rightarrow \theta\end{matrix}}{\Delta, \Gamma', \omega, \Delta', \Sigma \Rightarrow \theta}\ (Cut)$$

  The conclusions can be derived by $(Weakening - L)$ from the right premise without the cut rule.

Subcase 1b The left premise is not the axiom, while the right premise is the axiom.

- The cut-type is principal in the right premise. ($\alpha \in \mathbb{A}$)

$$\frac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \alpha\end{matrix} \quad \Delta, \alpha, \Sigma \Rightarrow \alpha}{\Delta, \Gamma, \Sigma \Rightarrow \alpha}\ (Cut)$$

  The conclusion can be derived by $(Weakening - L)$ from the left premise without the cut rule.

- The cut-type is not principal in the right premise.

$$\dfrac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, a, \Sigma \Rightarrow a}{\Delta, \Gamma, a, \Sigma \Rightarrow a}\ (Cut) \qquad \dfrac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \omega}{\Delta, \Gamma, \Sigma \Rightarrow \omega}\ (Cut)$$

  The conclusions are axioms, so they can be directly derived without the cut rule.

Case 2

None of the two premises is the axiom and the cut-type is not principal in at least one of the premises.

Subcase 2a The cut-type is not principal in the right premise.
Although we need to consider all rules ended up as the right premise with the cut-type in the context, it turns out that we only need to apply the cut rule before every rule. Here we only show two cases, and all other cases can be proved similarly.

$(\wedge \Rightarrow)$

$$\dfrac{\Gamma \Rightarrow \alpha \quad \dfrac{\Delta, \alpha, \sigma, \tau, \Sigma \Rightarrow \theta}{\Delta, \alpha, \sigma \wedge \tau, \Sigma \Rightarrow \theta}\ (\wedge \Rightarrow)}{\Delta, \Gamma, \sigma \wedge \tau, \Sigma \Rightarrow \theta}\ (Cut) \qquad \dfrac{\dfrac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \sigma, \tau, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \sigma, \tau, \Sigma \Rightarrow \theta}\ (Cut)}{\Delta, \Gamma, \sigma \wedge \tau, \Sigma \Rightarrow \theta}\ (\wedge \Rightarrow)$$

$(\rightarrow\Rightarrow\rightarrow)$

$$\dfrac{\Gamma \Rightarrow \alpha \quad \dfrac{\sigma' \Rightarrow \sigma \quad \tau' \Rightarrow \tau}{\Delta, \alpha, \sigma \rightarrow \tau, \Sigma \Rightarrow \sigma' \rightarrow \tau'}\ (\rightarrow\Rightarrow\rightarrow)}{\Delta, \Gamma, \sigma \rightarrow \tau, \Sigma \Rightarrow \sigma' \rightarrow \tau'}\ (Cut) \qquad \dfrac{\sigma' \Rightarrow \sigma \quad \tau' \Rightarrow \tau}{\Delta, \Gamma, \sigma \rightarrow \tau, \Sigma \Rightarrow \sigma' \rightarrow \tau'}$$

Subcase 2b The cut-type is principal in the right premise, while it is not principal in the left premise.
In this subcase, the rule ended up as the right premise can only be $(\wedge \Rightarrow)$, $(\vee \Rightarrow)$ and $(\rightarrow\Rightarrow\rightarrow)$, while the rule ended up as the left premise can only be $(\wedge \Rightarrow)$ and $(\vee \Rightarrow)$.
This subcase can be similarly proved as Subcase 2a, here we only show the proof in which the left premise is $(\wedge \Rightarrow)$.

$$\dfrac{\dfrac{\Gamma', \sigma, \tau, \Delta' \Rightarrow \alpha \wedge \beta}{\Gamma', \sigma \wedge \tau, \Delta' \Rightarrow \alpha \wedge \beta} \quad \dfrac{\Delta, \alpha, \beta, \Sigma \Rightarrow \theta}{\Delta, \alpha \wedge \beta, \Sigma \Rightarrow \theta}}{\Delta, \Gamma', \sigma \wedge \tau, \Delta', \Sigma \Rightarrow \theta}\ (Cut)$$

It can be transformed into the following proof.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma', \sigma, \tau, \Delta' \Rightarrow \alpha \wedge \beta\end{matrix} \quad \dfrac{\begin{matrix}\vdots \\ \Delta, \alpha, \beta, \Sigma \Rightarrow \theta\end{matrix}}{\Delta, \alpha \wedge \beta, \Sigma \Rightarrow \theta}}{\Delta, \Gamma', \sigma, \tau, \Delta', \Sigma \Rightarrow \theta}\,(Cut)}{\Delta, \Gamma', \sigma \wedge \tau, \Delta', \Sigma \Rightarrow \theta}$$

Case 3 None of the two premises is the axiom, and the cut-type is principal in both premises.

Subcase 3a The last rule applied in the right premise is $(\wedge \Rightarrow)$.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma \Rightarrow \beta\end{matrix}}{\Gamma \Rightarrow \alpha \wedge \beta} \quad \dfrac{\begin{matrix}\vdots \\ \Delta, \alpha, \beta, \Sigma \Rightarrow \theta\end{matrix}}{\Delta, \alpha \wedge \beta, \Sigma \Rightarrow \theta}\,(\wedge \Rightarrow)}{\Delta, \Gamma, \Sigma \Rightarrow \theta}\,(Cut)$$

This subcase can be proved as follows.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \beta\end{matrix} \quad \dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \Delta, \alpha, \beta, \Sigma \Rightarrow \theta\end{matrix}}{\Delta, \Gamma, \beta, \Sigma \Rightarrow \theta}\,(Cut)}{\Delta, \Gamma, \Gamma, \Sigma \Rightarrow \theta}\,(Cut)}{\Delta, \Gamma, \Sigma \Rightarrow \theta}\,3.37(2),(3)$$

Subcase 3b The last rule applied in the right premise is $(\vee \Rightarrow)$. This case can be proved as above, so we omit the proof here.

Subcase 3c The last rule applied in the right premise is $(\rightarrow\Rightarrow\rightarrow)$. This case can be further divided into three cases as the rule applied in the left premise can be $(\rightarrow\Rightarrow\rightarrow)$, $(\Rightarrow\rightarrow \wedge)$, $(\Rightarrow \vee \rightarrow)$ and $(\Rightarrow\rightarrow)$.

$(\rightarrow\Rightarrow\rightarrow)$ In this case, the proof is as follows.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \alpha \Rightarrow \alpha'\end{matrix} \quad \begin{matrix}\vdots \\ \beta' \Rightarrow \beta\end{matrix}}{\Gamma', \alpha' \rightarrow \beta', \Delta' \Rightarrow \alpha \rightarrow \beta} \quad \dfrac{\begin{matrix}\vdots \\ \alpha'' \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \beta \Rightarrow \beta''\end{matrix}}{\Delta, \alpha \rightarrow \beta, \Sigma \Rightarrow \alpha'' \rightarrow \beta''}}{\Delta, \Gamma', \alpha' \rightarrow \beta', \Delta', \Sigma \Rightarrow \alpha'' \rightarrow \beta''}\,(Cut)$$

It can be transformed into the following proof.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \alpha'' \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \alpha \Rightarrow \alpha'\end{matrix}}{\alpha'' \Rightarrow \alpha'}\,(Cut) \quad \dfrac{\begin{matrix}\vdots \\ \beta'' \Rightarrow \beta\end{matrix} \quad \begin{matrix}\vdots \\ \beta \Rightarrow \beta'\end{matrix}}{\beta'' \Rightarrow \beta'}\,(Cut)}{\Delta, \Gamma', \alpha' \rightarrow \beta', \Delta', \Sigma \Rightarrow \alpha'' \rightarrow \beta''}\,(\rightarrow\Rightarrow\rightarrow)$$

($\Rightarrow\to\wedge$) In this case, the proof is as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha\to\sigma \quad \Gamma\Rightarrow\alpha\to\tau \quad \sigma\wedge\tau\Rightarrow\beta}{\Gamma\Rightarrow\alpha\to\beta} \qquad \dfrac{\alpha'\Rightarrow\alpha \quad \beta\Rightarrow\beta'}{\Delta,\alpha\to\beta,\Sigma\Rightarrow\alpha'\to\beta'}}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}\ (Cut)$$

It can be transformed without any applications of the cut rule into the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \alpha\to\sigma \quad \alpha'\Rightarrow\alpha}{\Gamma\Rightarrow\alpha'\to\sigma} \quad \dfrac{\Gamma\Rightarrow\alpha\to\tau \quad \alpha'\Rightarrow\alpha}{\Gamma\Rightarrow\alpha'\to\tau} \quad \sigma\wedge\tau\Rightarrow\beta}{\Gamma\Rightarrow\alpha'\to\beta} \qquad \beta\Rightarrow\beta'}{\Gamma\Rightarrow\alpha'\to\beta'}}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}\ (\mathrm{Weakening-L})$$

($\Rightarrow\vee\to$) In this case, the proof is as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow \sigma\to\beta \quad \Gamma\Rightarrow\tau\to\beta \quad \alpha\Rightarrow\sigma\vee\tau}{\Gamma\Rightarrow\alpha\to\beta} \qquad \dfrac{\alpha'\Rightarrow\alpha \quad \beta\Rightarrow\beta'}{\Delta,\alpha\to\beta,\Sigma\Rightarrow\alpha'\to\beta'}}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}\ (Cut)$$

It can be transformed into the following proof with one application of the cut rule, which has lower rank and level.

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \sigma\to\beta \quad \Gamma\Rightarrow\tau\to\beta \quad \dfrac{\alpha'\Rightarrow\alpha \quad \alpha\Rightarrow\sigma\vee\tau}{\alpha'\Rightarrow\sigma\vee\tau}\ (Cut)}{\Gamma\Rightarrow\alpha'\to\beta} \qquad \beta\Rightarrow\beta'}{\Gamma\Rightarrow\alpha'\to\beta'}}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}\ (\mathrm{Weakening-L})$$

($\Rightarrow\to$) In this case, the proof is as follows.

$$\dfrac{\dfrac{\omega\Rightarrow\beta}{\Gamma\Rightarrow\alpha\to\beta} \qquad \dfrac{\alpha'\Rightarrow\alpha \quad \beta\Rightarrow\beta'}{\Delta,\alpha\to\beta,\Sigma\Rightarrow\alpha'\to\beta'}}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}\ (Cut)$$

It can be transformed into the following proof.

$$\dfrac{\dfrac{\omega\Rightarrow\beta \quad \beta\Rightarrow\beta'}{\omega\Rightarrow\beta'}\ (Cut)}{\Delta,\Gamma,\Sigma\Rightarrow\alpha'\to\beta'}$$

Note: As you can see, all converted cut-types have lower levels in this proof. The reason why we still need the double induction as in the canonical proof method [6] is in Subcase 3a, in which the higher cut is replaced by the I.H with a depth-unknown proof so that we can not eliminate the lower one with the I.H of level. □

Theorem 3.41. *$L_T \vdash \Gamma \Rightarrow \theta$ if and only if $\bigwedge \Gamma \leq \theta$.*

Proof. Straightforward. □

Definition 3.42. *The set of **prime types** $\mathbb{P}(\subseteq \mathbb{T})$ can be defined as follows.*

$$\mathbb{P} := \mathbb{A} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{P} \wedge \mathbb{P}$$

Lemma 3.43. *If $\alpha \in \mathbb{T}$, then there exists a non-empty finite set $\{\sigma_i \mid i \in I, \sigma_i \in \mathbb{P}\}$ such that $\alpha \sim \bigvee_{i \in I} \sigma_i$.*

Proof. We prove this lemma by induction on the complexity of α.

($\alpha \equiv \mathbb{A}$) This case is straightforward from the definition.

From I.H, we have $\beta \sim \bigvee_{j_1 \in J_1} \sigma_{j_1}$ and $\gamma \sim \bigvee_{j_2 \in J_2} \sigma_{j_2}$.

($\alpha \equiv \beta \wedge \gamma$) From Lemma 3.33, we have $\alpha \equiv \beta \wedge \gamma \sim \bigvee_{j_1 \in J_1, j_2 \in J_2} (\sigma_{j_1} \wedge \sigma_{j_2})$

($\alpha \equiv \beta \wedge \gamma$) We have $\alpha \equiv \beta \vee \gamma \sim \bigvee_{i \in J_1 \cup J_2} \sigma_i$ from the I.H,.

($\alpha \equiv \beta \to \gamma$) We have $\alpha \equiv \beta \to \gamma \sim \bigvee_{j_1 \in J_1} \sigma_{j_1} \to \bigvee_{j_2 \in J_2} \sigma_{j_2} (\in \mathbb{P})$ from the I.H,.

□

Proposition 3.44. *If $\sigma \in \mathbb{P}$ and $\sigma \leq \alpha \vee \beta$, then $\sigma \leq \alpha$ or $\sigma \leq \beta$.*

Proof. By Theorem 3.41, it suffices to show that for a non-empty finite sequence Γ of prime types, $L_T \vdash \Gamma \Rightarrow \alpha \vee \beta$ implies $L_T \vdash \Gamma \Rightarrow \alpha$ or $L_T \vdash \Gamma \Rightarrow \beta$. We will prove this proposition by induction on the depth of the proof of $L_T \vdash \Gamma \Rightarrow \alpha$. The only cases ended up with $L_T \vdash \Gamma \Rightarrow \alpha$ are ($\wedge \Rightarrow$), ($\vee \Rightarrow$), ($\Rightarrow \vee 1$) and ($\Rightarrow \vee 2$).

($\Rightarrow \vee 1$), ($\Rightarrow \vee 2$) Straightforward.

($\wedge \Rightarrow$) This case is straightforward from the I.H.

($\vee \Rightarrow$) In this case, the proof ends up as follows.

$$\frac{\begin{array}{cc} \vdots & \vdots \\ \Gamma, \sigma, \Delta \Rightarrow \alpha \vee \beta & \Gamma, \tau, \Delta \Rightarrow \alpha \vee \beta \end{array}}{\Gamma, \sigma \vee \tau, \Delta \Rightarrow \alpha \vee \beta}$$

But $\sigma \vee \tau$ is not a prime type, so we do not need to consider this case.

$\square$

Proposition 3.45. *If* $\bigwedge_{i\in I}(\alpha_i \to \beta_i) \leq \alpha \to \beta$ *and* $\beta \not\sim \omega$, *then there exist two finite sets* J *and* $\{I_j \mid j \in J, I_j \subseteq I\}$ *such that*

$$\alpha \leq \bigvee_{j\in J}\bigwedge_{i\in I_j} \alpha_i \textit{ and } \bigvee_{j\in J}\bigwedge_{i\in I_j} \beta_i \leq \beta.$$

Proof. By Theorem 3.41, it suffices to show that $L_T \vdash \alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m \Rightarrow \alpha \to \beta$ implies the same conclusion. We can prove this by induction on the depth of $\alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m \Rightarrow \alpha \to \beta$, and the only cases need to be treated are $(\to\Rightarrow\to)$, $(\Rightarrow\to\wedge)$ and $(\Rightarrow\vee\to)$.

Note: We do not need to treat $(\Rightarrow\to)$ case because of the restriction on β.

$(\to\Rightarrow\to)$

$$\frac{\begin{matrix}\vdots & \vdots \\ \alpha \Rightarrow \alpha_k & \beta_k \Rightarrow \beta\end{matrix}}{\alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m \Rightarrow \alpha \to \beta}\ (\star)(\to\Rightarrow\to)$$

$(\star)$ $1 \leq k \leq m$
By Theorem 3.41, we have $\alpha \leq \alpha_k$ and $\beta_k \leq \beta$ from the assumptions. We simply set $J := \{1\}$ and $I_1 := k$.

$(\Rightarrow\to\wedge)$

$$\frac{\begin{matrix}\vdots & \vdots & \vdots \\ \Sigma \Rightarrow \alpha \to \gamma_1 & \Sigma \Rightarrow \alpha \to \gamma_2 & \gamma_1 \wedge \gamma_2 \Rightarrow \beta\end{matrix}}{\Sigma \Rightarrow \alpha \to \beta}\ (\star)(\Rightarrow\to\wedge)$$

$(\star)$ $\Sigma = \alpha_1 \to \beta_1, \cdots, \alpha_m \to \beta_m$
From I.H, there exists four finite sets $J_1, J_2, \{I_{j_1} \mid j_1 \in J_1, I_{j_1} \subseteq I\}$, and $\{I_{j_2} \mid j_2 \in J_2, I_{j_2} \subseteq I\}$, such that

$$\alpha \leq \bigvee_{j_1\in J_1}\bigwedge_{i\in I_{j_1}} \alpha_i \text{ and } \bigvee_{j_1\in J_1}\bigwedge_{i\in I_{j_1}} \beta_i \leq \gamma_1,$$
$$\alpha \leq \bigvee_{j_2\in J_2}\bigwedge_{i\in I_{j_2}} \alpha_i \text{ and } \bigvee_{j_2\in J_2}\bigwedge_{i\in I_{j_2}} \beta_i \leq \gamma_2.$$

From Lemma 3.33 and $(\wedge - mono)$, we have

$$(\bigvee_{j_1\in J_1}\bigwedge_{i\in I_{j_1}} \beta_i) \wedge (\bigvee_{j_2\in J_2}\bigwedge_{i\in I_{j_2}} \beta_i) \sim \bigvee_{(j_1,j_2)\in J}\bigwedge_{i\in I_{(j_1,j_2)}} \beta_i$$
$$\bigvee_{(j_1,j_2)\in J}\bigwedge_{i\in I_{(j_1,j_2)}} \beta_i \leq \gamma_1 \wedge \gamma_2 \leq \beta$$

and

$$(\bigvee_{j_1\in J_1}\bigwedge_{i\in I_{j_1}} \alpha_i) \wedge (\bigvee_{j_2\in J_2}\bigwedge_{i\in I_{j_2}} \alpha_i) \sim \bigvee_{(j_1,j_2)\in J}\bigwedge_{i\in I_{(j_1,j_2)}} \alpha_i$$

$$\alpha \sim \alpha \wedge \alpha \leq \bigvee_{(j_1,j_2)\in J} \bigwedge_{i\in I_{(j_1,j_2)}} \alpha_i.$$

In this case, $J := J_1 \times J_2$ and $I_{(j_1,j_2)} := I_{j_1} \cup I_{j_2}$.

$(\Rightarrow \vee \rightarrow)$

$$\dfrac{\begin{array}{ccc} \vdots & \vdots & \vdots \\ \Sigma \Rightarrow \gamma_1 \rightarrow \beta & \Sigma \Rightarrow \gamma_2 \rightarrow \beta & \alpha \Rightarrow \gamma_1 \vee \gamma_2 \end{array}}{\Sigma \Rightarrow \alpha \rightarrow \beta} \ (\star)(\Rightarrow \vee \rightarrow)$$

$(\star)$ $\Sigma = \alpha_1 \rightarrow \beta_1, \cdots, \alpha_m \rightarrow \beta_m$

From I.H, there exists four finite sets $J_1, J_2, \{I_{j_1} \mid j_1 \in J_1, I_{j_1} \subseteq I\}$, and $\{I_{j_2} \mid j_2 \in J_2, I_{j_2} \subseteq I\}$, such that

$$\gamma_1 \leq \bigvee_{j_1\in J_1} \bigwedge_{i\in I_{j_1}} \alpha_i \text{ and } \bigvee_{j_1\in J_1} \bigwedge_{i\in I_{j_1}} \beta_i \leq \beta,$$
$$\gamma_2 \leq \bigvee_{j_2\in J_2} \bigwedge_{i\in I_{j_2}} \alpha_i \text{ and } \bigvee_{j_2\in J_2} \bigwedge_{i\in I_{j_2}} \beta_i \leq \beta.$$

By $(\vee - mono)$, we have

$$\alpha \leq \gamma_1 \vee \gamma_2 \leq (\bigvee_{j_1\in J_1} \bigwedge_{i\in I_{j_1}} \beta_i) \vee (\bigvee_{j_2\in J_2} \bigwedge_{i\in I_{j_2}} \beta_i)$$
$$(\bigvee_{j_1\in J_1} \bigwedge_{i\in I_{j_1}} \beta_i) \vee (\bigvee_{j_2\in J_2} \bigwedge_{i\in I_{j_2}} \beta_i) \leq \beta \vee \beta \sim \beta$$

In this case, $J := J_1 \cup J_2$.

□

As in the intersection system, we need to restrain β so that the assumption will not explode by Lemma 3.38.

Lemma 3.46.

1. $\Gamma \vdash_{TA^-} M : \alpha \Rightarrow \Gamma \upharpoonright_{FV(M)} \vdash_{TA^-} M : \alpha$ *(The free variable lemma)*

2. $\Gamma, x : \alpha \vdash_{TA^-} M : \gamma$ *and* $\beta \leq \alpha \Rightarrow \Gamma, x : \beta \vdash_{TA^-} M : \gamma$

3. $\Gamma \vdash_{TA^-} x : \beta \Leftrightarrow \bigwedge \Gamma_x \leq \beta$ *or* $\beta \sim \omega$, *where* $\Gamma_x = \{\alpha \mid x : \alpha \in \Gamma\}$

Proof.

1. One can easily prove this lemma by induction on the derivation of $\Gamma \vdash_{TA^-} M : \alpha$.

2. One can easily prove this lemma by $(\leq)$.

3. The proof for this lemma is trivial, so we omit here.

□

Definition 3.47. Γ *is **prime basis** if* $\Gamma \vdash_{TA^-} x : \alpha \vee \beta$ *implies* $\Gamma \vdash_{TA^-} x : \alpha$ *or* $\Gamma \vdash_{TA^-} x : \beta$.

Lemma 3.48.

1. *Every deduction in* TA^- *can be replaced by a* $(\vee E)^-$*-last deduction with the same assumptions and conclusion.*

2. *If* Γ *is a prime basis and* $\Gamma \vdash_{TA^-} M : \theta$*, then there exists a* $(\vee E)^-$*-free deduction of the same derivation.*

Proof.

1. We push the $(\vee E)^-$ step down below all other rules which means that $(\to I)$, $(\to E)$, $(\wedge I)$ and $(\leq)$ need to be treated.
   $(\wedge I)$ case: The proof ends up as follows.

$$\dfrac{\begin{matrix}\Delta_0\\ \vdots\\ M:\alpha\end{matrix} \qquad \dfrac{\begin{matrix}\Delta_1\\ \vdots\\ x:\sigma\vee\tau\end{matrix}\quad \begin{matrix}\Delta_2,[x:\sigma]\\ \vdots\\ M:\beta\end{matrix}\quad \begin{matrix}\Delta_3,[x:\tau]\\ \vdots\\ M:\beta\end{matrix}}{M:\beta}\,(\vee E)^-}{M:\alpha\wedge\beta}\,(\wedge I)$$

   It is instinctive to think that we can treat this case as simply move the $(\vee E)^-$ application below, but it turns out to be a problematic proof as follows.

$$\dfrac{\begin{matrix}\Delta_1\\ \vdots\\ x:\sigma\vee\tau\end{matrix} \qquad \dfrac{\begin{matrix}\Delta_0',[x:\sigma]\\ \vdots\\ M:\alpha\end{matrix}\quad \begin{matrix}\Delta_2,[x:\sigma]\\ \vdots\\ M:\beta\end{matrix}}{M:\alpha\wedge\beta} \qquad \dfrac{\begin{matrix}\Delta_0',[x:\tau]\\ \vdots\\ M:\alpha\end{matrix}\quad \begin{matrix}\Delta_3,[x:\tau]\\ \vdots\\ M:\beta\end{matrix}}{M:\alpha\wedge\beta}}{M:\alpha\wedge\beta}\,(\vee E)^-$$

   As you can see, when $\Delta_{0x} \neq \emptyset$, the transformed proof has a different assumption set compared to the original one. In order to solve this, we take the conjunction form of the Δ_{0x} so that it will be canceled while the original assumption remains.

$$\dfrac{\dfrac{\dfrac{x:\gamma \quad \begin{matrix}\Delta_1\\ \vdots\\ x:\sigma\vee\tau\end{matrix}}{x:\gamma\wedge(\sigma\vee\tau)}}{x:(\gamma\wedge\sigma)\vee(\gamma\wedge\tau)} \qquad \dfrac{\begin{matrix}\Gamma,[x:\gamma\wedge\sigma]\\ \vdots\\ M:\alpha\end{matrix}\quad \begin{matrix}\Delta_2,[x:\gamma\wedge\sigma]\\ \vdots\\ M:\beta\end{matrix}}{M:\alpha\wedge\beta} \qquad \begin{matrix}\Gamma,\Delta_3,[x:\gamma\wedge\tau]\\ \vdots\\ M:\alpha\wedge\beta\end{matrix}}{M:\alpha\wedge\beta}\,(\vee E)^-$$

   Note: $\Delta_0 \Leftrightarrow \Gamma, x:\gamma$, where $\gamma \equiv \bigwedge \Delta_{0x}$.

   It is easy to see that this problem occurs when the last applied rule has two premises with different assumption set, so $(\leq)$ do not have this

problem which means it can be treated simply be pushing down the $(\vee E)^-$.
$(\to E)$ case can be treated with the same trick as above, so we omit here.

$(\to I)$ case:

($x \not\equiv y$) The proof ends as follows.

$$\dfrac{\dfrac{\begin{matrix}\Delta_0,[x:\alpha]\\ \vdots \\ y:\sigma\vee\tau\end{matrix} \qquad \begin{matrix}\Delta_1,[x:\alpha],[y:\sigma]\\ \vdots \\ M:\beta\end{matrix} \qquad \begin{matrix}\Delta_2,[x:\alpha],[y:\tau]\\ \vdots \\ M:\beta\end{matrix}}{M:\beta}\,(\vee E)^-}{\lambda x.M:\alpha\to\beta}\,(\to I)$$

This case can be treated easily with the free variable lemma as follows.

$$\dfrac{\begin{matrix}\Delta_0\\ \vdots \\ y:\sigma\vee\tau\end{matrix} \qquad \dfrac{\begin{matrix}\Delta_1,[x:\alpha],[y:\sigma]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M:\alpha\to\beta} \qquad \dfrac{\begin{matrix}\Delta_2,[x:\alpha],[y:\tau]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M:\alpha\to\beta}}{\lambda x.M:\alpha\to\beta}$$

($x \equiv y$) The proof ends as follows.

$$\dfrac{\dfrac{\begin{matrix}\Delta_0,[x:\alpha]\\ \vdots \\ x:\sigma\vee\tau\end{matrix} \qquad \begin{matrix}\Delta_1,[x:\sigma]\\ \vdots \\ M:\beta\end{matrix} \qquad \begin{matrix}\Delta_2,[x:\tau]\\ \vdots \\ M:\beta\end{matrix}}{M:\beta}\,(\vee E)^-}{\lambda x.M:\alpha\to\beta}\,(\to I)$$

This case can be treated with a little trick with Lemma 3.46(3) as follows.

$$\dfrac{(\sigma\to\beta)\wedge(\tau\to\beta)\leq\gamma\to\beta \qquad \dfrac{\dfrac{\begin{matrix}\Delta_1,[x:\sigma]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M:\sigma\to\beta} \quad \dfrac{\begin{matrix}\Delta_2,[x:\tau]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M:\tau\to\beta}}{\lambda x.M:(\sigma\to\beta)\wedge(\tau\to\beta)}}{\lambda x.M:\gamma\to\beta}\,(\leq)$$

We can prove $\gamma\equiv\alpha$ as follows.

$$\dfrac{\dfrac{\Delta_0, x:\alpha\vdash_{TA^-} x:\sigma\vee\tau}{x:\alpha\vdash_{TA^-} x:\sigma\vee\tau}\,3.46(1)}{\alpha\leq\sigma\vee\tau}$$

2. For every deduction in TA^- ending up with $(\vee E)^-$, the proof is as follows.

$$\dfrac{\begin{matrix} & [x:\sigma] & [x:\tau] \\ \vdots & \vdots & \vdots \\ x:\sigma\vee\tau & M:\theta & M:\theta \end{matrix}}{M:\theta}\ (\vee E)^-$$

If we restrain the basis to be prime, we actually can convert the proof into the following convenient one without $(\vee E)^-$ completely.

$$\dfrac{\begin{matrix} \Gamma & \\ \vdots & \\ x:\bigwedge\Gamma_x & \dfrac{\Gamma\vdash_{TA^-} x:\sigma}{\bigwedge\Gamma_x\le\sigma}\ 3.46(3)(\star) \end{matrix}}{x:\sigma}\ (\le)$$
$$\vdots$$
$$M:\theta$$

$(\star)$ The case with $\sigma\sim\omega$ naturally stands.

$\Gamma\vdash_{TA^-} x:\tau$ case is omitted because of its similarity with the above case.

□

Lemma 3.49. *(The generation lemma)*
Γ *is a prime basis.*

1. $\Gamma\vdash_{TA^-} MN:\alpha\Rightarrow\exists\beta\in\mathbb{T}[\Gamma\vdash M:\beta\to\alpha$ *and* $\Gamma\vdash N:\beta]$.

2. $\Gamma\vdash_{TA^-}\lambda x.M:\gamma\Rightarrow\exists\sigma_1,\cdots,\sigma_n\in\mathbb{P},\beta_1,\cdots,\beta_n\in\mathbb{T}[\forall i[\Gamma,x:\sigma_i\vdash_{TA^-} M:\beta_i]$ *and* $\bigwedge_i(\sigma_i\to\beta_i)\le\gamma](1\le i\le n)$.

3. $\Gamma\vdash_{TA^-}\lambda x.M:\alpha\Rightarrow\exists\sigma,\tau\in\mathbb{T}[\alpha\equiv\sigma\to\tau$ *and* $\Gamma,x:\sigma\vdash M:\tau]$.

Proof.

1. We can prove this lemma by turning all deduction to the $(\vee E)^-$-free deduction by Lemma 3.48, then everything follows as in the intersection system.

2. By induction on the depth of the $(\vee E)^-$-free derivation of $\Gamma\vdash_{TA^-}\lambda x.M:\gamma$. $(\le)$, $(\omega)$and $(\wedge I)$ are trivial, so we only treat the $(\to I)$ case. The proof ends up as lower left and can be proved as lower right.

$$\dfrac{\begin{matrix}[x:\alpha]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M:\alpha\to\beta(\equiv\gamma)} \qquad \dfrac{x:\sigma_i \quad \overline{\sigma_i\le\bigvee_i\sigma_i\le\alpha}\ (3.43)}{x:\alpha} \\ \vdots \\ M:\beta$$

3. By (2), we have $\sigma_1, \cdots, \sigma_n \in \mathbb{P}, \beta_1, \cdots, \beta_n \in \mathbb{T}$ such that

$$\forall m[\Gamma, x : \sigma_m \vdash_{TA^-} M : \beta_m] \text{ and } \bigwedge_m (\sigma_m \to \beta_m) \leq \sigma \to \tau (1 \leq m \leq n).$$

Then by Proposition 3.45, we have

$$\sigma \leq \bigvee_J \bigwedge_{I_j} \sigma_i \text{ and } \bigvee_J \bigwedge_{I_j} \beta_i \leq \tau.$$

By these four, we can prove this lemma as follows.

Firstly, we need to prove that for all $j \in J$ that

$$\Gamma, x : \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \tau$$

as follows.

$$\dfrac{\bigwedge_{I_j} \beta_i \leq \bigvee_J \bigwedge_{I_j} \beta_i \leq \tau \qquad \dfrac{\dfrac{\Gamma, x : \sigma_m \vdash_{TA^-} M : \beta_m \quad \bigwedge_{I_j} \sigma_i \leq \sigma_m}{\Gamma, x : \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \beta_m}(3.46)(2) \quad \cdots}{\Gamma, x : \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \bigwedge_{I_j} \beta_i}(\wedge I)}{\Gamma, x : \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \tau}$$

Secondly, we can prove the remaining by Lemma 3.46(2) and $(\vee E)^-$ as follows.

$$\dfrac{\sigma \leq \bigvee_J \bigwedge_{I_j} \sigma_i \qquad \dfrac{\dfrac{\sigma \leq \bigvee_J \bigwedge_{I_j} \sigma_i \quad x : \sigma}{x : \bigvee_J \bigwedge_{I_j} \sigma_i}(\leq) \quad \begin{matrix} 1st\ j \in J \\ \vdots \\ \Gamma, x : \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \tau \end{matrix} \quad \cdots}{\Gamma, x : \bigvee_J \bigwedge_{I_j} \sigma_i \vdash_{TA^-} M : \tau}(\vee E)^-}{\Gamma, x : \sigma \vdash_{TA^-} M : \tau}(3.46)(2)$$

□

4 Semantics

4.1 The Filter model

Definition 4.1. *(Type interpretation)*

- *Let ξ: $\{\psi_i\} \to \mathscr{P}(D)$, so ξ is a* ***type environment*** *from all type variables to power set of D.*

- *The interpretation of $\sigma \in \mathbb{T}$ in a lambda model $\mathscr{M}$ via a type environment ξ, denoted as $[\![\sigma]\!]_\xi^{\mathscr{M}} \in \mathscr{P}(D)$, can be defined as follows.*
  - $[\![\omega]\!]_\xi^{\mathscr{M}} = D$
  - $[\![\psi_i]\!]_\xi^{\mathscr{M}} = \xi(\psi_i)$
  - $[\![\sigma \to \tau]\!]_\xi^{\mathscr{M}} = \{d \in D \mid \forall e \in [\![\sigma]\!]_\xi^{\mathscr{M}} [d \cdot e \in [\![\tau]\!]_\xi^{\mathscr{M}}]\}$
  - $[\![\sigma \wedge \tau]\!]_\xi^{\mathscr{M}} = [\![\sigma]\!]_\xi^{\mathscr{M}} \cap [\![\tau]\!]_\xi^{\mathscr{M}}$
- *Let ρ be a term environment in D.*
- $\mathscr{M}, \rho, \xi \models M : \sigma$ *if and only if* $[\![M]\!]_\rho^{\mathscr{M}} \in [\![\sigma]\!]_\xi^{\mathscr{M}}$.
- $\mathscr{M}, \rho, \xi \models \Gamma$ *if and only if* $\mathscr{M}, \rho, \xi \models x : \sigma$ *for all* $x : \sigma \in \Gamma$.
- $\Gamma \models M : \sigma$ *if and only if* $\forall \mathscr{M}, \rho, \xi \models \Gamma[\mathscr{M}, \rho, \xi \models M : \sigma]$.

Lemma 4.2. $\sigma \leq \tau \Rightarrow \forall \mathscr{M}, \xi[[\![\sigma]\!]_\xi^{\mathscr{M}} \subseteq [\![\tau]\!]_\xi^{\mathscr{M}}]$.

Proof. Induction on the definition of $\leq$. The only two non-trivial cases can be proved as follows.

- We take an element $x \in [\![(\alpha \to \beta) \wedge (\alpha \to \gamma)]\!]_\xi^{\mathscr{M}}$, so by the definition we have the first line.

$$\dfrac{\dfrac{\forall d \in [\![\alpha]\!]_\xi^{\mathscr{M}}[d \cdot x \in [\![\beta]\!]_\xi^{\mathscr{M}}, d \cdot x \in [\![\gamma]\!]_\xi^{\mathscr{M}}]}{\forall d \in [\![\alpha]\!]_\xi^{\mathscr{M}}[d \cdot x \in [\![\beta]\!]_\xi^{\mathscr{M}} \cap [\![\gamma]\!]_\xi^{\mathscr{M}}]}}{x \in [\![\alpha \to (\beta \wedge \gamma)]\!]_\xi^{\mathscr{M}}}$$

- We take an element $x \in [\![\alpha \to \beta]\!]_\xi^{\mathscr{M}}$, so by the definition we have the first line.

$$\dfrac{\dfrac{\dfrac{\forall d \in [\![\alpha]\!]_\xi^{\mathscr{M}}[d \cdot x \in [\![\beta]\!]_\xi^{\mathscr{M}}]}{[\![\alpha']\!]_\xi^{\mathscr{M}} \subseteq [\![\alpha]\!]_\xi^{\mathscr{M}}, [\![\beta]\!]_\xi^{\mathscr{M}} \subseteq [\![\beta']\!]_\xi^{\mathscr{M}}} \; I.H}{\forall d \in [\![\alpha']\!]_\xi^{\mathscr{M}}[d \cdot x \in [\![\beta']\!]_\xi^{\mathscr{M}}]}}{x \in [\![\alpha' \to \beta']\!]_\xi^{\mathscr{M}}}$$

□

Lemma 4.3. *(Soundness).* $\Gamma \vdash_\wedge M : \sigma \Rightarrow \Gamma \models M : \sigma$.

Proof. Induction on the derivation of $M : \sigma$.

Basis:

- $x : \sigma \in \Gamma$. This case is trivial.
- $\sigma \equiv \omega$. This case is trivial.

Induction Steps:

We take a lambda model $\mathscr{M}$, a term environment ρ and a type environment ξ such that they satisfy $\mathscr{M}, \rho, \xi \models \Gamma$.

- The last rule applied is ($\to$ I).

$$\dfrac{\begin{array}{c}\Gamma, [x : \alpha]\\ \vdots\\ M_1 : \beta\end{array}}{(M \equiv)\lambda x.M_1 : \alpha \to \beta(\equiv \sigma)}\ (\to I)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{M}, \rho, \xi \models \Gamma \qquad \dfrac{[\forall a \in [\![\alpha]\!]^{\mathscr{M}}_{\xi}]^1}{\mathscr{M}, \rho[x := a], \xi \models x : \alpha}}{\mathscr{M}, \rho[x := a], \xi \models \Gamma, x : \alpha}}{[\![M_1]\!]^{\mathscr{M}}_{\rho[x:=a]} \in [\![\beta]\!]^{\mathscr{M}}_{\xi}}\ I.H}{\forall a \in [\![\alpha]\!]^{\mathscr{M}}_{\xi} [([\![\lambda x.M_1]\!]^{\mathscr{M}}_{\rho} \cdot a =)[\![M_1]\!]^{\mathscr{M}}_{\rho[x:=a]} \in [\![\beta]\!]^{\mathscr{M}}_{\xi}]}\ 1}{\dfrac{[\![\lambda x.M_1]\!]^{\mathscr{M}}_{\rho} \in [\![\alpha \to \beta]\!]^{\mathscr{M}}_{\xi}}{\mathscr{M}, \rho, \xi \models \lambda x.M_1 : \alpha \to \beta}}$$

- The last rule applied is ($\to$ E).

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ M_1 : \alpha \to \sigma\end{array} \quad \begin{array}{c}\Gamma\\ \vdots\\ N_1 : \alpha\end{array}}{(M \equiv)M_1 N_1 : \sigma}\ (\to E)$$

$$\dfrac{\dfrac{\dfrac{\mathscr{M}, \rho, \xi \models \Gamma}{[\![M_1]\!]^{\mathscr{M}}_{\rho} \in [\![\alpha \to \sigma]\!]^{\mathscr{M}}_{\xi},\ [\![N_1]\!]^{\mathscr{M}}_{\rho} \in [\![\alpha]\!]^{\mathscr{M}}_{\xi}}\ I.H}{\forall a \in [\![\alpha]\!]^{\mathscr{M}}_{\xi} [[\![M_1]\!]^{\mathscr{M}}_{\rho} \cdot a \in [\![\sigma]\!]^{\mathscr{M}}_{\xi}],\ [\![N_1]\!]^{\mathscr{M}}_{\rho} \in [\![\alpha]\!]^{\mathscr{M}}_{\xi}}\ def}{\dfrac{[\![M_1]\!]^{\mathscr{M}}_{\rho} \cdot [\![N_1]\!]^{\mathscr{M}}_{\rho} \in [\![\sigma]\!]^{\mathscr{M}}_{\xi}}{\dfrac{[\![M_1 N_1]\!]^{\mathscr{M}}_{\rho} \in [\![\sigma]\!]^{\mathscr{M}}_{\xi}}{\mathscr{M}, \rho, \xi \models M_1 N_1 : \sigma}}}$$

- The last rule applied is ($\wedge$ I).

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ M : \alpha\end{array} \quad \begin{array}{c}\Gamma\\ \vdots\\ M : \beta\end{array}}{M : \alpha \wedge \beta(\equiv \sigma)}\ (\wedge I)$$

This case is trivial.

- The last rule applied is $(\leq)$.

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ M : \alpha\end{matrix} \quad \alpha \leq \sigma}{M : \sigma}\ (\leq)$$

This case can be easily proved by Lemma 4.2.

□

As you can easily see, the simple type assignment system is a subset of the intersection type assignment system, so soundness stands also for the former one.

Definition 4.4.

- *$\mathscr{F}$ =$\{d \mid d$ is a filter$\}$.*

- *For d_1, $d_2 \in \mathscr{F}$, we define the relation $\cdot$ as follows.*

$$d_1 \cdot d_2 = \{\beta \in \mathbb{T} \mid \exists \alpha \in d_2[\alpha \to \beta \in d_1]\}.$$

- *Let ρ be a term environment over $\mathscr{F}$. Then we define Γ_ρ as follows.*

$$\Gamma_\rho = \{x : \alpha \mid \alpha \in \rho(x)\}.$$

- *We define $[\![M]\!]_\rho^{\mathscr{M}}$ for $M \in \Lambda$ as follows.*

$$[\![M]\!]_\rho^{\mathscr{M}} = \{\alpha \mid \Gamma_\rho \vdash_\wedge M : \alpha\} (\in \mathscr{F} \text{ by Lemma 3.23(1)}).$$

We need to confirm the relation $\cdot$ is defined on $\mathscr{F}$ properly, so we shall prove the following lemma.

Lemma 4.5. $d_1, d_2 \in \mathscr{F} \Rightarrow d_1 \cdot d_2 \in \mathscr{F}$.

Proof. It suffices to prove $d_1 \cdot d_2$ is a filter.

- $\omega \in d_1 \cdot d_2$, because $\omega \leq \omega \to \omega \in d_1$, so it is a non-empty set.

- $\beta_1, \beta_2 \in d_1 \cdot d_2 \Rightarrow \beta_1 \wedge \beta_2 \in d_1 \cdot d_2$.
From the definition of $\cdot$ and filter, we have $\exists \alpha_1, \alpha_2 \in d_2[\alpha_1 \to \beta_1, \alpha_2 \to \beta_2 \in d_1]$ and $\alpha_1 \wedge \alpha_2 \in d_2$. By the definition of $\leq$, we have $((\alpha_1 \to \beta_1) \wedge (\alpha_2 \to \beta_2))(\in d_1) \leq ((\alpha_1 \wedge \alpha_2) \to \beta_1) \wedge ((\alpha_1 \wedge \alpha_2) \to \beta_2) \leq ((\alpha_1 \wedge \alpha_2) \to (\beta_1 \wedge \beta_2)) \in d_1$.

- $\alpha \le \beta$ and $\alpha \in d_1 \cdot d_2 \Rightarrow \beta \in d_1 \cdot d_2$.
  From the definition of $\cdot$ and $\le$, we have $\exists \gamma \in d_2[\gamma \to \alpha (\le \gamma \to \beta) \in d_1]$.

$\square$

Theorem 4.6. $\left\langle \mathscr{F}, \ \cdot \ , [\![\ ]\!]^{\mathscr{M}} \right\rangle$ *is a lambda model.*

Proof. As you can easily see from the definitions above, the relation $\cdot$ and $[\![\ ]\!]^{\mathscr{M}}$ are properly defined over $\mathscr{F}$. It suffices to check the 6 equations in the definition of lambda model.

- We take $\sigma \in [\![x]\!]^{\mathscr{M}}_{\rho}$, then from the definition we have the first line.
  $$\dfrac{\Gamma_\rho \in \mathscr{F} \quad \dfrac{\Gamma_\rho \vdash_\wedge x : \sigma}{\sigma \in \ \mathit{filter\ generated\ by}\ \{\alpha \mid x : \alpha \in \Gamma_\rho\}}\ 3.23(2)}{\sigma \in \Gamma_\rho (= \rho(x))}$$
  The converse is trivial.

- We take $\sigma \in [\![MN]\!]^{\mathscr{M}}_{\rho}$, then from the definition we have the first line.
  $$\dfrac{\dfrac{\dfrac{\Gamma_\rho \vdash_\wedge MN : \sigma}{\exists \alpha \in \mathbb{T}[\Gamma_\rho \vdash_\wedge M : \alpha \to \sigma,\ \Gamma_\rho \vdash_\wedge N : \alpha]}\ 3.25(1)}{\alpha \to \sigma \in [\![M]\!]^{\mathscr{M}}_{\rho},\ \alpha \in [\![N]\!]^{\mathscr{M}}_{\rho}}\ def}{\sigma \in [\![M]\!]^{\mathscr{M}}_{\rho} \cdot [\![N]\!]^{\mathscr{M}}_{\rho}}\ def$$
  The converse is trivial.

- This case can be proved as λ-term is considered modulo α-equality.

- We take $\sigma \in [\![M]\!]^{\mathscr{M}}_{\rho[x:=d]}$, then from the definition we have the first line.
  $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_{\rho[x:=d]} \vdash_\wedge M : \sigma}{\Gamma'_\rho, \{x : \alpha \mid \alpha \in d\} \vdash_\wedge M : \sigma}\ def}{\Gamma'_\rho, x : \beta \vdash_\wedge M : \sigma}\ (\star)(\text{for some } \beta \in d)}{\Gamma'_\rho \vdash_\wedge \lambda x.M : \beta \to \sigma}\ (\to I)}{\Gamma_\rho \vdash_\wedge \lambda x.M : \beta \to \sigma}}{\beta \to \sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_{\rho}}\ def}{\sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} \cdot d}\ def$$
  $(\star)$ Actually, we can easily prove the following proposition.

  Proposition 4.7. $\Gamma, \{x : \alpha \mid \alpha \in d\} \vdash_\wedge M : \sigma$ *if and only if* $\Gamma, x : \beta \vdash_\wedge M : \sigma$, *for* $d \in \mathscr{F}$, $\beta \in d$.

  Proof. The proof is trivial, so we omit here. $\square$

Actually, restraining the set to filter is not necessary. We can prove this proposition over random set which only need to be closed under $\wedge$.

For the converse, we simply take $\sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} \cdot d$, then from the definition we have the first line.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \to \sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} (\beta \in d)}{\Gamma_{\rho} \vdash_{\wedge} \lambda x.M : \beta \to \sigma}\ def}{\Gamma'_{\rho} \vdash_{\wedge} \lambda x.M : \beta \to \sigma}\ \text{(the free variable lemma)}}{\Gamma'_{\rho}, x : \beta \vdash_{\wedge} M : \sigma}\ 3.25(3)}{\Gamma'_{\rho}, \{x : \alpha \mid \alpha \in d\} \vdash_{\wedge} M : \sigma}}{\dfrac{\Gamma_{\rho[x:=d]} \vdash_{\wedge} M : \sigma}{\sigma \in [\![M]\!]^{\mathscr{M}}_{\rho[x:=d]}}\ def}\ def$$

- This case can be proved by Lemma 3.25(2).
- This case can be proved by the free variable lemma.

□

Definition 4.8.

- $\xi_0(\psi_i) = \{d \in \mathscr{F} \mid \psi_i \in d\}$.
- $\rho_{\Gamma}(x) = \{\alpha \in \mathbb{T} \mid \Gamma \vdash_{\wedge} x : \alpha\}\ (\in \mathscr{F})$.

Lemma 4.9.

1. $\forall \alpha \in \mathbb{T}[[\![\alpha]\!]^{\mathscr{M}}_{\xi_0} = \{d \in \mathscr{F} \mid \alpha \in d\}]$.
2. $\Gamma \vdash_{\wedge} M : \alpha \Leftrightarrow \Gamma_{\rho_{\Gamma}} \vdash_{\wedge} M : \alpha$.
3. $\mathscr{F}, \rho_{\Gamma}, \xi_0 \models \Gamma$.

Proof.

1. By induction on the complexity of α.

   $\alpha \equiv \psi_i, \omega$ This case is proved from definition.

$\alpha \equiv \sigma \to \tau$ From the definition, we have:

$[\![\sigma \to \tau]\!]^{\mathscr{M}}_{\xi_0} = \{d \in \mathscr{F} \mid \forall e \in [\![\sigma]\!]^{\mathscr{M}}_{\xi_0}[d \cdot e \in [\![\tau]\!]^{\mathscr{M}}_{\xi_0}]\}$.

Then from the I.H, we have:

$\forall d_1 \in [\![\sigma]\!]^{\mathscr{M}}_{\xi_0}[\sigma \in d_1]$ and $\forall d_2 \in [\![\tau]\!]^{\mathscr{M}}_{\xi_0}[\tau \in d_2]$.

So $\tau \in d \cdot e$, and from the definition of $\cdot$, we have some $\sigma' \in e$ such that $\sigma' \to \tau \in d$. In order to show $\sigma \to \tau \in d$, we first take e as the filter generated by σ which is in $[\![\sigma]\!]^{\mathscr{M}}_{\xi_0}$. Then we have $\exists n \geq 1, \exists \beta_1, \cdots, \beta_n \in \{\sigma, \omega\}[\beta_1 \wedge \cdots \wedge \beta_n \leq \sigma']$ by Lemma 3.22. By the definition of $\leq$, we have $\sigma' \to \tau \leq (\beta_1 \wedge \cdots \wedge \beta_n) \to \tau \leq (\sigma \wedge \cdots \wedge \sigma) \to \tau \leq \sigma \to \tau$. So we have $\sigma \to \tau \in d$ by the definition of filter.

$\alpha \equiv \sigma \wedge \tau$ This case can be proved from definition.

2. This lemma is trivial, so we omit the proof here.
3. This lemma is trivial, so we omit the proof here.

□

By the new constructed type environment ξ_0 and the special basis, we can easily prove that β-equality holds in this type assignment system through semantic equality as follows.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_\wedge M : \alpha}{\Gamma_{\rho_\Gamma} \vdash_\wedge M : \alpha}}{\alpha \in [\![M]\!]^{\mathscr{M}}_{\rho_\Gamma} = [\![N]\!]^{\mathscr{M}}_{\rho_\Gamma}}}{\dfrac{\Gamma_{\rho_\Gamma} \vdash_\wedge N : \alpha}{\Gamma \vdash_\wedge N : \alpha}}$$

Theorem 4.10. *(Completeness Theorem)*

$$\Gamma \models M : \sigma \Rightarrow \Gamma \vdash_\wedge M : \sigma.$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathscr{F}, \rho_\Gamma, \xi_0 \models M : \sigma}\ 4.9(3) \qquad \Gamma \models M : \sigma}{[\![M]\!]^{\mathscr{M}}_{\rho_\Gamma} \in [\![\sigma]\!]^{\mathscr{M}}_{\xi_0}}\ def}{\sigma \in [\![M]\!]^{\mathscr{M}}_{\rho_\Gamma}}\ 4.9(1)}{\Gamma_{\rho_\Gamma} \vdash_\wedge M : \sigma}\ def}{\Gamma \vdash_\wedge M : \sigma}\ 4.9(2)$$

□

In [1], Barendregt et al. proved that this intersection type assignment system is conservative over the simple type assignment system, so the completeness theorem also stands in the simple type assignment system.

4.2 The call-by-value filter model

We will now prove the completeness of TA^{-}.

Definition 4.11. *(Type interpretation)*

- *Let* $\xi\colon \{\psi_i\} \to \Omega K$*, so* ξ *is a* ***type environment*** *from all type variables to power set of* K*.*
- $\Omega K \subseteq \mathscr{P}(K)$*.*
- *We define* K *as the smallest subset of* D *satisfying the following condition.*

$$X \to Y := \{p \in K \mid \forall u \ \varepsilon \ X[p \cdot u \in Y]\}$$
$$\forall X, Y \in \Omega K[K, X \cap Y, X \cup Y, X \to Y \in \Omega K]$$

- *We define the relation* ε *as a subset of* $D \times \Omega K$ *satisfying following conditions.*

  For all $u \in D$ *and* $p \in K$*:*

  1. $u \ \varepsilon \ K$*.*
  2. $u \ \varepsilon \ X$ *and* $X \subseteq Y$ *implies* $u \ \varepsilon \ Y$*.*
  3. $u \ \varepsilon \ X$ *and* $u \ \varepsilon \ Y$ *implies* $u \ \varepsilon \ X \cap Y$*.*
  4. $p \ \varepsilon \ X \cup Y$ *implies* $p \ \varepsilon \ X$ *or* $p \ \varepsilon \ Y$*.*
  5. *The following three conditions are equivalent for* $v \in D$*:*
     (a) $v \ \varepsilon \ X \to Y$
     (b) $v \cdot q \ \varepsilon \ Y$ *for all* $q \ \varepsilon \ X$ *with* $q \ \varepsilon \ K$*.*
     (c) $v \cdot u \ \varepsilon \ Y$ *for all* $u \ \varepsilon \ X$

- *The interpretation of* $\sigma \in \mathbb{T}$ *in a call-by-value lambda model* $\mathscr{M}$ *via a type environment* ξ*, denoted as* $[\![\sigma]\!]_\xi^{\mathscr{M}} \in \Omega K$*, can be defined as follows.*
  - $[\![\omega]\!]_\xi^{\mathscr{M}} = K$
  - $[\![\psi_i]\!]_\xi^{\mathscr{M}} = \xi(\psi_i)$
  - $[\![\sigma \to \tau]\!]_\xi^{\mathscr{M}} = [\![\sigma]\!]_\xi^{\mathscr{M}} \to [\![\tau]\!]_\xi^{\mathscr{M}}$
  - $[\![\sigma \wedge \tau]\!]_\xi^{\mathscr{M}} = [\![\sigma]\!]_\xi^{\mathscr{M}} \cap [\![\tau]\!]_\xi^{\mathscr{M}}$
  - $[\![\sigma \vee \tau]\!]_\xi^{\mathscr{M}} = [\![\sigma]\!]_\xi^{\mathscr{M}} \cup [\![\tau]\!]_\xi^{\mathscr{M}}$

- *Let* ρ *be a term environment in* D*,* $\mathscr{M}$ *be a call-by-value lambda model.*

- $\mathscr{M}, \rho, \xi \models M : \sigma$ *if and only if* $[\![M]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\sigma]\!]_\xi^{\mathscr{M}}$.
- $\mathscr{M}, \rho, \xi \models \Gamma$ *if and only if* $\mathscr{M}, \rho, \xi \models x : \sigma$ *for all* $x : \sigma \in \Gamma$.
- $\Gamma \models M : \sigma$ *if and only if* $\forall \mathscr{M}, \rho, \xi \models \Gamma[\mathscr{M}, \rho, \xi \models M : \sigma]$.

Lemma 4.12. $\sigma \leq \tau \Rightarrow \forall \mathscr{M}, \xi[[\![\sigma]\!]_\xi^{\mathscr{M}} \subseteq [\![\tau]\!]_\xi^{\mathscr{M}}]$.

Proof. This lemma can be proved by a similar proof as Lemma 4.2. We here only discuss the non-trivial case $(\alpha \to \gamma) \wedge (\beta \to \gamma) \leq \alpha \vee \beta \to \gamma$.

$$\dfrac{\dfrac{p \in [\![(\alpha \to \gamma) \wedge (\beta \to \gamma)]\!]_\xi^{\mathscr{M}}}{p \in [\![\alpha \to \gamma]\!]_\xi^{\mathscr{M}} \cap [\![\beta \to \gamma]\!]_\xi^{\mathscr{M}}}\, def \qquad \dfrac{q \; \varepsilon \; [\![\alpha \vee \beta]\!]_\xi^{\mathscr{M}}}{p \; \varepsilon \; [\![\alpha]\!]_\xi^{\mathscr{M}} \text{ or } p \; \varepsilon \; [\![\beta]\!]_\xi^{\mathscr{M}}}\, def}{\dfrac{\dfrac{p \cdot q \; \varepsilon \; [\![\gamma]\!]_\xi^{\mathscr{M}}}{p \in [\![\alpha \vee \beta]\!]_\xi^{\mathscr{M}} \to [\![\gamma]\!]_\xi^{\mathscr{M}}}\, def}{p \in [\![\alpha \vee \beta \to \gamma]\!]_\xi^{\mathscr{M}}}\, def}$$

□

Lemma 4.13. *(Soundness).* $\Gamma \vdash_{TA} M : \sigma \Rightarrow \Gamma \models M : \sigma$.

Proof. We prove this lemma by induction on the derivation of $M : \sigma$.

Axiom: We have $[\![M]\!]_\rho^{\mathscr{M}} (\in D) \; \varepsilon \; [\![\omega]\!]_\xi^{\mathscr{M}} (= K)$ by the definition of ε .

Induction Steps:

$(\wedge I)$ The proof ends up as lower left, and can be proved as lower right.

$$\dfrac{\begin{matrix}\vdots \\ M : \alpha\end{matrix} \quad \begin{matrix}\vdots \\ M : \beta\end{matrix}}{M : \alpha \wedge \beta} \qquad \dfrac{\overline{\overline{[\![M]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\alpha]\!]_\xi^{\mathscr{M}}}}\, I.H \qquad \overline{\overline{[\![M]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\beta]\!]_\xi^{\mathscr{M}}}}\, I.H}{[\![M]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\alpha \wedge \beta]\!]_\xi^{\mathscr{M}}}\, def$$

$(\leq)$ This case can be proved by Lemma 4.12.

$(\to E)$ The proof ends up as lower left, and can be proved as lower right.

$$\dfrac{\begin{matrix}\vdots \\ M : \alpha \to \beta\end{matrix} \quad \begin{matrix}\vdots \\ N : \alpha\end{matrix}}{MN : \beta} \qquad \dfrac{\dfrac{\dfrac{\overline{\overline{[\![M]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\alpha \to \beta]\!]_\xi^{\mathscr{M}}}}\, I.H}{[\![M]\!]_\rho^{\mathscr{M}} \cdot p \; \varepsilon \; [\![\beta]\!]_\xi^{\mathscr{M}} (p \; \varepsilon \; [\![\alpha]\!]_\xi^{\mathscr{M}})}\, def \qquad \overline{\overline{[\![N]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\alpha]\!]_\xi^{\mathscr{M}}}}\, I.H}{[\![M]\!]_\rho^{\mathscr{M}} \cdot [\![N]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\beta]\!]_\xi^{\mathscr{M}}}}{[\![MN]\!]_\rho^{\mathscr{M}} \; \varepsilon \; [\![\beta]\!]_\xi^{\mathscr{M}}}\, def$$

($\to I$) The proof ends up as lower left, and can be proved as lower right.

$$\dfrac{\begin{matrix}[x:\alpha]\\ \vdots \\ M:\beta\end{matrix}}{\lambda x.M : \alpha \to \beta} \qquad \dfrac{\dfrac{}{\llbracket M \rrbracket^{\mathscr{M}}_{\rho[x:=p]} \;\varepsilon\; \llbracket \beta \rrbracket^{\mathscr{M}}_{\xi} (p \;\varepsilon\; \llbracket \alpha \rrbracket^{\mathscr{M}}_{\xi})} I.H}{\llbracket \lambda x.M \rrbracket^{\mathscr{M}}_{\rho} \;\varepsilon\; \llbracket \alpha \to \beta \rrbracket^{\mathscr{M}}_{\xi}} def$$

($\vee E$) The proof ends up as lower left, and can be proved as lower right.

$$\dfrac{N : \alpha \vee \beta \quad \begin{matrix}[x:\alpha]\\ \vdots \\ M:\gamma\end{matrix} \quad \begin{matrix}[x:\beta]\\ \vdots \\ M:\gamma\end{matrix}}{M[x := N] : \gamma} \qquad \dfrac{\dfrac{}{\llbracket M \rrbracket^{\mathscr{M}}_{\rho[x:=p]} \;\varepsilon\; \llbracket \gamma \rrbracket^{\mathscr{M}}_{\xi} (p \;\varepsilon\; \llbracket \alpha \rrbracket^{\mathscr{M}}_{\xi})} I.H \quad \dfrac{\dfrac{}{\llbracket N \rrbracket^{\mathscr{M}}_{\rho} \;\varepsilon\; \llbracket \alpha \vee \beta \rrbracket^{\mathscr{M}}_{\xi}} I.H}{\llbracket N \rrbracket^{\mathscr{M}}_{\rho} \;\varepsilon\; \llbracket \alpha \rrbracket^{\mathscr{M}}_{\xi}}}{\llbracket M \rrbracket^{\mathscr{M}}_{\rho[x:=\llbracket N \rrbracket^{\mathscr{M}}_{\rho}]} \;\varepsilon\; \llbracket \gamma \rrbracket^{\mathscr{M}}_{\xi}}$$

Note: $\llbracket N \rrbracket^{\mathscr{M}}_{\rho} \;\varepsilon\; \llbracket \beta \rrbracket^{\mathscr{M}}_{\xi}$ case can be treated similarly. □

Definition 4.14.

- *A **prime filter** p is a filter with the following property.*

$$\alpha \vee \beta \in p \Rightarrow \alpha \in p \text{ or } \beta \in p$$

- $\mathscr{F}_P = \{p \mid p \text{ is a prime filter}\}$.
- *For d_1, $d_2 \in \mathscr{F}$, we define the relation $\cdot$ as follows.*

$$d_1 \cdot d_2 = \{\beta \in \mathbb{T} \mid \exists \alpha \in d_2[\alpha \to \beta \in d_1]\}.$$

- *Let ρ be a term environment over $\mathscr{F}_P$. Then we define Γ_ρ as follows.*

$$\Gamma_\rho = \{x : \alpha \mid \alpha \in \rho(x)\}.$$

- *We define $\llbracket M \rrbracket^{\mathscr{M}}_{\rho}$ for $M \in \Lambda$ as follows.*

$$\llbracket M \rrbracket^{\mathscr{M}}_{\rho} = \{\alpha \mid \Gamma_\rho \vdash_{TA^-} M : \alpha\}.$$

Theorem 4.15. $\left\langle \mathscr{F}, \mathscr{F}_P, \;\cdot\;, \llbracket\;\rrbracket^{\mathscr{M}} \right\rangle$ *is a call-by-value lambda model.*

Proof. It suffices to verify the seven clauses in the definition under this structure.

1. We take $\sigma \in \llbracket x \rrbracket^{\mathscr{M}}_{\rho}$, then from the definition we have the first line.

$$\dfrac{\Gamma_\rho \vdash_{TA^-} x : \sigma}{\sigma \in \rho(x)} (\star)$$

($\star$) By induction on the derivation of $\Gamma_\rho \vdash_{TA^-} x : \sigma$, the only non-trivial case is when the last applied rule is $(\vee E)^-$ as follows.

$$\dfrac{\begin{matrix}(x \not\equiv y) & & \\ & [y:\alpha] & [y:\beta] \\ \vdots & \vdots & \vdots \\ y : \alpha \vee \beta & x : \sigma & x : \sigma\end{matrix}}{x : \sigma} \qquad \dfrac{\begin{matrix}(x \equiv y) & & \\ & [x:\alpha] & [x:\beta] \\ \vdots & \vdots & \vdots \\ x : \alpha \vee \beta & x : \sigma & x : \sigma\end{matrix}}{x : \sigma}$$

The left case can be easily proved by I.H with the free variable lemma, so we only prove the right one.

$$\dfrac{\dfrac{\dfrac{\overline{\overline{\alpha \vee \beta \in \rho(x)}}\ I.H}{\alpha \in \rho(x)\ or\ \beta \in \rho(x)}\ \rho(x) \in \mathscr{F}_P}{\Gamma_\rho \vdash_{TA^-} x : \sigma}}{\sigma \in \rho(x)}\ I.H$$

The converse is trivial.

2. We take $\sigma \in [\![MN]\!]^{\mathscr{M}}_\rho$, then from the definition we have the first line.

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_{TA^-} MN : \sigma}{\Gamma \vdash_{TA^-} M : \alpha \to \sigma}\ (3.49)}{\alpha \to \sigma \in [\![M]\!]^{\mathscr{M}}_\rho} \quad \dfrac{\dfrac{\Gamma \vdash_{TA^-} MN : \sigma}{\Gamma \vdash_{TA^-} N : \alpha}\ (3.49)}{\alpha \in [\![N]\!]^{\mathscr{M}}_\rho}}{\sigma \in [\![M]\!]^{\mathscr{M}}_\rho \cdot [\![N]\!]^{\mathscr{M}}_\rho}$$

The converse is trivial.

3. This case is trivial, because one only need to prove the following proposition.

$$\Gamma_\rho \vdash_{TA^-} M : \alpha \Rightarrow \Gamma[x := y] \vdash_{TA^-} M[x := y] : \alpha$$

4. We take $\sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_\rho \cdot k$, then from the definition we have the first line.

$$\dfrac{\dfrac{\Gamma_\rho \vdash_{TA^-} \lambda x.M : \alpha \to \sigma\ (\alpha \in k)}{\Gamma_\rho \restriction_{FV(\lambda x.M)}, x : \alpha \vdash_{TA^-} M : \sigma}\ 3.49}{\sigma \in [\![M]\!]^{\mathscr{M}}_{\rho[x:=k]}}\ def$$

For the converse, we take $\sigma \in [\![M]\!]^{\mathscr{M}}_\rho[x := k]$, then from the definition we have the first line.

$$\dfrac{\dfrac{\dfrac{\Gamma_\rho \restriction_{FV(\lambda x.M)}, \{x : \beta \mid \beta \in k\} \vdash_{TA^-} M : \sigma}{\Gamma_\rho \restriction_{FV(\lambda x.M)}, x : \alpha \vdash_{TA^-} M : \sigma}\ (\star)}{\Gamma_\rho \restriction_{FV(\lambda x.M)} \vdash_{TA^-} \lambda x.M : \alpha \to \sigma}}{\sigma \in [\![\lambda x.M]\!]^{\mathscr{M}}_\rho \cdot k}\ def$$

($\star$) One can easily prove that The Proposition 4.7 still stands under TA^- system.

5. We take $\sigma \in [\![\lambda x.M]\!]_\rho^{\mathscr{M}}$, then from the definition we have the first line.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma_\rho \restriction_{FV(\lambda x.M)} \vdash_{TA^-} \lambda x.M : \sigma}{\Gamma_\rho \restriction_{FV(\lambda x.M)}, x : \sigma_i \vdash_{TA^-} M : \beta_i, \bigwedge_{i=1}^{n} (\sigma_i \to \beta_i) \leq \sigma}\ 3.49(2)}{\beta_i \in [\![M]\!]_{\rho[x:=\uparrow\sigma_i]}^{\mathscr{M}} = [\![N]\!]_{\rho[x:=\uparrow\sigma_i]}^{\mathscr{M}}}}{\Gamma_\rho \restriction_{FV(\lambda x.N)}, x : \sigma_i \vdash_{TA^-} N : \beta_i, \bigwedge_{i=1}^{n} (\sigma_i \to \beta_i) \leq \sigma}\ def}{\Gamma_\rho \restriction_{FV(\lambda x.N)} \vdash_{TA^-} \lambda x.N : \sigma}\ (\wedge I), (\leq)}{\Gamma_\rho \vdash_{TA^-} \lambda x.N : \sigma}\ \text{Weakening}}{\sigma \in [\![\lambda x.N]\!]_\rho^{\mathscr{M}}}\ def$$

We must verify that $\uparrow \sigma_i$ is a prime filter. Suppose $\alpha \vee \beta \in \uparrow \sigma_i$, then by the definition, we have $\sigma_i \wedge \omega \wedge \cdots \wedge \sigma_i \leq \alpha \vee \beta$. By Proposition 3.44, we have $\sigma_i \wedge \omega \wedge \cdots \wedge \sigma_i \leq \alpha$ or $\sigma_i \wedge \omega \wedge \cdots \wedge \sigma_i \leq \beta$.

6. This case can be proved by the free variable lemma.

7. $(M \equiv x)$ case is trivial, so we only treat $(M \equiv \lambda x.N)$ case here. Suppose $\alpha \vee \beta \in [\![\lambda x.N]\!]_\rho^{\mathscr{M}}$, then by the definition, we have the first line.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma_\rho \restriction_{FV(\lambda x.N)} \vdash_{TA^-} \lambda x.N : \alpha \vee \beta}{\Gamma_\rho \restriction_{FV(\lambda x.N)}, x : \sigma_i \vdash_{TA^-} N : \beta_i, \bigwedge_{i=1}^{n} (\sigma_i \to \beta_i) \leq \alpha \vee \beta}\ 3.49(2)}{\Gamma_\rho \restriction_{FV(\lambda x.N)}, x : \sigma_i \vdash_{TA^-} N : \beta_i, \bigwedge_{i=1}^{n} (\sigma_i \to \beta_i) \leq \alpha (or\ \beta)}\ 3.44}{\Gamma_\rho \vdash_{TA^-} \lambda x.N : \alpha (or\ \beta)}\ (\to I), (\leq)}{\alpha \in [\![\lambda x.N]\!]_\rho^{\mathscr{M}}\ or\ \beta \in [\![\lambda x.N]\!]_\rho^{\mathscr{M}}}\ def$$

□

Lemma 4.16. *For $u, v \in \mathscr{F}$ and $p \in \mathscr{F}_P$,*

1. $\alpha \notin u \Rightarrow \exists q \in \mathscr{F}_P[u \subseteq q, \alpha \notin q]$.
2. $u \cdot v \subseteq p \Rightarrow \exists q \in \mathscr{F}_P[u \subseteq q, q \cdot v \subseteq p]$.
3. $u \cdot v \subseteq p \Rightarrow \exists q \in \mathscr{F}_P[v \subseteq q, u \cdot q \subseteq p]$.

Proof. We enumerate with infinite repetition all union types, and label them as $\alpha_0 \vee \beta_0, \alpha_1 \vee \beta_1 \cdots$. Then we inductively construct a sequence $u_0, u_1, u_2 \cdots$ of filters such that $u_0 \subseteq u_1 \subseteq \cdots$.

1. We can add an extra restriction to the sequence that all elements satisfy $\alpha \notin u_k$. When $k = 0$ which means that $\alpha \notin u_0(= u)$, it naturally stands. Suppose we have constructed u_k, then it leaves two possibilities for $\uparrow (u_k \cup \{\alpha_k \vee \beta_k\})$:
$$\alpha \in\uparrow (u_k \cup \{\alpha_k \vee \beta_k\})$$
or
$$\alpha \notin\uparrow (u_k \cup \{\alpha_k \vee \beta_k\}).$$
Under the latter case, we can prove by contradiction that either $\alpha \notin\uparrow (u_k \cup \{\alpha_k\})$ or $\alpha \notin\uparrow (u_k \cup \{\beta_k\})$ as follows.
$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \in\uparrow (u_k \cup \{\alpha_k\}, \alpha \in\uparrow (u_k \cup \{\beta_k\}}{\bigwedge_{i=1}^{n} \gamma_i \wedge \alpha_k \leq \alpha, \bigwedge_{j=1}^{m} \delta_j \wedge \beta_k \leq \alpha}\, def(\star)}{\bigwedge_{i=1}^{n} \gamma_i \wedge \bigwedge_{j=1}^{m} \delta_j \wedge (\alpha_k \vee \beta_k) \leq \alpha}}{\alpha \in\uparrow (u_k \cup \{\alpha_k \vee \beta_k\})}}{\bot}$$
$(\star)$ $\gamma_i \in u_k$, $\delta_j \in u_k$.
So we define u_{k+1} as follows.
$$u_{k+1} = \begin{cases} \uparrow (u_k \cup \{\alpha_k\} & \alpha \notin\uparrow (u_k \cup \{\alpha_k\} \\ \uparrow (u_k \cup \{\beta_k\} & \alpha \notin\uparrow (u_k \cup \{\beta_k\} \\ u_k & \text{Otherwise} \end{cases}$$
Finally, we define that $q := \bigcup_{k=0}^{\infty} u_k$, then $u(= u_0) \subseteq q$, $\alpha \notin q$ by the definition. The proof of q being prime is the same as the following proof, so omitted here.

2. We can add an extra restriction to the sequence that all elements satisfy $u_k \cdot v \subseteq p$. When $k = 0$ which means that $u_0(= u) \cdot v \subseteq p$, it naturally stands. Suppose we have constructed u_k, then it leaves two possibilities for $\uparrow (u_k \cup \{\alpha_k \vee \beta_k\}) \cdot v$:
$$\uparrow (u_k \cup \{\alpha_k \vee \beta_k\}) \cdot v \subseteq p$$
or
$$\uparrow (u_k \cup \{\alpha_k \vee \beta_k\}) \cdot v \nsubseteq p.$$
Under the former case, we can prove by contradiction that either $\uparrow (u_k \cup \{\alpha_k\}) \cdot v \subseteq p$ or $\uparrow (u_k \cup \{\beta_k\}) \cdot v \subseteq p$ as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists \sigma, \tau \notin p[\sigma \in\uparrow (u_k \cup \{\alpha_k\}) \cdot v, \tau \in\uparrow (u_k \cup \{\beta_k\}) \cdot v]}{\bigwedge_{i=1}^{n} \gamma_i \wedge \alpha_k \leq \sigma' \to \sigma,\ \bigwedge_{j=1}^{m} \delta_j \wedge \beta_k \leq \tau' \to \tau} def(\star)}{\bigwedge_{i=1}^{n} \gamma_i \wedge \bigwedge_{j=1}^{m} \delta_j \wedge (\alpha_k \vee \beta_k) \leq \sigma' \wedge \tau' \to \sigma \vee \tau} (\star\star)}{\sigma \vee \tau \in\uparrow (u_k \cup \{\alpha_k \vee \beta_k\}) \cdot v \subseteq p (\in \mathscr{F}_P)}}{\sigma (or\ \tau) \in p} def}{\bot}$$

($\star$) $\gamma_i \in u_k$, $\delta_j \in u_k$, $\sigma' \in v, \tau' \in v$.

($\star\star$) Suppose we have $\alpha_1 \wedge \beta_1 \leq \sigma_1 \to \tau_1$ and $\alpha_2 \wedge \beta_2 \leq \sigma_2 \to \tau_2$, we can prove that $(\alpha_1 \wedge \alpha_2) \wedge (\beta_1 \vee \beta_2) \leq \sigma_1 \wedge \sigma_2 \to \tau_1 \vee \tau_2$ as follows.

$$\begin{aligned}
(\alpha_1 \wedge \alpha_2) \wedge (\beta_1 \vee \beta_2) &\leq ((\alpha_1 \wedge \alpha_2) \wedge \beta_1) \vee ((\alpha_1 \wedge \alpha_2) \wedge \beta_2) \\
&\leq (\alpha_1 \wedge \beta_1) \vee (\alpha_2 \wedge \beta_2) \\
&\leq (\sigma_1 \to \tau_1) \vee (\sigma_2 \to \tau_2) \\
&\leq (\sigma_1 \wedge \sigma_2 \to \tau_1 \vee \tau_2) \vee (\sigma_1 \wedge \sigma_2 \to \tau_1 \vee \tau_2) \\
&\sim \sigma_1 \wedge \sigma_2 \to \tau_1 \vee \tau_2
\end{aligned}$$

So we define u_{k+1} as follows.

$$u_{k+1} = \begin{cases} \uparrow (u_k \cup \{\alpha_k\} & \uparrow (u_k \cup \{\alpha_k\}) \cdot v \subseteq p \\ \uparrow (u_k \cup \{\beta_k\} & \uparrow (u_k \cup \{\beta_k\}) \cdot v \subseteq p \\ u_k & \text{Otherwise} \end{cases}$$

Finally, we define that $q := \bigcup_{k=0}^{\infty} u_k$, then $u (= u_0) \subseteq q$, $q \cdot v \subseteq p$ by the definition. It suffices to show that q is prime as follows.

$$\dfrac{\dfrac{\dfrac{\exists n[\alpha_n \vee \beta_n = \alpha_k \vee \beta_k, \alpha_n \vee \beta_n \in u_n](\star)}{\uparrow (u_n \cup \{\alpha_n \vee \beta_n\}) \cdot v \subseteq p}}{\alpha_n (or\ \beta_n) \in u_{n+1}} def}{\alpha_n (or\ \beta_n) \in q}$$

($\star$) One should notice that the reason why we need infinite repetition of all union types lies here. Because when the sequence reaches $\alpha_n \vee \beta_n$ the first time, it may not be in the u_n, therefore we need to loop until $\alpha_n \vee \beta_n$ is included in the $u_{n'}$ so that when the sequence reaches $\alpha_n \vee \beta_n$ next time the deduction above will work.

3. This lemma can be proved similarly as above, so omitted here, but one should notice that we define v as the sequence this time which means that $u_0 = v$.

$\square$

Definition 4.17.

- *Let* Γ *be a prime basis.*

$$\rho_\Gamma := \{\alpha \mid \Gamma \vdash_{TA^-} x : \alpha\}$$

- *The type environment* ξ *is defined as follows.*

$$\xi(\alpha) := \{p \in \mathscr{F}_P \mid \alpha \in p\}$$

- $\mathscr{P}_{up}(T)$ *is defined as the set of all upward closed subsets of* T *with respect to* $\subseteq$*.*

- $\varepsilon : \mathscr{F} \times \mathscr{P}_{up}(\mathscr{F}_P)$

$$u \ \varepsilon \ X := \forall p \in \mathscr{F}_P[u \subseteq p \Rightarrow p \in X]$$

Note: Under this convenient definition, it is easy to prove that when $u \in \mathscr{F}_P$*:*

$$u \in X \Leftrightarrow u \ \varepsilon \ X.$$

Lemma 4.18. ε *is defined properly.*

Proof. Induction on the definition of the type interpretation concerning ε. Here we only treat the non-trivial case 5.

(5b)⇒(5a)

$$\dfrac{\dfrac{\dfrac{\dfrac{(5b) v \cdot q \ \varepsilon \ Y (q \ \varepsilon \ X, q \in \mathscr{F}_P) \qquad \dfrac{\dfrac{[u \ \varepsilon \ X]^2, [r \in \mathscr{F}_P, p \cdot u \subseteq r]^1}{\exists q \in \mathscr{F}_P[u \subseteq q, p \cdot q \subseteq r]} 4.16 \quad \dfrac{[p \in \mathscr{F}_P, v \subseteq p]^3}{v \subseteq p}}{v \cdot q \subseteq p \cdot q \subseteq r}}{r \in Y}}{p \cdot u \ \varepsilon \ Y} 1}{p \in X \to Y} 2}{(5a) v \ \varepsilon \ X \to Y} 3$$

(5a)⇒(5c)

$$\dfrac{\dfrac{\dfrac{\dfrac{(5a) v \ \varepsilon \ X \to Y \qquad \dfrac{[u \ \varepsilon \ X]^1, [r \in \mathscr{F}_P]^2, [v \cdot u \subseteq r]^3}{\exists p \in \mathscr{F}_P[[v \subseteq p]^1, [p \cdot u \subseteq r]^2]} 4.16}{p \ \varepsilon \ X \to Y}}{p \cdot u \ \varepsilon \ Y} 1}{r \in Y} 2}{(5c) v \cdot u \ \varepsilon \ Y} 3$$

(5c)⇒(5b)

$$\frac{\forall u \mathrel{\varepsilon} X[v \cdot u \mathrel{\varepsilon} Y(u \in \mathscr{F})]}{\forall q \mathrel{\varepsilon} X[v \cdot q \mathrel{\varepsilon} Y(q \in \mathscr{F}_P)]}$$

□

Lemma 4.19. *For every* $u \in \mathscr{F}$, $\alpha \in \mathbb{T}$, $u \mathrel{\varepsilon} [\![\alpha]\!]^{\mathscr{M}}_{\xi} \Leftrightarrow \alpha \in u$.

Proof. We shall prove the following proposition firstly.

$$\alpha \in u \Leftrightarrow \forall p \in \mathscr{F}_P[u \subseteq p \Rightarrow \alpha \in p](\text{asis})$$

(⇒) This case is trivial.

(⇐) This case can be proved by Lemma 4.16(1), which is its contraposition .

We prove this lemma by induction on the complexity of α.

Basis:

($\alpha \equiv \omega$) This case is trivial.

($\alpha \equiv x$)

$$\cfrac{\cfrac{\cfrac{\alpha \in u}{\forall p \in \mathscr{F}_P[u \subseteq p \Rightarrow \alpha \in p]}(asis)}{p \in [\![\alpha]\!]^{\mathscr{M}}_{\xi}}def}{u \mathrel{\varepsilon} [\![\alpha]\!]^{\mathscr{M}}_{\xi}} \qquad \cfrac{\cfrac{u \mathrel{\varepsilon} [\![\alpha]\!]^{\mathscr{M}}_{\xi}}{\forall p \in \mathscr{F}_P[u \subseteq p \Rightarrow p \in [\![\alpha]\!]^{\mathscr{M}}_{\xi}(\Rightarrow \alpha \in p)]}}{\alpha \in u}$$

Induction Steps:

($\alpha \equiv \alpha_1 \wedge \alpha_2$)

$$\cfrac{\cfrac{\cfrac{\alpha_1 \wedge \alpha_2 \in u}{\alpha_1 \in u, \alpha_2 \in u}(\leq)}{u \mathrel{\varepsilon} [\![\alpha_1]\!]^{\mathscr{M}}_{\xi}, u \mathrel{\varepsilon} [\![\alpha_2]\!]^{\mathscr{M}}_{\xi}}I.H}{u \mathrel{\varepsilon} [\![\alpha_1 \wedge \alpha_2]\!]^{\mathscr{M}}_{\xi}}def \qquad \cfrac{\cfrac{\cfrac{u \mathrel{\varepsilon} [\![\alpha_1 \wedge \alpha_2]\!]^{\mathscr{M}}_{\xi}}{u \mathrel{\varepsilon} [\![\alpha_1]\!]^{\mathscr{M}}_{\xi}, u \mathrel{\varepsilon} [\![\alpha_2]\!]^{\mathscr{M}}_{\xi}}}{\alpha_1 \in u, \alpha_2 \in u}I.H}{\alpha_1 \wedge \alpha_2 \in u}$$

($\alpha \equiv \alpha_1 \vee \alpha_2$)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\alpha_1 \vee \alpha_2 \in u}{\forall p \in \mathscr{F}_P[u \subseteq p \Rightarrow \alpha_1 \vee \alpha_2 \in p]}(asis)}{\alpha_1 \in p \text{ or } \alpha_2 \in p}(def)}{\alpha_1 \in u \text{ or } \alpha_2 \in u}(asis)}{u \mathrel{\varepsilon} [\![\alpha_1]\!]^{\mathscr{M}}_{\xi} \text{ or } u \mathrel{\varepsilon} [\![\alpha_2]\!]^{\mathscr{M}}_{\xi}}I.H}{u \mathrel{\varepsilon} [\![\alpha_1 \vee \alpha_2]\!]^{\mathscr{M}}_{\xi}}def \qquad \cfrac{\cfrac{\cfrac{u \mathrel{\varepsilon} [\![\alpha_1 \vee \alpha_2]\!]^{\mathscr{M}}_{\xi}}{u \mathrel{\varepsilon} [\![\alpha_1]\!]^{\mathscr{M}}_{\xi} \text{ or } u \mathrel{\varepsilon} [\![\alpha_2]\!]^{\mathscr{M}}_{\xi}}}{\alpha_1 \in u \text{ or } \alpha_2 \in u}I.H}{\alpha_1 \vee \alpha_2 \in u}(\leq)$$

($\alpha \equiv \alpha_1 \rightarrow \alpha_2$)

$$\dfrac{\dfrac{u\ \varepsilon\ [\![\alpha_1 \to \alpha_2]\!]_\xi^{\mathscr{M}}, p \in \mathscr{F}_P, u \subseteq p}{p \in [\![\alpha_1]\!]_\xi^{\mathscr{M}} \to [\![\alpha_2]\!]_\xi^{\mathscr{M}}} \quad \dfrac{\dfrac{}{\alpha_1 \in \uparrow(\alpha_1)}}{\uparrow(\alpha_1)\ \varepsilon\ [\![\alpha_1]\!]_\xi^{\mathscr{M}}}}{\dfrac{\dfrac{p\cdot \uparrow(\alpha_1)\ \varepsilon\ [\![\alpha_2]\!]_\xi^{\mathscr{M}}}{\alpha_2 \in p\cdot \uparrow(\alpha_1)}\ I.H}{\dfrac{\beta \to \alpha_2 \in p}{\alpha_1 \to \alpha_2 \in u}\ (\star)}}$$

($\star$) By definition, we have $\exists \tau_1, \cdots, \tau_n \in \{\alpha_1, \omega\}[\tau_1 \wedge \cdots \wedge \tau_n \leq \beta]$, then we can derive that $\alpha_1 \sim \alpha_1 \wedge \cdots \wedge \alpha_1 \leq \tau_1 \wedge \cdots \wedge \tau_n$ so that $\alpha_1 \leq \beta$.

$$\dfrac{\dfrac{\dfrac{\alpha_1 \to \alpha_2 \in u \quad \dfrac{}{\forall v\ \varepsilon\ [\![\alpha_1]\!]_\xi^{\mathscr{M}} [\alpha_1 \in v]}\ I.H}{\alpha_2 \in u \cdot v}}{u \cdot v\ \varepsilon\ [\![\alpha_2]\!]_\xi^{\mathscr{M}}}\ I.H}{u\ \varepsilon\ [\![\alpha_1 \to \alpha_2]\!]_\xi^{\mathscr{M}}}$$

□

Lemma 4.20. *If* $\Gamma \nvdash_{TA^-} M : \alpha$*, then there exists a prime basis* Δ *such that* $\Gamma \subseteq \Delta$ *and* $\Delta \nvdash_{TA^-} M : \alpha$*.*

Proof. We can prove this lemma by constructing a sequence as the proof of Lemma 4.16, so we omit the detail here and start with the discussion about Δ_k. Suppose we have constructed Δ_k, then either

$\Delta_k \vdash_{TA^-} x_k : \beta_k \vee \gamma_k$ or $\Delta_k \nvdash_{TA^-} x_k : \beta_k \vee \gamma_k$.

In the former case, we can prove that either

$\Delta_k \cup \{x_k : \beta_k\} \nvdash_{TA^-} M : \alpha$ or $\Delta_k \cup \{x_k : \gamma_k\} \nvdash_{TA^-} M : \alpha$

by contradiction with $(\vee E)^-$. So we define Δ_{k+1} as follows.

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{x_k : \beta_k\} & \Delta_k \cup \{x_k : \beta_k\} \nvdash_{TA^-} M : \alpha \\ \Delta_k \cup \{x_k : \gamma_k\} & \Delta_k \cup \{x_k : \gamma_k\} \nvdash_{TA^-} M : \alpha \\ \Delta_k & \text{Otherwise} \end{cases}$$

Finally, we define that $\Delta := \bigcup_{k=0}^{\infty} \Delta_k$, then everything follows as the proof of Lemma 4.16.

□

Theorem 4.21. *(Completeness Theorem)*

$$\Gamma \models M : \sigma \Rightarrow \Gamma \vdash_{TA^-} M : \sigma.$$

Proof. We prove this theorem by its contra-position as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \not\vdash_{TA^-} M : \alpha}{\Gamma \subseteq \Delta, \Delta \not\vdash_{TA^-} M : \alpha}\ 4.20}{\Gamma_{\rho_\Delta} \not\vdash_{TA^-} M : \alpha}\ (\star)}{\alpha \notin [\![M]\!]^{\mathscr{M}}_{\rho_\Delta}}}{[\![M]\!]^{\mathscr{M}}_{\rho_\Delta} \not\varepsilon\ [\![\alpha]\!]^{\mathscr{M}}_{\xi}}\ 4.19}{\Gamma \not\models M : \sigma}\ (\star\star)}{}$$

($\star$) For all $x : \gamma \in \Gamma_{\rho_\Delta}$, we have $\gamma \in \xi_\Delta(x)$, then $\Delta \vdash_{TA^-} x : \gamma$ by definition.
($\star\star$) We prove $\mathscr{M}, \rho_\Delta, \xi \models \Gamma$ as follows

$$\dfrac{\dfrac{\dfrac{\forall\{x : \gamma\} \in \Gamma(\subseteq \Delta) \quad [\![x]\!]^{\mathscr{M}}_{\rho_\Delta} := \{\alpha \mid \Delta \vdash_{TA^-} x : \alpha\}}{\gamma \in [\![x]\!]^{\mathscr{M}}_{\rho_\Delta}}}{\forall p \in \mathscr{F}_P[[\![x]\!]^{\mathscr{M}}_{\rho_\Delta} \subseteq p \Rightarrow p \in [\![\gamma]\!]^{\mathscr{M}}_{\xi}\ \ (\gamma \in p)]}}{[\![x]\!]^{\mathscr{M}}_{\rho_\Delta}\ \varepsilon\ [\![\gamma]\!]^{\mathscr{M}}_{\xi}}$$

□

Corollary 4.22.

- *The following are equivalent.*
  - $\Gamma \vdash_{TA} M : \alpha$.
  - $\Gamma \vdash_{TA^-} M : \alpha$.
  - $\Gamma \models M : \alpha$.
- *TA system is invariant under v-equality defined as follows.*

$$\dfrac{M : \alpha \quad M =_v N}{N : \alpha}$$

Appendix A The original proof for Lemma 3.20

Proposition A.1. $\alpha_1 \wedge \cdots \wedge \alpha_n \leq \beta$ *with* $n \geq 1, \beta \not\sim \omega \Rightarrow$
$\exists k \leq n[\alpha_k \not\sim \omega]$.

Proof. We prove this proposition by contradiction. Suppose $\forall k \leq n[\alpha_k \sim \omega]$, then we have $\alpha_1 \wedge \cdots \wedge \alpha_n \geq \omega$, then $\beta \geq \omega$ by (trans), then $\beta \sim \omega$ by definition which finally leads to a contradiction. □

Lemma A.2. $(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \leq \sigma \to \tau$ *and* $\tau \not\sim \omega$, *then there are* $i_1, \cdots, i_k \in \{1, \cdots, n\}$ *such that* $\mu_{i_1} \wedge \cdots \wedge \mu_{i_k} \geq \sigma$ *and* $\nu_{i_1} \wedge \cdots \wedge \nu_{i_k} \leq \tau$.

Proof. It suffices to show that following proposition holds.

For $n, n', m, m' \geq 0$ that for all $l \in \{1, \cdots, n'\}$
$[(\mu_1 \to \nu_1) \wedge \cdots \wedge (\mu_n \to \nu_n) \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_m} \wedge \omega \wedge \cdots \wedge \omega \leq$
$(\sigma_1 \to \tau_1) \wedge \cdots \wedge (\sigma_{n'} \to \tau_{n'}) \wedge \varphi'_{j_1} \wedge \cdots \wedge \varphi'_{j_{m'}} \wedge \omega \wedge \cdots \wedge \omega]$
and $\tau_l \not\sim \omega \Rightarrow$
$\exists i_1, \cdots, i_k \in \{1, \cdots, n\}$ $[\mu_{i_1} \wedge \cdots \wedge \mu_{i_k} \geq \sigma_l$ and $\nu_{i_1} \wedge \cdots \wedge \nu_{i_k} \leq \tau_l]$.

By induction on the definition of $\leq$.

- $(\alpha \leq \alpha)$. $\forall l \in \{1, \cdots, n'\}[\mu_l \equiv \sigma_l, \nu_l \equiv \tau_l]$. $n = n', k = 1, i_1 = l$.
  $\tau_l \not\sim \omega \Rightarrow [\mu_{i_1} \geq \sigma_l$ and $\nu_{i_1} \leq \tau_l]$, by (ref).
- $(\omega \leq \omega \to \omega)$. $\neg(\exists l[\tau_l \not\sim \omega])$, so this case is trivial.
- $(\alpha \leq \omega)$. $\neg(\exists l[\tau_l \not\sim \omega])$, so this case is trivial.
- $(\alpha \leq \alpha \wedge \alpha)$. Reduce to $(\alpha \leq \alpha)$.
- $(\alpha \wedge \beta \leq \alpha(\beta))$. Reduce to $(\alpha \leq \alpha)$.
- $((\alpha \to \beta) \wedge (\alpha \to \gamma) \leq \alpha \to (\beta \wedge \gamma))$.

  $(\beta \wedge \gamma \sim \omega)$ $\neg(\exists l[\tau_l \not\sim \omega])$, so this case is trivial.

  $(\beta \wedge \gamma \not\sim \omega)$ $l = 1, k = 2, i_1 = 1, i_2 = 2$. By (ref) and $(\alpha \leq \alpha \wedge \alpha)$, we have $\mu_{i_1} \wedge \mu_{i_2} \geq \sigma_l$ and $\nu_{i_1} \wedge \nu_{i_2} \leq \tau_l$. $(\alpha \wedge \alpha \geq \alpha$ and $\beta \wedge \gamma \leq \beta \wedge \gamma)$.
- ($\wedge$-mono)
  From the definition, we have

  (1) $\forall l_\alpha \in \{1, \cdots, n'_\alpha\}[\alpha \leq \alpha'$ and $\tau_{l_\alpha} \not\sim \omega \Rightarrow \exists i_1 \cdots i_\alpha \in \{1, \cdots, n_\alpha\}$
  $\mu_{i_1} \wedge \cdots \wedge \mu_{i_\alpha} \geq \sigma_{l_\alpha}$, $\nu_{i_1} \wedge \cdots \wedge \nu_{i_\alpha} \leq \tau_{l_\alpha}]$.

  (2) $\forall l_\beta \in \{1, \cdots, n'_\beta\}[\beta \leq \beta'$ and $\tau_{l_\beta} \not\sim \omega \Rightarrow \exists i_1 \cdots i_\beta \in \{1, \cdots, n_\beta\}$
  $\mu_{i_1} \wedge \cdots \wedge \mu_{i_\beta} \geq \sigma_{l_\beta}$, $\nu_{i_1} \wedge \cdots \wedge \nu_{i_\beta} \leq \tau_{l_\beta}]$.

  $n = n_\alpha + n_\beta$, $n' = n'_\alpha + n'_\beta$.
  We combine the two set as follows.
  $\{1, \cdots, n\} = \{1, \cdots, n_\alpha, 1 + n_\alpha, \cdots, n_\beta + n_\alpha\}$(n' is the same).
  $\forall l \in \{1, \cdots, n'\}[\alpha \wedge \beta \leq \alpha' \wedge \beta'$ and $\tau_l \not\sim \omega] \Rightarrow$

  $(l \leq n_\alpha)$ By (1), $i_1 = i_1, \cdots, i_k = i_\alpha$.

  $(n_\alpha < l \leq n_\alpha + n_\beta)$ By (2), $i_1 = i_1 + n_\alpha, \cdots, i_k = i_\beta + n_\alpha$.
- ($\to$-mono) $l = 1$.
  Suppose $\tau_l \not\sim \omega$, then we can use the assumption, $k = 1, i_k = 1$.
  $(\alpha \equiv)\mu_1 \geq \sigma_l(\equiv \alpha')$ and $(\beta \equiv)\nu_1 \leq \tau_l(\equiv \beta')$.

- ($\alpha \leq \beta \leq \gamma \Rightarrow \alpha \leq \gamma$) From the definition we have

  (1) $\forall l_\beta \in \{1, \cdots, n_\beta\}[\alpha \leq \beta$ and $\tau_{l_\beta} \not\sim \omega \Rightarrow \exists i_1 \cdots i_\alpha \in \{1, \cdots, n_\alpha\}$
  $\mu_{i_1} \wedge \cdots \wedge \mu_{i_\alpha} \geq \sigma_{l_\beta}$, $\nu_{i_1} \wedge \cdots \wedge \nu_{i_\alpha} \leq \tau_{l_\beta}]$.

  (2) $\forall l_\gamma \in \{1, \cdots, n_\gamma\}[\beta \leq \gamma$ and $\tau_{l_\gamma} \not\sim \omega \Rightarrow \exists i_1 \cdots i_\beta \in \{1, \cdots, n_\beta\}$
  $\mu_{i_1} \wedge \cdots \wedge \mu_{i_\beta} \geq \sigma_{l_\gamma}$, $\nu_{i_1} \wedge \cdots \wedge \nu_{i_\beta} \leq \tau_{l_\gamma}]$.

  $\{1, \cdots, n\} = \{1, \cdots, n_\alpha\}$, $\{1, \cdots, n'\} = \{1, \cdots, n_\gamma\}$.
  $\forall l \in \{1, \cdots, n'\}[\alpha \leq \gamma$ and $\tau_l \not\sim \omega] \Rightarrow$

$$\cfrac{\cfrac{\cfrac{\tau_l (\equiv \tau_{l_\gamma}) \not\sim \omega}{\mu_{i_1} \wedge \cdots \wedge \mu_{i_\beta} \geq \sigma_l \ , \ \nu_{i_1} \wedge \cdots \wedge \nu_{i_\beta} \leq \tau_l}\ (2)}{\mu_{i_k} \geq \mu_{i_1} \wedge \cdots \wedge \mu_{i_\beta} (\mu_{i_k} \not\sim \omega)}\ A.1 \qquad \cfrac{}{\mu_{i_1} \wedge \cdots \wedge \mu_{i_\alpha} \geq \mu_{i_k}}\ (1)}{\mu_{i_1} \wedge \cdots \wedge \mu_{i_\alpha} \geq \sigma_l}\ (trans)$$

  We can construct such set as follows.
  $\forall n \in \{1, \cdots, \beta\}$
  $[\nu_{i_n} \not\sim \omega \Rightarrow \nu_n \equiv \nu_{i_1} \wedge \cdots \wedge \nu_{i_\alpha}]$
  By (1), we have $\nu_{i_1} \wedge \cdots \wedge \nu_{i_\alpha} \leq \nu_{i_n}$.
  $[\nu_{i_n} \sim \omega \Rightarrow \nu_n \equiv \nu_{i_\alpha}]$
  By (ω-top), we have $\nu_{i_\alpha} \leq \nu_{i_n}$.
  So we have
  $\nu_1 \wedge \cdots \wedge \nu_\beta \leq \nu_{i_1} \wedge \cdots \wedge \nu_{i_\beta} \leq \tau_l$.

□

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