On qualitative multi-attribute group decision making and its consensus measure: A probability based perspective

Hong-Bin Yan[a,*], Tieju Ma[a], Van-Nam Huynh[b]

[a] *School of Business, East China University of Science and Technology Meilong Road 130, Shanghai 200237, P.R. China*
[b] *School of Knowledge Science, Japan Advanced Institute of Science and Technology 1-1 Asahidai, Nomi City, Ishikawa, 923-1292, Japan*

Abstract

This paper focuses on qualitative multi-attribute group decision making (MAGDM) with linguistic information in terms of single linguistic terms and/or flexible linguistic expressions. To do so, we propose a new linguistic decision rule based on the concepts of random preference and stochastic dominance, by a probability based interpretation of weight information. The importance weights and the concept of fuzzy majority are incorporated into both the multi-attribute and collective decision rule by the so-called weighted ordered weighted averaging operator with the input parameters expressed as probability distributions over a linguistic term set. Moreover, a probability based method is proposed to measure the consensus degree between individual and collective overall random preferences based on the concept of stochastic dominance, which also takes both the importance weights and the fuzzy majority into account. As such, our proposed approaches are based on the ordinal semantics of linguistic terms and voting statistics. By this, on one hand, the strict constraint of the uniform linguistic term set in linguistic decision making can be released; on the other hand, the difference and variation of individual opinions can be captured. The proposed approaches can deal with qualitative MAGDM with single linguistic terms and flexible linguistic expressions. Two application examples taken from the literature are used to illuminate the proposed techniques by comparisons with existing studies. The results show that our proposed approaches are comparable with existing studies.

Keywords: Linguistic MAGDM; Random preference; Weights; Stochastic dominance; Consensus measure.

1. Introduction

A group decision making (GDM) problem is defined as a decision problem where several experts (judges, decision makers, etc) provide their judgments over a set of alternatives (options, candidates, etc). The aim is to reconcile the differences of opinions expressed by individual experts to find an alternative (or set of alternatives) that is most acceptable by the group of experts as a whole [58, 66]. As an important branch of GDM, multi-attribute GDM (MAGDM) deals with decisions where several experts express their opinions on a set of possible options with respect to multiple attributes and attempt to find a common solution. In practice, both GDM and MAGDM require subjective assessments by a set of experts to solve complex and unstructured problems [19], which are often vaguely qualitative and cannot be estimated by exact numerical values. Such phenomena may arise from the following two facts [3]: first, the information may be qualitative due to its nature, and can be stated only in linguistic terms (for example when evaluating the comfort

*Corresponding author, Tel.: +86-21-64250013.
Email addresses: hbyan@ecust.edu.cn (Hong-Bin Yan), tjma@ecust.edu.cn (Tieju Ma), huynh@jaist.ac.jp (Van-Nam Huynh)

or design of a car, terms like "good", "poor" can be used); second, in other cases, precise quantitative information may not be stated because either it is unavailable or the cost of its computation is too high, so an "approximate value" may be tolerated (for example when evaluating a car's speed, linguistic terms like "fast", "slow" can be used instead of numeric values). In this sense, the fuzzy linguistic approach [79, 80, 81] enhances the feasibility, flexibility, and reliability of decision models when the decision problems are too complex or ill-defined to be described properly by conventional quantitative expressions [21].

In practice, experts usually use single linguistic terms to provide their opinions. In this case, two categories of models have been proposed in the literature [25, 51]: the approximate model based on the extension principle [e.g., 9, 20, 31, 75] and the term index based models [3, 8, 25]. As one model using term indices, the two-tuple linguistic model [25] has been widely studied in the literature [42], perhaps due to its no information loss, straightforwardness and convenience in calculation. In some cases, the experts may have a set of possible linguistic terms about the attributes or alternatives [52, 67]. To provide more flexible and richer linguistic expressions, three types of models have been proposed in the literature, namely the interval linguistic model [69], the model based on absolute order of magnitude spaces [60], and the hesitant fuzzy linguistic term set (HFLTS) [52]. Over the past decades, we have witnessed a lot of studies focusing on (MA)GDM with single linguistic terms and flexible linguistic expressions. As one process of linguistic MAGDM, the selection process refers to obtaining the solution set of alternatives and involves two different steps: the aggregation and exploitation [21, 29], which have been widely studied and reviewed in Subsec. 2.2. Despite their great advances, most existing studies are based on the term indices, which are only suitable to the case of symmetric and balanced linguistic term sets. Although the unbalanced linguistic term set can be transformed into a uniform one by some techniques [10, 26] or directly processed by an ordinal technique [16], it is still a tedious work which may create an obstacle to use of linguistic approaches in decision making. Furthermore, the term index based models cannot represent the differences and variations of individual opinions. Since the evaluation in (MA)GDM is quite subjective and highly individualistic, it may be inappropriate to perform computations without further considering the variation and difference in individual opinions. Finally, different approaches have been proposed to solve (MA)GDM with single linguistic terms and/or flexible linguistic expressions. Since a single linguistic term may be viewed as a special case of a linguistic interval, a unified approach to MAGDM with single linguistic terms or flexible linguistic expressions may provide ease of use to the users in practice.

As it is well known, another process of the usual resolution method for a (MA)GDM problem is the consensus process [28], which consists of obtaining the maximum degree of consensus or agreement among the experts on their preferences. For an overview of consensus models, please see [6, 27]. It is preferable that the experts reach a high degree of consensus before applying the selection process. Thus, how to find a group consensus to represent a common opinion of the group is a valuable and important topic [58]. With single linguistic terms and flexible linguistic expressions, different approaches have been proposed to address the issue of consensus measure in (MA)GDM, as reviewed in Subsec. 2.3. Unfortunately, most existing studies are also based on the term indices. For example, Sun and Ma [58] have extended Xu's [70] work to propose a new consensus measure with a threshold value based on the term indices. Since the consensus measure is closely related to the linguistic representation models, similar problems may arise as the ones in the process of aggregation and exploitation, i.e., strict constraint of symmetric and balanced linguistic term set, unable to capture the differences and variations of individual opinions. For example, the consensus measure defined in [48] depends greatly on the cardinality of a linguistic term set, which means that different cardinalities may generate different consensus degrees.

Our final motivation comes from the weight information in (MA)GDM. As a basic element underlying (MA)GDM, the concept of *fuzzy majority* is accepted by most of its members/attributes in practice, since it is quite difficult for the solution to be accepted by all members/attributes [32, 33, 34, 49]. The ordered weighted averaging (OWA) operator [72] and its extensions [3, 25] have been widely applied in linguistic

(MA)GDM [e.g., 11, 63, 69] to model the fuzzy majority. In (MA)GDM, the experts and/or attributes can be treated unequally considering their possible importance differences, each of which reflects the reliability of each information source; the weight information in the OWA operator reflects the reliability of each value [59]. In this sense, it may be important and necessary to incorporate these two types of weight information into linguistic MAGDM simultaneously [49, 59]. Unfortunately, most existing studies consider either the importance weights or the fuzzy majority, but have not take both of them into consideration simultaneously. There are limited studies involving both the fuzzy majority and importance weights, see [3, 22, 49]. However, these models are still based on the term indices and cannot deal with (MA)GDM with flexible linguistic expressions. With the importance weights and fuzzy majority used, different results may be yielded by the process of aggregation. Consequently, these two types of weight information may also be necessary and important to be incorporated into the consensus measure and will influence the final consensus result, which is missed in the literature.

Due to the above observations, the main focus of this paper is to propose alternative approaches to qualitative MAGDM with linguistic expressions and its consensus measure, based on the ordinal semantics of the linguistic term set [29, 36, 37]. The main contributions of this paper are two-fold. First, regarding the process of aggregation and selection, we propose a new linguistic decision rule for MAGDM problems by means of the concepts of random preference and stochastic dominance, which is based on a probability interpretation of weight information. The importance weights and fuzzy majority have been both incorporated into the multi-attribute decision rule and collective decision rule by means of the so-called weighted ordered weighted averaging (WOWA) operator with the input parameters expressed as probability distributions. Second, a new method is proposed to measure the consensus degree between individual and collective overall random preferences based on the concept of stochastic dominance, which involves the importance weights and the fuzzy majority. By this, on one hand, the strict constraint of the symmetric and balanced linguistic term set in linguistic decision making can be released; on the other hand, the difference and variation of individual opinions can be captured. Moreover, the proposed approaches can deal with qualitative MAGDM with both single linguistic terms and flexible linguistic expressions.

The outline of this paper is as follows. Sec. 2 begins with a brief review of approaches and consensus measures in linguistic (MA)GDM, and follows by presenting a general scheme of MAGDM problems. Sec. 3 proposes a probability based approach to aggregation and exploitation in linguistic MAGDM. Sec. 4 applies the proposed approach to two MAGDM problems with single linguistic terms and flexible linguistic expressions by comparisons with existing studies. Sec. 5 proposes a new consensus measure based on the concept of stochastic dominance, which takes the importance weights and fuzzy majority into account simultaneously. Comparisons with existing studies are also provided. Finally, Sec. 6 presents some concluding remarks.

2. Literature review and problem formulation

After reviewing the linguistic term set, linguistic approaches, and consensus measures in linguistic decision making, this section presents a general scheme of MAGDM problems with linguistic information.

2.1. Fuzzy linguistic approach in decision making

By scanning the literature, one can find extensive applications of linguistic approaches to many different areas such as new product development [30, 75], Kansei evaluation [74], quality function deployment [76, 77, 78], supply chain management [7, 62], energy planning [13], etc. Essentially, in any linguistic approach to solving a decision making problem, the term set of a linguistic variable [79, 80, 81] and its associated semantics must be defined first to supply the users with an instrument by which they can naturally express their opinions. An important aspect to analyze in this process is the granularity of uncertainty, i.e., the level of discrimination or the cardinality of the linguistic term set. The cardinality of the linguistic term set must be small enough so as not to impose useless precision on the users, and it must be rich enough in order to allow a discrimination of the assessments in a limited number of degrees [3].

Syntactically, there are two main approaches to generating a linguistic term set. The first one is based on a context-free grammar [79, 80, 81]. This approach may yield an infinite term set. A similar approach is to consider primary linguistic terms (e.g., high, low) as generators, and linguistic hedges (e.g., very, rather, more, or less) as unary operations. Then the linguistic term set can be obtained algebraically [47]. However, according to observations in [45], the generated language does not have to be infinite, and in practice human beings can reasonably manage to keep about seven terms in mind. A second approach is to directly supply a finite term set and consider all terms as primary ones, distributed on a scale on which a total order is defined [3, 24, 26]. Formally, let

$$\mathcal{S} = \{S_0, S_1, \ldots, S_G\} \tag{1}$$

be a finite and totally ordered discrete linguistic term set, where $S_i < S_j \iff i < j$.

Remark 1. In the literature, the linguistic term set $\mathcal{S}$ should satisfy the negation operator: $\mathrm{Neg}(S_i) = S_j$ such that $i = G - j$, which indicates that the linguistic term set S is a uniform scale, i.e., symmetric and balanced [e.g., 25, 66, 67]. Without loss of generality, we shall assume that the linguistic term set $\mathcal{S}$ can be a uniform or non-uniform scale.

Regarding the semantic aspect, once the mechanism of generating a linguistic term set has been determined, its associated semantics must be defined accordingly. In the literature, there are three main possibilities for defining the semantics of the linguistic term set (see [21, 29] for more details): (1) Semantics based on fuzzy membership functions and a semantic rule. Usually, this semantic approach is used when the term set is generated by means of a generative grammar. (2) Semantics based on the ordered structure of the term set, which is based on a finite linguistic term set accompanied with an ordered structure which intuitively represents the semantical order of linguistic terms. (3) The third semantic approach is a mixed representation of the previous two approaches, that is, an ordered structure of the primary linguistic terms and a fuzzy set representation of linguistic terms. In this paper, we adopt the ordered structure based semantics of the linguistic term set.

2.2. Approaches to decision making with linguistic information

In this subsection, we review different approaches to the aggregation and exploitation in MAGDM with linguistic information in terms of single linguistic terms or flexible linguistic expressions.

2.2.1. Linguistic decision making with single linguistic terms

When using linguistic approaches to solving decision problems, we need linguistic representation models, which can be roughly divided into two categories [25, 51]: the approximate model based on the extension principle [e.g., 9, 20, 31, 75] and the term index based models [3, 8, 25]. The models using term indices make computations based on the indices of linguistic terms and can be divided into two types: the symbolic model [3, 8] and the two-tuple linguistic model [25]. In linguistic decision making, one has to face the problem of aggregation of linguistic information, which heavily depends on the semantic description of the linguistic term set. As mentioned in [25, 51], the results yielded by the approximate model do not exactly match any of the initial linguistic terms, so a process of linguistic approximation must be applied. This process causes loss of information and hence a lack of precision. Moreover, the approximate model makes operations on the fuzzy numbers that support the semantics of the linguistic terms, which will create the burden of quantifying a qualitative concept [25, 29] and complex mathematical computations [78].

Consequently, we have witnessed a large number of studies on (MA)GDM problems based on the indices of linguistic terms over the past decades. Under the symbolic model, Herrera et al. [24] have presented a linguistic OWA operator; Bordogna et al. [3] have presented an OWA based linguistic MAGDM framework. However, the result yielded by such a model does not exactly match any of the initial linguistic terms and may cause loss of information [25, 51]. To avoid the information loss inherent in the symbolic model, Herrera and Martínez [25] have further proposed a two-tuple linguistic model and developed a two-tuple OWA operator. Perhaps due to its no information loss, straightforwardness and convenience in calculation, the two-tuple linguistic model has received great attention in the literature, as reviewed in [42]. For example, Herrera and Martínez [26] have proposed a model based on linguistic two-tuples for dealing with multi-granular hierarchical linguistic contexts in GDM. Huynh and Nakamori [30] have proposed a linguistic two-tuple screening model in new product development. Dhouib [9] has integrated an extended version of MACBETH methodology and two-tuple linguistic model to solve waste tire related environmental problem and its recycling alternatives. Recently, Merigó et al. [44] have proposed a linguistic probabilistic weighted average (WA) aggregation operator in linguistic MAGDM to consider subjective and objective information in the same formulation.

2.2.2. Linguistic decision making with flexible linguistic expressions

In qualitative decision making, when experts face decision situations with a high degree of uncertainty, they often hesitate among different linguistic terms and would like to use more complex linguistic expressions [52]. Several attempts have therefore tried to provide more flexible and richer expressions which can include more than one linguistic term. For a recent overview of fuzzy modeling of complex linguistic preferences in decision making, please see [50].

As a flexible linguistic expression in decision making, the linguistic interval has received great attention in the literature. Xu in [69] has proposed an uncertain (also called interval) linguistic decision making approach, where the evaluation information provided by experts may be between two linguistic phrases. With the linguistic intervals, Xu [69] has developed the uncertain linguistic OWA operator and uncertain linguistic hybrid aggregation operator, and applied them to MAGDM problems with interval linguistic information. Later, Xu [71] has developed an approach based on the (induced) uncertain linguistic operator to GDM with uncertain multiplicative linguistic preference relations. Xu et al. [68] have proposed a two-stage approach to MAGDM problems under an uncertain linguistic environment. Wang et al. [61] have proposed a decision making method for MAGDM problems with uncertain linguistic information, in which the importance weights of experts are obtained by a cloud model.

To help experts elicit interval linguistic information, the absolute order of magnitude spaces [60] has been widely adapted in uncertain linguistic decision making. For example, Agell et al. [1] have presented a new approach to representing and synthesizing the information given by a group of evaluators, which is based on comparing distances against an optimal reference point. Falcó et al. [15] have proposed a three-stage approach to GDM with linguistic intervals, which is based on the distance function between linguistic expressions. Roselló et al. [55] have presented a mathematical framework and methodology for GDM under multi-granular and multi-attribute linguistic assessments.

In their pioneering work, Rodríguez et al. [52] have proposed the concept of HFLTS to improve the elicitation of linguistic expressions by using a context-free grammar [4], which increases the flexibility of the model by eliciting comparative linguistics expressions. Since its introduction, the HFLTS has attracted more and more scholars' attention in the literature. Similar with (MA)GDM with single linguistic terms, there are three main schools of approaches to (MA)GDM with HFLTSs. Based on the fuzzy extension principle, Liu and Rodríguez [40] have presented a new representation of the HFLTSs by means of a fuzzy envelope to carry out the process of computing with words (CWW), which aggregates linguistic terms in an HFLTS into a trapezoidal fuzzy number. There are also some works based on the indices of linguistic terms in an HFLTS. Beg and Rashid [2] have extended the technique for order preference by similarity to an ideal solution (TOPSIS) method for HFLTS. Wei et al. [63] have developed some comparison methods and studied the WA operators for HFLTSs by possibility degree formulas. Zhu and Xu [84] have developed a hesitant fuzzy linguistic preference relation concept, which requires that the HFLTSs concerned must have the same length to carry out the computations correctly [67]. Li et al. [38] have proposed a MAGDM evaluation approach for the individual research output by context-free grammar judgment description. The two-tuple linguistic model [25] has also been extended to (MA)GDM with HFLTSs [46, 53]. Taking a different track, Wu and Xu [67] have developed a possibility distribution based approach for MAGDM with HFLTSs; at the same time they [66] have also studied GDM with hesitant fuzzy linguistic preference relations.

2.2.3. Summary

It is clearly concluded that existing studies have made great contributions to qualitative (MA)GDM with single linguistic terms and flexible linguistic expressions. However, there are still several limitations in these studies. First, most studies are based on the term indices, which are only suitable to the case of symmetric and balanced linguistic term sets [10]. Within a fuzzy approach, although the unbalanced linguistic term set can be transformed into the uniform one by some cardinal proposals [10, 26], it is a tedious work which may create an obstacle to use of linguistic approaches in decision making. Under the ordinal semantics of linguistic term set, García-Lapresta and Pérez-Román [16] have introduced ordinal proximity measures in the setting of unbalanced qualitative scales by comparing the proximities between linguistic terms. However, their work needs many pairwise comparisons among linguistic terms in a pre-experiment, which is also a tedious work in practice. Moreover, the studies on the non-uniform scale [10, 16, 26] do not consider flexible linguistic expressions. In this sense, a unified approach to the uniform or non-uniform linguistic scale may provide ease of use to users in practice.

Second, despite their no information loss and ease of use in practice, most models using term indices cannot represent the differences and variations of individual opinions. Since the evaluation in (MA)GDM is quite subjective and highly individualistic, it may be inappropriate to perform computations without further considering the variation in individual evaluations, see the illustrative comparisons in [29]. Third, it may be important and necessary to incorporate both the importance weights and the concept of fuzzy majority into linguistic MAGDM problems [49]. Unfortunately, most studies consider either the importance weights or the fuzzy majority, but have not take both of them into consideration simultaneously. There are limited studies involving both the fuzzy majority and importance weights, see [3, 22]. However, they still have the same problems as the ones from the term index based models as well as cannot deal with (MA)GDM with flexible linguistic expressions. Finally, as we have seen, different approaches have been proposed to solve the decision making with single linguistic terms or flexible linguistic expressions. Since the single linguistic term is a special case of a linguistic interval [52, 67], a unified approach to MAGDM with either single linguistic terms or flexible linguistic expressions may provide ease of use to users in practice.

2.3. The consensus measure in qualitative (MA)GDM with linguistic expressions

Consensus is another fundamental issue widely employed in (MA)GDM. Some notable consensus models have been developed for (MA)GDM problems under linguistic environments. Herrera et al. [23] have introduced a consensus model for GDM using linguistic preference relations based on the use of a fuzzy consensus majority. Bordogna et al. [3] have proposed a linguistic consensus model for GDM based on the OWA operator. Herrera-Viedma et al. [28] have introduced a model of consensus support system to assist the experts in all phases of the consensus reaching process of GDM problems with multi-granular linguistic preference relations. Xu [70] have defined the concepts of deviation degree and similarity degree between two linguistic values, and the ones between two linguistic preference relations. Dong et al. [11] have introduced another deviation measure using a different distance metric. Wu and Xu [65] have further proposed two consensus models based on the deviation measures given by Xu [70] and Dong et al. [11]. Dong et al. [12] have proposed a consensus operator as a generalization of the OWA operator and provide an alternative GDM consensus model with linguistic information. Pang and Liang in [48] have proposed a closeness measure by the distance function of two linguistic values. Sun and Ma [58] have extended Xu's [70] work to propose a new consensus measure with a threshold value. Dong et al. [10] have proposed a consensus-based GDM model with multi-granular unbalanced two-tuple linguistic preference relations.

In the context of (MA)GDM with flexible linguistic expressions, some consensus measures have also been proposed. Xu et al. [68] have proposed a consensus measure based on the distance function of interval linguistic information. Within the context of absolute order of magnitude spaces [60], Roselló et al. [54] have presented a proposal to assess the consensus among different evaluators who use ordinal scales in GDM and evaluation processes, by means of the quantitative entropy. García-Lapresta and Pérez-Román [17] have presented a consensus measure based on the distance between two linguistic intervals. Within the context of HFLTS, Zhu and Xu [84] have investigated the consistency of linguistic preference relations expressed in terms of HFLTSs. Liao et al. [39] have discussed the distance and similarity measures for decision problems with HFLTSs. Wu and Xu in [67] have proposed the consensus measure based on the similarity matrix between two possibility distributions, which is still based on the term indices; at the same time, they [66] have also studied the consistency and consensus in GDM with hesitant fuzzy linguistic preference relations.

In essence, the consensus measures in (MA)GDM with linguistic expressions are closely related to the linguistic representation models in the aggregation and exploitation. As we have seen, most existing consensus measures are based on the operations of term indices. Consequently, similar problems may arise as the ones in the process of aggregation and exploitation, i.e., strict constraint of symmetric and balanced linguistic term set, unable to capture the differences and variations of individual opinions. In addition, few studies have incorporated both the importance weights and the concept of fuzzy majority into the consensus measure in linguistic (MA)GDM, simultaneously. Finally, a unified approach to consensus measure with either single linguistic terms or flexible linguistic expressions may provide ease of use to users in practice.

2.4. A general scheme of MAGDM problems

Before going into detail, we first introduce some basic notations which will be used through the rest of this paper. Let $\mathcal{S} = \{S_0, S_1, \ldots, S_G\}$ be a finite and totally ordered discrete term set. The HFLTS based approach provides experts greater flexibility to elicit comparative linguistic expressions and is close to human being's cognitive model [40], therefore the HFLTS is used to represent the flexible linguistic expressions, defined as follows [52].

Definition 1. Let $\mathcal{S}$ be a linguistic term set and $\mathbf{G}$ be a context-free grammar. Given a comparative linguistic expression θ generated by the context-free grammar, a transformation function $\mathcal{F}_{\mathbf{G}} : \theta \longrightarrow \mathcal{H}(\theta)$ is needed to derive an HFLTS, which is an ordered finite subset of the consecutive linguistic terms of $\mathcal{S}$.

For example, a comparative linguistic expression may be "*between* S_1 *and* S_3", then an HFLTS is derived as $\{S_1, S_2, S_3\}$. Based on the above definition, the empty HFLTS and the full HFLTS for a linguistic term set $\mathcal{S}$ are defined by $\emptyset$ and $\mathcal{S}$, respectively.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_M\}$ be a discrete set of options (alternatives, candidates), $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ be the set of attributes and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be the weighting vector of attributes where $\sum_{n=1}^{N}\mu_n = 1, \mu_n \geq 0$. Let $\mathcal{E} = \{E_1, E_2, \ldots, E_K\}$ be the set of experts and $\nu = (\nu_1, \nu_2, \ldots, \nu_K)$ be the weighting vector of experts where $\sum_{k=1}^{K}\nu_k = 1, \nu_k \geq 0$. The general scheme of MAGDM problems considered in this paper is shown in Table 1, where x_{mn}^k is the linguistic assessment of alternative A_m on attribute C_n provided by expert E_k, in terms of a single linguistic term from $\mathcal{S}$ or an HFLTS derived from a comparative linguistic expression which is elicited by the context-free grammar [4]. Note that the comparative linguistic expressions by the context-free grammar are used to help experts elicit flexible linguistic expressions. Thus, it is natural to assume that the HFLTSs derived from experts' comparative linguistic expressions are non-empty. The empty set is not considered in our current work.

Table 1: Decision matrix on attribute C_n with linguistic expressions, where $n = 1, \ldots, N$.

Options	Experts			
	E_1	E_2	$\ldots$	E_K
A_1	x_{1n}^1	x_{1n}^2	$\ldots$	x_{1n}^K
A_2	x_{2n}^1	x_{2n}^2	$\ldots$	x_{2n}^K
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_M	x_{Mn}^1	x_{Mn}^2	$\ldots$	x_{Mn}^K

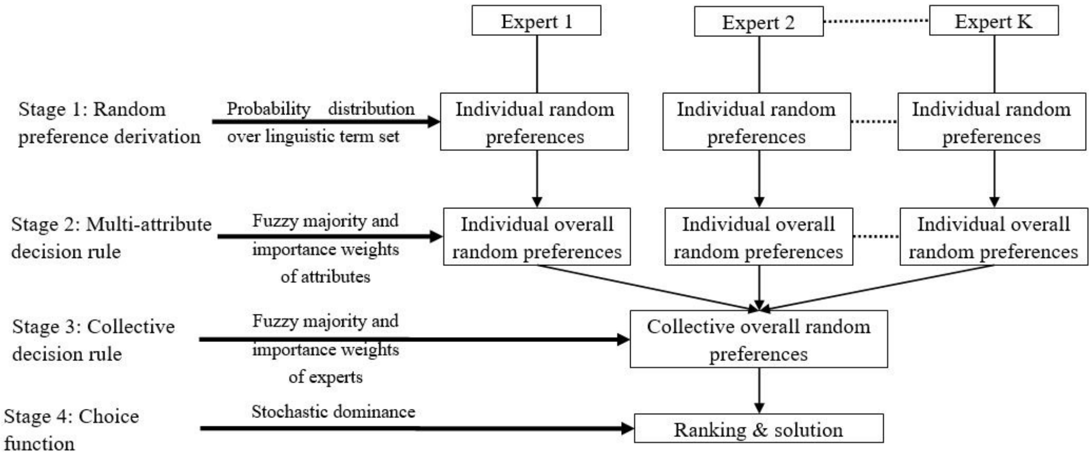

Figure 1: Flowchart of the proposed approach to aggregation and exploitation.

3. Aggregation and exploitation: A probability based approach

In linguistic decision analysis, a solution scheme must comply aggregation and exploitation phases [21, 29]. As mentioned in [24], there are two types of basic approaches to aggregation and exploitation: the direct approach to deriving a solution based on individual decision matrices and the indirect approach to providing the solution based on an overall decision matrix. In this section, we focus on the direct approach to MAGDM with linguistic information, which consists of the following four steps (as depicted in Fig. 1): random preference derivation, multi-attribute decision rule, collective decision rule, and choice function.

3.1. Random preference derivation from HFLTS

In our qualitative MAGDM context, the linguistic assessment x_{mn}^k can be either a single linguistic term or an HFLTS derived from a comparative linguistic expression, which is elicited by the context-free grammar [4]. As pointed out in [52], a single linguistic term can be expressed by a comparative linguistic expression and thus be viewed as an HFLTS, i.e., the single linguistic term $S_g \in \mathcal{S}$ is expressed as $\{S_g\}$. The linguistic assessment x_{mn}^k will be expressed in terms of HFLTS and re-denoted as $\mathcal{H}_{mn}^k$, where $m = 1, \ldots, M, n = 1, \ldots, N, k = 1, \ldots, K$. The HFLTS, $\mathcal{H}_{mn}^k$, is a subset of the linguistic term set $\mathcal{S}$, which represents expert E_k's uncertain judgment for alternative A_m on attribute C_n. For the sake of convenience, the family of all the HFLTSs defined over a linguistic term set $\mathcal{S}$ is denoted by $\Omega(\mathcal{S})$.

Returning back to our qualitative MAGDM context in Table 1, when expert E_k provides his/her judgment for the performance of alternative A_m with respect to attribute C_n, a probability distribution of his/her opinion on the family of all the possible HFLTSs, $\Omega(\mathcal{S})$, can be derived as

$$p_{\Omega(\mathcal{S})}\left(\mathcal{H}|A_m, C_n, E_k\right) = \begin{cases} 1, & \text{if } \mathcal{H} = \mathcal{H}_{mn}^k; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

where $m = 1,\ldots,M, n = 1,\ldots,N, k = 1,\ldots,K$. The probability distribution $p_{\Omega(\mathcal{S})}\left(\mathcal{H}|A_m, C_n, E_k\right)$ is nothing but a basic probability assignment in the sense of Shafer [56]. We then can fortunately use the so-called pignistic transformation method [57] to obtain the least prejudiced distribution over the linguistic term set $\mathcal{S}$ for alternative A_m on attribute C_n under expert E_k as follows:

$$\begin{aligned} p_{\mathcal{S}}(S_g|A_m, C_n, E_k) &= \frac{p_{\Omega(\mathcal{S})}\left(\mathcal{H}|A_m, C_n, E_k\right)}{|\mathcal{H}_{mn}^k|} \\ &= \begin{cases} 1/\left|\mathcal{H}_{mn}^k\right|, & \text{if } S_g \in \mathcal{H}_{mn}^k; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

where $m = 1,\ldots,M, n = 1,\ldots,N, k = 1,\ldots,K, g = 0,\ldots,G$. For example, with an HFLTS $\mathcal{H} = \{S_0, S_1, S_2\}$, a probability distribution over the linguistic term set $\mathcal{S}$ is derived as

$$p_{\mathcal{S}}(S_g) = \begin{cases} 1/3, & \text{if } S_g \in \mathcal{H}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the terminology "pignistic probability distribution" has been used in the context of belief modeling [57]. Here, we borrow this terminology from [36], which we think is more appropriate for our context. Such a probability distribution can be viewed as the prior probability that the expert E_k believes that the linguistic term $S_g \in \mathcal{S}$ is appropriate enough to describe the performance of alternative A_m on attribute C_n. For notational convenience, $p_{\mathcal{S}}(S_g|A_m, C_n, E_k)$ will be denoted by $p_{mn}^k(S_g)$. Under such a formulation, for each alternative A_m, each expert E_k generates a vector of N individual random preferences, denoted by $(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k)$, with respect to the N attributes such that

$$X_{mn}^k = \left[p_{mn}^k(S_0), p_{mn}^k(S_1), \ldots, p_{mn}^k(S_G)\right], \tag{4}$$

where $m = 1,\ldots,M, n = 1,\ldots,N, k = 1,\ldots,K$.

Remark 2. Here, we borrow the terminology of "random preference" from the theory of random preference [41], which we think is more appropriate for our context. The central assumption of the random preference theory is that each individual has a set of preference orderings and a probability distribution over that set. Our research is built on the ordinal semantics of a linguistic term set and the voting statistics [37], which derives associated probability distributions over that set.

Remark 3. It should be noted here that Zhang et al. [83] have proposed a concept of distribution assessment in a linguistic term set, in which the distribution assessment is specified by the expert. Wu and Xu [66, 67] have defined the concept of a possibility distribution from an HFLTS. The background in [66, 67, 83] are quite different from that in this paper. In addition, a probability based interpretation for the linguistic expression is proposed in this paper. Finally, the works [66, 67, 83] are also based on the indices of linguistic terms. As we shall see later, our research is based on the ordinal semantics of linguistic terms.

3.2. Multi-attribute decision rule

Recall that a set of attributes $\mathcal{C}$ is involved in our GDM context. We shall assume a subjective probability distribution $p_{\mathcal{C}}$ defined over the set of attributes, which essentially underlies the calculating basis for the following proposed multi-attribute decision rule.

From a practical point of view, given an alternative, if there is an ideal attribute, say C_{I}, which the expert completely believes in to represent the alternative, then it is enough to use the ideal attribute C_{I} in GDM. However most decision making problems involve multiple attributes. In this sense, $p_{\mathcal{C}}$ may be interpreted as the probability that the expert randomly selects attribute C_n from the set of attributes $\mathcal{C}$ as a sufficient information source for the purpose of decision making. In other words, the set of attributes $\mathcal{C}$ plays the role of states of the world and the weighting vector $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ associated with the attribute set $\mathcal{C}$ plays the role of subjective probabilities assigned to the states such that

$$p_{\mathcal{C}}(C_n) = \mu_n, n = 1, 2, \ldots, N. \tag{5}$$

Such an interpretation has its solid foundation in the striking similarity between decision making under uncertainty and multi-attribute decision making [14]. In the sequel, we shall propose our multi-attribute decision rule based on the probabilistic interpretation of weights.

3.2.1. Incorporating importance weights into multi-attribute decision rule

Taking the prior probability distributions (individual random preferences) $X_{mn}^k(m = 1, \ldots, M, n = 1, \ldots, N, k = 1, \ldots, K)$ into consideration, together with the importance weights of attributes, a posterior probability distribution of alternative A_m under expert E_k over the linguistic term set $\mathcal{S}$ can be obtained as follows:

$$\begin{aligned} X_m^k &= \mathcal{F}_{\mathrm{WA}}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k; \mu_1, \mu_2, \ldots, \mu_N\right) \\ &= \bigoplus_{n=1}^{N} \left[X_{mn}^k \odot \mu_n\right] \\ &= \left[p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)\right] \end{aligned} \tag{6}$$

where

$$p_m^k(S_g) = \sum_{n=1}^{N} p_{mn}^k(S_g) \cdot \mu_n,$$

and $g = 0, \ldots, G, m = 1, \ldots, M, k = 1, \ldots, K$. The symbols $\oplus$ and $\odot$ are, respectively, the addition operation and product operation of random preferences, which is in fact the weighted combination of probability distributions, see [18]. The derived random preference X_m^k is used to represent the uncertain performance of alternative A_m with respect to expert E_k, i.e., individual overall random preference. The importance weights reflect the reliabilities of attributes [59]. The aforementioned multi-attribute decision rule in (6) is in fact the WA aggregation, which may be linguistically stated as "each expert prefers that *important* attributes are satisfied by the alternatives."

3.2.2. Incorporating fuzzy majority into multi-attribute decision rule

In addition to the importance weights of attributes, we also want to incorporate the concept of majority, which is a basic element underlying decision making. The term "majority" indicates that an alternative satisfies most of its attributes, since in practice it is quite difficult for the alternative to satisfy all the attributes. The concept of "fuzzy majority" is used to make the strict concept of majority more vague so as to make it closer to its real human perception [34]. A natural manifestation of such a "soft" majority is the so-called linguistic quantifier Q, e.g., *most*, *at least half*, *as many as possible*. With the linguistic quantifier, a "fuzzy majority" quantified statement for our multi-attribute decision rule can be linguistically written as "each expert prefers that Q (of) attributes are satisfied by the alternatives."

Such a fuzzy majority guided linguistic statement can be solved by an aggregation function. Fortunately enough, Yager [72] proposed a special class of aggregation operators, called OWA operator, which seems to provide an even better and general aggregation in the sense of being able to simply and uniformly model a large class of fuzzy linguistic quantifiers, see Appendix B. The original OWA operator has been first defined to aggregate a set of crisp values and later transformed into the case with input parameters expressed as fuzzy numbers [75]. Here, the OWA operator will be transformed to the case with input parameters expressed as probability distributions over a linguistic term set, defined as follows.

Definition 2. Let $\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right)$ be the vector of individual random preferences of alternative A_m with respect to the N attributes under expert E_k, an OWA operator of dimension N is a mapping $\mathcal{F}_{\text{OWA}} : \mathcal{S}^N \to \mathcal{S}$ if $\mathcal{F}$ is associated with an OWA weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ such that: $\omega_n \in [0,1]$, $\sum_{n=1}^{N} \omega_n = 1$, and

$$\begin{aligned} X_m^k &= \mathcal{F}_{\text{OWA}}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right) \\ &= \bigoplus_{n=1}^{N} \left[X_{m\sigma(n)}^k \odot \omega_n\right] \\ &= \left[p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)\right], \end{aligned} \tag{7}$$

where $\left(X_{m\sigma(1)}^k, X_{m\sigma(2)}^k, \ldots, X_{m\sigma(N)}^k\right)$ is the permutation of $\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right)$ such that $X_{m\sigma(n-1)}^k \geq X_{m\sigma(n)}^k$ for all $n = 2, \ldots, N$.

The fuzzy majority in terms of linguistic quantifiers can be represented by means of fuzzy sets [82], i.e. any relative quantifier can be expressed as a fuzzy subset Q of the unit interval $[0,1]$. In this representation for any proportion, $r \in [0,1]$, $Q(r)$ indicates the degree to which r satisfies the concept conveyed by the linguistic quantifier Q. Yager in [73] further defined a Regular Increasing Monotone (RIM) quantifier to represent the linguistic quantifier Q. The definition of RIM quantifier and its examples can be referred to Appendix B. With the quantifier function defined, Yager [73] proposed a method for obtaining the OWA weighting vector via linguistic quantifiers, especially the RIM quantifiers, which can provide information aggregation procedures guided by verbally expressed concepts. By using the OWA operator and an RIM function Q, a posterior probability distribution of an alternative A_m under expert E_k over the linguistic term set $\mathcal{S}$ can be obtained as

$$\begin{aligned} X_m^k &= \mathcal{F}_{\text{OWA}}^{Q}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right) \\ &= \bigoplus_{n=1}^{N} \left[X_{m\sigma(n)}^k \odot \left[Q\left(\frac{n}{N}\right) - Q\left(\frac{n-1}{N}\right)\right]\right] \\ &= \left[p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)\right], \end{aligned} \tag{8}$$

where $(X_{m\sigma(1)}^k, X_{m\sigma(2)}^k, \ldots, X_{m\sigma(N)}^k)$ is the permutation of $\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right)$ such that $X_{m\sigma(n-1)}^k \geq X_{m\sigma(n)}^k$ for all $n = 2, \ldots, N$.

Central to the linguistic quantifier guided OWA operator is the permutation of the vector of probability distributions $\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right)$, which falls into the category of comparison and ranking techniques of probability distributions. In this paper, the stochastic dominance based approach (see Appendix A) is used to rank the vector of probability distributions, defined as follows.

Definition 3. Given two individual random preferences X_{mn}^k and X_{ml}^k expressed as probability distributions over a linguistic term set $\mathcal{S}$, the stochastic dominance degree of X_{mn}^k over X_{ml}^k is defined as

$$D_{X_{mn}^k \succ X_{ml}^k} = \Pr(X_{mn}^k \geq X_{ml}^k) - 0.5\Pr(X_{mn}^k = X_{ml}^k). \tag{9}$$

The comparison operation of X_{mn}^k over X_{ml}^k is defined as follows.

- If $D_{X^k_{mn} \succ X^k_{ml}} > 0.5$, then it indicates that X^k_{mn} is preferred to X^k_{ml} such that $X^k_{mn} > X^k_{ml}$.
- If $D_{X^k_{mn} \succ X^k_{ml}} = 0.5$, then there is indifference between X^k_{mn} and X^k_{ml} such that $X^k_{mn} = X^k_{ml}$.
- If $D_{X^k_{mn} \succ X^k_{ml}} < 0.5$, then it indicates that X^k_{ml} is preferred to X^k_{mn} such that $X^k_{mn} < X^k_{ml}$.

To facilitate the permutation process, a ranking index is defined for each individual random preference as

$$\mathrm{Ind}^k_{mn} = \frac{1}{N-1} \sum\nolimits_{l=1, l \neq n}^{N} \left[\Pr(X^k_{mn} \geq X^k_{ml}) - 0.5 \Pr(X^k_{mn} = X^k_{ml}) \right], \tag{10}$$

where $m = 1, \ldots, M, n = 1, \ldots, N, k = 1, \ldots, K$. With the vector $(\mathrm{Ind}^k_{m1}, \mathrm{Ind}^k_{m2}, \ldots, \mathrm{Ind}^k_{mN})$ of ranking indices obtained, permutation of individual random preferences $\left(X^k_{m1}, X^k_{m2}, \ldots, X^k_{mN}\right)$ can be easily obtained.

3.2.3. Incorporating both importance weights and fuzzy majority into multi-attribute decision rule

Taking both the importance weights of attributes and the concept of fuzzy majority into consideration, a linguistic statement for our multi-attribute decision rule can be expressed as

"each expert prefers that Q (of) *important* attributes are satisfied by the alternatives." $(\bigstar_1)$

Such a linguistically quantified statement can be, fortunately enough, dealt with by the WOWA operator [59]. The original WOWA has also been first used to aggregate crisp values. Similar with Definition 2, here the WOWA operator will be transformed to the case with input parameters expressed as probability distributions over a linguistic term set, defined as follows.

Definition 4. Let $\left(X^k_{m1}, X^k_{m2}, \ldots, X^k_{mN}\right)$ be the vector of individual random preferences of alternative A_m with respect to the N attributes under expert E_k and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be the importance weights associated with the set of attributes $\mathcal{C}$. A WOWA operator of dimension N with respect to the vector of individual random preferences is a mapping, $\mathcal{F}_{\mathrm{WOWA}} : \mathcal{S}^N \to \mathcal{S}$, defined as

$$\begin{aligned} X^k_m &= \mathcal{F}_{\mathrm{WOWA}}\left(X^k_{m1}, X^k_{m2}, \ldots, X^k_{mN}; \mu_1, \mu_2, \ldots, \mu_N\right) \\ &= \bigoplus_{n=1}^{N} X^k_{m\sigma(n)} \odot W_n \\ &= \left[p^k_m(S_0), p^k_m(S_1), \ldots, p^k_m(S_G)\right], \end{aligned} \tag{11}$$

where $\left(X^k_{m\sigma(1)}, X^k_{m\sigma(2)}, \ldots, X^k_{m\sigma(N)}\right)$ is the permutation of $(X^k_{m1}, X^k_{m2}, \ldots, X^k_{mN})$ via Definition 3 such that $X^k_{m\sigma(n-1)} \geq X^k_{m\sigma(n)}$ for all $n = 2, \ldots, N$, and the weight W_n is defined as

$$W_n = W^*\left(\sum\nolimits_{l \leq n} \mu_{\sigma(l)}\right) - W^*\left(\sum\nolimits_{l < n} \mu_{\sigma(l)}\right), \tag{12}$$

with W^* a monotonically non-decreasing function that interpolates the points $\left(n/N, \sum_{l \leq n} \mu_{\sigma(n)}\right)$ together with the point $(0, 0)$. The value $\mu_{\sigma(l)}$ means the permutation of $(\mu_1, \mu_2, \ldots, \mu_N)$ according to the permutated individual random preferences $\left(X^k_{m\sigma(1)}, X^2_{m\sigma(n)}, \ldots, X^k_{m\sigma(N)}\right)$.

In this paper, W^* is replaced with an RIM linguistic quantifier Q introduced in Appendix B. Together with the WOWA operator, the individual overall random preference of alternative A_m under expert E_k is derived as

$$\begin{aligned} X^k_m &= \mathcal{F}^Q_{\mathrm{WOWA}}\left(X^k_{m1}, X^k_{m2}, \ldots, X^k_{mN}; \mu_1, \mu_2, \ldots, \mu_N\right) \\ &= \bigoplus_{n=1}^{N} X^k_{m\sigma(n)} \odot \left[Q\left(\sum\nolimits_{l \leq n} \mu_{\sigma(l)}\right) - Q\left(\sum\nolimits_{l < n} \mu_{\sigma(l)}\right)\right] \\ &= \left[p^k_m(S_0), p^k_m(S_1), \ldots, p^k_m(S_G)\right]. \end{aligned} \tag{13}$$

Interestingly enough, our multi-attribute decision rule in (13) generalizes the method in (6) and the one in (8) as follows.

- When the linguistic quantifier "identity" is used, then $Q(x) = x$ and the individual overall random preference becomes

$$\begin{aligned} X_m^k &= \mathcal{F}_{\mathrm{WOWA}}^{\mathrm{I}}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k; \mu_1, \mu_2, \ldots, \mu_N\right) \\ &= \bigoplus_{n=1}^{N} X_{m\sigma(n)}^k \odot \left[\sum_{l \le n} \mu_{\sigma(l)} - \sum_{l<n} \mu_{\sigma(l)}\right] \\ &\triangleq \mathcal{F}_{\mathrm{WA}}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k; \mu_1, \mu_2, \ldots, \mu_N\right). \end{aligned}$$

- If $\mu_N = \frac{1}{N}, n = 1, 2 \ldots, N$, i.e., all the attributes are equivalently important, then

$$\begin{aligned} X_m^k &= \mathcal{F}_{\mathrm{WOWA}}^{Q}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k; \mu_1, \mu_2, \ldots, \mu_N\right) \\ &= \bigoplus_{n=1}^{N} X_{m\sigma(n)}^k \odot \left[Q\left(\frac{n}{N}\right) - Q\left(\frac{n-1}{N}\right)\right] \\ &\triangleq \mathcal{F}_{\mathrm{OWA}}^{Q}\left(X_{m1}^k, X_{m2}^k, \ldots, X_{mN}^k\right). \end{aligned}$$

Remark 4. Essentially, the probabilistic aggregations are based on the assumption that there is mutual independence among the alternatives, among the experts, and among the attributes. As pointed out in [3], in any linguistic decision analysis, the procedure of asking each expert to provide his/her absolute linguistic evaluations for a set of alternatives is based on the mutual independence among the set of alternatives. The pool of experts is called to provide their opinions on each attribute separately, therefore mutual independence among the experts and among attributes is assumed naturally.

Remark 5. Fuzzy majority in terms of a linguistic quantifier is applied in the multi-attribute decision rule. Different experts may specify different linguistic quantifiers according to their knowledge and experiences [7]. For the purpose of illustrative convenience, here, we assume the set of experts will specify the same linguistic quantifier in this step.

3.3. Collective decision rule

Also note that a set of experts $\mathcal{E}$ is called to express their judgments regarding the alternatives, on one hand, to collect enough information for the decision making problem from various points of view; and on the other hand, to reduce the subjectivity of the decision making problem. Similar with the multi-attribute decision rule, we shall also assume a subjective probability distribution $p_{\mathcal{E}}$ defined over the set of experts $\mathcal{E}$. In this regards, $p_{\mathcal{E}}(E_k)$, for each $k = 1, 2, \ldots, K$, may be interpreted as the probability that the expert E_k would be randomly selected as a sufficient information source for the purpose of decision making. In addition, a weighting vector $\nu = (\nu_1, \nu_2, \ldots, \nu_K)$ is also often associated with the set of experts such that $\nu_k \in [0, 1]$ and $\sum_{k=1}^{K} \nu_k = 1$. In this sense, the set of experts plays the role of states of the world and the weighting vector ν serves as the subjective probabilities assigned to the states such that

$$p_{\mathcal{E}}(E_k) = \nu_k, k = 1, 2, \ldots, K. \tag{14}$$

After the multi-attribute decision rule in Subsec. 3.2, each expert E_k has generated a vector of M individual overall random preferences with respect to the M alternatives as $(X_1^k, X_2^k, \ldots, X_M^k)$, each of which can be viewed as the uncertain performance of alternative A_m under expert E_k such that $X_m^k = \left[p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)\right]$. With the importance weights of experts and the concept of fuzzy majority in mind, the collective decision rule may be linguistically stated as

"Q (of) *important* experts are satisfied by the alternatives." ($\bigstar_2$)

In this case the goal is to obtain an opinion which can be considered as the opinion of a majority, what we can call the majority opinion.

Similar with the multi-attribute decision rule, the collective overall random preferences can be derived by means of the WOWA operator and the RIM linguistic quantifier Q as follows.

$$\begin{aligned} X_m &= \mathcal{F}^Q_{\text{WOWA}}\left(X^1_m, X^2_m, \ldots, X^K_m; \nu_1, \nu_2, \ldots, \nu_K\right) \\ &= \bigoplus_{k=1}^{K}\left[X_m^{\sigma(k)} \odot \left[Q\left(\sum_{l\le k}\nu_{\sigma(l)}\right) - Q\left(\sum_{l<k}\nu_{\sigma(l)}\right)\right]\right] \\ &= \left[p_m(S_0), p_m(S_1), \ldots, p_m(S_G)\right], \end{aligned} \tag{15}$$

where $\left(X_m^{\sigma(1)}, X_m^{\sigma(2)}, \ldots, X_m^{\sigma(K)}\right)$ is the permutation of $\left(X^1_m, X^2_m, \ldots, X^K_m\right)$ such that $X_m^{\sigma(k-1)} \ge X_m^{\sigma(k)}$ for all $k = 2, \ldots, K$. The value $\nu_{\sigma(k)}$ means the permutation of $(\nu_1, \nu_2, \ldots, \nu_K)$ according to the permutated individual overall random preferences $\left(X_m^{\sigma(1)}, X_m^{\sigma(2)}, \ldots, X_m^{\sigma(K)}\right)$. Similar with Definition 3, the permutation of $\left(X^1_m, X^2_m, \ldots, X^K_m\right)$ is also based on the stochastic dominance degree. Obviously, the following properties can be easily obtained from our collective decision rule:

- when the linguistic quantifier "identity" is used, then the aggregation function in (15) is the WA aggregation method;
- if $\nu_k = \frac{1}{K}, k = 1, 2 \ldots, K$, i.e., all the experts are equivalently important, then the aggregation function in (15) reduces to the OWA aggregation method.

3.4. Choice function

After the collective decision rule, we have a vector of M collective overall random preferences with respect to the M alternatives as $(X_1, X_2, \ldots, X_M)$, each of which can be viewed as the uncertain performance of alternative A_m with a probability distribution

$$X_m = \left[p_m(S_0), p_m(S_1), \ldots, p_m(S_G)\right]$$

over the linguistic term set $\mathcal{S}$. By accepting the mutual independence among all alternatives, we are now able to define a choice function based on the *stochastic dominance*, as introduced in Appendix A.

Definition 5. Let $\{A_1, A_2, \ldots, A_M\}$ be the set of alternatives with a vector of collective overall random preferences $(X_1, X_2, \ldots, X_M)$, each of which is represented by a probability distribution over the linguistic term set $\mathcal{S} = \{S_0, S_1, \ldots, S_G\}$. Given two alternatives A_m and A_l, the stochastic dominance degree of A_m over A_l is defined as

$$D_{ml} = \Pr(X_m \ge X_l) - 0.5\Pr(X_m = X_l).$$

Then we have the following properties:

- when $0.5 < D_{ml} \le 1$, it indicates that A_m is preferred to A_l, i.e., X_m is greater than X_l;
- when $D_{ml} = 0.5$, there is no difference between A_m and A_l, i.e., X_m is equivalent to X_l;
- when $0 \le D_{ml} < 0.5$, it indicates that A_l is slightly preferred to A_m, i.e., X_m is less than X_l.

The overall stochastic dominance degree of alternative A_m can be obtained as

$$V_m = \frac{1}{M-1}\sum_{l=1, l\ne m}^{M} D_{ml}, \tag{16}$$

where $m = 1, \ldots, M$. The vector $\mathbf{V} = (V_1, V_2, \ldots, V_M)$ will be used to rank the alternatives.

3.5. Summary

As a conclusion, the proposed approach to the aggregation and exploitation in our linguistic MAGDM problem can be summarized as the following steps.

- **Step 1) Random preference derivation.** To derive an individual random preference for each alternative on each attribute with respect to each expert via (3) as $X_{mn}^k = \left[p_{mn}^k(S_0), p_{mn}^k(S_1), \ldots, p_{mn}^k(S_G)\right]$, where $m = 1, \ldots, M$, $n = 1, \ldots, N$, $k = 1, \ldots, K$.

- **Step 2) Multi-attribute decision rule.** To derive an individual overall random preference for each alternative with respect to each expert via (13) as $X_m^k = \left[p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)\right]$, where $m = 1, \ldots, M$, $k = 1, \ldots, K$.

- **Step 3) Collective decision rule.** To derive a collective overall random preference for each alternative via (15) as $X_m = [p_m(S_0), p_m(S_1), \ldots, p_m(S_G)]$, where $m = 1, \ldots, M$.

- **Step 4) Choice function.** Ranking the alternatives via (16).

4. Comparative illustrative examples

In this section, two examples with single linguistic terms and flexible linguistic expressions will be used to illustrate the effectiveness and efficiency of our approach by comparisons with existing studies.

4.1. Qualitative MAGDM with single linguistic terms

First, let us suppose a risk investment company wants to invest a sum of money in the best option [64]. This investment problem involves the evaluation of four possible options denoted as $A = \{A_1, A_2, A_3, A_4\}$ according to seven attributes: C_1–the ability of sale, C_2–the ability of management, C_3–the ability of production, C_4–the ability of technology, C_5–the ability of financing, C_6–the ability to resist venture, and C_7–the consistency of corporation strategy. A set of three experts $\mathcal{E} = \{E_1, E_2, E_3\}$ was selected and asked to evaluate the four options on the seven attributes by using the following linguistic term set

$$\begin{aligned} \mathcal{S} = \{&S_0 = \text{Extremely poor}, S_1 = \text{Very poor}, S_2 = \text{Poor}, S_3 = \text{Slightly poor}, S_4 = \text{Fair}, \\ &S_5 = \text{Slightly good}, S_6 = \text{Good}, S_7 = \text{Very good}, S_8 = \text{Extremely good}\}, \end{aligned} \tag{17}$$

the decision matrix is shown in Table 2. In [64], the importance weights of the seven attributes were derived by a maximizing deviation method and expressed as $\mu = (0.1154, 0.0216, 0.2452, 0.0481, 0.1875, 0.1178, 0.2644)$. Moreover, the three experts were assumed to be equivalently important such that $\nu = (1/3, 1/3, 1/3)$.

Now let us use our approach to solve this problem, which is summarized as follows.

Step 1) Random preference derivation. From the information given to the problem, we obtain an individual random preference for each alternative on each attribute with respect to each expert via (3). Since single linguistic terms were provided by the experts, each individual random preference is a probability distribution with a probability 1.0 on the selected linguistic term. For example, the evaluation of alternative A_1 on attribute C_1 with respect to expert E_1 is "S_5 = Slightly good", the associated individual random preference is derived as $X_{11}^1 = [0, 0, 0, 0, 0, 1, 0, 0, 0]$.

Step 2) Multi-attribute decision rule. Assume each expert prefers that "*as many as possible important* attributes should be satisfied by the alternatives". Taking expert E_1 as an example, the result yielded by the multi-attribute decision rule with respect to alternative A_1 is derived via (13) as $X_1^1 = [0, 0, 0, 0, 0.096, 0.606, 0.298, 0, 0]$, which indicates that the aggregated result is a probability distribution with the highest probability 0.606 on "S_5 = Slightly good". Similarly, the results of the four alternatives yielded by the multi-attribute decision rule under linguistic quantifier "as many as possible" are derived, as shown in Table 3.

Table 2: Decision matrix with single linguistic terms: The risk investment problem [64].

Experts	Options	Attributes						
		C_1	C_2	C_3	C_4	C_5	C_6	C_7
E_1	A_1	S_5	S_7	S_7	S_4	S_5	S_6	S_6
	A_2	S_7	S_6	S_4	S_6	S_7	S_5	S_3
	A_3	S_6	S_6	S_7	S_5	S_8	S_7	S_6
	A_4	S_6	S_6	S_3	S_5	S_7	S_5	S_6
E_2	A_1	S_5	S_6	S_7	S_4	S_6	S_6	S_8
	A_2	S_4	S_5	S_4	S_5	S_6	S_6	S_5
	A_3	S_7	S_5	S_6	S_6	S_8	S_8	S_5
	A_4	S_4	S_5	S_4	S_5	S_4	S_5	S_3
E_3	A_1	S_4	S_6	S_6	S_5	S_7	S_6	S_3
	A_2	S_6	S_5	S_5	S_6	S_4	S_6	S_3
	A_3	S_6	S_5	S_6	S_6	S_6	S_7	S_6
	A_4	S_4	S_5	S_3	S_5	S_4	S_4	S_5

Table 3: Results of multi-attribute decision rule under linguistic quantifier "as many as possible": The risk investment problem.

Options	Experts		
	E_1	E_2	E_3
A_1	$[0, 0, 0, 0, 0.096, 0.606, 0.298, 0, 0]$	$[0, 0, 0, 0, 0.096, 0.231, 0.654, 0.019, 0]$	$[0, 0, 0, 0, 0.231, 0.096, 0.673, 0, 0]$
A_2	$[0, 0, 0, 0.529, 0.471, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0.721, 0.279, 0, 0, 0]$	$[0, 0, 0, 0.529, 0.375, 0.096, 0, 0, 0]$
A_3	$[0, 0, 0, 0, 0, 0.096, 0.803, 0.101, 0]$	$[0, 0, 0, 0, 0, 0.572, 0.428, 0, 0]$	$[0, 0, 0, 0, 0, 0.043, 0.957, 0, 0]$
A_4	$[0, 0, 0, 0.49, 0, 0.51, 0, 0, 0]$	$[0, 0, 0, 0.529, 0.471, 0, 0, 0, 0]$	$[0, 0, 0, 0.49, 0.51, 0, 0, 0, 0]$

Step 3) Collective decision rule. Assume linguistic quantifier "most" is utilized in this step. Since all the three experts are equivalently important, the linguistic quantifier guided statement can be written as "*most* experts are satisfied by the alternatives". Accordingly, the collective overall random preferences of the four alternatives can be obtained via (15) as

$$X_1=[0,\ 0,\ 0,\ 0,\ 0.186,\ 0.241,\ 0.572,\ 0.001,\ 0], \qquad X_2=[0,\ 0,\ 0,\ 0.494,\ 0.424,\ 0.083,\ 0,\ 0,\ 0]$$
$$X_3=[0,\ 0,\ 0,\ 0,\ 0,\ 0.188,\ 0.806,\ 0.007,\ 0], \qquad X_4=[0,\ 0,\ 0,\ 0.501,\ 0.465,\ 0.034,\ 0,\ 0,\ 0]$$

which are probability distributions over the linguistic term set $\mathcal{S}$ in (17).

Step 4) Choice function. With the choice criterion of stochastic dominance degree, it is easy to obtain the choice values as $\mathbf{V} = (0.748, 0.196, 0.876, 0.180)$ via (16), which indicates that $A_3 \succ A_1 \succ A_2 \succ A_4$.

4.1.1. Solution based on two-tuple linguistic model

As reviewed in Subsec. 2.2, there are three types of approaches to decision making with single linguistic terms, namely, the approximate one based on fuzzy extension principle, the symbolic one, and the two-tuple linguistic model. Since it outperforms other linguistic models [51], the two-tuple linguistic model will be used to compare with our approach and briefly recalled as follows.

In the two-tuple linguistic model, information is represented by means of two-tuples of the form (S_g, α), where $S_g \in \mathcal{S}$ and $\alpha \in [-0.5, 0.5)$, i.e., linguistic information is encoded in the space $\mathcal{S} \times [-0.5, 0.5)$. Under such a representation, if a value representing the result of a linguistic aggregation operation, then the two-tuple that expresses the equivalent information to β is obtained by means of the following transformation:

$$\begin{aligned} \Delta : [0, G] &\to \mathcal{S} \times [-0.5, 0.5) \\ \beta &\mapsto (S_g, \alpha) \end{aligned} \tag{18}$$

with $g = \text{round}(\beta)$ and $\alpha = \beta - g$. Inversely, a linguistic two-tuple $(S_g, \alpha) \in \mathcal{S} \times [-0.5, 0.5)$ can be equivalently represented by a numerical value in $[0, G]$ by the following transformation

$$\begin{aligned} \Delta^{-1} : \mathcal{S} \times [-0.5, 0.5) &\to [0, G] \\ (S_g, \alpha) &\mapsto \Delta^{-1}(S_g, \alpha) = g + \alpha \end{aligned} \tag{19}$$

Furthermore, traditional numerical aggregation operators have been extended for dealing with linguistic two-tuples in [25, 30]. For example, let $y = ((r_1, \alpha_1), \ldots, (r_N, \alpha_N))$ be a vector of linguistic two-tuples, the two-tuple arithmetic mean is computed as

$$\overline{y} = \Delta\left(\sum_{n=1}^{N} \frac{1}{N} \Delta^{-1}(r_n, \alpha_n)\right). \tag{20}$$

The comparison of linguistic two-tuples is defined as follows. Let (S_{g_1}, α_1) and (S_{g_2}, α_2) be two linguistic two-tuples, if $g_1 < g_2$, then (S_{g_1}, α_1) is less than (S_{g_2}, α_2); if $g_1 = g_2$,

- if $\alpha_1 = \alpha_2$, then (S_{g_1}, α_1) and (S_{g_2}, α_2) represent the same information;
- if $\alpha_1 < \alpha_2$, then (S_{g_1}, α_1) is less than (S_{g_2}, α_2);
- if $\alpha_1 > \alpha_2$, then (S_{g_1}, α_1) is greater than (S_{g_2}, α_2).

Now let us apply the two-tuple linguistic model to the above problem. For the sake of illustration, linguistic quantifier "identity" is assumed in both the multi-attribute decision rule and the collective decision rule. The procedure of the two-tuple linguistic model is summarized as follows.

Step 1) Conversion function. Transform the linguistic values in Table 2 into the form of two-tuples as $y_{mn}^k = (x_{mn}^k, 0)$, where $m = 1, \ldots, M$, $n = 1, \ldots, N$, $k = 1, \ldots, K$.

Step 2) Multi-attribute decision rule. With the importance weights of attributes, the individual overall performance values are defined as $\overline{y}_m^k = \Delta(\sum_{n=1}^{N} \mu_n \cdot \Delta^{-1}(y_{mn}^k))$ and obtained as follows.

$$\begin{array}{lll} \overline{y}_1^1 = (S_6, -0.1323), & \overline{y}_1^2 = (S_7, -0.4376), & \overline{y}_1^3 = (S_6, 0.173) \\ \overline{y}_2^1 = (S_5, -0.0985), & \overline{y}_2^2 = (S_5, -0.0553), & \overline{y}_2^3 = (S_5, -0.435) \\ \overline{y}_3^1 = (S_7, -0.3101), & \overline{y}_3^2 = (S_6, 0.44), & \overline{y}_3^3 = (S_6, 0.0962) \\ \overline{y}_4^1 = (S_5, 0.0216), & \overline{y}_4^2 = (S_4, -0.0769), & \overline{y}_4^3 = (S_4, 0.0889) \end{array}$$

Step 3) Collective decision rule. With the importance weights of experts, the collective overall performance values are defined as $\overline{y}_m = \Delta(\sum_{k=1}^{K} \nu_k \cdot \Delta^{-1}(\overline{y}_m^k))$ and obtained as $\overline{y}_1 = (S_6, 0.2011)$, $\overline{y}_2 = (S_5, -0.1963)$, $\overline{y}_3 = (S_6, 0.4087)$, $\overline{y}_4 = (S_4, 0.3446)$.

Step 4) Choice function. According to the comparison rules of linguistic two-tuples, the ranking of the four alternatives is $A_3 \succ A_1 \succ A_2 \succ A_4$.

4.1.2. Comparative analysis

With linguistic quantifier "identity" used in both the multi-attribute decision rule and the collective decision rule, our approach derives the same ranking as the one generated by the two-tuple linguistic model. Detailed comparisons will be conducted in terms of uniform scale, variation of individual opinions, weight information, comprehension and interpretation, as summarized in Table 4.

- **Uniform scale.** The two-tuple linguistic model is not suitable to the case of asymmetric and unbalanced linguistic term set [51]. Instead, the strict constraint of symmetric and balanced linguistic term set is released in our approach.

Table 4: Comparative analysis of aggregation and exploitation: Single linguistic terms.

Criteria	Two-tuple model	Proposed approach
Uniform scale	Uniform only	Uniform or non-uniform
Variation of individual opinions	No	Yes
Weight information	Extendable	Both importance weights and fuzzy majority
Comprehension	Easy to understand	Understandable
Interpretation	Fuzzy	Probability

- **Variation of individual opinions.** Assume that the seven attributes are equivalently important, i.e., $\mu_n = 1/7(n = 1, \dots, 7)$. Taking alternative A_2 as an example, the results of the multi-attribute decision rule yielded by the two-tuple linguistic model are $\overline{y}_2^1 = (S_5, 0.4286)$, $\overline{y}_2^2 = (S_5, 0)$, $\overline{y}_2^3 = (S_5, 0)$, which indicate that experts E_2 and E_3 generate the same individual overall performance value as $(S_5, 0)$. The results of the multi-attribute decision rule yielded by our approach are obtained as

$$
\begin{aligned}
X_2^1&=[0, \quad 0, \quad 0, \quad 0.1429, \quad 0.1429, \quad 0.1429, \quad 0.2857, \quad 0.2857, \quad 0] \\
X_2^2&=[0, \quad 0, \quad 0, \quad 0, \quad 0.2857, \quad 0.4286, \quad 0.2857, \quad 0, \quad 0] \\
X_2^3&=[0, \quad 0, \quad 0, \quad 0.1429, \quad 0.1429, \quad 0.2857, \quad 0.4286, \quad 0, \quad 0]
\end{aligned}
$$

  which indicate that experts E_2 and E_3 generate different individual overall random preferences. Thus, the two-tuple linguistic model cannot capture such differences and variations.

- **Weights information.** Since the two-tuple linguistic model is based on term indices, which may be viewed as crisp values [25, 30], the original WOWA operator introduced in Appendix C can be easily extended for dealing with linguistic two-tuples. Our approach can directly take both the importance weights and the fuzzy majority into account simultaneously.

- **Comprehension and interpretation.** The linguistic decision making falls into the process of CWW. As mentioned by Mendel in [43], the output from CWW must be at least a word and not just a number. The CWW also produces a decision or output based on these words. The two-tuple linguistic model results with a linguistic two-tuple expressed by a linguistic term and a numerical value, which assign (inherent) fuzzy semantics and syntax. Due to its straightforwardness, the result by the two-tuple linguistic model is easy to understand. Our approach yields a probability distribution over a linguistic term set, which reflects the uncertainty of decision making and is understandable by decision makers.

4.2. Qualitative MAGDM with flexible linguistic expressions

Now let us consider a qualitative MAGDM problem with flexible linguistic expressions: evaluation of university faculty for tenure promotion [63]. The attributes used at some universities were: C_1–teaching, C_2–research, and C_3–service, whose importance weighting vector was $\mu = (0.4, 0.3, 0.3)$. There were five candidates $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$ to be evaluated. A set of three experts $\mathcal{E} = \{E_1, E_2, E_3\}$ was selected and asked to evaluate the alternatives on the seven attributes with the following linguistic term set

$$\mathcal{S} = \{S_0 = \text{Nothing}, S_1 = \text{Very low}, S_2 = \text{Low}, S_3 = \text{Medium}, S_4 = \text{High}, S_5 = \text{Very high}, S_6 = \text{Perfect}\}, \tag{21}$$

the decision matrix is shown in Table 5. The procedure of our approach is summarized as follows.

Step 1) Random preference derivation. From the information given to the problem, we obtain an individual random preference for each alternative on each attribute with respect to each expert via (3). For example, the evaluation of candidate A_1 on attribute C_1 with respect to expert E_1 is $\{S_4, S_5\}$, which generates the individual random preference as $X_{11}^1 = [0, 0, 0, 0, 0.5, 0.5, 0]$.

Table 5: Decision matrix with flexible linguistic expressions: The tenure promotion [63].

Experts $\mathcal{E}$	Candidates $\mathcal{A}$	Attributes $\mathcal{C}$		
		C_1	C_2	C_3
E_1	A_1	$\{S_4, S_5\}$	S_3	S_4
	A_2	S_2	S_5	S_3
	A_3	S_1	$\{S_3, S_4\}$	$\{S_1, S_2\}$
	A_4	$\{S_5, S_6\}$	S_4	S_3
	A_5	S_1	$\{S_1, S_2\}$	S_5
E_2	A_1	$\{S_5, S_6\}$	S_2	$\{S_3, S_4\}$
	A_2	$\{S_3, S_4\}$	$\{S_4, S_5\}$	S_2
	A_3	S_2	$\{S_2, S_3\}$	S_1
	A_4	$\{S_5, S_6\}$	$\{S_4, S_5, S_6\}$	$\{S_3, S_4, S_5\}$
	A_5	S_2	S_1	S_4
E_3	A_1	S_5	$\{S_4, S_5\}$	S_6
	A_2	S_4	$\{S_3, S_4\}$	S_3
	A_3	S_3	$\{S_1, S_2\}$	S_2
	A_4	S_5	S_6	S_4
	A_5	$\{S_1, S_2\}$	S_3	S_4

Table 6: Results of multi-attribute decision rule under linguistic quantifier "identity": The tenure promotion.

Candidates	Experts		
	E_1	E_2	E_3
A_1	$[0, 0, 0, 0.3, 0.5, 0.2, 0]$	$[0, 0, 0.3, 0.15, 0.15, 0.2, 0.2]$	$[0, 0, 0, 0, 0.15, 0.55, 0.3]$
A_2	$[0, 0, 0.4, 0.3, 0, 0.3, 0]$	$[0, 0, 0.3, 0.2, 0.35, 0.15, 0]$	$[0, 0, 0, 0.45, 0.55, 0, 0]$
A_3	$[0, 0.55, 0.15, 0.15, 0.15, 0, 0]$	$[0, 0.3, 0.55, 0.15, 0, 0, 0]$	$[0, 0.15, 0.5, 0.4, 0, 0, 0]$
A_4	$[0, 0, 0, 0.3, 0.3, 0.2, 0.2]$	$[0, 0, 0, 0.1, 0.2, 0.4, 0.3]$	$[0, 0, 0, 0, 0.3, 0.4, 0.3]$
A_5	$[0, 0.55, 0.15, 0, 0, 0.3, 0]$	$[0, 0.3, 0.4, 0, 0.3, 0, 0]$	$[0, 0.2, 0.2, 0.3, 0.3, 0, 0]$

Step 2) Multi-attribute decision rule. Similar with [63], linguistic quantifier "identity" is used in this step. The individual overall random preferences are obtained via (13) and shown in Table 6, each of which is a probability distribution over the linguistic term set $\mathcal{S}$.

Step 3) Collective decision rule. Since the importance weights of the three experts were not provided in this example, it is natural to assume that the three experts are equivalently important such that $\nu = (1/3, 1/3, 1/3)$. Similar with [63], linguistic quantifier "as many as possible" is utilized in this step. Then, the linguistic statement can be written as "*as many as possible* experts are satisfied by the candidates". We can obtain the collective overall random preferences via (15) as follows.

$$X_1=[0,\ 0,\ 0.1,\ 0.25,\ 0.3833,\ 0.2,\ 0.0667],\quad X_2=[0,\ 0,\ 0.3667,\ 0.2667,\ 0.1167,\ 0.25,\ 0],$$
$$X_3=[0,\ 0.4667,\ 0.2833,\ 0.15,\ 0.1,\ 0,\ 0],\quad X_4=[0,\ 0,\ 0,\ 0.2333,\ 0.2667,\ 0.2667,\ 0.2333],$$
$$X_5=[0,\ 0.4667,\ 0.2333,\ 0,\ 0.1,\ 0.2,\ 0].$$

Step 4) Choice function. According to the choice criterion of stochastic dominance degree, it is easy to obtain a vector of choice values via (16) as $\mathbf{V} = (0.6677, 0.5248, 0.2016, 0.7987, 0.3072)$, which indicates that $A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$.

4.2.1. Solutions based on existing studies

As a comparative analysis, we briefly recall four existing methods: interval linguistic model [46, 53, 69], fuzzy envelope model [40], symbolic hesitant model [63], and possibility distribution based model [67].

(1) Interval linguistic model. The two-tuple linguistic model has been applied to (MA)GDM with HFLTSs [46, 53]. Xu in [69] has proposed an interval linguistic MAGDM model, which involves both the importance weights and fuzzy majority. To be consistent with the comparison study of MAGDM with single linguistic terms, the two-tuple linguistic model will be incorporated into the interval linguistic model, summarized as follows.

Step 1) Conversion function. The HFLTS is transformed into a linguistic interval in terms of two-tuples, denoted as $y_{mn}^k = [y_{mn}^k(L), y_{mn}^k(R)]$. For example, the evaluation of candidate A_4 on attribute C_2 with respect to expert E_2 was provided as $\{S_4, S_5, S_6\}$ and transformed into $[(S_4, 0), (S_6, 0)]$.

Step 2) Multi-attribute decision rule. With the importance weights of attributes and linguistic quantifier "identity", the individual overall linguistic intervals are derived via the uncertain hybrid linguistic average operator and obtained as follows.

$$\begin{array}{llll}
A_1: & \overline{y}_1^1 = [(S_4, -0.3), (S_4, 0.1)], & \overline{y}_1^2 = [(S_4, -0.5), (S_4, 0.2)], & \overline{y}_1^3 = [(S_5, 0), (S_5, 0.3)] \\
A_2: & \overline{y}_2^1 = [(S_3, 0.2), (S_3, 0.2)], & \overline{y}_2^2 = [(S_3, 0), (S_4, -0.3)], & \overline{y}_2^3 = [(S_3, 0.4), (S_4, -0.3)] \\
A_3: & \overline{y}_3^1 = [(S_2, -0.4), (S_2, 0.2)], & \overline{y}_3^2 = [(S_2, -0.3), (S_2, 0)], & \overline{y}_3^3 = [(S_2, 0.1), (S_2, 0.4)], \\
A_4: & \overline{y}_4^1 = [(S_4, 0.1), (S_5, -0.5)], & \overline{x}_4^2 = [(S_4, 0.1), (S_6, -0.3)], & \overline{y}_4^3 = [(S_5, 0), (S_5, 0)], \\
A_5: & \overline{y}_5^1 = [(S_2, 0.2), (S_3, -0.5)] & \overline{y}_5^2 = [(S_2, 0.3), (S_2, 0.3)] & \overline{y}_5^3 = [(S_3, -0.5), (S_3, -0.1)]
\end{array}$$

Step 3) Collective decision rule. With the importance weights of experts and linguistic quantifier "as many as possible", the collective overall linguistic intervals are derived via the uncertain hybrid linguistic average operator and obtained as follows.

$$\begin{array}{lll}
\overline{y}_1 = [(S_4, -0.4333), (S_4, 0.1667)], & \overline{y}_2 = [(S_3, 0.1333), (S_4, -0.3)], & \overline{y}_3 = [(S_2, -0.3333), (S_2, 0.0667)], \\
\overline{y}_4 = [(S_5, -0.3), (S_5, -0.1667)], & \overline{y}_5 = [(S_2, 0.4333), (S_3, -0.3)]. &
\end{array}$$

Step 4) Choice function. A possibility based method is defined to rank the linguistic intervals as follows.

$$\text{Poss}(\overline{y}_m \geq \overline{y}_l) = \min\left\{\max\left(\frac{\Delta^{-1}(\overline{y}_m(R)) - \Delta^{-1}(\overline{y}_l(L))}{\Delta^{-1}(\overline{y}_m(R)) - \Delta^{-1}(\overline{y}_m(L)) + \Delta^{-1}(\overline{y}_l(R)) - \Delta^{-1}(\overline{y}_l(L))}, 0\right), 1\right\}, \tag{22}$$

where $m, l = 1, \ldots, M$. By pairwise comparisons, the choice values are derived as $(3.3857, 2.6143, 0.5, 4.5, 1.5)$, which indicates $A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$.

(2) Fuzzy envelope model. Liu and Rodríguez [40] have developed a fuzzy envelope model, in which the linguistic terms in an HFLTS are aggregated into a fuzzy envelope based on the extension principle. The following fuzzy numbers are used to represent the fuzzy semantics of the linguistic term set in (21).

$$\mathcal{S} = \{(0, 0, 0.17), (0, 0.17, 0.33), (0.17, 0.33, 0.5), (0.33, 0.5, 0.67), (0.5, 0.67, 0.83), (0.67, 0.83, 1), (0.83, 1, 1)\}. \tag{23}$$

Here, the fuzzy envelope model is revised to fit our context as follows.

Step 1) Aggregation function. Aggregate the linguistic terms in an HFLTS into a fuzzy envelope based on the fuzzy numbers defined in (23). The fuzzy envelope is a trapezoidal fuzzy number such that $\widetilde{y}_{mn}^k = (a_{mn}^k, b_{mn}^k, c_{mn}^k, d_{mn}^k)$. For example, the evaluation of candidate A_1 on attribute C_1 with respect to expert E_1 was provided as $\{S_4, S_5\}$ and is transformed into a fuzzy envelope as $\widetilde{y}_{11}^1 = (0.5, 0.67, 0.83, 1.0)$.

Step 2) Multi-attribute decision rule. With the importance weights of attributes, the individual overall fuzzy envelopes are defined as $\widetilde{y}_m^k = \sum_{n=1}^N \mu_n \cdot \widetilde{y}_{mn}^k$. For example, the individual overall fuzzy envelope of alternative A_1 with respect to expert E_1 is derived as $\widetilde{y}_m^k = (0.50, 0.619, 0.683, 0.85)$.

Step 3) Collective decision rule. With the importance weights of experts, the collective overall fuzzy envelopes of the four alternatives are defined as $\widetilde{y}_m = \sum_{k=1}^K \nu_k \cdot \widetilde{y}_m^k$ and obtained as $\widetilde{y}_1 = (0.511, 0.697, 0.755, 0.883)$, $\widetilde{y}_2 = (0.366, 0.533, 0.589, 0.755)$, $\widetilde{y}_3 = (0.134, 0.300, 0.366, 0.533)$, $\widetilde{y}_4 = (0.567, 0.803, 0.832, 0.933)$, $\widetilde{y}_4 = (0.223, 0.390, 0.428, 0.593)$.

Step 4) Choice function. Obtain the distances of each alternative relative to the fuzzy positive ideal solution and fuzzy negative ideal solution, respectively. The closeness indices are calculated as $\text{CC}_1 = 0.711, \text{CC}_2 = 0.561, \text{CC}_3 = 0.333, \text{CC}_4 = 0.784, \text{CC}_5 = 0.409$, which indicate $A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$.

(3) Symbolic hesitant model. Recently, Wei et al. in [63] have proposed operators and comparisons of HFLTSs. According to different situations where importance weights are known or unknown, the hesitant linguistic WA operator or the hesitant linguistic OWA operator is used, which is summarized as follows.

Step 1) Multi-attribute decision rule. Since the importance weights of attributes are provided, the hesitant linguistic WA operator is utilized to derive the individual overall HFLTSs of the five candidates with respect to the three experts and obtained as follows.

$$
\begin{array}{llllll}
E_1: & \mathcal{H}_1^1=\{S_4\}, & \mathcal{H}_2^1=\{S_3\}, & \mathcal{H}_3^1=\{S_2\}, & \mathcal{H}_4^1=\{S_4,S_5\}, & \mathcal{H}_5^1=\{S_2,S_3\} \\
E_2: & \mathcal{H}_1^2=\{S_4\}, & \mathcal{H}_2^2=\{S_3,S_4\}, & \mathcal{H}_3^2=\{S_2\}, & \mathcal{H}_4^2=\{S_4,S_5,S_6\}, & \mathcal{H}_5^2=\{S_3\} \\
E_3: & \mathcal{H}_1^3=\{S_5,S_6\}, & \mathcal{H}_2^3=\{S_3,S_4\}, & \mathcal{H}_3^3=\{S_2\}, & \mathcal{H}_4^3=\{S_5\}, & \mathcal{H}_5^3=\{S_3\}
\end{array}
$$

Step 2) Collective decision rule. Since the weights of experts are unknown, the hesitant linguistic OWA operator with linguistic quantifier "as many as possible" is used to derive the collective overall HFLTSs of the five candidates and obtained as $\mathcal{H}_1=\{S_4\}$, $\mathcal{H}_2=\{S_3\}$, $\mathcal{H}_3=\{S_2\}$, $\mathcal{H}_4=\{S_4,S_5\}$, $\mathcal{H}_5=\{S_2,S_3\}$.

Step 3) Choice function. With a possibility based method to compare the HFLTSs using the term indices, the ranking of the five candidates is obtained as $A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$.

(4) Possibility distribution based model. Wu and Xu in their very recent pioneering work [67] have proposed a possibility distribution based hesitant linguistic model. Let $\mathcal{H}=\{S_L,\ldots,S_R\}$ be an HFLTS with several possible linguistic terms, where S_L and S_R are the lower and upper bounds of $\mathcal{H}$, respectively. Then a possibility distribution $\Pi=[\pi(S_0),\ldots,\pi(S_G)]$ is generated from the $\mathcal{H}$, defined as

$$
\pi(S_g)=\begin{cases} \frac{1}{R-L+1}, & \text{if } g=L,L+1,\ldots,R; \\ 0, & \text{otherwise.} \end{cases} \tag{24}
$$

Based on this possibility distribution, the mean and variance values of the possibility distribution Π are used to compare and rank the possibility distributions, defined as follows.

$$
\mathrm{EV}=\sum_{g=0}^{G} g\cdot\pi(S_g),\quad \mathrm{Var}=\sum_{g=0}^{G}(g-\mathrm{EV})^2\cdot\pi(S_g) \tag{25}
$$

Given two HFLTSs, $\mathcal{H}_1$ and $\mathcal{H}_2$, the comparison of $\mathcal{H}_1$ over $\mathcal{H}_2$ is performed by the following formulas.

- If $\mathrm{EV}_1<\mathrm{EV}_2$, then $\mathcal{H}_1<\mathcal{H}_2$.
- If $\mathrm{EV}_1=\mathrm{EV}_2$, then (1) if $\mathrm{Var}_1<\mathrm{Var}_2$, then $\mathcal{H}_1>\mathcal{H}_2$; (2) if $\mathrm{Var}_1=\mathrm{Var}_2$, then $\mathcal{H}_1=\mathcal{H}_2$.

Then, the procedure of the possibility distribution based model is summarized as follows.

Step 1) Conversion function. The flexible linguistic expression $\mathcal{H}_{mn}^k$ is transformed into the corresponding possibility distribution Π_{mn}^k via (24), where $m=1,\ldots,M, n=1,\ldots,N, k=1,\ldots,K$.

Step 2) Multi-attribute decision rule. Since the importance weights of attributes are provided, a hesitant linguistic WA operator is utilized to derive the individual overall performance Π_m^k, where $m=1,\ldots,M, k=1,\ldots,K$. The mean and variance values in (25) are used to permutate the possibility distributions.

Step 3) Collective decision rule. Since the importance weights of experts are unknown, a hesitant linguistic OWA operator is used to derive the collective overall performance Π_m, where $m=1,\ldots,M$. The mean and variance values in (25) are used to permutate the possibility distributions.

Step 4) Choice function. By the mean and variance values of the possibility distributions in (25), the choice values with respect to the five candidates are obtain as follows.

$$
\begin{array}{lllll}
\mathrm{EV}_1=3.8667 & \mathrm{EV}_2=3.25 & \mathrm{EV}_3=1.8667 & \mathrm{EV}_4=4.5 & \mathrm{EV}_5=2.3167 \\
\mathrm{Var}_1=1.7156 & \mathrm{Var}_2=1.4208 & \mathrm{Var}_3=0.7156 & \mathrm{Var}_4=1.1833 & \mathrm{Var}_5=1.9831
\end{array}
$$

which generate the ranking list as $A_4 \succ A_1 \succ A_2 \succ A_5 \succ A_3$.

Table 7: Comparative analysis of aggregation and exploitation: Flexible linguistic expressions.

Criteria	Our approach	Interval linguistic model	Fuzzy envelope model	Symbolic hesitant model	Possibility distribution based model
Uniform scale	Uniform or non-uniform	Uniform only	Uniform or non-uniform	Uniform only	Uniform only
Quantification	No need	No Need	Need	No need	No need
Variation of individual opinions	Yes	No	Yes	No	Yes
Weight information	Both importance weights and fuzzy majority	Both importance weights and fuzzy majority	Only importance weights	Either importance weights or fuzzy majority	Either importance weights or fuzzy majority
Comprehension	Easy to understand	Easy to understand	Difficult to understand	Understandable	Easy to understand
Interpretation	Probability	Fuzzy	Fuzzy	HFLTS	Possibility

4.2.2. Comparative analysis

It is obvious that the ranking result generated by our approach is consistent with the ones generated by the four existing approaches. Comparisons with the four approaches will be conducted in terms of uniform scale, quantification of qualitative concepts, variations of individual opinions, weight information, comprehension and interpretation, as summarized in Table 7.

- **Uniform scale.**
  - Our approach can deal with uniform or non-uniform linguistic term set, since it is based on the ordinal semantics of linguistic term set. The fuzzy envelope model is also suitable to the uniform or non-uniform scale since it makes computations on the fuzzy numbers.
  - The revised interval linguistic model is based on the two-tuple linguistic model, which is strictly constrained by the symmetric and balanced linguistic term set. Therefore, the interval linguistic model is only suitable to the uniform scale. The symbolic hesitant model makes operations and comparisons based on the term indices, which is only suitable to the uniform scale. The possibility distribution based model is also based on the indices of linguistic terms. As pointed out by the authors in [67, Discussion part], a drawback of the possibility distribution based model is that it depends on the numerical scale of a linguistic term set. If the linguistic term set is uniformly and symmetrically distributed, it is easy to get the numerical scale; otherwise, it has proved difficult.
- **Quantification.**
  - Similar with the approximate model based on the extension principle in the context of decision making with single linguistic terms, the fuzzy envelope model makes operations on fuzzy numbers that support the fuzzy semantics of linguistic terms, which will create the burden of quantifying a qualitative concept [25, 29] and complex mathematical computations [78].
  - Our approach eliminates the burden of quantifying a qualitative concept, since it is based on the ordinal semantics of the linguistic term set. There is no need to quantify a qualitative concept in the revised interval linguistic model, symbolic hesitant model, and the possibility distribution based model, since they are based on the term indices.
- **Variation of individual opinions.**

– As already illustrated in Subsubsec. 4.1.2, the two-tuple linguistic model cannot reflect the variations and differences of individual opinions. The revised interval linguistic model cannot capture such variations and differences, since it is based on the two-tuple linguistic model. The symbolic hesitant model [63] is similar with the symbolic model [3, 8], and consequently will generate the problems of information loss and lack of precision. Moreover, this model cannot represent the differences and variations of individual opinions. For example, the aggregation results of candidate A_1 with respect to experts E_1 and E_2 after the multi-attribute decision rule are $H_1^1 = \{S_4\}$ and $H_1^2 = \{S_4\}$, respectively. In other words, there is no difference between the aggregated result of A_1 under E_1 and the one under E_2. However, the information in Table 5 is quite different and our approach generates the individual overall random preferences as $X_1^1 = [0, 0, 0, 0.3, 0.5, 0.2, 0]$ and $X_1^2 = [0, 0, 0.3, 0.15, 0.15, 0.2, 0.2]$, which are quite different.

– Our approach can represent the variations and differences of individual opinions. The fuzzy envelope model can also capture such variations and difference, since it makes computations on fuzzy numbers associated with the linguistic terms. The possibility distribution based model derives a possibility distribution from an HFLTS and thus can reflect the variations and differences of individual opinions.

- **Weight information.**

  – The importance weights and fuzzy majority are incorporated into both our approach and the revised interval linguistic model simultaneously.

  – The fuzzy envelope model only considers the importance weights, it cannot take the fuzzy majority into account. Both the symbolic hesitant model and possibility distribution based model can model either the importance weights or the fuzzy majority in linguistic MAGDM; unfortunately, they cannot take both of them into account simultaneously.

- **Comprehension and interpretation.**

  – The fuzzy envelope model results with a fuzzy number, where the accuracy outweighs interpretability; thus the result of this model is difficult to understand. The symbolic hesitant model results with an HFLTS which has a subset of linguistic terms without numerical values; the result of this model is not easy to understand.

  – Our approach results with a probability distribution over the linguistic term set, which involves several linguistic terms and their associated probabilities. The interval linguistic model results with a linguistic interval expressed by linguistic two-tuples, which assign fuzzy semantics and syntax. The possibility distribution based model results with a possibility distribution over the linguistic term set, which involves several linguistic terms and their associated possibilities. The results of our approach, interval linguistic model, and the possibility distribution based model are easy to understand.

4.3. Summary

As a conclusion, our approach to the procedure of aggregation and exploitation performs computations based on the ordinal semantics of linguistic information and the probability distributions over the linguistic term set, it is quite natural in terms of interpretation. As such, the strict constraint of symmetric and balanced linguistic term set can be released. The quantification of qualitative concepts can also be eliminated. Moreover, it can capture the differences and variations of individual opinions as well as incorporates both the importance weights and the concept of fuzzy majority simultaneously. Compared with the two-tuple linguistic model in the context (MA)GDM with single linguistic terms, the result of our approach is a little complex and understandable. However, in the context of (MA)GDM with flexible linguistic expressions, our approach may be the most suitable one. Finally, it can be easily seen that a unified approach is provided to qualitative MAGDM with both single linguistic terms and flexible linguistic expressions.

5. Consensus measure: A probability based approach

Recently, evaluating the effect of linguistic MAGDM has become an important research topic [48, 58]. Effective decision making indices are required to evaluate the results of MAGDM with linguistic expressions [67]. It is very rare when all experts in a group share the same opinion about the alternatives, since a diversity of opinions commonly exists. The consensus reaching process is a necessity of all (MA)GDM processes, because to achieve a general consensus about selected options is a desirable goal. Consensus is traditionally meant as a full and unanimous agreement of all individuals' opinions, which is an ideal consensus and very difficult to achieve. It is thus quite natural to look for the highest consensus, that is, the maximum possible consensus. Consensus makes it possible for a group to reach a final decision that all group members can support among these differing opinions. There are two categories for computing consensus measures [67]: (1) consensus measure between individual and collective preferences [48, 58]; (2) consensus measure among the experts [17, 67]. This paper follows the former definition for the consensus measure. In this section, based on the procedure of aggregation and exploitation introduced in Sec. 3, a probability based approach is proposed to evaluate the consensus degree in MAGDM with linguistic expressions.

5.1. A probability based approach to consensus measure

Based on the proposed approach to aggregation and exploitation introduced in Sec. 3, the concept of deviation degree of any two random preferences will be first presented and defined as follows.

Definition 6. Let X_1 and X_2 be two independent discrete random variables with respective probability distributions p_1 and p_2 defined over a finite and totally ordered discrete linguistic term set $\mathcal{S} = \{S_0, S_1, \ldots, S_G\}$ with $S_0 < S_1 < \cdots < S_G$, where

$$\sum_{g=0}^{G} p_1\left(S_g\right) = 1 \text{ and } \sum_{g=0}^{G} p_2\left(S_g\right) = 1.$$

Based on the concept of stochastic dominance introduced in Appendix A, the stochastic dominance degrees of X_1 over X_2 and X_2 over X_1 are derived as

$$\begin{cases} D_{12} = \Pr(X_1 \geq X_2) - 0.5\Pr(X_1 = X_2) \\ D_{21} = \Pr(X_2 \geq X_1) - 0.5\Pr(X_2 = X_1) \end{cases} \tag{26}$$

which have the following properties: (1) $D_{12} + D_{21} = 1$; (2) $0 \leq D_{12} \leq 1$; (3) $0 \leq D_{21} \leq 1$.

Then, the deviation degree of X_1 relative to X_2 is defined as

$$\begin{aligned} \mathrm{Dev}(X_1, X_2) &= |D_{12} - D_{21}| \\ &\triangleq |\Pr(X_1 \geq X_2) - \Pr(X_2 \geq X_1)| \end{aligned} \tag{27}$$

Based on the properties of the stochastic dominance introduced in Appendix A, it is easily to obtain the following properties of the deviation degree:

(1) $0 \leq \mathrm{Dev}(X_1, X_2) \leq 1$;
(2) $\mathrm{Dev}(X_1, X_2) = 1 \Longrightarrow \Pr(X_1 \geq X_2) = 1$ or $\Pr(X_2 \geq X_1) = 1$;
(3) $\mathrm{Dev}(X_1, X_2) = 0 \Longleftrightarrow X_1 = X_2$, i.e., the deviation degree between X_1 and X_2 is zero if and only if X_1 is equivalent to X_2; and

(4) $\mathrm{Dev}(X_1, X_2) = \mathrm{Dev}(X_2, X_1)$.

We now present a new consensus measure based on the deviation degree of any two random preferences. After the multi-attribute decision rule in Subsec. 3.2, expert E_k generates an individual overall random preference for alternative A_m such that

$$X_m^k = [p_m^k(S_0), p_m^k(S_1), \ldots, p_m^k(S_G)],$$

where $m = 1, \ldots, M, k = 1, \ldots, K$. Also, after the collective decision rule in Subsec. 3.3, a collective overall random preference with respect to alternative A_m is obtain as

$$X_m = [p_m(S_0), p_m(S_1), \ldots, p_m(S_G)],$$

where $m = 1, \ldots, M$. We now present the definition of consensus degree between individual and collective overall random preferences as follows.

Definition 7. Let X_m^k be the individual overall random preference of alternative A_m with respect to expert E_k, and X_m be the collective overall random preference of alternative A_m, respectively. Then, the individual consensus degree of X_m^k relative to X_m is defined by

$$\begin{aligned} \rho_{mk} &= 1 - \mathrm{Dev}\left(X_m^k, X_m\right) \\ &= 1 - \left|\Pr\left(X_m^k \geq X_m\right) - \Pr\left(X_m \geq X_m^k\right)\right| \end{aligned} \tag{28}$$

where $m = 1, \ldots, M, k = 1, \ldots, K$.

Such a consensus measure between X_m^k and X_m is also based on the deviation degree and has a definite implication. Obviously, the smaller the value of $\mathrm{Dev}(X_m^k, X_m)$, the greater the value ρ_{mk}, and the more similar these two random preferences X_m^k, X_m. Since $\mathrm{Dev}(X_m^k, X_m) \in [0, 1]$, it is easy to get $\rho_{mk} \in [0, 1]$. It is easily found that:

- if $\rho_{mk} = 1$, then expert E_k fully agrees with the collective overall random preference with respect to alternative A_m, which indicates that there is no difference between X_m^k and X_m such that $X_m^k = X_m$;
- if $\rho_{mk} = 0$, then expert E_k fully disagrees with the collective overall random preference with respect to alternative A_m, i.e., the two random preferences X_m^k and X_m are absolutely different such that $\Pr\left(X_m^k \geq X_m\right) = 1$ or $\Pr\left(X_m \geq X_m^k\right) = 1$.

Next, we aggregate these individual consensus degrees to obtain an individual overall consensus degree of each expert relative to the collective overall random preference, defined as follows.

Definition 8. Let $(\rho_{1k}, \rho_{2k}, \ldots, \rho_{Mk})$ be a vector of individual consensus degrees of expert E_k with respect to the M alternatives, then the individual overall consensus degree of expert E_k relative to the collective overall random preferences is defined as

$$\rho_k = \frac{1}{M} \sum_{m=1}^{M} \rho_{mk}, \tag{29}$$

where $k = 1, \ldots, K$. It is obvious that $\rho_k \in [0, 1]$ is also based on the concept of deviation degree between two random preferences. Moreover, the closer ρ_k to 0, the poorer the consensus; conversely, the closer ρ_k to 1, the better the consensus. In particular, if $\rho_k = 1$, expert E_k can be said to fully agree with the collective overall random preferences.

Recall that the set of experts is associated with an importance weighting vector $(\nu_1, \nu_2, \ldots, \nu_K)$. Moreover, the concept of fuzzy majority is also incorporated in the collective decision rule. Similar with the decision rule in Sec. 3, the linguistic quantifier guided collective decision rule with respect to the consensus measure can be verbally expressed as

"The collective random preferences are agreed by Q (of) *important* experts." ($\bigstar_3$)

In the following, we present the definition of collective overall consensus degree based on the WOWA operator.

Definition 9. Let $(\rho_1, \rho_2, \ldots, \rho_K)$ be the vector of individual overall consensus degrees with respect to the set of experts $\mathcal{E} = \{E_1, E_2, \ldots, E_K\}$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_K)$ be the importance weights associated with the experts, and Q be a linguistic quantifier used in the collective decision rule. Then the collective overall consensus degree is obtained as

$$\begin{aligned} \rho &= \mathcal{F}^{Q}_{\mathrm{WOWA}}(\rho_1, \rho_2, \ldots, \rho_K; \nu_1, \nu_2, \ldots, \nu_K) \\ &= \sum_{k=1}^{K} \rho_{\sigma(k)} \cdot \left[Q\left(\sum_{l \leq k} \nu_{\sigma(l)} \right) - Q\left(\sum_{l<k} \nu_{\sigma(l)} \right) \right] \end{aligned} \tag{30}$$

where $(\rho_{\sigma(1)}, \rho_{\sigma(2)}, \ldots, \rho_{\sigma(K)})$ is the permutation of $(\rho_1, \rho_2, \ldots, \rho_K)$ such that $\rho_{\sigma(k-1)} \geq \rho_{\sigma(k)}$ for all $k = 2, \ldots, K$. The value $\nu_{\sigma(k)}$ means the permutation of $(\nu_1, \nu_2, \ldots, \nu_K)$ according to the permutated vector of individual overall consensus degrees $\left(\rho_{\sigma(1)}, \rho_{\sigma(2)}, \ldots, \rho_{\sigma(K)}\right)$.

With respect to the consensus analysis of the evaluation results, the experts may assign a consensus level required for the solution in advance [28]. This analysis is intended to make consistent decisions [58, 67]. When the consensus degree obtained in the analysis reaches the consensus level, the evaluation results are sufficiently accurate and reliable for consistent decision making [58], i.e., the consensus reaching process is completed if the experts accept the evaluation results. If this is not the case, the process must return to the initial stage to gather additional information on the evaluation problem, i.e., to revise the linguistic judgements or change the linguistic quantifiers.

5.2. Comparative analysis of consensus measure: MAGDM with single linguistic terms

In this section, the MAGDM example with single linguistic terms [64] in Subsec. 4.1 will be used to illustrate the effectiveness and efficiency of the our consensus measure by comparisons with existing studies.

First, let us consider the consensus measure of the example used in Subsec. 4.1 by our approach. With linguistic quantifier "as many as possible" in the multi-attribute decision rule, the individual overall random preferences are shown in Table 3.

Step 1) As for the collective decision rule where linguistic quantifier "most" is used, the collective overall random preferences can be obtained. The individual overall consensus degrees of the three experts relative to the collective overall random preferences are derived via (28)-(29) and obtained as $\rho_1 = 0.8312, \rho_2 = 0.7217, \rho_3 = 0.9420$.

Step 2) With the importance weights of the three experts $\nu = (1/3, 1/3, 1/3)$ and the linguistic quantifier "most" in the collective decision rule, we can obtain the collective overall consensus degree via (30) as 0.8094.

With combinations of different quantifiers in the processes of multi-attribute decision rule and collective decision rule, different collective overall consensus degrees can be obtained, as shown in Table 8. Suppose the set of experts has assigned a consensus level CL, if $0.8094 \geq \mathrm{CL}$, then the consensus reaching process is completed since the three experts accept the evaluation results. Otherwise, this decision problem has to return back to the initial stage to gather additional information on the evaluation problem. The experts may revise their evaluations on the performances of alternatives or change the linguistic quantifiers.

5.2.1. Consensus degrees based on existing approaches

Here, we shall briefly recall two existing methods using term indices: Pang and Liang [48]'s measure (Pang-Liang for short) and Sun and Ma [58]'s measure (Sun-Ma for short). The results yielded by the two-tuple linguistic model in Subsubsec. 4.1.1 are used, where the individual overall performance with respect to

Table 8: Collective overall consensus degrees under different scenarios: The risk investment problem.

Multi-attribute decision rule	Collective decision rule					
	There exist	At least half	Identity	Most	As many as possible	For all
There exist	0.75	0.6389	0.5556	0.6156	0.4167	0.25
At least half	0.7787	0.7072	0.6905	0.6809	0.547	0.441
Identity	0.895	0.8639	0.8696	0.8689	0.797	0.7438
Most	0.7902	0.7537	0.7765	0.7988	0.6975	0.667
As many as possible	0.8006	0.825	0.8112	0.8094	0.7616	0.7488
For all	1.0	0.8889	0.8889	0.9256	0.8333	0.75

expert E_k is obtained as $(\overline{y}_1^k, \overline{y}_2^k, \ldots, \overline{y}_M^k)$ and the collective overall performance is derived as $(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_M)$, respectively. Then, the consensus measures by Pang-Liang [48] and Sun-Ma [58] are summarized as follows.

(1) Pang-Liang's measure [48]

Step 1) The individual overall consensus degree of expert E_k relative to the collective overall performance is defined as

$$\rho_k = \frac{\sqrt{\sum_{m=1}^{M} (\delta_m^k - |\mathcal{S}| - 1)^2}}{\sqrt{\sum_{m=1}^{M} (\delta_m^k - |\mathcal{S}| - 1)^2} + \sqrt{\sum_{m=1}^{M} (\delta_m^k)^2}} \tag{31}$$

where δ_m^k is the absolute difference between $\overline{y}_m^k$ and $\overline{y}_m$ such that $\delta_m^k = |\Delta^{-1}(\overline{y}_m^k) - \Delta^{-1}(\overline{y}_m)|$, and $|\mathcal{S}|$ is the cardinality of the linguistic term set $\mathcal{S}$. The individual overall consensus degrees of the three experts are obtained as $\rho_1 = 0.9497, \rho_2 = 0.9644, \rho_3 = 0.9707$.

Step 2) The collective overall consensus degree is defined as $\rho = \sum_{k=1}^{K} \rho_k \cdot \nu_k$ and obtained as $\rho = 0.9616$.

(2) Sun-Ma's measure [58]

Step 1) To derive the min and max information for the individual and collective overall performance as

$$\begin{aligned} &\overline{y}_m^k(\min) = \min\left\{\overline{y}_m^k | m = 1, \ldots, M\right\}, \quad \overline{y}_m^k(\max) = \max\left\{\overline{y}_m^k | m = 1, \ldots, M\right\} \\ &\overline{y}_m(\min) = \min\left\{\overline{y}_m | m = 1, \ldots, M\right\}, \quad \overline{y}_m(\max) = \max\left\{\overline{y}_m | m = 1, \ldots, M\right\} \end{aligned} \tag{32}$$

where $k = 1, \ldots, K$.

Step 2) The individual overall consensus degree of expert E_k relative to the collective overall performance is defined as $\rho_k = 1 - \left| \mathrm{Poss}_{\overline{y}_m^k \geq \overline{y}_m} - \mathrm{Poss}_{\overline{y}_m \geq \overline{y}_m^k} \right|$, where

$$\mathrm{Poss}_{\overline{y}_m^k \geq \overline{y}_m} = \min\left\{1, \max\left\{\frac{\Delta^{-1}(\overline{y}_m^k(\max)) - \Delta^{-1}(\overline{y}_m(\min))}{\Delta^{-1}(\overline{y}_m^k(\max)) - \Delta^{-1}(\overline{y}_m^k(\min)) + \Delta^{-1}(\overline{y}_m(\max)) - \Delta^{-1}(\overline{y}_m(\min))}\right\}\right\}, \tag{33}$$

the value $\mathrm{Poss}_{\overline{y}_m^k \geq \overline{y}_m}$ means the possibility that the individual overall performance $\overline{y}_m^k$ is greater than or equal to collective overall performance $\overline{y}_m$. The individual overall consensus degrees of the three experts are obtained as $\rho_1 = 0.7824, \rho_2 = 0.9431, \rho_3 = 0.8816$.

Step 3) The collective overall consensus degree is defined as $\rho = \sum_{k=1}^{K} \rho_k \cdot \nu_k$ and obtained as $\rho = 0.869$.

5.2.2. Comparative analysis

When linguistic quantifier "identity" is used by both the multi-attribute decision rule and collective decision rule in our approach, the collective overall consensus degree is obtained as $\rho = 0.8696$ (see Table 8), which is almost the same as the one generated by Sun-Ma's measure. Whereas, the consensus degree generated by Pang-Liang's measure is higher with a degree 0.9616. Comparisons with these two existing consensus measures will be conducted in terms of uniform scale, variation of individual opinions, and weight information, as summarized in Table 9.

Table 9: Comparative analysis of consensus measure: Single linguistic terms.

Criteria	Pang-Liang [48]	Sun-Ma [58]	Our measure
Uniform scale	Uniform only	Uniform only	Uniform or non-uniform
Variation of individual opinions	No	No	Yes
Weight information	Only importance weights	Only importance weights	Both importance weights and fuzzy majority

- **(1) Uniform scale.** The above two consensus measures are based on the distance function of linguistic two-tuples by using term indices. Since the two-tuple model is strictly constrained by the uniform scale, these two consensus measures are only suitable to the uniform scale. Moreover, Pang-Liang's measure depends heavily on the cardinality of the linguistic term set.
- **(2) Variation of individual opinions.** In the above two consensus measures, the individual and collective overall performance values are expressed in terms of linguistic two-tuples, which cannot represent the differences and variations of individual opinions. Therefore, these two consensus measures can not capture such differences and variations.
- **(3) Weight information.** Both of these two approaches neglect the concept of fuzzy majority in the consensus measure.

Instead, our consensus measure can release the constraint of uniform scale, represent differences and variations of individual opinions, and take both the importance weights and fuzzy majority into account.

5.3. Comparative analysis of consensus measure: MAGDM with flexible linguistic expressions

In this section, the MAGDM example [63] with flexible linguistic expressions in Subsec. 4.2 will be utilized to illustrate the effectiveness and efficiency of the proposed consensus measure by comparisons with existing studies. First, let us consider the consensus measure of the example in Subsec. 4.2 by our approach.

Step 1) With the importance weights of attributes and linguistic quantifier "identity" in the multi-attribute decision rule, the individual and collective overall random preferences can be obtained. Then, the individual overall consensus degrees of the three experts are derived via (28)-(29) as $\rho_1 = 0.9603, \rho_2 = 0.9207, \rho_3 = 0.6798$.

Step 2) With the importance weights of experts and linguistic quantifier "as many as possible" in the collective decision rule, we can obtain the collective overall consensus degree via (30) as 0.7601, which is quite low.

5.3.1. Consensus measure based on existing approaches

In the literature, different consensus measures have been proposed in different research contexts [e.g., 48, 58, 66, 67]. There is no direct consensus measure between individual and collective preferences in our context of MAGDM with flexible linguistic expressions. To illustrate the effectiveness and efficiency of our proposed measure, two consensus measures [67, 68] will be used and slightly revised to fit our context.

(1) Consensus measure based on interval linguistic model. Xu et al. in [68] have built their consensus measure based on the deviation degree of any two linguistic intervals. In their measure, the smaller the deviation degree, the higher consensus degree. With the results obtained by the interval linguistic model in Subsubsec. 4.2.1, Xu et al.'s deviation measure is used to build the consensus measure and summarized as follows.

Table 10: Comparative analysis of consensus measure: Flexible linguistic expressions.

Criteria	Xu et al.'s measure [68]	Wu and Xu's measure [67]	Our measure
Uniform scale	Uniform only	Uniform only	Uniform or non-uniform
Variation of individual opinions	No	Yes	Yes
Weight information	Only importance weights	Only importance weights	Both importance weighs and fuzzy majority

Step 1) The individual overall deviation degree of expert E_k relative to the collective overall performance is defined as

$$\mathrm{Dev}_k = \frac{1}{M}\sum_{m=1}^{M} \mathrm{Dev}\left(\overline{y}_m^k, \overline{y}_m\right), \tag{34}$$

where $k = 1, \ldots, K$, and

$$\mathrm{Dev}\left(\overline{y}_m^k, \overline{y}_m\right) = \frac{\left|\Delta^{-1}(\overline{y}_m^k(L)) - \Delta^{-1}(\overline{y}_m(L))\right| + \left|\Delta^{-1}(\overline{y}_m^k(R)) - \Delta^{-1}(\overline{y}_m(R))\right|}{2|\mathcal{S}|} \tag{35}$$

By this, the individual overall deviation degrees of the three experts are obtained as $\mathrm{Dev}_1 = 0.16667, \mathrm{Dev}_2 = 0.1667, \mathrm{Dev}_3 = 0.3095$. It is easily seen that $\mathrm{Dev}_1 = \mathrm{Dev}_2 < \mathrm{Dev}_3$, whereas our approach generates $\rho_1 > \rho_2 > \rho_3$. In this sense, our approach provides more differentiated information than the consensus measure based on the deviation degree of interval linguistic information.

Step 2) The collective overall deviation degree is defined as $\mathrm{Dev} = \sum_{k=1}^{K} \mathrm{Dev}_k \cdot \nu_k$ and obtained as 0.2143.

(2) Consensus measure based on possibility distribution model. Wu and Xu in [67] have built the consensus measure based on the similarity between two possibility distributions.

Step 1) The individual overall similarity degree of expert E_k relative to the collective overall performance is defined as

$$\rho_k = \frac{1}{M}\sum_{m=1}^{M} \rho_{mk}, k = 1, \ldots, K, \tag{36}$$

where

$$\rho_{mk} = 1 - \frac{1}{G}\left|\sum_{g=0}^{G} g \cdot \pi_m^k(S_g) - \sum_{g=0}^{G} g \cdot \pi_m(S_g)\right| \tag{37}$$

By this, the individual overall similarity degrees of the three experts are obtained as $\rho_1 = 0.9883, \rho_2 = 0.9817, \rho_3 = 0.905$, which is consistent with ranking list generated by our approach, i.e., $\rho_1 > \rho_2 > \rho_3$.

Step 2) The collective overall similarity degree is defined as $\rho = \sum_{k=1}^{K} \rho_k \cdot \nu_k$ and obtained as $\rho = 0.9583$.

5.3.2. Comparative analysis

It is seen that the collective overall consensus degree obtained by Wu and Xu's measure [67] is much higher than the ones derived by our consensus measure and the one based on interval linguistic model. Comparisons with these two existing measures are conducted in terms of uniform scale, variation of individual opinions, and weight information, as summarized in Table 10. The above two measures are based on the distance function of linguistic values by using term indices. It can be easily concluded that these two consensus measures are strictly constrained by the symmetric and balanced linguistic term set. Xu et al.'s measure cannot reflect the difference and variation of individual opinions, since it is based on the two-tuple linguistic model. Wu and Xu's measure can capture such difference and variation since it is based on the possibility distribution over the linguistic term set. Finally, these two measures neglect the concept of fuzzy majority.

5.4. Summary

As a conclusion, our consensus measure performs computations based on the ordinal semantics of linguistic information and probability distributions over the linguistic term set. As such, the strict constraint of symmetric and balanced linguistic term set can be released. Our consensus measure can capture the difference and variation of individual opinions as well as incorporate both the importance weights and the concept of fuzzy majority. Finally, it is easily seen that a unified consensus measure is proposed for qualitative MAGDM with both single linguistic terms and flexible linguistic expressions.

6. Concluding remarks

In summary, in this paper we have studied qualitative MAGDM with linguistic information in terms of single linguistic terms and/or flexible linguistic expressions. First, we have proposed a new linguistic decision rule for MAGDM problems by making use of the random preferences and the concept of stochastic dominance, which is based on a probability interpretation of weight information. The importance weights and the concept of fuzzy majority have been incorporated into both the multi-attribute decision rule and collective decision rule by the so-called WOWA operator with the input parameters expressed as probability distributions. Moreover, a new method has been proposed to measure the consensus degree between the individual overall random preferences and the collective overall random preference based on the concept of stochastic dominance, which takes both the importance weights and the fuzzy majority into account. As such, our proposed approaches are based on the ordinal semantics of linguistic terms and voting statistics. By this, on one hand, the strict constraint of symmetric and balanced linguistic term set in linguistic decision making can be released; on the other hand, the difference and variation of individual opinions can be captured. Two application examples taken from the literature have been used to illuminate the proposed approaches, which show that our proposed approaches are comparable with existing studies.

Perhaps the biggest drawback to the proposed approaches is the information scarce problem, since probability distributions are derived from experts' judgments. Fortunately, in linguistic decision analysis, several experts are always selected and asked to provide their assessments to reduce the subjectiveness and collect enough information. The first direct potential research is to apply our approaches to social networks [5] or online reviews of products, since there are thousands of users and linguistic evaluations, which are suitable to a probability based perspective. Moreover, this paper adopts the absolute linguistic judgments to represent the performance of alternatives. It is necessary and possible to consider the case where the performance and/or importance weights are expressed by linguistic expressions in terms of preference relations and other new preference structures [5]. In addition, a fundamental aspect of (MA)GDM is the importance of looking for approaches to reach consensus [28, 67], in which a consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator [28]. The third interesting direction is to investigate the dynamic and iterative consensus reaching process based on our proposed consensus measure in great detail. Finally, our proposed approach to aggregation results with a probability distribution over a linguistic term set, which may be viewed as a problem of decision making under uncertainty. In essence, the exploitation phase orders the collective overall preferences according to a given criterion. Our approach is based on the stochastic dominance degree. Recently, the decision maker's psychological preferences have played an important role in practice [46, 53]. Therefore, the final interesting direction is to investigate the decision makers' psychological preferences [35] in decision making with linguistic expressions.

Acknowledgement

We would like to appreciate constructive comments and valuable suggestions from the three anonymous referees. This study was partly supported by the National Natural Sciences Foundation of China (NSFC) under grant nos. 71471063, 71125002, 71571069, and 71431004; sponsored by the Innovation Program of Shanghai Municipal Education Commission under grant no. 14ZS060; and supported by the Fundamental Research Funds for the Central Universities in China under grant no. WN1516009.

Appendix A. Stochastic dominance degrees of random variables

In this appendix, we will introduce an approach to derive stochastic dominance degrees from uncertain profiles, which is based on our previous work [78].

Let Z_1 and Z_2 be two independent discrete random variables with respective probability distributions p_1 and p_2 defined over a finite set of linguistic terms $\mathcal{S} = \{S_0, S_1, \ldots, S_G\}$ with $S_0 < S_1 < \cdots < S_G$, where $\sum_{g=0}^{G} p_1(S_g) = 1$ and $\sum_{g=0}^{G} p_2(S_g) = 1$. Let $S_{g_1} \in \mathcal{S}$ and $S_{g_2} \in \mathcal{S}$ be possible outcomes of Z_1 and Z_2, respectively. Let $\Pr(Z_1 \geq Z_2)$, $\Pr(Z_1 = Z_2)$, and $\Pr(Z_1 \leq Z_2)$ denote the probabilities of $Z_1 \geq Z_2$, $Z_1 = Z_2$, and $Z_1 \leq Z_2$, respectively. Since the two random variables Z_1 and Z_2 are stochastically independent, we have the following functions.

$$\begin{aligned}
\Pr(Z_1 \geq Z_2) &= \sum_{g_1=1}^{G} \sum_{g_2=1}^{g_1} p_1(S_{g_1}) \cdot p_2(S_{g_2}) \\
\Pr(Z_1 = Z_2) &= \sum_{g_1=0}^{G} p_1(S_{g_1}) \cdot p_2(S_{g_1}) \\
\Pr(Z_1 \leq Z_2) &= \sum_{g_1=0}^{G} \sum_{g_2=g_1}^{G} p_1(S_{g_1}) \cdot p_2(S_{g_2})
\end{aligned} \tag{A.1}$$

Accordingly, we have

$$\begin{aligned}
\Pr(Z_1 > Z_2) &= \Pr(Z_1 \geq Z_2) - \Pr(Z_1 = Z_2) \\
\Pr(Z_1 < Z_2) &= \Pr(Z_1 \leq Z_2) - \Pr(Z_1 = Z_2)
\end{aligned} \tag{A.2}$$

The case $Z_1 = Z_2$ can be regarded as a situation where $Z_1 \geq Z_2$ and $Z_1 \leq Z_2$ occur with the same probability simultaneously.

Due to the above analysis, we give the definition of stochastic dominance degree of two random variables with discrete probability distributions defined over a set of linguistic terms as follows. Let Z_1 and Z_2 be two independent discrete random variables with (discrete) probability distributions p_1 and p_2 over a finite set of linguistic labels $\mathcal{S} = \{S_0, S_1, \ldots, S_G\}$ with $S_0 < S_1 < \cdots < S_G$, then the stochastic dominance degrees of Z_1 over Z_2 (Z_2 over Z_1) are given by

$$\begin{aligned}
D_{12} = D_{Z_1 \succ Z_2} &= \Pr(Z_1 \geq Z_2) - 0.5 \Pr(Z_1 = Z_2) \\
D_{21} = D_{Z_2 \succ Z_1} &= \Pr(Z_2 \geq Z_1) - 0.5 \Pr(Z_1 = Z_2)
\end{aligned} \tag{A.3}$$

Extending two random variables to a vector of N random variables $\mathcal{Z} = (Z_1, Z_2, \ldots, Z_N)$, we are able to derive a matrix $\mathbf{D}$ of stochastic dominance degrees of the N discrete random variables as

$$\mathbf{D} = \begin{array}{cccccc}
 & Z_1 & Z_2 & \ldots & Z_N \\
Z_1 & D_{11} & D_{12} & \ldots & D_{1N} \\
Z_2 & D_{21} & D_{22} & \ldots & D_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_N & D_{N1} & D_{N2} & \ldots & D_{NN}
\end{array} \tag{A.4}$$

Such a matrix of stochastic dominance degrees has the following interesting properties: (1) $D_{nl} + D_{ln} = 1, \forall n, l = 1, \ldots, N$; (2) $D_{nn} = 0.5, n = 1, \ldots, N$; (3) $\sum_{n=1}^{N} \sum_{l=1}^{N} D_{nl} = N^2/2$; and (4) $0 \leq D_{nl} \leq 1, \forall n, l = 1, \ldots, N$.

Interestingly, the matrix $\mathbf{D}$ of stochastic dominance degrees with respect to a vector of N random variables $\mathcal{Z} = (Z_1, Z_2, \ldots, Z_N)$ satisfies the following properties of fuzzy preference relations:

(1) when $D_{nl} = 1$, it indicates that Z_n is absolutely preferred to Z_l, i.e., indicates the maximum degree of preference of Z_n over Z_l;
(2) when $0.5 < D_{nl} < 1$, it indicates that Z_n is slightly preferred to Z_l;
(3) when $D_{nl} = 0.5$, there is indifference between Z_n and Z_l;
(4) when $0 < D_{nl} < 0.5$, it indicates that Z_l is slightly preferred to Z_n;
(5) when $D_{nl} = 0$, it indicates that Z_l is absolutely preferred to Z_n.

Therefore, the matrix of stochastic dominance degrees of the random variable set $\mathcal{Z}$ is in fact a matrix of fuzzy preference relations formulated as $\mu_{\mathbf{D}} : \mathcal{Z}\times\mathcal{Z} \longrightarrow [0, 1]$ with $\mu_{\mathbf{D}} : (Z_n, Z_l) = D_{nl}$, where $n, l = 1, \ldots, N$, and D_{nl} reflects the degree of fuzzy preference of Z_n over Z_l. Moreover, it is obvious that the matrix of fuzzy preference relations satisfies the condition of *fuzzy reciprocity* such that $D_{nl} + D_{ln} = 1, \forall n, l = 1, \ldots, N$.

Appendix B. The OWA operator and linguistic quantifiers

The notion of OWA operator was first introduced in [72] regarding the problem of aggregating multi-attribute values to form an overall decision function. Since its invention, the OWA operator has been extensively studied, and has been found useful in many applications of information fusion and decision making. By definition, an OWA operator of dimension N is a mapping $\mathcal{F} : R^N \rightarrow R$ associated with a weighting vector $\omega = (\omega_1, \ldots, \omega_N)$ such that: 1) $\omega_n \in [0, 1]$ and 2) $\sum_{n=1}^N \omega_n = 1$, and

$$\mathcal{F}_{\text{OWA}}(V_1, V_2, \ldots, V_N) = \sum\nolimits_{n=1}^{N} \omega_n \cdot V_{\sigma(n)}$$

where $V_{\sigma(n)}$ is the nth largest element in the vector $(V_1, V_2, \ldots, V_N)$ such that $V_{\sigma(1)} \geq V_{\sigma(2)} \geq \cdots \geq V_{\sigma(N)}$. The symbol R means the universe/domain of discourse (a set of real numbers, e.g. $R = [0, 1]$). Obviously, the key step of this aggregation is reordering of arguments V_n in a non-increasing order so that the weight ω_n is associated with the ordered position of the argument, rather than associated with the argument itself.

The OWA operator provides a type of aggregation operator between the“AND” and the “OR” aggregations. As suggested by [72], there exist at least two methods for obtaining the OWA weight information $\omega_n, n = 1, \ldots, N$. The first approach is to use some kind of learning mechanism, i.e., we use some sample data, arguments, and associated aggregate values, and try to fit the weights to this collection of sample data. The second approach is to give some semantics or meaning to the weights. Then, based on these semantics, we can directly provide the values for the weights. For the purpose of this paper, let us introduce the semantics based on fuzzy linguistic quantifiers for the weights.

The fuzzy linguistic quantifiers were introduced by Zadeh [82]. According to Zadeh, there are basically two types of quantifiers: absolute and relative. Here, we focus on the relative quantifiers typified by terms such as *most*, *at least half*, and *as many as possible*. A fuzzy subset Q of the universe domain $[0, 1]$ is called a relative quantifier if $Q(0) = 0, Q(1) = 1,$ and $Q(x) \geq Q(y)$ for $x \geq y$. Then, Yager [72] proposed to compute the OWA weights based on the linguistic quantifier Q as $\omega_n = Q\left(\frac{n}{N}\right) - Q\left(\frac{n-1}{N}\right), n = 1, \ldots, N$. Table B.11 provides typical examples of linguistic quantifiers associated with their membership functions.

Appendix C. The WOWA operator

The OWA operator only considers the weights of values in the aggregation. In order to incorporate the importance weights of information sources, Torra [59] has proposed a WOWA operator, defined as follows.

Let $(V_1, V_2, \ldots, V_N)$ be a set of values, $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be the importance weights of the values such that: $\mu_n \in [0, 1]$ and $\sum_{n=1}^N \mu_n = 1$. In this case, a mapping $\mathcal{F}_{\text{WOWA}} : R^N \rightarrow R$ is a WOWA operator of dimension N if

$$\mathcal{F}_{\text{WOWA}}(V_1, V_2, \ldots, V_N; \mu_1, \mu_2, \ldots, \mu_N) = \sum\nolimits_{n=1}^{N} W_n \cdot V_{\sigma(n)} \tag{C.1}$$

Table B.11: Typical examples of linguistic quantifiers associated with their membership functions.

Linguistic quantifier	Membership function
there exists	$Q(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$
at least half	$Q(x) = \begin{cases} 2x & \text{if } 0 \le x \le 0.5 \\ 1 & \text{if } 0.5 < x \le 1 \end{cases}$
identity	$Q(x) = x$
most	$Q(x) = \begin{cases} 0 & \text{if } 0 \le x \le 0.3 \\ 2x - 0.6 & \text{if } 0.3 < x \le 0.8 \\ 1 & \text{if } 0.8 < x \le 1 \end{cases}$
as many as possible	$Q(x) = \begin{cases} 0 & \text{if } 0 \le x \le 0.5 \\ 2x - 1 & \text{if } 0.5 < x \le 1 \end{cases}$
for all	$Q(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \ne 1 \end{cases}$

where $(V_{\sigma(1)}, V_{\sigma(2)}, \ldots, V_{\sigma(N)})$ is a permutation of $(V_1, V_2, \ldots, V_N)$ such that $V_{\sigma(n-1)} \ge V_{\sigma(n)}$ $\forall n = 2, \ldots, N$, i.e., $V_{\sigma(n)}$ is the nth largest element in the vector $(V_1, V_2, \ldots, V_N)$, and the weight W_n is defined as

$$W_n = W^* \left(\sum_{l \le n} \mu_{\sigma(l)} \right) - W^* \left(\sum_{l < n} \mu_{\sigma(l)} \right) \tag{C.2}$$

with W^* a monotone increasing function that interpolates the points $\left(n/N, \sum_{l \le n} \mu_{\sigma(n)} \right)$ together with the point $(0, 0)$. The value $\mu_{\sigma(l)}$ means the permutation according to $(V_{\sigma(1)}, V_{\sigma(2)}, \ldots, V_{\sigma(N)})$.

When W^* is replaced with an RIM linguistic quantifier Q, then

$$W_n = Q \left(\sum_{l \le n} \mu_{\sigma(l)} \right) - Q \left(\sum\nolimits_{l < n} \mu_{\sigma(l)} \right), n = 1, \ldots, N,$$

which indicates that the WOWA operator becomes the importance weighted quantifier guided aggregation [73]. With the WOWA operator and the fuzzy linguistic quantifier, the overall value is derived by

$$\begin{aligned} V &= \mathcal{F}_{\mathrm{WOWA}}(V_1, V_2, \ldots, V_N; \mu_1, \mu_2, \ldots, \mu_N) \\ &= \sum\nolimits_{n=1}^{N} \left[Q \left(\sum_{l \le n} \mu_{\sigma(l)} \right) - Q \left(\sum_{l < n} \mu_{\sigma(l)} \right) \right] \cdot V_m^{\sigma(n)} \end{aligned} \tag{C.3}$$

where $\left(V_{\sigma(1)}, V_{\sigma(2)}, \ldots, V_{\sigma(N)}\right)$ is a permutation of $(V_1, V_2, \ldots, V_N)$ such that $V_{\sigma(n-1)} \ge V_{\sigma(n)}, \forall n = 2, \ldots, N$.

[1] Agell, N., Sánchez, M., Prats, F., & Roselló, L. (2012). Ranking multi-attribute alternatives on the basis of linguistic labels in group decisions. *Information Sciences*, *209*, 49–60.

[2] Beg, I., & Rashid, T. (2013). TOPSIS for hesitant fuzzy linguistic term sets. *International Journal of Intelligent Systems*, *28*, 1162–1171.

[3] Bordogna, G., Fedrizzi, M., & Pasi, G. (1997). A linguistic modeling of consensus in group decision making based on OWA operators. *IEEE Transactions on Systems Man and Cybernetics, Part A: Systems and Humans*, *27*, 126–133.

[4] Bordogna, G., & Pasi, G. (1993). A fuzzy linguistic approach generalizing Boolean information retrieval: A model and its evaluation. *Journal of the American Society for Information Science*, *44*, 70–82.

[5] Cabrerizo, F. J., Chiclana, F., Al-Hmouz, R., Morfeq, A., Balamash, A. S., & Herrera-Viedma, E. (2015). Fuzzy decision making and consensus: Challenges. *Journal of Intelligent & Fuzzy Systems*, *29*, 1109–1118.

[6] Cabrerizo, F. J., Moreno, J. M., Pérez, I. J., & Herrera-Viedma, E. (2010). Analyzing consensus approaches in fuzzy group decision making: advantages and drawbacks. *Soft Computing*, *14*, 451–463.

[7] Chuu, S.-J. (2011). Interactive group decision-making using a fuzzy linguistic approach for evaluating the flexibility in a supply chain. *European Journal of Operational Research*, *213*, 279–289.

[8] Delgado, M., Verdegay, J., & Vila, M. (1993). On aggregation operations of linguistic labels. *International Journal of Intelligent Systems*, *8*, 351–370.

[9] Dhouib, D. (2014). An extension of MACBETH method for a fuzzy environment to analyze alternatives in reverse logistics for automobile tire wastes. *Omega*, *42*, 25–32.

[10] Dong, Y., Li, C.-C., Xu, Y., & Gu, X. (2015). Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations. *Group Decision and Negotiation*, *24*, 217–242.

[11] Dong, Y., Xu, Y., & Li, H. (2008). On consistency measures of linguistic preference relations. *European Journal of Operational Research*, *189*, 430–445.

[12] Dong, Y., Xu, Y., Li, H., & Feng, B. (2010). The OWA-based consensus operator under linguistic representation models using position indexes. *European Journal of Operational Research*, *203*, 455–463.

[13] Doukas, H. (2013). Modelling of linguistic variables in multicriteria energy policy support. *European Journal of Operational Research*, *227*, 227–238.

[14] Dubois, D., Grabisch, M., Modave, F., & Prade, H. (2000). Relating decision under uncertainty and multicriteria decision making models. *International Journal of Intelligent Systems*, *15*, 967–979.

[15] Falcó, E., García-Lapresta, J. L., & Roselló, L. (2014). Allowing agents to be imprecise: A proposal using multiple linguistic terms. *Information Sciences*, *258*, 249–265.

[16] García-Lapresta, J. L., & Pérez-Román, D. (2015). Ordinal proximity measures in the context of unbalanced qualitativescales and some applications to consensus and clustering. *Applied Soft Computing*, *35*, 864–872.

[17] García-Lapresta, J. L., & Pérez-Román, D. (2016). Consensus-based clustering under hesitant qualitative assessments. *Fuzzy Sets and Systems*, *292*, 261–273.

[18] Genest, C., & Zidek, J. V. (1986). Combining probability distributions: A critique and an annotated bibliography. *Statistical Science*, *1*, 114–135.

[19] Gong, Z., Zhang, H., Forrest, J., Li, L., & Xu, X. (2015). Two consensus models based on the minimum cost and maximum return regarding either all individuals or one individual. *European Journal of Operational Research*, *240*, 183–192.

[20] Hatami-Marbini, A., & Tavana, M. (2011). An extension of the Electre I method for group decision-making under a fuzzy environment. *Omega*, *39*, 373–386.

[21] Herrera, F., & Herrera-Viedma, E. (2000). Linguistic decision analysis: steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems*, *115*, 67–82.

[22] Herrera, F., Herrera-Viedma, E., & Martínez, L. (2000). A fusion aproach for managing multigranularity linguistic term sets in decision making. *Fuzzy Sets and Systems*, *114*, 43–58.

[23] Herrera, F., Herrera-Viedma, E., & Verdegay, J. L. (1996). A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets and Systems*, *78*, 73–87.

[24] Herrera, F., Herrera-Viedma, E., & Verdegay, J. L. (1996). Direct approach processes in group decision making using linguistic OWA operators. *Fuzzy Sets and Systems*, *79*, 175–190.

[25] Herrera, F., & Martínez, L. (2000). A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, *8*, 746–752.

[26] Herrera, F., & Martínez, L. (2001). A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, *31*, 227–234.

[27] Herrera-Viedma, E., Cabrerizo, F. J., Kacprzyk, J., & Pedrycz, W. (2014). A review of soft consensus models in a fuzzy environment. *Information Fusion*, *17*, 4–13.

[28] Herrera-Viedma, E., Martínez, L., Mata, F., & Chiclana, F. (2005). A consensus support systems model for group decision making problems with multigranular linguistic preference relations. *IEEE Transactions on Fuzzy Systems*, *13*, 644–658.

[29] Huynh, V.-N., & Nakamori, Y. (2005). A satisfactory-oriented approach to multi-expert decision-making under linguistic assessments. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, *35*, 184–196.

[30] Huynh, V.-N., & Nakamori, Y. (2011). A linguistic screening evaluation model in new product development. *IEEE Transactions on Engineering Management*, *58*, 165–175.

[31] Jiménez, A., Mateos, A., & Sabio, P. (2013). Dominance intensity measure within fuzzy weight oriented MAUT: An application. *Omega*, *41*, 397–405.

[32] Kacprzyk, J. (1986). Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems*, *18*, 105–118.

[33] Kacprzyk, J., Fedrizzi, M., & Nurmi, H. (1992). Group decision making and consensus under fuzzy preferences and fuzzy majority. *Fuzzy Sets and Systems*, *49*, 21–31.

[34] Kacprzyk, J., Zadrozny, S., Fedrizzi, M., & Nurmi, H. (2008). Fuzzy sets and their extensions: Representation, aggregation and models. chapter On group decision making, consensus reaching, voting and voting paradoxes under fuzzy preferences and a fuzzy majority: a survey and some perspectives. (pp. 263–295). Heidelberg: Physica-Verlag.

[35] Kahneman, D., & Tversky, A. (1979). Prospect Theory: An analysis of decision under risk. *Econometrica*, *47*, 263–291.

[36] Lawry, J. (2001). A methodology for computing with words. *International Journal of Approximate Reasoning*, *28*, 51–89.

[37] Lawry, J. (2004). A framework for linguistic modeling. *Artificial Intelligence*, *155*, 1–39.
[38] Li, Z., Xu, J., Lev, B., & Gang, J. (2015). Multi-criteria group individual research output evaluation based on context-free grammar judgments with assessing attitude. *Omega*, *57*, 282–293.
[39] Liao, H., Xu, Z., & Zeng, X.-J. (2014). Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. *Information Sciences*, *271*, 125–142.
[40] Liu, H., & Rodríguez, R. (2014). A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making. *Information Sciences*, *258*, 220–238.
[41] Loomes, G., & Sugden, R. (1995). Incorporating a stochastic element into decision theories. *European Economic Review*, *39*, 641–648.
[42] Martínez, L., & Herrera, F. (2012). An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, *207*, 1–18.
[43] Mendel, J. M., Zadeh, L. A., Trillas, E., Yager, R., Lawry, J., Hagras, H., & Guadarrama, S. (2010). What computing with words means to me. *IEEE Computational Intelligence Magazine*, *5*, 20–26.
[44] Merigó, J. M., Palacios-Marqués, D., & Zeng, S. (2016). Subjective and objective information in linguistic multi-criteria group decision making. *European Journal of Operational Research*, *248*, 522–531.
[45] Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, *63*, 81–97.
[46] Montes, R., Sánchez, A. M., Villara, P., & Herrera, F. (2015). A web tool to support decision making in the housing market usinghesitant fuzzy linguistic term sets. *Applied Soft Computing*, *35*, 949–957.
[47] Nguyen, C. H., & Huynh, V. N. (2002). An algebraic approach to linguistic hedges in Zadeh's fuzzy logic. *Fuzzy Sets and Systems*, *129*, 229–254.
[48] Pang, J., & Liang, J. (2012). Evaluation of the results of multi-attribute group decision-making with linguistic information. *Omega*, *40*, 294–301.
[49] Pasi, G., & Yager, R. R. (2006). Modeling the concept of majority opinion in group decision making. *Information Sciences*, *176*, 390–414.
[50] Rodríguez, R. M., Labella, Á., & Martínez, L. (2016). An overview on fuzzy modelling of complex linguistic preferences in decision making. *International Journal of Computational Intelligence Systems*, *9*, 81–94.
[51] Rodríguez, R. M., & Martínez, L. (2013). An analysis of symbolic linguistic computing models in decision making. *International Journal of General Systems*, *42*, 121–136.
[52] Rodríguez, R. M., Martínez, L., & Herrera, F. (2012). Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on Fuzzy Systems*, *20*, 109–119.
[53] Rodríguez, R. M., Martínez, L., & Herrera, F. (2013). A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sciences*, *241*, 28–42.
[54] Roselló, L., Prats, F., Agell, N., & Sánchez, M. (2010). Measuring consensus in group decisions by means of qualitative reasoning. *International Journal of Approximate Reasoning*, *51*, 441–452.
[55] Roselló, L., Sánchez, M., Agell, N., Prats, F., & Mazaira, F. A. (2014). Using consensus and distances between generalized multi-attribute linguistic assessments for group decision-making. *Information Fusion*, *17*, 83–92.
[56] Shafer, G. (1976). *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.
[57] Smets, P., & Kennes, R. (1994). The transferable belief model. *Artificial Intelligence*, *66*, 191–234.
[58] Sun, B., & Ma, W. (2015). An approach to consensus measurement of linguistic preference relations in multi-attribute group decision making and application. *Omega*, *51*, 83–92.
[59] Torra, V. (1997). The weighted OWA operator. *International Journal of Intelligent Systems*, *12*, 153–166.
[60] Travé-Massuyés, L., & Piera, N. (1989). The orders of magnitude models as qualitative algebras. In *Proceedings of the 11th International Joint Conference on Artificial Intelligence-Volume 2* IJCAI'89 (pp. 1261–1266). San Francisco, CA, USA: Morgan Kaufmann Publishers Inc.
[61] Wang, J.-Q., Peng, J.-J., Zhang, H.-Y., Liu, T., & Chen, X.-H. (2015). An uncertain linguistic multi-criteria group decision-making method based on a cloud model. *Group Decision and Negotiation*, *24*, 171–192.
[62] Wang, S.-Y., Chang, S.-L., & Wang, R.-C. (2009). Assessment of supplier performance based on product-development strategy by applying multi-granularity linguistic term sets. *Omega*, *37*, 215–226.
[63] Wei, C., Zhao, N., & Tang, X. (2014). Operators and comparisons of hesitant fuzzy linguistic term sets. *IEEE Transactions on Fuzzy Systems*, *22*, 575–585.
[64] Wu, Z., & Chen, Y. (2007). The maximizing deviation method for group multiple attribute decision making under linguistic environment. *Fuzzy Sets and Systems*, *158*, 1608–1617.
[65] Wu, Z., & Xu, J. (2012). Consensus reaching models of linguistic preference relations based on distance functions. *Soft Computing*, *16*, 577–590.
[66] Wu, Z., & Xu, J. (2016). Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. *Omega*, *http://dx.doi.org/10.1016/j.omega.2015.12.005*.
[67] Wu, Z., & Xu, J. (2016). Possibility distribution-based approach for MAGDM with hesitant fuzzy linguistic information. *IEEE Transactions on Cybernetics*, *46*, 694–705.

[68] Xu, J., Wu, Z., & Zhang, Y. (2014). A consensus based method for multi-criteria group decision making under uncertain linguistic setting. *Group Decision and Negotiation*, *23*, 127–148.

[69] Xu, Z. (2004). Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. *Information Sciences*, *168*, 171–184.

[70] Xu, Z. (2005). Deviation measures of linguistic preference relations in group decision making. *Omega*, *33*, 249–254.

[71] Xu, Z. (2006). An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. *Decision Support Systems*, *41*, 488–499.

[72] Yager, R. R. (1988). On ordered weighted averaging operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics*, *18*, 183–190.

[73] Yager, R. R. (1996). Quantifier guided aggregation using OWA operators. *International Journal of Intelligent Systems*, *11*, 49–73.

[74] Yan, H. B., Huynh, V. N., & Nakamori, Y. (2012). A group nonadditive multiattribute consumer-oriented Kansei evaluation model with an application to traditional crafts. *Annals of Operations Research*, *195*, 325–354.

[75] Yan, H.-B., & Ma, T. (2015). A fuzzy group decision making approach to new product concept screening at the fuzzy front end. *International Journal of Production Research*, *53*, 4021–4049.

[76] Yan, H.-B., & Ma, T. (2015). A group decision-making approach to uncertain quality function deployment based on fuzzy preference relation and fuzzy majority. *European Journal of Operational Research*, *241*, 815–829.

[77] Yan, H.-B., Ma, T., & Huynh, V.-N. (2014). Coping with group behaviors in uncertain quality function deployment. *Decision Sciences*, *45*, 1025–1052.

[78] Yan, H.-B., Ma, T., & Li, Y. (2013). A novel fuzzy linguistic model for prioritising engineering design requirements in quality function deployment under uncertainties. *International Journal of Production Research*, *51*, 6336–6355.

[79] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning–Part I. *Information Sciences*, *8*, 199–249.

[80] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning–Part II. *Information Sciences*, *8*, 301–357.

[81] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning–Part III. *Information Sciences*, *9*, 43–80.

[82] Zadeh, L. A. (1983). A computational approach to fuzzy quantifiers in natural languages. *Computers and Mathematics with Applications*, *9*, 149–184.

[83] Zhang, G., Dong, Y., & Xu, Y. (2014). Consistency and consensus measures for linguistic preference relations based on distribution assessments. *Information Fusion*, *17*, 46–55.

[84] Zhu, B., & Xu, Z. (2014). Consistency measures for hesitant fuzzy linguistic preference relations. *IEEE Transactions on Fuzzy Systems*, *22*, 35–45.