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A study of left residuated lattices and logics without contraction and exchange rules

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1 Introduction

Both classical logic (**Cl**) and intuitionistic logic (**Int**), when formulated as Gentzen systems, possess three structural rules, which are: exchange, contraction and weakening. Logics that lack some (or all) structural rules are known as substructural logics and have recently been intensively investigated. Examples of substructural logics include Lambek calculus (which does not have any structural rules), linear logic (which lacks contraction and weakening), or BCK logic (which does not have contraction). Studying properties of substructural logic in contrast to properties of **Cl** and **Int** can clarify the role the structural rules play. One of the tools for such investigations is algebraic semantics, which offers the advantage of providing a uniform framework for many different substructural logics and makes comparisons between them easier. In this paper, we will deal with logics extending $\mathbf{FL_w}$, which stands for "Full Lambek calculus + weakening", and thus is the smallest logic lacking both exchange and contraction.

Algebraic semantics for extensions of $\mathbf{FL_w}$ is provided by (noncommutative) residuated lattices. Residuated lattices have been studied since 1930, but only recently they have been investigated in connection with substructural logics, and even in this connection they have been largely confined to the case of commutative residuated lattices (see, e.g., [1],[3]), which are algebraic semantics of extensions of $\mathbf{FL_{ew}}$, i.e., logics without contraction (but with exchange). As for algebraic semantic of $\mathbf{FL_w}$, investigations are still in initial stage and many problems remain open.

In this paper, we will introduce left residuated lattices, which are the algebraic counterpart of $\mathbf{FL'_w}$, which in turn is a reduct of $\mathbf{FL_w}$. We study basic properties of left residuated lattices. Next, we turn to a classification of left residuated lattices with C_n.

2 Logics without contraction and exchange rules

The systems $\mathbf{FL_w}$ and $\mathbf{FL'_w}$ are introduced here. The sequent calculus $\mathbf{FL_w}$ is obtained from the usual sequent calculus for intuitionistic logic **Int** by eliminating contraction and exchange rules. In $\mathbf{FL_w}$ it is natural to introduce two implications, since it does not have exchange. To simplify the discussion, we consider only one implication. We call this restricted system $\mathbf{FL'_w}$.

3 Left residuated lattices

In this section, we introduce left residuated lattices. The class of left residuated lattices gives algebraic semantics for $\mathbf{FL'_w}$, in the same way as the class of *Boolean algebras* does it for classical logic. Since the algebraic counterpart of exchange is commutativity of the monoid operation, in a left residuated lattice commutativity is not assumed.

We define left residuated lattices and filters on them, then we introduce subdirectly irreducible and simple left residuated lattices.

An algebra $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is a left residuated lattice if it satisfies the following conditions:

1. $\langle M, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with the greatest element 1 and the least 0,

2. $\langle M, \cdot, 1\rangle$ is a monoid,

3. $c \cdot a \leq b \Leftrightarrow c \leq a \rightarrow b$ (left-residuation),

4. $w \cdot (x \vee y) \cdot z = (w \cdot x \cdot z) \vee (w \cdot y \cdot z)$.

A nonempty subset F of a left residuated lattice $\mathbf{M}$ is a filter if for $a, b \in M$ it satisfies:

1. $1 \in F$,
2. $a, a \to b \in F \quad \textit{imply} \quad b \in F$,
3. $a \in F \quad \textit{implies} \quad (a \to b) \to b \in F$.

There exists a lattice isomorphism between the set of all filters of $\mathbf{M}$ and the set of all congruences of $\mathbf{M}$ (see [4]).

Before passing on to subdirectly irreducible and simple left residuated lattices, we will need a rather technical definition. Let A_m^S be a subset of M defined by induction as follows:

For any subset S of M,

$$\begin{aligned}
D_0^S &= S \\
A_0^S &= \{w_1 \cdots w_k | w_i \in D_0^s, k \geq 1\} \\
D_1^S &= \left\{(x \to y) \to y | x \in A_0^S, y \in M\right\} \\
A_1^s &= \{w_1 \cdots w_k | w_i \in D_1^s, k \geq 1\} \\
&\vdots \\
D_{n+1}^S &= \{(x \to y) \to y | x \in A_n^s, y \in M\} \\
A_{n+1}^S &= \left\{w_1 \cdots w_k | w_i \in D_{n+1}^S, k \geq 1\right\}.
\end{aligned}$$

Now we can characterize subdirectly irreducible left residuated lattices as follows.

A left residuated lattice $\mathbf{M}$ is subdirectly irreducible if and only if there exists an element $c(< 1)$ such that for any $x < 1$ there exists $z \in A_m^x$ $(m \geq 0)$ for which $z \leq c$ holds.

We are also able to characterize simple left residuated lattices as follows.

A left residuated lattice $\mathbf{M}$ is simple if and only if for any $x < 1$ in M there exists a positive integer m such that $0 \in A_m^x$.

4 Left residuated lattices with C_n

For any $x, y \in M,\ n \geq 1$,

$$y^n \leq (y \to x) \to x \quad (C_n).$$

The identity C_n has been introduced in [5]. In presence of C_n filters of left residuated lattices can be defined exactly as for commutative residuated lattices. Thus, subdirectly irreducible and left residuated lattices with C_n are characterized exactly as in the commutative case (cf. [1], [3]).

Let the variety of left residuated lattices satisfying C_n be denoted by $\mathcal{NC}_n$. We have that $\mathcal{NC}_n \subsetneq \mathcal{NC}_{n+1}$. This yields a strictly ascending chain of varieties, and therefore a certain classification of left residuated lattices.

5 Conclusions and remarks

We have characterized subdirectly irreducible and simple left residuated lattices. We have shown that similar characterization of left residuated lattices satisfying C_n can be obtained in a much simpler way. We have also proposed a classification of left residuated lattices.

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