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Description	

Recent studies on the proof-theoretic strength of Ramsey's theorem for pairs

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1 Introduction

Calibrating the strength of Ramsey's theorem is one of the central topics in the study of reverse mathematics. Our target is infinite Ramsey's theorem on $\mathbb{N}$. Within the second-order arithmetic, we consider Ramsey's theorem for n-tuples and k-colors (RT^n_k) which asserts that every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous subset, and we write RT^n for the statement $\forall k \mathrm{RT}^n_k$.

The strength of Ramsey's theorem was precisely analyzed by means of computability theoretic methods, which led the comparison of Ramsey's theorem with the big five systems in the setting of reverse mathematics. In [15], Jockusch showed that there exists a computable coloring for $[\mathbb{N}]^3$ whose homogeneous set always computes the halting problem. This idea together with a standard proof of Ramsey's theorem is formalized by Simpson [24] within the second-order arithmetic, namely, if $n \geq 3$, Ramsey's theorem for n-tuples is equivalent to ACA_0. The status of Ramsey's theorem for pairs was open for a long time, until Seetapun [23] proved that RT^2_2 is strictly weaker than ACA_0 over RCA_0. On the relation between WKL_0 and RT^2_2, Jockusch [15] showed that WKL_0 does not imply RT^2_2. The converse direction was very difficult, but finally, Liu [19] showed that $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ does not imply WKL_0 by a clever forcing method. Furthermore, there are numerous studies on Ramsey's theorem for pairs and related combinatorial principles mainly from the view point of computability theory. See Hirschfeldt [12] for a gentle introduction to the reverse mathematics studies for Ramsey's theorem.

In this manuscript, we mainly focus on the proof-theoretic strength of Ramsey's theorem for pairs. There are long series of studies on this topic by various people and various methods. In [14], Hirst showed that RT^2_2 implies the Σ^0_2-bounding principle ($\mathrm{B}\Sigma^0_2$), and

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RT^2 implies $\mathrm{B}\Sigma^0_3$ over $\mathsf{RCA_0}$. On the other hand, Cholak, Jockusch and Slaman [6] showed that $\mathsf{WKL_0} + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$ is a Π^1_1-conservative extension of $\mathrm{I}\Sigma^0_2$, and $\mathsf{WKL_0} + \mathrm{RT}^2 + \mathrm{I}\Sigma^0_3$ is a Π^1_1-conservative extension of $\mathrm{I}\Sigma^0_3$. Thus, the first-order strength of RT^2_2 is in between $\mathrm{B}\Sigma^0_2$ and $\mathrm{I}\Sigma^0_2$, and the first-order strength of RT^2 is in between $\mathrm{B}\Sigma^0_3$ and $\mathrm{I}\Sigma^0_3$. After this work, many advanced studies are done to investigate the first-order strength of Ramsey's theorem and related combinatorial principles. One of the most important methods for these studies is adapting computability-theoretic techniques for combinatorial principles in nonstandard models of arithmetic. By this method, Chong, Slaman and Yang [9, 8] analyzed slightly weaker but important combinatorial principles ADS, CAC and SRT^2_2 (see e.g., [12] for these principles), and finally they showed that RT^2_2 does not imply $\mathrm{I}\Sigma^0_2$ over $\mathsf{RCA_0}$ in [7]. More recently, Chong, Kreuzer and Yang [unpublished] showed that $\mathsf{WKL_0} + \mathrm{SRT}^2_2 + \mathrm{WF}(\omega^\omega)$ is Π^0_3-conservative over $\mathsf{RCA_0} + \mathrm{WF}(\omega^\omega)$, where $\mathrm{WF}(\omega^\omega)$ asserts the well-foundedness of ω^ω.

Another important approach is calibration of the proof-theoretic strength of variations of the Paris-Harrington principle which is deduced from infinite Ramsey's theorem by using the idea of the ordinal analysis. One of the most important result of this line is a sharp upper bounds for the Paris-Harrington principle by Ketonen and Solovay [16]. More recently, Bovykin/Weiermann [5] showed that indicators defined by Paris's density notion can approach the proof-theoretic strength of various versions of Ramsey's theorem, and by a similar method, the author [28] showed that $\mathrm{RT}^n_k + \mathsf{WKL}^*_0$ is fairly weak and is a Π^0_2-conservative extension of RCA^*_0, where RCA^*_0 is $\mathsf{RCA_0}$ with only Σ^0_0-induction and the exponentiation. There are many more studies from this view point, e.g., by Kotlarski, Weiermann, et al. [26, 18, 4].

Here, we will overview the recent results on the exact strength of RT^2_2 and RT^2, namely, $\mathrm{RT}^2_2 + \mathsf{WKL_0}$ is a Π^0_3-conservative extension of $\mathsf{RCA_0}$, and $\mathrm{RT}^2 + \mathsf{WKL_0}$ is a Π^1_1-conservative extension of $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$. The main tool for the former result is Paris's density notion plus the ordinal analysis, while the latter result is derived by computability-theoretic arguments in nonstandard models.

2 The proof-theoretic strength of RT^2_2

In this section, we see the proof-theoretic strength of Ramsey's theorem for pairs and two colors (RT^2_2) based on [21]. A formula φ is said to be $\tilde{\Pi}^0_n$ if it is of the form $\varphi \equiv \forall X \theta$ where θ is Π^0_n. The main theorem of this section is the following.

Theorem 2.1 (Patey/Yokoyama)**.** $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathsf{RCA_0}$.

Recall that $\mathsf{RCA_0} + \mathrm{RT}^2_2$ implies $\mathrm{B}\Sigma^0_2$ and $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2$ is $\tilde{\Pi}^0_3$-conservative over $\mathrm{I}\Sigma^0_1$. Thus, the theorem says that $\mathrm{I}\Sigma^0_1$ is the exact $\tilde{\Pi}^0_3$-part of $\mathsf{WKL_0} + \mathrm{RT}^2_2$. This answers the long-standing open question of determining the Π^0_2-consequences of RT^2_2 posed e.g., in Seetapun and Slaman [23, Question 4.4] Cholak, Jockusch and Slaman [6, Question 13.2]. Indeed, one can see that RT^2_2 does not imply the totality of Ackermann function nor the consistency of $\mathrm{I}\Sigma^0_1$. Moreover, one can formalize the proof of this theorem within PRA, which means that $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ is equiconsitent with PRA over PRA.

Now, we overview the idea of the proof. The first step to this theorem is the indicator argument with the density notion introduced by Kirby and Paris [17, 20]

Definition 2.1 (RCA_0)**.**

- A finite set $X \subseteq \mathbb{N}$ is said to be 0-*dense* if $|X| > \min X$.
- A finite set X is said to be $m+1$-*dense* if for any $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is m-dense and P-homogeneous.

Note that "X is m-dense" can be expressed by a Σ^0_0-formula. Let $m\mathrm{PH}^2_2$ be the assertion "for any infinite set $X \subseteq \mathbb{N}$, there exists a finite set $F \subseteq X$ such that X is m-dense." The following theorem is a generalization of the theorem by Bovykin/Weiermann in [5].

Theorem 2.2. $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathsf{RCA}_0 + \{m\mathrm{PH}^2_2 \mid m \in \omega\}$.

Thus, what we need for the main theorem is proving $m\mathrm{PH}^2_2$ within RCA_0 for any $m \in \omega$. For this, we will decompose the density notion by α-largeness notion with ordinals $\alpha < \omega^\omega$. (Here, we use the symbols $\omega, \omega^2, \dots$ for the internal ordinals.)

Definition 2.2 (RCA_0, see [11] for the general definition)**.** Let $\alpha < \omega^\omega$.

- If $\alpha = 0$, then any set is said to be α-large.
- If $\alpha = \beta + 1$, then X is said to be α-large if $X \setminus \{\min X\}$ is β-large.
- If $\alpha = \beta + \omega^{n+1}$, then X is said to be α-large if $X \setminus \{\min X\}$ is $(\beta + \omega^n \cdot \min X)$-large.

Now we will work on finite combinatorics for Ramsey's theorem based on α-largeness notion. For a given $n \in \omega$, we want to find large enough $m \in \omega$ so that for any ω^m-large set $X \subseteq \mathbb{N}$ and for any coloring $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is P-homogeneous and ω^n-large. For this, the key notions are "transitivity" and "grouping".

Definition 2.3 (RCA_0)**.** Let $\alpha, \beta < \omega^\omega$. Let $X \subseteq \mathbb{N}$ and let $P : [X]^2 \to 2$.

- A set $Y \subseteq X$ is said to be *transitive for* P if for any $x, y, z \in Y$ such that $x < y < z$, $P(x, y) = P(y, z) \to P(x, y) = P(x, z)$. If X is transitive for P, then P is said to be a transitive coloring on X.
- A sequence of finite sets $\langle F_i \subseteq X \mid i < l\rangle$ is said to be an (α, β)-*grouping for* P if
  - $\forall i < j < l\, F_i < F_j$,

- $\forall i < l\, F_i$ is α-large,
- for any $H \subseteq_{\text{fin}} \mathbb{N}$, if $H \cap F_i \neq \emptyset$ for any $i < l$, then H is β-large, and,
- $\forall i < j < l\, \exists c < 2\, \forall x \in F_i, \forall y \in F_j\, P(x, y) = c$.

By transitivity, one can decompose the construction of a homogeneous set into two parts, *i.e.*, first find a large enough transitive subset for a given coloring, and then find a homogeneous set for transitive coloring. The idea of this decomposition is essentially due to Bovykin/Weiermann[5] and Hirschfeldt/Shore[13]. In fact, finding a large homogeneous set for a transitive coloring is much easier than the general case since two homogeneous set can be combined easily by transitivity. On the other hand, constructing a large enough transitive set for a given coloring is harder. For this, we use the idea of grouping. A grouping for P is a family of finite sets such that for any pair of sets from the family, the color between them is fixed. If a family of transitive set forms a grouping, then one can combine them as follows: if a family of finite sets $\langle F_i \subseteq X \mid i < l\rangle$ is a grouping for $P : [X]^2 \to 2$ such that each of F_i is transitive for P and there is a unified color $c < 2$ such that the color between F_i and F_j is c for any $i < j < l$, then the union $\bigcup_{i<l} F_i$ is transitive for P. By these considerations, we have the following combinatorics.

Lemma 2.3 ($\mathsf{RCA_0}$)**.** *Let $n \in \omega$. Let $X \subseteq_{\text{fin}} \mathbb{N}$ and $\min X > 3$. Then we have the following.*

1. *Ketonen/Solovey[16, Section 6], see also Pelupessy[22]: if X is ω^{n+4}-large, then any coloring $P : [X]^2 \to n$ has an ω-large homogeneous set.*

2. *If X is ω^{2n+6}-large, then any transitive coloring $P : [X]^2 \to 2$ has an ω^n-large homogeneous set.*

3. *$\langle F_i \subseteq X \mid i < l\rangle$ is a (ω^n, ω)-grouping for $P : [X]^2 \to 2$ such that each of F_i is transitive for P and there is a unified color $c < 2$ such that the color between F_i and F_j is c for any $i < j < l$, then the union $\bigcup_{i<l} F_i$ is transitive for P which is ω^{n+1}-large.*

The last piece of the proof is the bound for grouping.

Theorem 2.4. *For any $n, k \in \omega$, there exists $m \in \omega$ such that $\mathsf{RCA_0}$ proves the following:*

> *if $X \subseteq_{\text{fin}} \mathbb{N}$ is ω^m-large and $\min X > 3$, then, for any coloring $P : [X]^2 \to 2$, there exists an (ω^n, ω^k)-grouping for P.*

In [21], this theorem is proved by considering the infinite version of grouping. Indeed, the existence of a large enough finite set which admits finite grouping for any coloring is an easy consequence of the infinite grouping principle, and the infinite grouping principle is

$\tilde{\Pi}^0_3$-conservative over RCA_0, which is shown by a variant of Mathias forcing introduced by Cholak/Jockusch/Slaman[6] and the resplendency argument by Barwise/Schlipf[2]. Recently, the theorem is reproved with a more direct method by Kołodziejczyk, Wong and the author.

Proof of Theorem 2.1. By Theorem 2.2, we only need to show that RCA_0 proves that any infinite set contains m-dense finite set for each $m \in \omega$.

In what follows, we only consider finite sets with their minimum greater than 3. We first show by induction that for any $n \in \omega$, there exists $m \in \omega$ such that RCA_0 proves that if a finite set $X \subseteq \mathbb{N}$ is ω^m-large, then any coloring on X has an ω^n-large transitive set. For the case $n = 1$, $m = 6$ is enough by Lemma 2.3.1. Assume now $n > 1$ and any coloring on an ω^{m_0}-large finite set has an ω^{n-1}-large transitive set. By Theorem 2.4, take $m \in \omega$ so that RCA_0 proves any coloring on an ω^m-large finite set has an (ω^{m_0}, ω^6)-grouping. Let $X \subseteq \mathbb{N}$ be ω^m-large, P be a coloring on X, and $\langle F_i \subseteq X \mid i < l \rangle$ be an (ω^{m_0}, ω^6)-grouping for P. Since each F_i is ω^{m_0}-large, there exists $H_i \subseteq F_i$ such that H_i is an ω^{m-1}-large transitive set for P. On the other hand, $\{\max F_i \mid i < l\}$ is ω^6-large, thus, there exists $\tilde{H} \subseteq \{\max F_i \mid i < l\}$ such that $\tilde{H}$ is ω-large and P is constant on $[\tilde{H}]^2$ by Lemma 2.3.1. Then, by Lemma 2.3.3, $H = \bigcup\{H_i \mid i < l, \max F_i \in \tilde{H}\}$ is an ω^n-large transitive set for P.

Now we see that for any $n \in \omega$, there exists $m \in \omega$ such that RCA_0 proves that if a finite set $X \subseteq \mathbb{N}$ is ω^m-large, then any coloring on X has an ω^n-large homogeneous set. This is an easy consequence of the above claim and Lemma 2.3.2. Thus, by induction, for any $n \in \omega$, there exists $m \in \omega$ such that RCA_0 proves that any ω^m-large finite set is n-dense. Finally, one can easily show that any infinite set contains ω^m-large finite subset for each $m \in \omega$ within RCA_0. □

3 The proof-theoretic strength of RT^2

In this section, we see the proof-theoretic strength of Ramsey's theorem for pairs and finitely many colors. Here, we write RT^2 for $\forall k \mathrm{RT}^2_k$. The full version of the proof for the following theorem will be available in [25].

Theorem 3.1 (Slaman/Yokoyama). $\mathsf{WKL}_0 + \mathrm{RT}^2$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_3$.

Since $\mathsf{RCA}_0 + \mathrm{RT}^2$ implies $\mathrm{B}\Sigma^0_3$, $\mathrm{B}\Sigma^0_3$ is the exact Π^1_1-part of $\mathsf{WKL}_0 + \mathrm{RT}^2$. Note that $\mathrm{B}\Sigma^0_3$ is Π^0_4-conservative over $\mathrm{I}\Sigma^0_2$. Thus, the proof-theoretic strength of $\mathsf{WKL}_0 + \mathrm{RT}^2$ is the same as $\mathrm{I}\Sigma^0_2$. In addition, the proof of this theorem is again formalizable within PRA, and $\mathsf{WKL}_0 + \mathrm{RT}^2$ is equiconsitent with $\mathrm{I}\Sigma^0_2$ over PRA.

The first step of the proof is the standard decomposition of RT^2 by the cohesiveness principle.

Theorem 3.2 (Cholak/Jockusch/Slaman[6]). *Over* RCA_0*,* RT^2 *is equivalent to* COH *plus* D^2*, where,*

- COH*: for any sequence of sets* $\langle R_n \subseteq \mathbb{N} \mid n \in \mathbb{N}\rangle$*, there exists an infinite set* $X \subseteq \mathbb{N}$ *such that* $\forall n(X \subseteq^* R_n \vee X \subseteq^* R_n^c)$*.*
- D^2*: for every* Δ^0_2*-partition* $\bigsqcup_{i<k} \mathcal{A}_i = \mathbb{N}$*, there exists an infinite set* $X \subseteq \mathbb{N}$ *such that* $X \subseteq \mathcal{A}_i$ *for some* $i < k$*.*

For Π^1_2-theories, two Π^1_1-conservative extensions can be amalgamated, *i.e.*, for given Π^1_2-theories T_0, T_1, T_2, if T_1 and T_2 are Π^1_1-conservative extensions of T_0, then $T_1 + T_2$ is also Π^1_1-conservative over T_0 (see [27]). Thus, we only need to check the conservation for WKL_0, COH and D^2 independently. A general conservation theorem for WKL_0 and COH over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_n$ are calibrated by Hájek[10] and Belanger[3], respectively.

Theorem 3.3 (Hájek, Belanger). $\mathsf{WKL}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_3$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_3$*.*

To obtain a conservation result for D^2, we will use the basis theorem for RT^2 from the computability theoretic view point.

Theorem 3.4 (Cholak/Jockusch/Slaman[6]). *For every* Δ^0_2*-partition* $\bigsqcup_{i<k} \mathcal{A}_i = \omega$*, there exists an infinite low*$_2$ *set* $X \subseteq \omega$ *such that* $X \subseteq \mathcal{A}_i$ *for some* $i < k$*.*

Here, a set $X \subseteq \omega$ is said to be low$_2$ if $X'' = \mathbf{0}''$. If X is low$_2$, then Σ^0_3 predicate relative to X is just Σ^0_3, thus X preserves $\mathrm{B}\Sigma^0_3$. Therefore, if the above theorem is formalizable within $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_3$, one can obtain a definable solution for each instance of D^2 which preserves $\mathrm{B}\Sigma^0_3$. This is actually possible, but not directly. Here, we will work within a nonstandard model $(M, S) \models \mathrm{B}\Sigma^0_3$, and consider a Δ^0_2-partition $\bigsqcup_{i<k} \mathcal{A}_i = M$ for some $k \in M$.

The first obstruction is that to construct a low$_2$ set, we essentially use $\mathbf{0}''$-primitive recursion, which requires $\mathrm{I}\Sigma^0_3$, but we only have $\mathrm{B}\Sigma^0_3$. To prove Theorem 3.4, one constructs an approximation of a solution $G_0 \subseteq G_1 \subseteq \dots$, and at each stage, decides one Σ^0_2-formula $\psi_e(G)$ by using the idea of Mathias forcing. However, because of the lack of $\mathrm{I}\Sigma^0_3$ in M, the construction stages may not cover the whole M, *i.e.*, $\{j \mid G_j \text{ exists}\}$ would form a proper Σ^0_3-cut of M. To overcome this situation, we can use Shore blocking argument, namely, we will decide finitely many Σ^0_2-formulas up to the use of the previous stage. Then, one can decide all Σ^0_2-formulas before the construction ends.

Another obstruction is an essential use of Σ^0_3-least number principle. In the original construction, one would first try constructing the solution on color 0, and if it fails, then try color 1 with using the information from the previous failure, and repeat this process. However, without $\mathrm{I}\Sigma^0_3$, one cannot repeat this for arbitrary many colors until the construction works since the number of color may be nonstandard. Thus, we have to construct possible solutions for all colors simultaneously. Then, $\mathrm{B}\Sigma^0_3$ is just enough to guarantee that the construction works for at least one color.

Formalizing these ideas, we have the following.

Theorem 3.5. *For any $(M;X) \models \mathrm{B}\Sigma^0_3$ and for every Δ^X_2-partition $\bigsqcup_{i<k} \mathcal{A}_i = M$, there exists an unbounded set $G \subseteq M$ which is Δ^X_3-definable in $(M;X)$ such that $G \subseteq \mathcal{A}_i$ for some $i < k$, and $(M;X,G) \models \mathrm{B}\Sigma^0_3$.*

Now, starting from a model $(M;X) \models \mathrm{B}\Sigma^0_3$, one can obtain $S \subseteq \mathcal{P}(M)$ with $X \in S$ such that $(M,S) \models \mathsf{RCA}_0 + \mathrm{D}^2 + \mathrm{B}\Sigma^0_3$ by using the above theorem repeatedly. Thus, we have the following.

Corollary 3.6. $\mathsf{RCA}_0 + \mathrm{D}^2$ *is a Π^1_1-conservative extension of* $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_3$.

Therefore, by the amalgamation of the conservation theorem mentioned avobe, we have Theorem 3.1.

4 Further studies

About the proof-theoretic/first-order strength of Ramsey's theorem for pairs, there are several more important questions to be considered.

4.1 The first-order part of RT^2_2

By Theorem 3.1, we already know that the first-order part of $\mathsf{WKL}_0 + \mathrm{RT}^2$ is $\mathrm{B}\Sigma^0_3$, but we still don't know what the first-order part of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ is.

Question 4.1. Is $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$?

Since $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ implies $\mathrm{B}\Sigma^0_2$, $\mathrm{B}\Sigma^0_2$ is the weakest possible system which may be the first-order part of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$. To prove Π^1_1-conservation, we usually consider the following version of ω-extension property.

Question 4.2. For given $(M,S) \models \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$ and $X \in S$, is there $\bar{S} \subseteq \mathcal{P}(M)$ such that $X \in \bar{S}$ and $(M,\bar{S}) \models \mathsf{WKL}_0 + \mathrm{RT}^2_2$?

One may assume that (M,S) is a countable recursively saturated model. Unfortunately, our proof of Theorem 2.1 does not provide any information about the possibility of the existence of such extension. On the other hand, one can generalize Theorem 2.2 and obtain a characterization for the $\tilde{\Pi}^0_4$-part of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$.

Definition 4.3 ($\mathsf{RCA_0}$)**.** Let $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{x\to\infty} f(x) = \infty$. Then, we define the notion of *f-m-density* as follows.

- A finite set X is said to be f-0-dense if $|X| > \min X$.
- A finite set X is said to be f-$m+1$-dense if for any coloring $P : [X]^2 \to 2$, there exists a P-homogeneous set $Y \subseteq X$ such that Y is f-m-dense and for any $x \in [\min Y, \max Y]$, $f(x) > \min X$.

As same as the usual density notion, "X is f-m-dense" can be expressed by a Σ^0_0 formula. Let $m\mathrm{PH}^{2+}_2$ be the assertion "for any $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{x\to\infty} f(x) = \infty$, and for any infinite set $X \subseteq \mathbb{N}$, there exists a finite set $F \subseteq X$ such that F is f-m-dense." Then we have a modification of Theorem 2.2 as follows.

Theorem 4.1. $\mathsf{WKL_0} + \mathrm{RT}^2_2$ *is a $\tilde{\Pi}^0_4$-conservative extension of* $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2 + \{m\mathrm{PH}^{2+}_2 \mid m \in \omega\}$.

Question 4.4. Is $m\mathrm{PH}^{2+}_2$ provable within $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2$ for any $m \in \omega$?

If the answer is positive, then we know that $\mathsf{WKL_0} + \mathrm{RT}^2_2$ is $\tilde{\Pi}^0_4$-conservative over $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2$.

4.2 Feasibility of the conservation results

Our conservation results are proved by model theoretic arguments. Unfortunately, that doesn't mean any feasibility of the conservation. For example, if we have a proof for a $\tilde{\Pi}^0_3$-sentence ψ from $\mathsf{WKL_0} + \mathrm{RT}^2_2$, then can we find a proof for ψ from $\mathsf{RCA_0}$ in a feasible way? Formally, we can ask the following.

Question 4.5. Is there a polynomial proof transformation for the $\tilde{\Pi}^0_3$-conservation between $\mathsf{RCA_0}$ and $\mathsf{WKL_0} + \mathrm{RT}^2_2$?

Question 4.6. Is there a polynomial proof transformation for the Π^1_1-conservation between $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$ and $\mathsf{WKL_0} + \mathrm{RT}^2$?

For the latter case, it is actually not so difficult to find a polynomial proof transformation. By the proof of Theorem 3.5, there is a canonical way to construct a Δ^0_3-definable solution for RT^2 which preserves $\mathrm{B}\Sigma^0_3$ within $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$. Thus, one can always use the solution for RT^2 within $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$ as if RT^2 is available, and WKL is also available within $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$ in a similar way (see [10]). This idea provides a direct interpretation of RT^2 within $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_3$.

For Question 4.5, the situation is more complicated. Our proof of Theorem 2.1 depends on the indicator argument, which essentially uses a nonstandard model and its initial segment which is not definable in the ground model, but in general, the use of nonstandard models may bring some conservation result with a super-exponential speed-up. Recently, Kołodziejczyk, Wong and the author studied this question and obtained a reformulation of the indicator argument by means of forcing. Generally speaking, if a model construction for a conservation theorem is provided by forcing, then one would often obtain a polynomial proof transformation as in Avigad[1]. In our case, a canonical polynomial proof transformation for the conservation between RCA_0 and $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ is available by a combination of forcing for the indicator argument plus quantitative proof for Theorem 2.4. Consequently, feasible versions of the conservation results are available in both cases.

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