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Master's Research Project Report

A study on relationships between some subrecursive function classes and complexity classes

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Abstract

In computer science, various classes of computational complexity have been studied, as represented by the class $\mathcal{F}$PTIME of polynomial time computable functions, and there are many unsolved problems such as the P vs NP problem. On the other hand, in computability theory, classes of subrecursive functions has been studied for a long time. Although many complexity classes are contained in the class of elementary functions $\mathcal{E}$, little research has been done on relationship between classes of subrecursive functions smaller than the class $\mathcal{E}$ and complexity classes. For example, the relationship between the class $\mathcal{E}^2$ or the class $\mathcal{M}^2$ and the class $\mathcal{F}$PTIME of polynomial time computable functions is not known at all, where the class $\mathcal{E}^2$ is the second class in the Grzegorczyk hierarchy using bounded recursion, and the class $\mathcal{M}^2$ is the second class in the hierarchy obtained by replacing bounded recursion with bounded minimisation.

In this thesis, we try to elucidate the relationship between the second class $\mathcal{E}^2$ in the Grzegorczyk hierarchy and the class $\mathcal{F}$PTIME of polynomial time computable functions. We define a class of functions $\mathcal{E}^{2+}$. The class $\mathcal{E}^{2+}$ is an extension of the second class $\mathcal{E}^2$ in the Grzegorczyk hierarchy to which course-of-values recursion whose values are bounded by 1 is added. And we also define simultaneiously and recursively two classes of functions $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$. Functions in the class $\mathcal{C}_{\mathbb{N}}$ take two data types as their arguments, natural numbers and binary strings, and their values are natural numbers, and functions in the class $\mathcal{C}_{\mathbb{W}}$ take the two data types as their arguments, and their values are binary strings. Then, we associate functions in $\mathcal{E}^{2+}$ with functions in $\mathcal{C}_{\mathbb{N}}$, and also associate functions in $\mathcal{F}$PTIME with functions in $\mathcal{C}_{\mathbb{W}}$. Using these relations, with respect to their set classes $\mathcal{E}^{2+}_*$ and PTIME, we show that PTIME is contained in $\mathcal{E}^{2+}_*$.

Furthermore, we try to elucidate the relationship between the second class $\mathcal{M}^2$ in the hierarchy of bounded minimisation and the function class $\mathcal{F}$LH of the logtime hierarchy. We define simultaneiously and recursively two classes of functions $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$. Functions in the class $\mathcal{D}_{\mathbb{N}}$ take two data types as their arguments, natural numbers and binary strings, and their values are natural numbers, and functions in the class $\mathcal{D}_{\mathbb{W}}$ take the two data types as their arguments, and their values are binary strings. Then, we associate functions in $\mathcal{M}^2$ with functions in $\mathcal{D}_{\mathbb{N}}$, and also associate functions in $\mathcal{F}$LH with functions in $\mathcal{D}_{\mathbb{W}}$. Using these relations, with respect to their set classes $\mathcal{M}^2_*$ and LH, we show that LH is contained in $\mathcal{M}^2_*$.

Keywords: Subrecursion; Grzegorczyk hierarchy; Implicit computational complexity; Complexity classes; Polynomial-time functions; Logtime hierarchy; Function algebra.

Contents

1 Introduction

1.1 Background

In computer science, various classes of computational complexity have been studied, as represented by the class of polynomial time computable functions, and there are many unsolved problems such as the P vs NP problem. On the other hand, in computability theory, classes of subrecursive functions has been studied for a long time (Grzegorczyk [4], Rose [7]). Although many complexity classes are contained in the class of elementary functions $\mathcal{E}$, little research has been done on relationship between classes of subrecursive functions smaller than the class $\mathcal{E}$ and complexity classes. For example, the relationship between the class $\mathcal{E}^2$ or the class $\mathcal{M}^2$ and the class $\mathcal{F}$PTIME of polynomial time computable functions is not known at all (Rose [7]), where the class $\mathcal{E}^2$ is the second class in the Grzegorczyk hierarchy using bounded recursion, and the class $\mathcal{M}^2$ is the second class in the hierarchy obtained by replacing bounded recursion with bounded minimisation (Grzegorczyk [4]).

1.2 Research purpose

The first purpose of this research is to elucidate the relationship between the second class $\mathcal{E}^2$ in the Grzegorczyk hierarchy using bounded recursion and the class $\mathcal{F}$PTIME of polynomial time computable functions. The second purpose of this research is to elucidate the relationship between the second class $\mathcal{M}^2$ in the hierarchy of bounded minimisation and the function class $\mathcal{F}$LH of the logtime hierarchy.

It is known that LH $\subsetneq$ PTIME, hence $\mathcal{F}$LH $\subsetneq$ $\mathcal{F}$PTIME. If we can take the correspondence between $\mathcal{E}^2$ and $\mathcal{F}$PTIME and the correspondence between $\mathcal{M}^2$ and $\mathcal{F}$LH, we may be able to solve the unsolved problem of whether the class $\mathcal{M}^2$ is properly contained in $\mathcal{E}^2$ or not (Rose [7]) by using the relationship $\mathcal{F}$LH $\subsetneq$ $\mathcal{F}$PTIME.

1.3 Methodology

In this research, we use a function algebra which does not depend on any specific computation model to represent a class of functions. If $\mathcal{X}$ is a set of initial functions and OP is a collection of recursive operators, then a function algebra $[\mathcal{X};\ \text{OP}]$ is the smallest set of functions containing $\mathcal{X}$ and closed under the operations of OP (Clote [2]).

Many classes of subecursive functions, including classes in the Grzegorczyk hierarchy, are defined as function algebras. It is also known that typical complexity classes can be characterized as function algebras by Clote and the like (Clote [2]). In particular, the characterizations of the class $\mathcal{F}$PTIME of polynomial time computable functions by function algebras include Cobham [3], Bellantoni and Cook [1], Ishihara [5] and the like.

On the other hand, previous researches on subrecursive functions have discussed functions on natural numbers, and previous researches on complexity classes have discussed functions on natural numbers or binary strings. These previous researches have assumed that natural numbers can be freely converted to binary strings and binary strings can be freely converted to natural numbers. This assumption holds for the class of elementary functions $\mathcal{E}$, which is equivalent to the third class $\mathcal{E}^3$ in the Grzegorczyk hierarchy and the third class $\mathcal{M}^3$ in the hierarchy of bounded minimisation, since it contains the exponential function. However, when we discuss smaller classes than $\mathcal{E}$, the validity of this assumption remains uncertain.

In this research, we deal with classes of functions defined on two data types, natural numbers and binary strings. This allows us to discuss without the assumption above and makes it possible to verify the validity of the assumption. We define simultaneiously and recursively two classes of functions. Functions in the one class take two data types as their arguments, natural numbers and binary strings, and their values are natural numbers, and functions in the other class take the two data types as their arguments, and their values are binary strings. We expect that the class of functions whose function values are natural numbers correspond to some class of subrecursive functions on natural numbers, and the class of functions whose function values are binary strings correspond to some complexity class on binary strings.

1.4 Organization

After this chapter, in chapter 2, we introduce a basic mathematical framework and also introduce function classes, comlexity classes and their related concepts. In chapter 3, we extend $\mathcal{E}^2$ to define $\mathcal{E}^{2+}$, and also define two intermediate classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$. Then we associate $\mathcal{E}^{2+}$ with $\mathcal{F}$PTIME via $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$. Consequently, we show that PTIME is contained in $\mathcal{E}^{2+}_*$. In chapter 4, we also define two intermediate classes $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$. Then we associate $\mathcal{M}^2$ with $\mathcal{F}$LH via $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$. Consequently, we show that LH is contained in $\mathcal{M}^2_*$. In Chapter 5, we state conclusions and directions for further research.

2 Preliminaries

In this chapter, we introduce some basic concepts and known results used in this thesis.

2.1 Definitions of $\mathbb{N}$ and $\mathbb{W}$, and some basic functions

To begin with, we define the set of natural numbers and the set of binary strings, and define some basic functions defined on them. Each of these functions gives a mathematical meaning caused by the function. Note that these functions do not depend on any specific function class.

Definition 2.1. Let $\mathbb{N}$ be the set of natural numbers inductively defined by

$$\frac{}{0}, \qquad \frac{x}{\mathrm{S}x},$$

where $\mathrm{S}x$ is a successor of x.
Let $\mathbb{W}$ be the set of finite binary strings inductively defined by

$$\frac{}{\varepsilon}, \qquad \frac{a}{a0}, \qquad \frac{a}{a1},$$

where ε is an empty string.

Definition 2.2. Let $\mathrm{prd}, +, \dot{-}, \cdot$ and 2^x be the predecessor function, the addition, the cut-off subtraction, the multiplication, and the exponential function, respectively, defined by

$$\begin{cases} \mathrm{prd}(0) = 0, \\ \mathrm{prd}(\mathrm{S}x) = x; \end{cases} \qquad \begin{cases} x + 0 = x, \\ x + \mathrm{S}y = \mathrm{S}(x + y); \end{cases}$$

$$\begin{cases} x \dot{-} 0 = x, \\ x \dot{-} \mathrm{S}y = \mathrm{prd}(x \dot{-} y); \end{cases} \qquad \begin{cases} x \cdot 0 = 0, \\ x \cdot \mathrm{S}y = (x \cdot y) + x; \end{cases}$$

$$\begin{cases} 2^0 = 1, \\ 2^{\mathrm{S}x} = 2 \cdot 2^x. \end{cases}$$

Definition 2.3. Let $I_i^m \in \mathbb{N}^m \to \mathbb{N}$ $(0 \le i < m)$ be the projection function,[1] defined by

$$I_i^m(x_0, \ldots, x_{m-1}) = x_i.$$

Let $s_0 \in \mathbb{N} \to \mathbb{N}$ and $s_1 \in \mathbb{N} \to \mathbb{N}$ be the binary successor functions, respectively defined by

$$s_0(x) = 2 \cdot x, \qquad s_1(x) = 2 \cdot x + 1.$$

[1] If f is a function from a set A to a set B, we use the notation $f \in A \to B$.

Let $|\cdot| \in \mathbb{N} \to \mathbb{N}$ be the length function of natural numbers which computes the length of a natural number represented in binary, defined by

$$|x| = \lceil \log(x+1) \rceil.$$

Using the same symbol, let $|\cdot| \in \mathbb{W} \to \mathbb{N}$ be the length function of binary strings, defined by

$$|\varepsilon| = 0, \qquad |a0| = \mathrm{S}(|a|), \qquad |a1| = \mathrm{S}(|a|).$$

Let $\# \in \mathbb{N}^2 \to \mathbb{N}$ be the smash function, defined by

$$x \# y = 2^{|x| \cdot |y|}.$$

Let $\mathrm{MOD2} \in \mathbb{N} \to \mathbb{N}$ be the modulo 2 function, defined by

$$\mathrm{MOD2}(x) = x \dot{-} 2 \cdot \lfloor x/2 \rfloor.$$

Let $\mathrm{msp} \in \mathbb{N}^2 \to \mathbb{N}$ be the most significant part function, defined by

$$\mathrm{msp}(x, y) = \lfloor x/2^{|y|} \rfloor.$$

Let $\mathrm{bin} \in \mathbb{N} \to \mathbb{W}$ be the binary representation of a natural number, defined by

$$\mathrm{bin}(0) = \varepsilon, \qquad \mathrm{bin}(\mathrm{S}x) = \begin{cases} \mathrm{bin}(\lfloor \mathrm{S}x/2 \rfloor)0 & \text{if } \mathrm{S}x \bmod 2 = 0, \\ \mathrm{bin}(\lfloor \mathrm{S}x/2 \rfloor)1 & \text{if } \mathrm{S}x \bmod 2 = 1. \end{cases}$$

Let $\mathrm{BIT} \in \mathbb{N} \times \mathbb{W} \to \mathbb{N}$ be the bit function, defined by

$$\begin{aligned} \mathrm{BIT}(0, \varepsilon) &= 0, & \mathrm{BIT}(\mathrm{S}x, \varepsilon) &= 0, \\ \mathrm{BIT}(0, a0) &= 0, & \mathrm{BIT}(0, a1) &= 1, \\ \mathrm{BIT}(\mathrm{S}x, a0) &= \mathrm{BIT}(x, a), & \mathrm{BIT}(\mathrm{S}x, a1) &= \mathrm{BIT}(x, a). \end{aligned}$$

Using the same symbol, Let $\mathrm{BIT} \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the bit function, defined by

$$\mathrm{BIT}(x, y) = \mathrm{MOD2}(\lfloor y/2^x \rfloor).$$

Let $\mathrm{bit} \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the (lowercase) bit function, defined by

$$\mathrm{bit}(x, y) = \mathrm{MOD2}(\lfloor y/2^{|x|} \rfloor).$$

Definition 2.4. Let $\mathrm{sg}, \overline{\mathrm{sg}}, \max, \min, \mathrm{cond}, \chi_=, \chi_\le$ and $\chi_<$ be the signum function, the inverse signum function, the maximum function, the minimum function, the conditional function, the characteristic function of =, the characteristic function of ≤ and the characteristic function of <, respectively, defined by

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad \overline{\mathrm{sg}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise;} \end{cases}$$
$$\max(x,y) = \begin{cases} x & \text{if } x \ge y, \\ y & \text{otherwise;} \end{cases} \qquad \min(x,y) = \begin{cases} x & \text{if } x \le y, \\ y & \text{otherwise;} \end{cases}$$
$$\mathrm{cond}(x,y,z) = \begin{cases} y & \text{if } x = 0, \\ z & \text{otherwise.} \end{cases} \qquad \chi_=(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise;} \end{cases}$$
$$\chi_\le(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{otherwise;} \end{cases} \qquad \chi_<(x,y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5. Let $\chi_\neg, \chi_\wedge, \chi_\vee, \chi_\to$ and $\chi_\oplus$ be the characteristic functions of $\neg, \wedge, \vee, \to$ and $\oplus$, respectively, defined by

$$\chi_\neg(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise;} \end{cases} \qquad \chi_\wedge(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise;} \end{cases}$$
$$\chi_\vee(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ or } y > 0, \\ 0 & \text{otherwise;} \end{cases} \qquad \chi_\to(x,y) = \begin{cases} 0 & \text{if } x > 0 \text{ and } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$\chi_\oplus(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y = 0, \text{ or } x = 0 \text{ and } y > 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Function classes, complexity classes and related concepts

We mainly deal with classes of functions. We sometimes call a class of functions a function class. We use a function algebra to represent a function class. According to Clote [2], we define a function algebra as follows:

Definition 2.6 (Clote [2])**.** An *operator* (also called an operation) is a mapping from functions to functions. If $\mathcal{X}$ is a set of functions (called *initial functions*) and OP is a collection of operators, then $[\mathcal{X};\ \text{OP}]$ denotes the smallest set of functions containing $\mathcal{X}$ and closed under the operations of OP. The set $[\mathcal{X};\ \text{OP}]$ is called a *function algebra*.

Many subrecursive function classes are defined as function algebras. We define some operators to define function classes of our interests.

Definition 2.7. 1. The function f is defined by *composition* (COMP) from functions $h, g_0, \ldots, g_{L-1}$ if

$$f(\vec{x}) = h(g_0(\vec{x}), \ldots, g_{L-1}(\vec{x})).$$

2. The function f is defined by *primitive recursion* (PR) from functions g, h if

$$f(0, \vec{y}) = g(\vec{y}),$$
$$f(x+1, \vec{y}) = h(x, \vec{y}, f(x, \vec{y})).$$

3. The function f is defined by *bounded recursion* (BR) from functions g, h, e if

$$f(0, \vec{y}) = g(\vec{y}),$$
$$f(x+1, \vec{y}) = h(x, \vec{y}, f(x, \vec{y})),$$

provided that $f(x, \vec{y}) \leq e(x, \vec{y})$ for all $x, \vec{y}$.

4. The function f is defined by *bounded minimisation* (BMIN) from a function g if

$$f(x, \vec{y}) = \begin{cases} \text{the least } z \leq x \text{ such that } g(z, \vec{y}) = 0 & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases}$$

5. The function f is defined by *bounded summation* (BSUM) from functions g, e if

$$f(x, \vec{y}) = \sum_{i=0}^{x} g(i, \vec{y}).$$

provided that $f(x, \vec{y}) \leq e(x, \vec{y})$ for all $x, \vec{y}$.

6. The function f is defined by *bounded product* (BPROD) from functions g, e if

$$f(x, \vec{y}) = \prod_{i=0}^{x} g(i, \vec{y}).$$

provided that $f(x, \vec{y}) \leq e(x, \vec{y})$ for all $x, \vec{y}$.

Definition 2.8. The class of primitive recursive functions is defined by

$$\mathcal{P}R = [\mathtt{0}, \mathtt{I}, \mathtt{S}; \mathrm{COMP}, \mathrm{PR}],$$

where $\mathtt{0}$ is constant 0, $\mathtt{I}$ is projection function, $\mathtt{S}$ is successor function, COMP is composition and PR is primitive recursion.

Definition 2.9. The class of elementary functions is defined by

$$\mathcal{E} = [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \dot{-}; \mathrm{COMP}, \mathrm{BSUM}, \mathrm{BPROD}],$$

where $\mathtt{0}$ is constant 0, $\mathtt{I}$ is projection function, $\mathtt{S}$ is successor function, $+$ is addtion, $\dot{-}$ is cut-off subtraction, COMP is composition, BSUM is bounded summation and BPROD is bounded product.

The elementary functions were introduced by Kalmár(1943) and by Csillag(1947). $\mathcal{E}$ is a basic class in the sense that it contains most of the useful number theoretic and mathematical functions. In most recursion theroy texts and papers where primitive recursive functions are used, it is in fact sufficient to use elementary functinos (Rose [7]).

Grzegorczyk [4] investigated a hierarchy of subclasses $\mathcal{E}^n$ of the class of primitive recursive functions $\mathcal{P}R$, defined as the closure of certain initial functions under composition and bounded recursion.

Definition 2.10. The Grzegorczyk hierarchy is a hierarchy of function classes defined by

$$\begin{aligned}
\mathcal{E}^0 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}; \mathrm{COMP}, \mathrm{BR}],\\
\mathcal{E}^1 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +; \mathrm{COMP}, \mathrm{BR}],\\
\mathcal{E}^2 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \mathrm{COMP}, \mathrm{BR}],\\
\mathcal{E}^3 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times, 2^x; \mathrm{COMP}, \mathrm{BR}],\\
&\vdots
\end{aligned}$$

where $\mathtt{0}$ is constant 0, $\mathtt{I}$ is projection function, $\mathtt{S}$ is successor function, $+$ is addition, $\times$ is multiplication, 2^x is exponetial function, COMP is composition and BR is bounded recursion.

For $n \geq 4$, $\mathcal{E}^n$ contains the initial functions in $\mathcal{E}^{n-1}$ and further contains $f_n(x) = f_{n-1}^{(x)}(1)$, where $f^{(0)}(x) = x$, $f^{(n+1)}(x) = f(f^{(n)}(x))$ and $f_3(x) = 2^x$, and is closed under composition and bounded recursion.

Grzegorczyk [4] also considered a hierarchy of function classes in which bounded recursion is replaced with bounded minimisation.

Definition 2.11. The hierarchy of bounded minimisation is a hierarchy of function classes defined by

$$\begin{aligned}
\mathcal{M}^0 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}; \text{COMP}, \text{BMIN}], \\
\mathcal{M}^1 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +; \text{COMP}, \text{BMIN}], \\
\mathcal{M}^2 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \text{COMP}, \text{BMIN}], \\
\mathcal{M}^3 &= [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times, x^y, 2^x; \text{COMP}, \text{BMIN}], \\
&\vdots
\end{aligned}$$

where $\mathtt{0}$ is constant 0, $\mathtt{I}$ is projection function, $\mathtt{S}$ is successor function, $+$ is addition, $\times$ is multiplication, x^y is x to the y-th power, 2^x is exponetial function, COMP is composition and BMIN is bounded minimisation.

For $n \geq 4$, $\mathcal{M}^n$ contains the initial functions in $\mathcal{M}^{n-1}$ and further contains $f_n(x) = f_{n-1}^{(x)}(1)$, where $f^{(0)}(x) = x$, $f^{(n+1)}(x) = f(f^{(n)}(x))$ and $f_3(x) = 2^x$, and is closed under composition and bounded minimisation.

We study an extension of the class $\mathcal{E}^2$ in the chapter 3, and the class $\mathcal{M}^2$ in the chapter 4.

Inclusion, equality and proper inclusion between classes of functions can be defined if their elements are functions from natural numbers to natural number.

Definition 2.12. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be classes of functions of types $\mathbb{N}^m \to \mathbb{N}$. $\mathcal{F}_1$ is included in $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \subseteqq \mathcal{F}_2$, if for any $f \in \mathcal{F}_1$ there exists $g \in \mathcal{F}_2$ such that $f(\vec{x}) = g(\vec{x})$ for each $\vec{x}$. $\mathcal{F}_1$ is equal to $\mathcal{F}_2$, denoted by $\mathcal{F}_1 = \mathcal{F}_2$, if $\mathcal{F}_1 \subseteqq \mathcal{F}_2$ and $\mathcal{F}_2 \subseteqq \mathcal{F}_1$. $\mathcal{F}_1$ is properly (or strictly) included in $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \subsetneqq \mathcal{F}_2$, if $\mathcal{F}_1 \subseteqq \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$, that is, $\mathcal{F}_1 \subseteqq \mathcal{F}_2$ and $\mathcal{F}_2 \nsubseteqq \mathcal{F}_1$.

We can relate a class of functions to a class of sets, or equivalently, a class of predicates, by means of characteristic functions.

Definition 2.13. For a set s or a predicate p, define their characteristic functions, respectively, by

$$\chi_s(\vec{x}) = \begin{cases} 1 & \vec{x} \in s, \\ 0 & \text{otherwise}, \end{cases} \qquad \chi_p(\vec{x}) = \begin{cases} 1 & p(\vec{x}) \text{ is true}, \\ 0 & \text{otherwise}. \end{cases}$$

Definition 2.14. Let $\mathcal{F}$ be a class of functions. Then the class of sets $\mathcal{F}_*$, or equivalently, the class of predicates $\mathcal{F}_*$, are defined respectively by

$$\chi_s \in \mathcal{F} \iff s \in \mathcal{F}_*, \qquad \chi_p \in \mathcal{F} \iff p \in \mathcal{F}_*.$$

Applying this definition to the classes in the Grzegorczyk hierarchy and the hierarchy of bounded minimisation, we obtain sequences of classes $\mathcal{E}^0_*, \mathcal{E}^1_*, \mathcal{E}^2_*, \mathcal{E}^3_*, \ldots$ and $\mathcal{M}^0_*, \mathcal{M}^1_*, \mathcal{M}^2_*, \mathcal{M}^3_*, \ldots$.

Next, we briefly describe complexity classes of our interests. A complexity class is a class of sets (or languages) which are accepted by some type of Turing machine within specified resource bounds. We assume that the sets accepted by Turing machines are over the alphabet $\{0, 1\}$.

Definition 2.15. (i) The PTIME (or P) is the class of all sets which are accepted by deterministic Turing machine in time $\mathcal{O}(p(n))$ for some polynomial $p(n)$, where n is the length of its input.
(ii) The *logtime hierarchy* LH is the class of all sets which are accepted by alternating Turing mahine with random access in time $\mathcal{O}(\log n)$ at most $\mathcal{O}(1)$ alternations, where n is the length of its input.
(iii) The *linear time hierarchy* LTH is the class of all sets which are accepted by alternating Turing mahine in time $\mathcal{O}(n)$ at most $\mathcal{O}(1)$ alternations, where n is the length of its input.[2]

It is known that LH is properly included in PTIME. LH is clearly included in LTH. It seems that LTH and PTIME are thought to be incomparable.

Note that though a complexity class is a class of sets, a set can be regarded as a predicate, hence a complexity class is regarded as a class of predicates.

In our discussion, we would like to deal mainly with function classes.

When we make Turing machine compute a function, we generally consider Turing machine with an output tape. That is, Turing machine with arguments of the function separated by some symbol on its input tape, computes the function and writes the value of the function on its output tape. If the resource bounds of this Turing machine corresponds to a complexity class $\mathcal{C}$, we can consider the function computed by this Turing machine to be in a function class $\mathcal{FC}$.

However, we adopt a bit different definition of $\mathcal{FC}$ according to Clote [2].

Definition 2.16 (Clote [2])**.** A function $f(\vec{x})$ is *polynomial growth* if

$$|f(\vec{x})| = \mathcal{O}(\max_{0 \le j < n} |x_j|^k) \quad \text{for some } k.$$

The *bitgraph* B_f satisfies $B_f(\vec{x}, i)$ if and only if the i-th bit of $f(\vec{x})$ is 1.
If $\mathcal{C}$ is a complexity class, then $\mathcal{FC}$ is the class of functions of polynomial growth whose bitgraph belongs to $\mathcal{C}$.

[2]For detailed descriptions of Turing machines, see, for example, Clote [2]. For more detailed descriptions of alternating Turing machines, see, for example, Chapter 3 of Balcázar, J. L., Díaz, J., Gabarró, J. *Structural Complexity II.* Springer-Verlag, 1990.

That is, if $\mathcal{C}$ is a complexity class, the function class $\mathcal{FC}$ is defined by

$$f(\vec{x}) \in \mathcal{FC} \iff \begin{cases} \text{(i) } |f(\vec{x})| = \mathcal{O}(\max_{0 \le j < n} |x_j|^k) & \text{for some } k, \\ \text{(ii) } B_f(\vec{x}, i) \in \mathcal{C} & \text{for } 0 \le i < |f(\vec{x})|. \end{cases}$$

By Definition 2.14 and 2.16, we obtain the following lemma:

Lemma 2.17. *Let $\mathcal{C}$ be a complexity class, then*

$$(\mathcal{FC})_* = \mathcal{C}.$$

Proof. For any set s,

$$\begin{aligned} s \in \mathcal{C} &\iff |\chi_s(\vec{x})| \le 1 \text{ and } B_{\chi_s}(\vec{x}, 0) \in \mathcal{C} \\ &\iff \chi_s(\vec{x}) \in \mathcal{FC} \\ &\iff s \in (\mathcal{FC})_*. \end{aligned}$$

□

Applying Definition 2.16 to PTIME, LH and LTH, we define the function classes $\mathcal{F}$PTIME, $\mathcal{F}$LH and $\mathcal{F}$LTH.

$\mathcal{F}$PTIME, $\mathcal{F}$LH and $\mathcal{F}$LTH are known to be represented by some function algebras. We further define some operators.

Definition 2.18. 1. The function f is defined by *bounded recursion on notation* (BRN) from functions g, h_0, h_1, e if

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}), \\ f(s_0(x), \vec{y}) &= h_0(x, \vec{y}, f(x, \vec{y})) \qquad (\text{if } x \ne 0), \\ f(s_1(x), \vec{y}) &= h_1(x, \vec{y}, f(x, \vec{y})) \end{aligned}$$

provided that $f(x, \vec{y}) \le e(x, \vec{y})$ for all $x, \vec{y}$.

2. The function f is defined by *full concatenation recursion on notation* (FCRN) from functions g, h_0, h_1 if

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}), \\ f(s_0(x), \vec{y}) &= s_{h_0(x, \vec{y}, f(x, \vec{y}))}(f(x, \vec{y})) \qquad (\text{if } x \ne 0), \\ f(s_1(x), \vec{y}) &= s_{h_1(x, \vec{y}, f(x, \vec{y}))}(f(x, \vec{y})). \end{aligned}$$

3. The function f is defined by *concatenation recursion on notation* (CRN) from functions g, h_0, h_1 if

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}), \\ f(s_0(x), \vec{y}) &= s_{h_0(x, \vec{y})}(f(x, \vec{y})) \qquad (\text{if } x \ne 0), \\ f(s_1(x), \vec{y}) &= s_{h_1(x, \vec{y})}(f(x, \vec{y})). \end{aligned}$$

Using these operators, $\mathcal{F}$PTIME and $\mathcal{F}$LH can be characterized as follows:

Theorem 2.19 (Cobham [3], see Rose [7], Clote [2])**.**

$$\mathcal{F}\text{PTIME} = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \#; \text{COMP}, \text{BRN}],$$

where $\mathtt{0}$ *is constant* 0*,* $\mathtt{I}$ *is projection function,* $\mathtt{s}_0$ *and* $\mathtt{s}_1$ *are binary successor functions,* $\#$ *is smash function,* COMP *is composition and* BRN *is bounded recursion on notation.*

Theorem 2.20 (Ishihara [5][3])**.**

$$\begin{aligned}\mathcal{F}\text{PTIME} &= [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{MOD2}, \mathtt{msp}, \#; \text{COMP}, \text{FCRN}] \\ &= [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{BIT}, |\cdot|, \#; \text{COMP}, \text{FCRN}],\end{aligned}$$

where $\mathtt{0}$ *is constant* 0*,* $\mathtt{I}$ *is projection function,* $\mathtt{s}_0$ *and* $\mathtt{s}_1$ *are binary successor functions,* $\#$ *is smash function,* $\mathtt{MOD2}$ *is modulo 2 function,* $\mathtt{msp}$ *is most significant part function,* $\mathtt{BIT}$ *is bit function,* $|\cdot|$ *is length function,* COMP *is composition and* FCRN *is full concatenation recursion on notation.*

Theorem 2.21 (Clote [2])**.**

$$\mathcal{F}\text{LH} = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{BIT}, |\cdot|, \#; \text{COMP}, \text{CRN}],$$

where $\mathtt{0}$ *is constant* 0*,* $\mathtt{I}$ *is projection function,* $\mathtt{s}_0$ *and* $\mathtt{s}_1$ *are binary successor functions,* $\mathtt{BIT}$ *is bit function,* $|\cdot|$ *is length function,* $\#$ *is smash function,* COMP *is composition and* CRN *is concatenation recursion on notation.*

Regarding $\mathcal{F}$LTH, the following characterization is known.

Theorem 2.22 (see **Corollary 3.57** in Clote [2])**.**

$$\mathcal{F}\text{LTH} = \mathcal{M}^2, \quad \text{LTH} = \mathcal{M}^2_*.$$

Among the function classes described above, the following relations are known. (Grzegorczyk [4], see Rose [7])

$$\begin{array}{ccccccccccc} & & & & \mathcal{F}\text{PTIME} & & & & & & \\ & & & & \not\supseteq\ \not\subseteq ? & & & & & & \\ \mathcal{E}^0 & \subsetneq & \mathcal{E}^1 & \subsetneq & \mathcal{E}^2 & \subsetneq & \mathcal{E}^3 & = & \mathcal{E} & \subsetneq & \mathcal{PR} \\ \cup\!\!\!\!\!\! \neq & & \cup\!\!\!\!\!\! \neq & & \cup\!\shortparallel & & \shortparallel & & & & \\ \mathcal{M}^0 & \subsetneq & \mathcal{M}^1 & \subsetneq & \mathcal{M}^2 & \subsetneq & \mathcal{M}^3 & & & & \\ & & & & \shortparallel & & & & & & \\ & & \mathcal{F}\text{LH} & \subseteq & \mathcal{F}\text{LTH} & & & & & & \end{array}$$

[3]There is one more characterization of $\mathcal{F}$PTIME in Ishihara [5].

In particular, it is not known whether $\mathcal{M}^2$ is strictly included in $\mathcal{E}^2$ or not. (Grzegorczyk [4], Rose [7])

And the relationship of $\mathcal{F}$PTIME and $\mathcal{E}^2$ is not fully known. $\mathcal{F}$PTIME is not a subclass of $\mathcal{E}^2$ because the function $x^{|y|}$ belongs to $\mathcal{F}$PTIME, but it does not belongs to $\mathcal{E}^2$. And it seems reasonable to assume that $\mathcal{E}^2$ is not a subclass of $\mathcal{F}$PTIME because, for instance, $\mathcal{E}^2$ is closed under summation but it seems most likely that $\mathcal{F}$PTIME is not closed. (Rose [7])

Among the classes of sets described above, the following relations are known (see Rose [7]).

$$
\begin{array}{ccccccccccc}
 & & & & & & & & \text{PTIME} & & \\
 & & & & & & & & \nparallel & & \\
 & & & & \mathcal{E}^0_* & \subseteqq & \mathcal{E}^1_* & \subseteqq & \mathcal{E}^2_* & \subsetneqq & \mathcal{E}^3_* \\
 & & & & \cup\!\shortparallel & & & & & & \\
\mathcal{M}^0_* & = & \mathcal{M}^1_* & \subsetneqq & \mathcal{M}^2_* & & & & & & \\
 & & & & \shortparallel & & & & & & \\
 & & \text{LH} & \subseteqq & \text{LTH} & & & & & &
\end{array}
$$

In particular, though we have $\mathcal{M}^2_* \subseteqq \mathcal{E}^0_* \subseteqq \mathcal{E}^1_* \subseteqq \mathcal{E}^2_*$, it is not known any of these inclusions is proper or not. (Rose [7])

And it is shown that PTIME $\neq \mathcal{E}^2_*$ by Book(1972), but the relationship of $\mathcal{E}^2_*$ and PTIME is not known. (Rose [7])

We present one more relationship between $\mathcal{E}^2$ (and $\mathcal{E}^2_*$) and some complexity class, but we omit its precise definition.

Theorem 2.23 (Ritchie (1963))**.**

$$\mathcal{F}\text{LINSPACE} = \mathcal{E}^2, \quad \text{LINSPACE} = \mathcal{E}^2_*.$$

Regarding $\mathcal{E}^2$, we use the following proposition.

Proposition 2.24 (see the proof of **Theorem 3.36** in Clote [2])**.** *The length function* $|\cdot|$ *and the bit function* $\texttt{BIT}$ *belong to* $\mathcal{E}^2$*.*

Regarding $\mathcal{M}^2$, we often use the following proposition.

Proposition 2.25 (**Corollary 3.54** in Clote [2])**.** *The function algebra* $[0, \texttt{I}, \texttt{s}_0, \texttt{s}_1, |\cdot|, \texttt{BIT}; \text{COMP}, \text{CRN}]$ *is contained in* $\mathcal{M}^2$*.*

2.3 Notational conventions

In principle, we usually use symbols $x, y, z, w, \dots$ for variables representing natural numbers, and use symbols i, j or z for variable representing bit numbers. We use symbols $a, b, c, \dots$ for variables representing binary strings. There is one exception. Though the argument of function $\mathtt{bin}(\cdot)$ is a natural number, we use symbol k for it like $\mathtt{bin}(k)$.

We change symbols for functions according to classes of functions to which they belong. We use symbols $f, g, h, e, \dots$ for functions in $\mathcal{E}^{2+}$ and $\mathcal{M}^2$, use symbols $r, t, u, v, \dots$ for functions in $\mathcal{F}$PTIME and $\mathcal{F}$LH, use symbols $\alpha, \beta, \gamma, \delta, \dots$ for functions in $\mathcal{C}_\mathbb{N}$, $\tilde{\mathcal{C}}$, $\mathcal{D}_\mathbb{N}$ and $\tilde{\mathcal{D}}$, and use symbols $\varphi, \psi, \chi, \dots$ for functions in $\mathcal{C}_\mathbb{W}$ and $\mathcal{D}_\mathbb{W}$.

Functions in $\mathcal{C}_\mathbb{N}, \mathcal{C}_\mathbb{W}, \tilde{\mathcal{C}}, \mathcal{D}_\mathbb{N}, \mathcal{D}_\mathbb{W}$ and $\tilde{\mathcal{D}}$ can take both natural numbers and binary strings as their arguments, and we distinguish between natural numbers and binary strings by placing a semicolon between them. For example, we write $\alpha(x, y; a, b) \in \mathcal{C}_\mathbb{N}$. Note that even if these functions take only ones of natural numbers or binary strings, we place a semicolon. For example, we write $\beta(x, y;) \in \mathcal{D}_\mathbb{N}$ or $\varphi(; a) \in \mathcal{C}_\mathbb{W}$.

We use a vector notations for a plurality of variables. For example, $\vec{x}$ is used for m variables $x_0, x_1, \dots, x_{m-1}$ and $\vec{a}$ is used for n variables $a_0, a_1, \dots, a_{n-1}$;

$$\vec{x} = x_0, x_1, \dots, x_{m-1}, \qquad \vec{a} = a_0, a_1, \dots, a_{n-1}.$$

When we apply a function to a vector variable, its meaning is a bit different from usual. Two functions $\mathtt{bin}$ and $|\cdot|$ are often applied to a vector variable. For example, $\mathtt{bin}(\vec{k})$ means

$$\mathtt{bin}(\vec{k}) = \mathtt{bin}(k_0), \mathtt{bin}(k_1), \dots, \mathtt{bin}(k_{m-1}),$$

which does not mean $\mathtt{bin}(k_0, k_1, \dots, k_{m-1})$. Similarly, $|\vec{a}|$ means

$$|\vec{a}| = |a_0|, |a_1|, \dots, |a_{n-1}|,$$

which does not mean $|a_0, a_1, \dots, a_{n-1}|$.

When we write the numbers of arguments (i.e., arity) of functions, we use lowercase letters ℓ, m, n. For example, we write $\alpha(\vec{x}; \vec{a}) \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$. If a function is defined by composition from some functions, we use uppercase letters L, M, N for an outer function and lowercase letters ℓ, m, n for inner functions. For example, if $\alpha(\vec{x}; \vec{a})$ is defined by

$$\alpha(\vec{x}; \vec{a}) = \gamma(\beta_0(\vec{x}; \vec{a}), \dots, \beta_{M-1}(\vec{x}; \vec{a}); \chi_0(\vec{x}; \vec{a}), \dots, \chi_{N-1}(\vec{x}; \vec{a})),$$

we may write $\alpha(\vec{x}; \vec{a}) \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, $\beta_i(\vec{x}; \vec{a}) \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ for $0 \le i < M$, $\chi_j(\vec{x}; \vec{a}) \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$ for $0 \le j < N$ and $\gamma(\vec{y}; \vec{b}) \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{N}$.

Lastly, we generally write an equation so that its left-hand side is an already-known function and its right-hand side is a function which is constructed or to be constructed. For example, we write as follows: For each $f \in \mathcal{M}^2$, there exists $\alpha \in \mathcal{D}_{\mathbb{N}}$ such that

$$f(\vec{x}) = \alpha(\vec{x};)$$

for each $\vec{x}$.

3 Relationship between $\mathcal{E}^{2+}$ and $\mathcal{F}$PTIME

In this capter, we study a relationship between $\mathcal{E}^{2+}$ and $\mathcal{F}$PTIME.

Let $\mathcal{E}^2$ be the the second class in the Grzegorczyk hierarchy, that is,

$$\mathcal{E}^2 = [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \mathrm{COMP}, \mathrm{BR}].$$

We extend $\mathcal{E}^2$ by adding course-of-values recursion whose values are bounded by 1 and we call it $\mathcal{E}^{2+}$.

Definition 3.1. The function f is defined by *1-bounded course-of-values recursion* (1-BCVR) from functions g, h if

$$f(0, \vec{y}) = g(\vec{x}),$$
$$f(x+1, \vec{y}) = h(x, \vec{y}, \langle f(0, \vec{y}), \dots, f(x, \vec{y}) \rangle),$$

provided that $f(x, \vec{y}) \leq 1$ for all $x, \vec{y}$.

Definition 3.2. $\mathcal{E}^{2+}$ is a class of functions defined by the following function algebra:

$$\mathcal{E}^{2+} = [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \mathrm{COMP}, \mathrm{BR}, \text{1-BCVR}],$$

where $\mathtt{0}$ is constant 0, $\mathtt{I}$ is projection function, $\mathtt{S}$ is successor function, + is addition, × is multiplication, COMP is composition, BR is bounded recursion and 1-BCVR is 1-bounded course-of-values recursion.

As are defined in the next section, let $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ be classes of functions defined simultaneously and recursively over both the set of natural numbers $\mathbb{N}$ and the set of binary strings $\mathbb{W}$ such that functions in $\mathcal{C}_{\mathbb{N}}$ maps them to $\mathbb{N}$ and functions in $\mathcal{C}_{\mathbb{W}}$ maps them to $\mathbb{W}$.

Then, we associate functions in $\mathcal{E}^{2+}$ with functions in $\mathcal{C}_{\mathbb{N}}$, and also associate functions in $\mathcal{F}$PTIME with functions in $\mathcal{C}_{\mathbb{W}}$ (via $\mathcal{C}_{\mathbb{N}}$).

Using these correspondences, with respect to their set classes $\mathcal{E}^{2+}_*$ and PTIME, we show that

$$\mathrm{PTIME} \subseteqq \mathcal{E}^{2+}_*$$

(Theorem 3.31).

3.1 Definitions of $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$, and some basic functions

To begin with, we define function classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$.

Definition 3.3. Classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, respectively, are generated simultaneously by the following clauses.

1. The *projection* functions $\mathtt{p}_{\mathbb{N}_i}^{m,n}$ and $\mathtt{p}_{\mathbb{W}_j}^{m,n}$ belong to $\mathcal{C}_\mathbb{N}$ and $\mathcal{C}_\mathbb{W}$, respectively:

$$\mathtt{p}_{\mathbb{N}_i}^{m,n}(x_0,\ldots,x_{m-1};\vec{a}) = x_i \quad (0 \le i < m),$$
$$\mathtt{p}_{\mathbb{W}_j}^{m,n}(\vec{x};a_0,\ldots,a_{n-1}) = a_j \quad (0 \le j < n);$$

2. the constant *zero* $\mathtt{0}$ belongs to $\mathcal{C}_\mathbb{N}$: $\mathtt{0} = 0$;

3. the *successor* function $\mathtt{S}$ belongs to $\mathcal{C}_\mathbb{N}$: $\mathtt{S}(x;) = \mathrm{S}x$;

4. the *addition* $+$ belongs to $\mathcal{C}_\mathbb{N}$: $+(x,y;) = x+y$;

5. the *multiplication* $\times$ belongs to $\mathcal{C}_\mathbb{N}$: $\times(x,y;) = x\cdot y$;

6. the *length* function $|\cdot| \in \mathbb{W} \to \mathbb{N}$ belongs to $\mathcal{C}_\mathbb{N}$: $|;a| = |a|$;

7. the *bit* function $\mathtt{BIT} \in \mathbb{N}\times\mathbb{W} \to \mathbb{N}$ belongs to $\mathcal{C}_\mathbb{N}$: $\mathtt{BIT}(z;a) = \mathrm{BIT}(z,a)$;

8. $\mathcal{C}_\mathbb{N}$ and $\mathcal{C}_\mathbb{W}$ are closed under *composition* (COMP):
if $\gamma,\beta_0,\ldots,\beta_{M-1} \in \mathcal{C}_\mathbb{N}$ and $\psi,\chi_0,\ldots,\chi_{N-1} \in \mathcal{C}_\mathbb{W}$ with $\gamma \in \mathbb{N}^M\times\mathbb{W}^N \to \mathbb{N}$, $\beta_i \in \mathbb{N}^m\times\mathbb{W}^n \to \mathbb{N}$, $\psi \in \mathbb{N}^M\times\mathbb{W}^N \to \mathbb{W}$ and $\chi_j \in \mathbb{N}^m\times\mathbb{W}^n \to \mathbb{W}$ for $0 \le i < M$ and $0 \le j < N$, then there exist $\alpha \in \mathcal{C}_\mathbb{N}$ and $\varphi \in \mathcal{C}_\mathbb{W}$ satisfying

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x},\vec{a}),\ldots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\ldots,\chi_{N-1}(\vec{x};\vec{a})),$$
$$\varphi(\vec{x};\vec{a}) = \psi(\beta_0(\vec{x},\vec{a}),\ldots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\ldots,\chi_{N-1}(\vec{x};\vec{a}));$$

9. $\mathcal{C}_\mathbb{N}$ is closed under *bounded recursion* (BR):
if $\beta,\gamma,\delta \in \mathcal{C}_\mathbb{N}$ with $\beta \in \mathbb{N}^m\times\mathbb{W}^n \to \mathbb{N}$, $\gamma \in \mathbb{N}^{m+2}\times\mathbb{W}^n \to \mathbb{N}$, and $\delta \in \mathbb{N}^{m+1}\times\mathbb{W}^n \to \mathbb{N}$, then there is $\alpha \in \mathcal{C}_\mathbb{N}$ satisfying

$$\alpha(\mathtt{0},\vec{x};\vec{a}) = \beta(\vec{x};\vec{a}),$$
$$\alpha(\mathtt{S}(z;),\vec{x};\vec{a}) = \gamma(z,\vec{x},\alpha(z,\vec{x};\vec{a});\vec{a}),$$

provided that $\alpha(z,\vec{x};\vec{a}) \le \delta(z,\vec{x};\vec{a})$ for all $z,\vec{x},\vec{a}$;

10. $\mathcal{C}_\mathbb{N}$ is closed under *boolean course-of-values recursion* (BCVR)[4]:
if $\beta,\gamma \in \mathcal{C}_\mathbb{N}$ with $\beta \in \mathbb{N}^m\times\mathbb{W}^n \to \mathbb{N}$ and $\gamma \in \mathbb{N}^{m+1}\times\mathbb{W}^{n+1} \to \mathbb{N}$, then there is $\alpha \in \mathcal{C}_\mathbb{N}$ satisfying

$$\alpha(\mathtt{0},\vec{x};\vec{a}) = \beta(\vec{x};\vec{a}),$$
$$\alpha(\mathtt{S}(z;),\vec{x};\vec{a}) = \gamma(z,\vec{x};b,\vec{a}),$$
$$\text{where } |;b| = z+1 \text{ and } \mathtt{BIT}(i;b) = \alpha(i,\vec{x};\vec{a}) \text{ for } 0 \le i \le z,$$

provided that $\alpha(z,\vec{x};\vec{a}) \le 1$ for all $z,\vec{x},\vec{a}$;

[4]This operator is a string version of 1-bounded course-of-values recursion.

11. $\mathcal{C}_{\mathbb{W}}$ is closed under *bounded comprehension* (BC):
if $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\varphi \in \mathcal{C}_{\mathbb{W}}$ satisfying

$$|;\varphi(z,\vec{x};\vec{a})| = z \quad \text{and}$$
$$\forall i < z\ [\mathtt{BIT}(i;\varphi(z,\vec{x};\vec{a})) = \mathtt{0} \leftrightarrow \alpha(i,\vec{x};\vec{a}) = \mathtt{0}].$$

Notation. We often use $x+1$, $x+y$ and $x \times y$ instead of $\mathtt{S}(x;)$, $+(x,y;)$ and $\times(x,y;)$, respectively.

In function algebras, the classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ are represented as follows:

$$\mathcal{C}_{\mathbb{N}} = [\mathrm{p}_{\mathbb{N}}{}_i^{m,n}, \mathtt{0}, \mathtt{S}, +, \times, |\cdot|, \mathtt{BIT}; \mathrm{COMP}, \mathrm{BR}, \mathrm{BCVR}],$$
$$\mathcal{C}_{\mathbb{W}} = [\mathrm{p}_{\mathbb{W}}{}_j^{m,n}; \mathrm{COMP}, \mathrm{BC}].$$

Note that any constant belongs to $\mathcal{C}_{\mathbb{N}}$ by repeatedly applying the successor function $\mathtt{S}$ to the constant $\mathtt{0}$.

Now, we introduce some useful functions and an operator to $\mathcal{C}_{\mathbb{N}}$ which will be used in the subsequent lemmas and propositions.

Lemma 3.4. *The following functions belong to* $\mathcal{C}_{\mathbb{N}}$ *:*

1. *the predecessor fucntion* $\mathtt{prd}$: $\mathtt{prd}(x;) = \mathrm{prd}(x)$;
2. *the cut-off subtraction* $\dot{-}$: $\dot{-}(x,y;) = x \dot{-} y$.

Proof. The predecessor function is written as:

$$\mathtt{prd}(\mathtt{0};) = \mathtt{0},$$
$$\mathtt{prd}(x+1;) = x,$$

and $\mathtt{prd}(x;) \leq x$ for any x, hence by bounded recursion, $\mathtt{prd}$ belongs to $\mathcal{C}_{\mathbb{N}}$.

The cut-off subtraction is written as:

$$\dot{-}(x,\mathtt{0};) = x,$$
$$\dot{-}(x,y+1;) = \mathtt{prd}(\dot{-}(x,y;);),$$

and $\dot{-}(x,y;) \leq x$ for any x, y, hence by bounded recursion, $\dot{-}$ belongs to $\mathcal{C}_{\mathbb{N}}$. □

Lemma 3.5. *The following functions belong to* $\mathcal{C}_{\mathbb{N}}$ *:*

1. *the signum function* $\mathtt{sg}$: $\mathtt{sg}(x;) = \mathrm{sg}(x)$;
2. *the inverse signum fucntion* $\overline{\mathtt{sg}}$: $\overline{\mathtt{sg}}(x;) = \overline{\mathrm{sg}}(x)$;

3. *the maximum function* $\texttt{max}$: $\texttt{max}(x, y;) = \max(x, y)$;

4. *the minimum function* $\texttt{min}$: $\texttt{min}(x, y;) = \min(x, y)$;

5. *the conditional function* $\texttt{cond}$: $\texttt{cond}(x, y, z;) = \mathrm{cond}(x, y, z)$;

6. *the characteristic function* $\chi_=$ *of* $=$: $\chi_=(x, y;) = \chi_=(x, y)$;

7. *the characteristic function* $\chi_\leq$ *of* $\leq$: $\chi_\leq(x, y;) = \chi_\leq(x, y)$;

8. *the characteristic function* $\chi_<$ *of* $<$: $\chi_<(x, y;) = \chi_<(x, y)$.

Proof. These functions are defined as follows:

$$
\begin{aligned}
\overline{\texttt{sg}}(x;) &= \texttt{S}(\texttt{0};) \dot{-} x,\\
\texttt{sg}(x;) &= \overline{\texttt{sg}}(\overline{\texttt{sg}}(x;);),\\
\texttt{max}(x, y;) &= x + (y \dot{-} x),\\
\texttt{min}(x, y;) &= x \dot{-} (x \dot{-} y),\\
\texttt{cond}(x, y, z;) &= \overline{\texttt{sg}}(x;) \times y + \texttt{sg}(x;) \times z,\\
\chi_=(x, y;) &= \overline{\texttt{sg}}((x \dot{-} y) + (y \dot{-} x);),\\
\chi_\leq(x, y;) &= \overline{\texttt{sg}}(x \dot{-} y;), \text{ and}\\
\chi_<(x, y;) &= \texttt{sg}(y \dot{-} x;).
\end{aligned}
$$

□

Notation. We often use $x = y$, $x \leq y$ and $x < y$ instead of $\chi_=(x, y;)$, $\chi_\leq(x, y;)$ and $\chi_<(x, y;)$, respectively.

Lemma 3.6. *The following logical functions belong to* $\mathcal{C}_\mathbb{N}$:

1. *the characteristic function* $\chi_\neg$ *of* $\neg$: $\chi_\neg(x;) = \chi_\neg(x)$;

2. *the characteristic function* $\chi_\wedge$ *of* $\wedge$: $\chi_\wedge(x, y;) = \chi_\wedge(x, y)$;

3. *the characteristic function* $\chi_\vee$ *of* $\vee$: $\chi_\vee(x, y;) = \chi_\vee(x, y)$;

4. *the characteristic function* $\chi_\rightarrow$ *of* $\rightarrow$: $\chi_\rightarrow(x, y;) = \chi_\rightarrow(x, y)$.

Proof. These functions are defined as follows:

$$
\begin{aligned}
\chi_\neg(x;) &= \overline{\texttt{sg}}(x;),\\
\chi_\wedge(x, y;) &= \texttt{cond}(x, \texttt{0}, \texttt{sg}(y;);),\\
\chi_\vee(x, y;) &= \texttt{cond}(x, \texttt{sg}(y;), \texttt{S}(\texttt{0};);), \text{ and}\\
\chi_\rightarrow(x, y;) &= \texttt{cond}(\overline{\texttt{sg}}(x;), \texttt{sg}(y;), \texttt{S}(\texttt{0};);).
\end{aligned}
$$

□

Notation. Similarly, we often use $\neg x$, $x \wedge y$, $x \vee y$ and $x \to y$ instead of $\chi_{\neg}(x;)$, $\chi_{\wedge}(x,y;)$, $\chi_{\vee}(x,y;)$ and $\chi_{\to}(x,y;)$, respectively.

Notation. In the definition of each function in the following discussion, if arguments of a function contain constants greater than $\mathtt{0}$, we often use $1, 2, 3, \ldots$ instead of $\mathtt{S}(\mathtt{0};)$, $\mathtt{S}(\mathtt{S}(\mathtt{0};);)$, $\mathtt{S}(\mathtt{S}(\mathtt{S}(\mathtt{0};););)$, ..., respectively.

Definition 3.7. The function $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by *bounded minimisation* (BMIN) from a function β if

$$\alpha(z, \vec{x}; \vec{a}) = \begin{cases} \text{the least } y \leq z \text{ such that } \beta(y, \vec{x}; \vec{a}) \neq \mathtt{0} & \text{if it exists,} \\ z + 1 & \text{otherwise.} \end{cases}$$

Proposition 3.8. *$\mathcal{C}_{\mathbb{N}}$ is closed under bounded minimisation.*

Proof. Suppose that β is in $\mathcal{C}_{\mathbb{N}}$ with $\beta \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$. Since $\mathcal{C}_{\mathbb{N}}$ is closed under bounded recursion, define α by

$$\begin{aligned} \alpha(\mathtt{0}, \vec{x}; \vec{a}) &= \mathtt{cond}(\beta(\mathtt{0}, \vec{x}; \vec{a}), 1, \mathtt{0}), \\ \alpha(z+1, \vec{x}; \vec{a}) &= \mathtt{cond}(\beta(z+1, \vec{x}; \vec{a}) = 0 \wedge \alpha(z, \vec{x}; \vec{a}) = z + 1, \\ &\qquad \alpha(z, \vec{x}; \vec{a}), \\ &\qquad \alpha(z, \vec{x}; \vec{a}) + 1;), \end{aligned}$$

and $\alpha(z, \vec{x}, \vec{a}) \leq z + 1$ for $\forall z, \vec{x}, \vec{a}$, hence $\alpha \in \mathcal{C}_{\mathbb{N}}$. Then α computes

$$\alpha(z, \vec{x}; \vec{a}) = \begin{cases} \text{the least } y \leq z \text{ such that } \beta(y, \vec{x}; \vec{a}) \neq \mathtt{0} & \text{if it exists,} \\ z + 1 & \text{otherwise.} \end{cases}$$

□

Notation. In using bounded minimisation, we write $\min\limits_{y \leq z}\{\beta(y, \vec{x}; \vec{a}) \neq \mathtt{0}\}$ for $\alpha(z, \vec{x}; \vec{a})$.

Lemma 3.9. *If α is in $\mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then the following $\beta_j (1 \leq j \leq 4)$ are also in $\mathcal{C}_{\mathbb{N}}$:*

$$\text{(i)} \qquad \beta_1(z, \vec{x}; \vec{a}) = \begin{cases} 1 & \textit{if } \forall y \leq z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}], \\ 0 & \textit{otherwise;} \end{cases}$$

$$\text{(ii)} \qquad \beta_2(z, \vec{x}; \vec{a}) = \begin{cases} 1 & \textit{if } \forall y \leq z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}], \\ 0 & \textit{otherwise;} \end{cases}$$

$$\text{(iii)} \qquad \beta_3(z, \vec{x}; \vec{a}) = \begin{cases} 1 & \textit{if } \exists y \leq z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}], \\ 0 & \textit{otherwise;} \end{cases}$$

$$\text{(iv)} \qquad \beta_4(z, \vec{x}; \vec{a}) = \begin{cases} 1 & \textit{if } \exists y \leq z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}], \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. (i) If $\forall y \le z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}]$, then by bounded minimisation, $\min_{y \le z}\{\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}\} = z + 1$, and if $\exists y \le z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}]$, then $\min_{y \le z}\{\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}\} \le z$. Hence we have

$$\beta_1(z, \vec{x}; \vec{a}) = \min_{y \le z}\{\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}\} \dot{-} z.$$

(ii) Since $\forall y \le z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}] \Leftrightarrow \forall y \le z[\overline{\mathtt{sg}}(\alpha(y, \vec{x}; \vec{a})) = \mathtt{0}]$, by using (i), we have

$$\beta_2(z, \vec{x}; \vec{a}) = \min_{y \le z}\{\overline{\mathtt{sg}}(\alpha(y, \vec{x}; \vec{a})) \neq \mathtt{0}\} \dot{-} z.$$

(iii) Since $\exists y \le z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}] \Leftrightarrow \neg[\forall y \le z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}]]$, by using (i), we have

$$\beta_3(z, \vec{x}; \vec{a}) = \neg\left(\min_{y \le z}\{\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}\} \dot{-} z\right).$$

(iv) Since $\exists y \le z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}] \Leftrightarrow \neg[\forall y \le z[\alpha(y, \vec{x}; \vec{a}) \neq \mathtt{0}]]$, by using (ii), we have

$$\beta_4(z, \vec{x}; \vec{a}) = \neg\left(\min_{y \le z}\{\overline{\mathtt{sg}}(\alpha(y, \vec{x}; \vec{a})) \neq \mathtt{0}\} \dot{-} z\right).$$

□

Notation. For each $\beta_j(z, \vec{x}; \vec{a})(1 \le j \le 4)$, we use the logical formula described in the condition when β_j's value is 1. For example, we use $\forall y \le z[\alpha(y, \vec{x}; \vec{a}) = \mathtt{0}]$ instead of $\beta_1(z, \vec{x}; \vec{a})$.

In the subsequent lemmas, we will construct $\mathtt{bin}(n;) \in \mathcal{C}_{\mathbb{W}}$ which computes the binary string of the binary representation of n.

Lemma 3.10. *The function $m|n \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ belongs to $\mathcal{C}_{\mathbb{N}}$:*

$$m|n = \begin{cases} 1 & \text{if } m \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $m|n \Leftrightarrow \exists i \le n[n = i \times m] \Leftrightarrow \exists i \le n[\chi_=(n, i \times m;) = 1] \Leftrightarrow \exists i \le n[\chi_=(n, i \times m;) \neq \mathtt{0}]$, we have

$$m|n = \exists i \le n[\chi_=(n, i \times m;) \neq \mathtt{0}],$$

which is definable in $\mathcal{C}_{\mathbb{N}}$ by lemma 3.9 (iii).

□

Lemma 3.11. *The function $\left\lfloor \frac{n}{m} \right\rfloor \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ belongs to $\mathcal{C}_{\mathbb{N}}$:*

$$\left\lfloor \frac{n}{m} \right\rfloor \text{ is a quotient when } n \text{ is divided by } m.$$

Proof. Since $n = m \times i + r$ $(0 \le r < m)$, i is the least number satisfying $n < m \times (i+1)$, hence we have

$$\begin{aligned}\left\lfloor \frac{n}{m} \right\rfloor &= \min_{i \le n}\{n < m \times (i+1)\} \\ &= \min_{i \le n}\{\chi_<(n, m \times (i+1);) = 1\} \\ &= \min_{i \le n}\{\chi_<(n, m \times (i+1);) \ne \mathtt{0}\},\end{aligned}$$

which is definable in $\mathcal{C}_{\mathbb{N}}$ by bounded minimisation. □

Lemma 3.12. *The function* $\mathtt{mod}(n, m;) \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$ *:*

$\mathtt{mod}(n, m;)$ *is a residue when* n *is divided by* m*.*

Proof. We have

$$\mathtt{mod}(n, m;) = n \dot{-} \left\lfloor \frac{n}{m} \right\rfloor \times m.$$

□

Lemma 3.13. *The function* $\mathtt{prime}(n;) \in \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$ *:*

$$\mathtt{prime}(n;) = \begin{cases} 1 & \textit{if } n \textit{ is a prime number,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The function

$$\mathtt{p}(k, n;) = \begin{cases} 1 & \text{if } k(\ge 1) \text{ divides } n \text{, then } k = 1 \text{ or } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

is definable in $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{p}(k, n;) = (\mathtt{0} < k \wedge k | n) \to (k = 1 \vee k = n).$$

Then, using $\mathtt{p}(k, n;)$, we have

$$\begin{aligned}\mathtt{prime}(n;) &= \forall k \le n[\mathtt{p}(k, n;) = 1] \wedge (1 < n) \\ &= \forall k \le n[\mathtt{p}(k, n;) \ne \mathtt{0}] \wedge (1 < n),\end{aligned}$$

which is definable in $\mathcal{C}_{\mathbb{N}}$ by lemma 3.9 (ii). Here, the condition $1 < n$ is to exclude the case that $\mathtt{prime}(\mathtt{0};)$ or $\mathtt{prime}(1;)$ holds. □

Lemma 3.14. *The function* $\mathtt{pow}(n;) \in \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$ *:*

$$\mathtt{pow}(n;) = \begin{cases} 1 & \textit{if } n \textit{ is a power of } 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The function

$$\mathtt{q}(k,n;) = \begin{cases} 1 & \text{if } n \text{ has a prime factor } k, \text{ then } k \text{ must be } 2, \\ 0 & \text{otherwise}, \end{cases}$$

is definable in $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{q}(k,n;) = (k|n \wedge \mathtt{prime}(k;)) \to k = 2.$$

Then, using $\mathtt{q}(k,n;)$, we have

$$\begin{aligned} \mathtt{pow}(n;) &= \forall k \leq n[\mathtt{q}(k,n;) = 1] \wedge (1 \leq n) \\ &= \forall k \leq n[\mathtt{q}(k,n;) \neq \mathtt{0}] \wedge (1 \leq n), \end{aligned}$$

which is definable in $\mathcal{C}_{\mathbb{N}}$ by lemma 3.9 (ii). Here, the condition $1 \leq n$ is to exclude the case that $\mathtt{pow}(\mathtt{0};)$ holds. □

Lemma 3.15. *The function* $\mathtt{lpw}(n;) \in \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{lpw}(n;) = \textit{ the least power of } 2 \textit{ exceeding } n.$$

Proof. The function

$$\mathtt{r}(n,k;) = \begin{cases} 1 & k \text{ is greater than } n \text{ and } k \text{ is a power of } 2, \\ 0 & \text{otherwise}, \end{cases}$$

is definable in $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{r}(n,k;) = (n < k) \wedge \mathtt{pow}(k).$$

If $n \geq 1$, there is a number 2^m such that $2^m \leq n$. Taking the maximum number 2^m satisfying $2^m \leq n$, we have $n < 2^{m+1} \leq 2n$. (If $2^{m+1} \leq n$, it contradicts 2^m being the maximum number satisfying $2^m \leq n$.) That is, taking such the number 2^{m+1} as k, there is a number k being a power of 2 such that $n < k \leq 2n$. If $n = 0$, $\mathtt{lpw}(\mathtt{0};) = 1$. Hence, using $\mathtt{r}(n,k;)$, we have

$$\begin{aligned} \mathtt{lpw}(n;) &= \min_{k \leq 2 \times n} \{\mathtt{r}(n,k) = 1\} \\ &= \min_{k \leq 2 \times n} \{\mathtt{r}(n,k) \neq \mathtt{0}\}, \end{aligned}$$

which is definable in $\mathcal{C}_{\mathbb{N}}$ by bounded minimisation. Notice that in the case $\mathtt{lpw}(\mathtt{0};)$, since $\mathtt{r}(\mathtt{0},\mathtt{0};) = \mathtt{0}$, we have $\mathtt{lpw}(\mathtt{0};) = 1$ by the definition of bounded minimisation. □

Lemma 3.16. *The function* $\mathtt{None}(n;) \in \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{None}(n;) = \textit{ the number of } 1\textit{'s in the binary representation of } n.$$

Proof. The function

$$\alpha(k,n;) = \begin{cases} 1 & \begin{array}{l} k \text{ is } 2^m \text{ for some } m \text{ and} \\ m\text{-th bit of the binary representation of } n \text{ is } 1, \end{array} \\ 0 & \text{otherwise,} \end{cases}$$

is definable in $\mathcal{C}_{\mathbb{N}}$:

$$\alpha(k,n;) = \texttt{pow}(k;) \wedge \left\{ \texttt{mod}\left(\left\lfloor \frac{n}{k} \right\rfloor, 2; \right) = 1 \right\}.$$

Then, the function

$$\begin{aligned} \beta(k,n;) &= \text{the number of } k (\leq n) \text{ satisfying } \alpha(k,n;) = 1 \\ &= \text{the number of 1's in the binary representation of } n \\ &\quad\ \text{whose positions are less than or equal to } m\text{-th} \\ &\quad\ \text{where } 2^m \text{ is the maximum number less than or equal to } k, \end{aligned}$$

is definable in $\mathcal{C}_{\mathbb{N}}$ using bounded recursion and $\alpha(k,n;)$:

$$\begin{aligned} \beta(\texttt{0},n;) &= \texttt{0}, \\ \beta(k+1,n;) &= \texttt{cond}(\alpha(k+1,n;), \beta(k,n;), \texttt{S}(\beta(k,n;););), \\ \beta(k,n;) &\leq n \quad \text{ for } \forall k, n. \end{aligned}$$

Then we have

$$\texttt{None}(n;) = \beta(n,n;).$$

□

Lemma 3.17. *The function* $|n;| \in \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$:

$$|n;| = \textit{ the length of the binary representation of } n.$$

Proof. We have

$$|n;| = \texttt{None}(\texttt{lpw}(n;) \dot{-} 1;).$$

Notice that $|0;| = 0$. □

Lemma 3.18. *The function* $\texttt{exp}(i,k;) \in \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$:

$$\texttt{exp}(i,k;) = \begin{cases} 1 & \textit{if } k = 2^i, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We have

$$\texttt{exp}(i,k;) = \texttt{pow}(k;) \wedge |k;| = i + 1.$$

□

Lemma 3.19. *The function* $\mathtt{BIT}(i,n;) \in \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ *belongs to* $\mathcal{C}_{\mathbb{N}}$:

$$\mathtt{BIT}(i,n;) = \textit{ the } i\textit{-th bit in the binary representation of } n \ .$$

Proof. The bounded exponential function

$$\mathtt{bexp}(i,n;) = \begin{cases} 2^i & \text{if } 2^i \le n, \\ n+1 & \text{otherwise.} \end{cases}$$

is definable in $\mathcal{C}_{\mathbb{N}}$ using bounded minimisation and $\mathtt{exp}(i,k;)$:

$$\mathtt{bexp}(i,n;) = \min_{k\le n}\{\mathtt{exp}(i,k;) \ne 0\}.$$

Then, using $\mathtt{bexp}(i,n;)$, we have

$$\begin{aligned}\mathtt{BIT}(i,n;) &= \mathtt{mod}\left(\left\lfloor \frac{n}{2^i}\right\rfloor, 2\right)\\ &= \mathtt{mod}\left(\left\lfloor \frac{n}{\mathtt{bexp}(i,n;)}\right\rfloor, 2\right).\end{aligned}$$

□

Lemma 3.20. *The function* $\mathtt{bin}(k;) \in \mathbb{N}\to\mathbb{W}$ *belongs to* $\mathcal{C}_{\mathbb{W}}$:

$$\mathtt{bin}(k;) = \textit{the binary string of the binary representation of } k.$$

Proof. In the definition of bounded comprehension, let $\alpha(i,k;) = \mathtt{BIT}(i,k;)$, then there is $\varphi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}&|;\varphi(z,k;)| = z,\\ &\forall i < z\ [\mathtt{BIT}(i;\varphi(z,k;)) = \mathtt{0} \leftrightarrow \alpha(i,k;) = \mathtt{0}].\end{aligned}$$

Then we have

$$\mathtt{bin}(k;) = \varphi(|k;|, k;).$$

Notice that $\mathtt{bin}(\mathtt{0};) = \varepsilon\ (\in \mathbb{W})$. □

In our notational conventions, we should write $\mathtt{bin}(k;)$, however, in the following discussion, we omit a semicolon and write $\mathtt{bin}(k)$ for readability.

3.2 Representation of $\mathcal{E}^{2+}$ functions by $\mathcal{C}_{\mathbb{N}}$ functions

In this section, we show that any function in $\mathcal{E}^{2+}$ is represented by some function in $\mathcal{C}_{\mathbb{N}}$, that is,

$$\forall f \in \mathcal{E}^{2+}\ \exists \alpha \in \mathcal{C}_{\mathbb{N}}\ [f(\vec{x}) = \alpha(\vec{x};)]$$

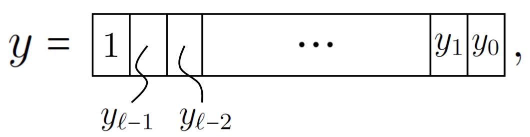
(Proposition 3.22).

To begin with, we show a lemma used in this proposition. This lemma asserts that for any function α in $\mathcal{C}_{\mathbb{N}}$ taking as an argument a coded natural number $y = \langle y_0, \dots, y_{\ell-1} \rangle$ with $y_i \leq 1$ for $0 \leq i < \ell$, there exists some function α' in $\mathcal{C}_{\mathbb{N}}$ taking as an argument a binary string b with $|b| = \ell$ and $\mathrm{BIT}(i, b) = y_i$ for $0 \leq i < \ell$ such that $\alpha(\vec{x}, y; \vec{a}) = \alpha'(\vec{x}; \vec{a}, b)$. The same assertion holds for any function in $\mathcal{C}_{\mathbb{W}}$.

In this case, we assume that there are two functions ρ and π in $\mathcal{C}_{\mathbb{N}}$ which compute the following values from a coded natural number $y = \langle y_0, \dots, y_{\ell-1} \rangle$:

- $\rho(y;) = \ell$: the number of elements coded in y,
- $\pi(i, y;) = y_i$: the value of the i-th element coded in y $(0 \leq i < \ell)$.

As a simple example, if natural numbers $y_0, \dots, y_{\ell-1}$ with $y_i \leq 1$ for $0 \leq i < \ell$ are coded as a natural number y in whose binary representation, the most significant bit is 1 and i-th bit is y_i for $0 \leq i < \ell$ such as



then we can use the functions $\rho(y;) = |y;| \dot{-} 1$ and $\pi(i, y;) = \mathtt{BIT}(i, y;)$. These functions are constructed in Lemma 3.17 and Lemma 3.19, respectively.

Since functions which directly operate y in the structue of $\alpha(\vec{x}, y; \vec{a})$ are only $\rho(y;) = \ell$ or $\pi(i, y;) = y_i$, these functions can be replaced with $|; b| = \ell$ or $\mathtt{BIT}(i; b) = y_i$ in the structure of $\alpha'(\vec{x}; \vec{a}, b)$, respectively.

Lemma 3.21. *Let y be in $\mathbb{N}$ such that $y = \langle y_0, \dots, y_{\ell-1} \rangle$ with $y_i \leq 1$ for $0 \leq i < \ell$, and let b be in $\mathbb{W}$ such that $|b| = \ell$ and $\mathrm{BIT}(i, b) = y_i$ for $0 \leq i < \ell$. Then* (i) *for any $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(\vec{x}, y; \vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, there exists $\alpha' \in \mathcal{C}_{\mathbb{N}}$ with $\alpha'(\vec{x}; \vec{a}, b) \in \mathbb{N}^m \times \mathbb{W}^{n+1} \to \mathbb{N}$ such that*

$$\alpha(\vec{x}, y; \vec{a}) = \alpha'(\vec{x}; \vec{a}, b),$$

and (ii) *for any $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi(\vec{x}, y; \vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{W}$, there exists $\varphi' \in \mathcal{C}_{\mathbb{W}}$ with $\varphi'(\vec{x}; \vec{a}, b) \in \mathbb{N}^m \times \mathbb{W}^{n+1} \to \mathbb{W}$ such that*

$$\varphi(\vec{x}, y; \vec{a}) = \varphi'(\vec{x}; \vec{a}, b).$$

Proof. By simultaneous induction on the structures of $\alpha \in \mathcal{C}_{\mathbb{N}}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}_i}^{m+1,n}$:
Note that $0 \le i < m$. Let $\alpha' = \mathtt{p}_{\mathbb{N}_i}^{m,n+1}$. Then

$$\alpha(\vec{x}, y; \vec{a}) = \mathtt{p}_{\mathbb{N}_i}^{m+1,n}(\vec{x}, y; \vec{a}) = x_i = \mathtt{p}_{\mathbb{N}_i}^{m,n+1}(\vec{x}; \vec{a}, b) = \alpha'(\vec{x}; \vec{a}, b).$$

Case $\varphi = \mathtt{p}_{\mathbb{W}_j}^{m+1,n}$:
Note that $0 \le j < n$. Let $\varphi' = \mathtt{p}_{\mathbb{W}_j}^{m,n+1}$. Then

$$\varphi(\vec{x}, y; \vec{a}) = \mathtt{p}_{\mathbb{W}_j}^{m+1,n}(\vec{x}, y; \vec{a}) = a_j = \mathtt{p}_{\mathbb{W}_j}^{m,n+1}(\vec{x}; \vec{a}, b) = \varphi'(\vec{x}; \vec{a}, b).$$

Case $\alpha = \mathtt{0}$:
Note that $m + 1 = n = 0$. Let $\alpha' = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $\alpha = \mathtt{S}$:
Note that $m + 1 = 1, n = 0$, and y does not appear in the argument of $\mathtt{S}$. Let $\alpha' = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1;)$.
Case $\alpha = +$:
Note that $m + 1 = 2, n = 0$, and y does not appear in the argument of $+$. Let $\alpha' = +$, then $x_1 + x_2 = x_1 + x_2$.
Case $\alpha = \times$:
Note that $m + 1 = 2, n = 0$, and y does not appear in the argument of $\times$. Let $\alpha' = \times$, then $x_1 \times x_2 = x_1 \times x_2$.
Case $\alpha = |\cdot|$:
Note that $m + 1 = 0, n = 1$, and y does not appear in the argument of $|\cdot|$. Let $\alpha' = |\cdot|$, then $|; a_1| = |; a_1|$.
Case $\alpha = \mathtt{BIT}$:
Note that $m+1 = 1, n = 1$, and y does not appear in the argument of $\mathtt{BIT}$. Let $\alpha' = \mathtt{BIT}$, then $\mathtt{BIT}(z; a_1) = \mathtt{BIT}(z; a_1)$.
Case $\alpha(\cdot) = \rho(\mathtt{p}_{\mathbb{N}m}^{m+1,n}(\cdot);)$:
Let $\alpha'(\cdot) = |; \mathtt{p}_{\mathbb{N}n}^{m,n+1}(\cdot)|$. Then

$$\begin{aligned} \alpha(\vec{x}, y; \vec{a}) &= \rho(\mathtt{p}_{\mathbb{N}m}^{m+1,n}(\vec{x}, y; \vec{a});) = \rho(y;) = \ell \\ &= |; b| = |; \mathtt{p}_{\mathbb{N}n}^{m,n+1}(\vec{x}; \vec{a}, b)| = \alpha'(\vec{x}; \vec{a}, b). \end{aligned}$$

Case $\alpha(\cdot) = \pi(\mathtt{p}_{\mathbb{N}_0}^{m+2,n}(\cdot), \mathtt{p}_{\mathbb{N}_{m+1}}^{m+2,n}(\cdot);)$:
Let $\alpha'(\cdot) = \mathtt{BIT}(\mathtt{p}_{\mathbb{N}_0}^{m+1,n+1}(\cdot); \mathtt{p}_{\mathbb{W}n}^{m+1,n+1}(\cdot))$. Then

$$\begin{aligned} \alpha(i, \vec{x}, y; \vec{a}) &= \pi(\mathtt{p}_{\mathbb{N}_0}^{m+2,n}(i, \vec{x}, y; \vec{a}), \mathtt{p}_{\mathbb{N}_{m+1}}^{m+2,n}(i, \vec{x}, y; \vec{a});) \\ &= \pi(i, y;) = y_i = \mathtt{BIT}(i; b) \\ &= \mathtt{BIT}(\mathtt{p}_{\mathbb{N}_0}^{m+1,n+1}(i, \vec{x}; \vec{a}, b); \mathtt{p}_{\mathbb{W}_n}^{m+1,n+1}(i, \vec{x}; \vec{a}, b)) \\ &= \alpha'(i, \vec{x}; \vec{a}, b). \end{aligned}$$

Induction step.
Case COMP($\in \mathcal{C}_{\mathbb{N}}$):
Suppose that

$$\alpha(\vec{x},y;\vec{a}) = \beta(\gamma_0(\vec{x},y;\vec{a}),\dots,\gamma_{M-1}(\vec{x},y;\vec{a});\psi_0(\vec{x},y;\vec{a}),\dots,\psi_{N-1}(\vec{x},y;\vec{a})),$$

whrere $\beta\,(\neq \rho,\pi),\gamma_0,\dots,\gamma_{M-1} \in \mathcal{C}_{\mathbb{N}}$, and $\psi_0,\dots,\psi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\gamma'_0,\dots,\gamma'_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\psi'_0,\dots,\psi'_{N-1} \in \mathcal{C}_{\mathbb{W}}$ such that

$$\begin{aligned}\gamma_i(\vec{x},y;\vec{a}) &= \gamma'_i(\vec{x};\vec{a},b),\\ \psi_j(\vec{x},y;\vec{a}) &= \psi'_j(\vec{x};\vec{a},b)\end{aligned}$$

for $0 \leq i < M$ and $0 \leq j < N$.
Since $\mathcal{C}_{\mathbb{N}}$ is closed under composition, define $\alpha' \in \mathcal{C}_{\mathbb{N}}$ by

$$\alpha'(\vec{x};\vec{a},b) = \beta(\gamma'_0(\vec{x};\vec{a},b),\dots,\gamma'_{M-1}(\vec{x};\vec{a},b);\psi'_0(\vec{x};\vec{a},b),\dots,\psi'_{N-1}(\vec{x};\vec{a},b)).$$

Then we have

$$\begin{aligned}\alpha(\vec{x},y;\vec{a}) &= \beta(\gamma_0(\vec{x},y;\vec{a}),\dots,\gamma_{M-1}(\vec{x},y;\vec{a});\psi_0(\vec{x},y;\vec{a}),\dots,\psi_{N-1}(\vec{x},y;\vec{a}))\\ &= \beta(\gamma'_0(\vec{x};\vec{a},b),\dots,\gamma'_{M-1}(\vec{x};\vec{a},b);\psi'_0(\vec{x};\vec{a},b),\dots,\psi'_{N-1}(\vec{x};\vec{a},b))\\ &= \alpha'(\vec{x};\vec{a},b).\end{aligned}$$

Case COMP($\in \mathcal{C}_{\mathbb{W}}$):
Suppose that

$$\varphi(\vec{x},y;\vec{a}) = \psi(\beta_0(\vec{x},y;\vec{a}),\dots,\beta_{M-1}(\vec{x},y;\vec{a});\chi_0(\vec{x},y;\vec{a}),\dots,\chi_{N-1}(\vec{x},y;\vec{a})),$$

whrere $\beta_0,\dots,\beta_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\psi,\chi_0,\dots,\chi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\beta'_0,\dots,\beta'_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\chi'_0,\dots,\chi'_{N-1} \in \mathcal{C}_{\mathbb{W}}$ such that

$$\begin{aligned}\beta_i(\vec{x},y;\vec{a}) &= \beta'_i(\vec{x};\vec{a},b),\\ \chi_j(\vec{x},y;\vec{a}) &= \chi'_j(\vec{x};\vec{a},b)\end{aligned}$$

for $0 \leq i < M$ and $0 \leq j < N$.
Since $\mathcal{C}_{\mathbb{W}}$ is closed under composition, define $\varphi' \in \mathcal{C}_{\mathbb{W}}$ by

$$\varphi'(\vec{x};\vec{a},b) = \psi(\beta'_0(\vec{x};\vec{a},b),\dots,\beta'_{M-1}(\vec{x};\vec{a},b);\chi'_0(\vec{x};\vec{a},b),\dots,\chi'_{N-1}(\vec{x};\vec{a},b)).$$

Then we have

$$\begin{aligned}\varphi(\vec{x},y;\vec{a}) &= \psi(\beta_0(\vec{x},y;\vec{a}),\dots,\beta_{M-1}(\vec{x},y;\vec{a});\chi_0(\vec{x},y;\vec{a}),\dots,\chi_{N-1}(\vec{x},y;\vec{a}))\\ &= \psi(\beta'_0(\vec{x};\vec{a},b),\dots,\beta'_{M-1}(\vec{x};\vec{a},b);\chi'_0(\vec{x};\vec{a},b),\dots,\chi'_{N-1}(\vec{x};\vec{a},b))\\ &= \varphi'(\vec{x};\vec{a},b).\end{aligned}$$

Case BR:

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(z, \vec{x}, y; \vec{a}) \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned} \alpha(0, \vec{x}, y; \vec{a}) &= \beta(\vec{x}, y; \vec{a}), \\ \alpha(z+1, \vec{x}, y; \vec{a}) &= \gamma(z, \vec{x}, \alpha(z, \vec{x}, y; \vec{a}), y; \vec{a}), \\ \alpha(z, \vec{x}, y; \vec{a}) &\leq \delta(z, \vec{x}, y; \vec{a}) \quad \text{for any } z, \vec{x}, y, \vec{a}, \end{aligned}$$

where $\beta, \gamma, \delta \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $\beta', \gamma', \delta' \in \mathcal{C}_{\mathbb{N}}$ such that

$$\begin{aligned} \beta(\vec{x}, y; \vec{a}) &= \beta'(\vec{x}; \vec{a}, b), \\ \gamma(z, \vec{x}, w, y; \vec{a}) &= \gamma'(z, \vec{x}, w; \vec{a}, b), \\ \delta(z, \vec{x}, y; \vec{a}) &= \delta'(z, \vec{x}; \vec{a}, b). \end{aligned}$$

Since $\mathcal{C}_{\mathbb{N}}$ is closed under bounded recursion, define α' by

$$\begin{aligned} \alpha'(0, \vec{x}; \vec{a}, b) &= \beta'(\vec{x}; \vec{a}, b), \\ \alpha'(z+1, \vec{x}; \vec{a}, b) &= \gamma'(z, \vec{x}, \alpha'(z, \vec{x}; \vec{a}, b); \vec{a}, b). \end{aligned}$$

Now, we show $\alpha(z, \vec{x}, y; \vec{a}) = \alpha'(z, \vec{x}; \vec{a}, b)$ by induction on z.
In the case $z = 0$,

$$\alpha(0, \vec{x}, y; \vec{a}) = \beta(\vec{x}, y; \vec{a}) = \beta'(\vec{x}; \vec{a}, b) = \alpha'(0, \vec{x}; \vec{a}, b).$$

In the case z, suppose that $\alpha(z, \vec{x}, y; \vec{a}) = \alpha'(z, \vec{x}; \vec{a}, b)$.
In the case $z + 1$,

$$\begin{aligned} \alpha(z+1, \vec{x}, y; \vec{a}) &= \gamma(z, \vec{x}, \alpha(z, \vec{x}, y; \vec{a}), y; \vec{a}) \\ &= \gamma'(z, \vec{x}, \alpha'(z, \vec{x}; \vec{a}, b); \vec{a}, b) \\ &= \alpha'(z+1, \vec{x}; \vec{a}, b). \end{aligned}$$

Therefore, we have $\alpha(z, \vec{x}, y; \vec{a}) = \alpha'(z, \vec{x}; \vec{a}, b)$ for all z.

Furthermore,

$$\begin{aligned} \alpha'(z, \vec{x}; \vec{a}, b) &= \alpha(z, \vec{x}, y; \vec{a}) \\ &\leq \delta(z, \vec{x}, y; \vec{a}) \\ &= \delta'(z, \vec{x}; \vec{a}, b) \quad \text{for all } z, \vec{x}, \vec{a}, b. \end{aligned}$$

Since α' is bounded from above by $\delta' \in \mathcal{C}_{\mathbb{N}}$, we conclude $\alpha' \in \mathcal{C}_{\mathbb{N}}$.
Case BCVR:

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(z, \vec{x}, y; \vec{a}) \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{N}$ is defined by boolean course-of-values recursion, then

$$\begin{aligned} \alpha(0, \vec{x}, y; \vec{a}) &= \beta(\vec{x}, y; \vec{a}), \\ \alpha(z+1, \vec{x}, y; \vec{a}) &= \gamma(z, \vec{x}, y; c, \vec{a}), \\ \text{where } |; c| = z{+}1 &\text{ and } \mathtt{BIT}(i; c) = \alpha(i, \vec{x}, y; \vec{a}) \text{ for } 0 \le i \le z, \\ \alpha(z, \vec{x}, y; \vec{a}) &\le 1 \quad \text{for any } z, \vec{x}, y, \vec{a}, \end{aligned}$$

where $\beta, \gamma \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $\beta', \gamma' \in \mathcal{C}_{\mathbb{N}}$ such that

$$\begin{aligned} \beta(\vec{x}, y; \vec{a}) &= \beta'(\vec{x}; \vec{a}, b), \\ \gamma(z, \vec{x}, y; c, \vec{a}) &= \gamma'(z, \vec{x}; c, \vec{a}, b). \end{aligned}$$

Since $\mathcal{C}_{\mathbb{N}}$ is closed under boolean course-of-values recursion, define α' by

$$\begin{aligned} \alpha'(0, \vec{x}; \vec{a}, b) &= \beta'(\vec{x}; \vec{a}, b), \\ \alpha'(z+1, \vec{x}; \vec{a}, b) &= \gamma'(z, \vec{x}; c, \vec{a}, b), \\ \text{where } |; c| = z{+}1 &\text{ and } \mathtt{BIT}(i; c) = \alpha'(i, \vec{x}; \vec{a}, b) \text{ for } 0 \le i \le z. \end{aligned}$$

Now, we show $\alpha(z, \vec{x}, y; \vec{a}) = \alpha'(z, \vec{x}; \vec{a}, b)$ by course-of-values induction on z.
In the case $z = 0$,

$$\alpha(0, \vec{x}, y; \vec{a}) = \beta(\vec{x}, y; \vec{a}) = \beta'(\vec{x}; \vec{a}, b) = \alpha'(0, \vec{x}; \vec{a}, b).$$

In the cases $w (\le z)$, suppose that $\alpha(w, \vec{x}, y; \vec{a}) = \alpha'(w, \vec{x}; \vec{a}, b)$.
In the case $z + 1$,

$$\begin{aligned} \alpha(z+1, \vec{x}, y; \vec{a}) &= \gamma(z, \vec{x}, y; c, \vec{a}), \\ \text{where } |; c| = z{+}1 &\text{ and } \mathtt{BIT}(i; c) = \alpha(i, \vec{x}, y; \vec{a}) \text{ for } 0 \le i \le z, \\ &= \gamma'(z, \vec{x}; c, \vec{a}, b), \\ \text{since } \mathtt{BIT}(i; c) &= \alpha(i, \vec{x}, y; \vec{a}) = \alpha'(i, \vec{x}; \vec{a}, b) \text{ for } 0 \le i \le z, \\ &= \alpha'(z+1, \vec{x}; \vec{a}, b). \end{aligned}$$

Therefore, we have $\alpha(z, \vec{x}, y; \vec{a}) = \alpha'(z, \vec{x}; \vec{a}, b)$ for all z.

Furthermore,

$$\alpha'(z, \vec{x}; \vec{a}, b) = \alpha(z, \vec{x}, y; \vec{a}) \le 1 \quad \text{for all } z, \vec{x}, \vec{a}, b.$$

Since α' is bounded from above by 1, we conclude $\alpha' \in \mathcal{C}_{\mathbb{N}}$.

Case BC:

Suppose that $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi(z, \vec{x}, y; \vec{a}) \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{W}$ is defined by bounded comprehension, then

$$
\begin{aligned}
&|; \varphi(z, \vec{x}, y; \vec{a})| = z, \\
&\forall j < z \, [\mathtt{BIT}(j; \varphi(z, \vec{x}, y; \vec{a})) = \mathtt{0} \leftrightarrow \alpha(j, \vec{x}, y; \vec{a}) = \mathtt{0}],
\end{aligned}
$$

where $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(j, \vec{x}, y; \vec{a}) \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{N}$.
By induction hypothesis, there exist $\alpha' \in \mathcal{C}_{\mathbb{N}}$ such that

$$\alpha(j, \vec{x}, y; \vec{a}) = \alpha'(j, \vec{x}; \vec{a}, b).$$

Since $\mathcal{C}_{\mathbb{W}}$ is closed under bounded comprehension, define $\varphi' \in \mathcal{C}_{\mathbb{W}}$ by

$$
\begin{aligned}
&|; \varphi'(z, \vec{x}; \vec{a}, b)| = z, \\
&\forall j < z \, [\mathtt{BIT}(j; \varphi'(z, \vec{x}; \vec{a}, b)) = \mathtt{0} \leftrightarrow \alpha'(j, \vec{x}; \vec{a}, b) = \mathtt{0}].
\end{aligned}
$$

Then we have

$$|; \varphi(z, \vec{x}, y; \vec{a})| = z = |; \varphi'(z, \vec{x}; \vec{a}, b)|.$$

In addition, for all $j (< z)$, we have

$$
\begin{aligned}
\mathtt{BIT}(j; \varphi(z, \vec{x}, y; \vec{a})) = \mathtt{0} &\leftrightarrow \alpha(j, \vec{x}, y; \vec{a}) = \mathtt{0} \\
&\leftrightarrow \alpha'(j, \vec{x}; \vec{a}, b) = \mathtt{0} \\
&\leftrightarrow \mathtt{BIT}(j; \varphi'(z, \vec{x}; \vec{a}, b)) = \mathtt{0}.
\end{aligned}
$$

Therefore, we have $\varphi(z, \vec{x}, y; \vec{a}) = \varphi'(z, \vec{x}; \vec{a}, b)$ for all z. □

Proposition 3.22. *For each $f \in \mathcal{E}^{2+}$ with $f \in \mathbb{N}^m \to \mathbb{N}$, there exists $\alpha \in \mathcal{C}_{\mathbb{N}}$ such that*

$$f(\vec{x}) = \alpha(\vec{x};)$$

for each $\vec{x}$.

Proof. By induction on the structure of $f \in \mathcal{E}^{2+}$.
Basis.
Case $f = \mathtt{0}$:

Note that $m = 0$. Let $\alpha = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.

Case $f = \mathtt{I}_i^m$:

Note that $0 \le i < m$. Let $\alpha = {\mathtt{p}_{\mathbb{N}}}_i^{m,0}$, then

$$f(\vec{x}) = \mathtt{I}_i^m(\vec{x}) = x_i = {\mathtt{p}_{\mathbb{N}}}_i^{m,0}(\vec{x};) = \alpha(\vec{x};).$$

Case $f = \mathtt{S}$:

Note that $m = 1$. Let $\alpha = \mathtt{S}$, then $\mathtt{S}(x_1) = \mathtt{S}(x_1;)$.

Case $f = +$:

Note that $m = 2$. Let $\alpha = +$, then $x_1 + x_2 = x_1 + x_2$.

Case $f = \times$:

Note that $m = 2$. Let $\alpha = \times$, then $x_1 \times x_2 = x_1 \times x_2$.

Induction step.

Case COMP:

Suppose that

$$f(\vec{x}) = h(g_0(\vec{x}), \ldots, g_{L-1}(\vec{x})),$$

whrere $h, g_0, \ldots, g_{L-1} \in \mathcal{E}^{2+}$. Then, by the induction hypothesis, there exist $\gamma, \beta_0, \ldots, \beta_{L-1} \in \mathcal{C}_{\mathbb{N}}$ such that

$$\begin{aligned} h(\vec{y}) &= \gamma(\vec{y};), \\ g_i(\vec{x}) &= \beta_i(\vec{x};) \end{aligned}$$

for $0 \le i < L$.

Since $\mathcal{C}_{\mathbb{N}}$ is closed under composition, define $\alpha \in \mathcal{C}_{\mathbb{N}}$ by

$$\alpha(\vec{x};) = \gamma(\beta_0(\vec{x};), \ldots, \beta_{L-1}(\vec{x};);).$$

Then we have

$$\begin{aligned} f(\vec{x}) &= h(g_0(\vec{x}), \ldots, g_{L-1}(\vec{x})) \\ &= \gamma(\beta_0(\vec{x};), \ldots, \beta_{L-1}(\vec{x};);) \\ &= \alpha(\vec{x};). \end{aligned}$$

Case BR:

Suppose that $f \in \mathcal{E}^{2+}$ is defined by bounded recursion, then

$$\begin{aligned} f(0, \vec{x}) &= g(\vec{x}), \\ f(z+1, \vec{x}) &= h(z, \vec{x}, f(z, \vec{x})), \\ f(z, \vec{x}) &\le e(z, \vec{x}) \quad \text{for any } z, \vec{x}, \end{aligned}$$

where $g, h, e \in \mathcal{E}^{2+}$.

By induction hypothesis, there exist $\beta, \gamma, \delta \in \mathcal{C}_{\mathbb{N}}$ such that

$$\begin{aligned} g(\vec{x}) &= \beta(\vec{x};), \\ h(z, \vec{x}, y) &= \gamma(z, \vec{x}, y;), \\ e(z, \vec{x}) &= \delta(z, \vec{x};). \end{aligned}$$

Since $\mathcal{C}_{\mathbb{N}}$ is closed under bounded recursion, define α by

$$\begin{aligned}\alpha(0,\vec{x};) &= \beta(\vec{x};),\\ \alpha(z+1,\vec{x};) &= \gamma(z,\vec{x},\alpha(z,\vec{x};);).\end{aligned}$$

Now, we show $f(z,\vec{x}) = \alpha(z,\vec{x};)$ by induction on z.
In the case $z = 0$,

$$f(0,\vec{x}) = g(\vec{x}) = \beta(\vec{x};) = \alpha(0,\vec{x};).$$

In the case z, suppose that $f(z,\vec{x};) = \alpha(z,\vec{x};)$.
In the case $z+1$,

$$\begin{aligned}f(z+1,\vec{x}) &= h(z,\vec{x},f(z,\vec{x}))\\ &= \gamma(z,\vec{x},\alpha(z,\vec{x};);)\\ &= \alpha(z+1,\vec{x};).\end{aligned}$$

Therefore, we have $f(z,\vec{x}) = \alpha(z,\vec{x};)$ for all z.

Furthermore,

$$\alpha(z,\vec{x};) = f(z,\vec{x}) \leq e(z,\vec{x}) = \delta(z,\vec{x};) \quad \text{for all } z,\vec{x}.$$

Since α is bounded from above by $\delta \in \mathcal{C}_{\mathbb{N}}$, we conclude $\alpha \in \mathcal{C}_{\mathbb{N}}$.
Case 1-BCVR:

Suppose that $f \in \mathcal{E}^{2+}$ is defined by 1-bounded course-of-values recursion, then

$$\begin{aligned}f(0,\vec{x}) &= g(\vec{x}),\\ f(z+1,\vec{x}) &= h(z,\vec{x},\langle f(0,\vec{x}),\ldots,f(z,\vec{x})\rangle),\\ f(z,\vec{x}) &\leq 1 \quad \text{for any } z,\vec{x},\end{aligned}$$

where $g,h \in \mathcal{E}^{2+}$.
By induction hypothesis, there exist $\beta,\gamma \in \mathcal{C}_{\mathbb{N}}$ such that

$$\begin{aligned}g(\vec{x}) &= \beta(\vec{x};),\\ h(z,\vec{x},y) &= \gamma(z,\vec{x},y;).\end{aligned}$$

Here, let y be in $\mathbb{N}$ such that $y = \langle y_0,\ldots,y_z\rangle$ with $y_i \leq 1$ for $0 \leq i \leq z$, and let b be in $\mathbb{W}$ such that $|b| = z+1$ and $\mathrm{BIT}(i,b) = y_i$ for $0 \leq i \leq z$. Then, by Lemma 3.21 (i), there exists $\gamma' \in \mathcal{C}_{\mathbb{N}}$ such that

$$\gamma(z,\vec{x},y;) = \gamma'(z,\vec{x};b).$$

Since $\mathcal{C}_{\mathbb{N}}$ is closed under boolean course-of-values recursion, define α by the following formula using β and γ':

$$\begin{aligned}
\alpha(0,\vec{x};) &= \beta(\vec{x};),\\
\alpha(z+1,\vec{x};) &= \gamma'(z,\vec{x};b),\\
&\text{where } |;b| = z+1 \text{ and } \mathtt{BIT}(i;b) = \alpha(i,\vec{x};) \text{ for } 0 \le i \le z.
\end{aligned}$$

Now, we show $f(z,\vec{x}) = \alpha(z,\vec{x};)$ by course-of-values induction on z.
In the case $z = 0$,

$$f(0,\vec{x}) = g(\vec{x}) = \beta(\vec{x};) = \alpha(0,\vec{x};).$$

In the case $w(\le z)$, suppose that $f(w,\vec{x};) = \alpha(w,\vec{x};)$.
In the case $z+1$,

$$\begin{aligned}
f(z+1,\vec{x}) &= h(z,\vec{x},\langle f(0,\vec{x}),\dots,f(z,\vec{x})\rangle)\\
&= \gamma(z,\vec{x},\langle f(0,\vec{x}),\dots,f(z,\vec{x})\rangle;)\\
&= \gamma(z,\vec{x},\langle \alpha(0,\vec{x};),\dots,\alpha(z,\vec{x};)\rangle;)\\
&= \gamma'(z,\vec{x};b)\\
&\quad \text{where } |;b| = z+1 \text{ and } \mathtt{BIT}(i;b) = \alpha(i,\vec{x};) \text{ for } 0 \le i \le z\\
&= \alpha(z+1,\vec{x};).
\end{aligned}$$

Therefore, we have $f(z,\vec{x}) = \alpha(z,\vec{x};)$ for all z.

Furthermore,

$$\alpha(z,\vec{x};) = f(z,\vec{x}) \le 1 \quad \text{for all } z,\vec{x}.$$

Since α is bounded from above by 1, we conclude $\alpha \in \mathcal{C}_{\mathbb{N}}$. □

3.3 Representation of $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ functions by $\mathcal{E}^{2+}$ functions

In this section, we would like to show that any function in $\mathcal{C}_{\mathbb{N}}$ is represented by some function in $\mathcal{E}^{2+}$ and that any function in $\mathcal{C}_{\mathbb{W}}$ is represented by some function in $\mathcal{E}^{2+}$, that is,

$$\forall \alpha \in \mathcal{C}_{\mathbb{N}}\ \exists f \in \mathcal{E}^{2+}\ [\alpha(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})],$$
$$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists g \in \mathcal{E}^{2+}\ [\varphi(\vec{x}; \mathtt{bin}(\vec{k})) = \mathrm{bin}(g(\vec{x}, \vec{k}))].$$

There is a problem here. If a function $\varphi \in \mathcal{C}_{\mathbb{W}}$ is defined by bounded comprehension, we have $|; \varphi(z, \vec{x}; \mathtt{bin}(\vec{k}))| = z$. Hence we must construct a fucntion $g \in \mathcal{E}^{2+}$ such that $g = \Theta(2^z)$. However, we cannot construct such a function in $\mathcal{E}^{2+}$ because $\mathcal{E}^{2+}$ does not contain exponential function. Therefore, the latter formula above does not hold.

For this reason, we consider an intermediate class $\tilde{\mathcal{C}}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, and for each function in $\mathcal{C}_{\mathbb{N}}$ we construct a function in $\tilde{\mathcal{C}}$ of the same values, and for each function in $\mathcal{C}_{\mathbb{W}}$ we construct two functions in $\tilde{\mathcal{C}}$, one giving its bit contents and the other giving its length, that is,

$$\forall \alpha \in \mathcal{C}_{\mathbb{N}}\ \exists \tilde{\alpha} \in \tilde{\mathcal{C}}\ [\alpha(\vec{x}; \vec{a}) = \tilde{\alpha}(\vec{x}; \vec{a})],$$
$$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists \tilde{\varphi} \in \tilde{\mathcal{C}}\ [\mathtt{BIT}(z; \varphi(\vec{x}; \vec{a})) = \tilde{\varphi}(z, \vec{x}; \vec{a})] \text{ and}$$
$$\exists \hat{\varphi} \in \tilde{\mathcal{C}}\ [|; \varphi(\vec{x}; \vec{a})| = \hat{\varphi}(\vec{x}; \vec{a})]$$

(Proposition 3.26). And then, for each function in $\tilde{\mathcal{C}}$ we construct a function in $\mathcal{E}^{2+}$ of the same values, that is,

$$\forall \tilde{\alpha} \in \tilde{\mathcal{C}}\ \exists f \in \mathcal{E}^{2+}\ [\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})]$$

(Proposition 3.27).

The ideas of thie section, especially, Definition 3.23, Lemma 3.24 and Proposition 3.26 are based on Ishihara [6].

To begin with, we define an intermediate class $\tilde{\mathcal{C}}$.

Definition 3.23. A class $\tilde{\mathcal{C}}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ is generated by the following clauses.

1. The *projection* functions ${\mathtt{p}_{\mathbb{N}}}_i^{m,n}$ belong to $\tilde{\mathcal{C}}$:

$${\mathtt{p}_{\mathbb{N}}}_i^{m,n}(x_0, \ldots, x_{m-1}; \vec{a}) = x_i \quad (0 \le i < m);$$

2. the constant *zero* $\mathtt{0}$ belongs to $\tilde{\mathcal{C}}$: $\mathtt{0} = 0$;

3. the *successor* function $\mathtt{S}$ belongs to $\tilde{\mathcal{C}}$: $\mathtt{S}(x;) = \mathrm{S}x$;
4. the *addition* + belongs to $\tilde{\mathcal{C}}$: $+(x, y;) = x + y$;
5. the *multiplication* × belongs to $\tilde{\mathcal{C}}$: $\times(x, y;) = x \cdot y$;
6. the *projective length* function $|\cdot|_j^{m,n}$ belongs to $\tilde{\mathcal{C}}$:

$$|(\vec{x}; a_0, \ldots, a_{n-1})|_j^{m,n} = |a_j| \quad (0 \le j < n);$$

7. the *projective bit* function $\mathtt{BIT}_j^{m+1,n}$ belongs to $\tilde{\mathcal{C}}$:

$$\mathtt{BIT}_j^{m+1,n}(z, \vec{x}; a_0, \ldots, a_{n-1}) = \mathrm{BIT}(z, a_j) \quad (0 \le j < n);$$

8. $\tilde{\mathcal{C}}$ is closed under *composition* (COMP):
   if $\beta_0, \ldots, \beta_{L-1}, \gamma \in \tilde{\mathcal{C}}$ with $\beta_i \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\gamma \in \mathbb{N}^L \times \mathbb{W}^n \to \mathbb{N}$ for $0 \le i < L$, then there exist $\alpha \in \tilde{\mathcal{C}}$ satisfying

$$\alpha(\vec{x}; \vec{a}) = \gamma(\beta_0(\vec{x}, \vec{a}), \ldots, \beta_{L-1}(\vec{x}; \vec{a}); \vec{a});$$

9. $\tilde{\mathcal{C}}$ is closed under *bounded recursion* (BR):
   if $\beta, \gamma, \delta \in \tilde{\mathcal{C}}$ with $\beta \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, $\gamma \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{N}$, and $\delta \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\alpha \in \tilde{\mathcal{C}}$ satisfying

$$\alpha(\mathtt{0}, \vec{x}; \vec{a}) = \beta(\vec{x}; \vec{a}),$$
$$\alpha(\mathtt{S}(z;), \vec{x}; \vec{a}) = \gamma(z, \vec{x}, \alpha(z, \vec{x}; \vec{a}); \vec{a}),$$

   provided that $\alpha(z, \vec{x}; \vec{a}) \le \delta(z, \vec{x}; \vec{a})$ for all $z, \vec{x}, \vec{a}$;

10. $\tilde{\mathcal{C}}$ is closed under *1-bounded course-of-values recursion* (1-BCVR):
    if $\beta, \gamma \in \tilde{\mathcal{C}}$ with $\beta \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, and $\gamma \in \mathbb{N}^{m+2} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\alpha \in \tilde{\mathcal{C}}$ satisfying

$$\alpha(\mathtt{0}, \vec{x}; \vec{a}) = \beta(\vec{x}; \vec{a}),$$
$$\alpha(\mathtt{S}(z;), \vec{x}; \vec{a}) = \gamma(z, \vec{x}, \langle \alpha(0, \vec{x}; \vec{a}), \ldots, \alpha(z, \vec{x}; \vec{a}) \rangle; \vec{a}),$$

    provided that $\alpha(z, \vec{x}; \vec{a}) \le 1$ for all $z, \vec{x}, \vec{a}$.

In function algebra, the class $\tilde{\mathcal{C}}$ is represented as follows:

$$\tilde{\mathcal{C}} = [\mathtt{p}_{\mathbb{N}_i}^{m,n}, \mathtt{0}, \mathtt{S}, +, \times, |\cdot|_j^{m,n}, \mathtt{BIT}_j^{m+1,n}; \mathrm{COMP}, \mathrm{BR}, \text{1-BCVR}].$$

We show two lemmas used in the proposition 3.26.

Lemma 3.24. *Let $\chi_0, \dots, \chi_{N-1}$ be in $\mathcal{C}_{\mathbb{W}}$ with $\chi_j \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$ for $0 \le j < N$, and suppose that there exist $\chi_0^{|\cdot|}, \dots, \chi_{N-1}^{|\cdot|}$ and $\chi_0^{\mathtt{BIT}}, \dots, \chi_{N-1}^{\mathtt{BIT}}$ in $\tilde{\mathcal{C}}$ such that*

$$|\chi_j(\vec{x};\vec{a})| = \chi_j^{|\cdot|}(\vec{x};\vec{a}), \qquad \mathrm{BIT}(z, \chi_j(\vec{x};\vec{a})) = \chi_j^{\mathtt{BIT}}(z,\vec{x};\vec{a})$$

for each $0 \le j < N$. Then for any $\alpha \in \tilde{\mathcal{C}}$ with $\alpha \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{N}$, there exists $\tilde{\alpha} \in \tilde{\mathcal{C}}$ with $\tilde{\alpha} \in \mathbb{N}^{M+m} \times \mathbb{W}^n \to \mathbb{N}$ such that

$$\alpha(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(\vec{y},\vec{x};\vec{a}).$$

Proof. By induction on the structure of $\alpha \in \tilde{\mathcal{C}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}_i}^{M,N}$:
Note that $0 \le i < M$. Let $\tilde{\alpha} = \mathtt{p}_{\mathbb{N}_i}^{M+m,n}$. Then

$$\begin{aligned}\alpha(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= \mathtt{p}_{\mathbb{N}_i}^{M,N}(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= y_i = \mathtt{p}_{\mathbb{N}_i}^{M+m,n}(\vec{y},\vec{x};\vec{a}) = \tilde{\alpha}(\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{0}$:
Note that $M = N = 0$. Let $\tilde{\alpha} = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $\alpha = \mathtt{S}$:
Note that $M = 1, N = 0$. Let $\tilde{\alpha} = \mathtt{S}$, then $\mathtt{S}(y_1;) = \mathtt{S}(y_1;)$.
Case $\alpha = +$:
Note that $M = 2, N = 0$. Let $\tilde{\alpha} = +$, then $y_1 + y_2 = y_1 + y_2$.
Case $\alpha = \times$:
Note that $M = 2, N = 0$. Let $\tilde{\alpha} = \times$, then $y_1 \times y_2 = y_1 \times y_2$.
Case $\alpha = |\cdot|_j^{M,N}$:
Note that $0 \le j < N$. Let $\tilde{\alpha}(\vec{y},\vec{x};\vec{a}) = \chi_j^{|\cdot|}(\vec{x};\vec{a})$. Then

$$\begin{aligned}\alpha(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= |(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a}))|_j^{M,N} \\ &= |\chi_j(\vec{x};\vec{a})| = \chi_j^{|\cdot|}(\vec{x};\vec{a}) = \tilde{\alpha}(\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{BIT}_j^{M+1,N}$:
Note that $0 \le j < N$. Let $\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}) = \chi_j^{\mathtt{BIT}}(z,\vec{x};\vec{a})$. Then

$$\begin{aligned}&\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \mathtt{BIT}_j^{M+1,N}(z,\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \mathrm{BIT}(z,\chi_j(\vec{x};\vec{a})) = \chi_j^{\mathtt{BIT}}(z,\vec{x};\vec{a}) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Induction step.

Case COMP:

Suppose that $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(\vec{y};\vec{b}) \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{N}$ is defined by composition, then

$$\alpha(\vec{y};\vec{b}) = \gamma(\beta_0(\vec{y};\vec{b}),\dots,\beta_{L-1}(\vec{y};\vec{b});\vec{b}),$$

whrere $\gamma, \beta_0, \dots, \beta_{L-1} \in \tilde{\mathcal{C}}$. Then, by the induction hypothesis, there exist $\tilde{\gamma}, \tilde{\beta}_0, \dots, \tilde{\beta}_{L-1} \in \tilde{\mathcal{C}}$ such that

$$\gamma(\vec{z};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\gamma}(\vec{z},\vec{x};\vec{a}),$$
$$\beta_j(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\beta}_j(\vec{y},\vec{x};\vec{a})$$

for $0 \le j < L$.
Since $\tilde{\mathcal{C}}$ is closed under composition, define $\tilde{\alpha} \in \tilde{\mathcal{C}}$ by

$$\tilde{\alpha}(\vec{y},\vec{x};\vec{a}) = \tilde{\gamma}(\tilde{\beta}_0(\vec{y},\vec{x};\vec{a}),\dots,\tilde{\beta}_{L-1}(\vec{y},\vec{x};\vec{a}),\vec{x};\vec{a}).$$

Then we have

$$\begin{aligned}
&\alpha(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \gamma(\beta_0(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),\dots,\beta_{L-1}(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}));\\
&\quad \chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \gamma(\tilde{\beta}_0(\vec{y},\vec{x};\vec{a}),\dots,\tilde{\beta}_{L-1}(\vec{y},\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \tilde{\gamma}(\tilde{\beta}_0(\vec{y},\vec{x};\vec{a}),\dots,\tilde{\beta}_{L-1}(\vec{y},\vec{x};\vec{a}),\vec{x};\vec{a})\\
&= \tilde{\alpha}(\vec{y},\vec{x};\vec{a}).
\end{aligned}$$

Case BR:

Suppose that $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(z,\vec{y};\vec{b}) \in \mathbb{N}^{M+1} \times \mathbb{W}^N \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned}
\alpha(0,\vec{y};\vec{b}) &= \beta(\vec{y};\vec{b}),\\
\alpha(z+1,\vec{y};\vec{b}) &= \gamma(z,\vec{y},\alpha(z,\vec{y};\vec{b});\vec{b}),\\
\alpha(z,\vec{y};\vec{b}) &\le \delta(z,\vec{y};\vec{b}) \quad \text{for any } z,\vec{y},\vec{b},
\end{aligned}$$

where $\beta, \gamma, \delta \in \tilde{\mathcal{C}}$.
By induction hypothesis, there exist $\tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}
\beta(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\beta}(\vec{y},\vec{x};\vec{a}),\\
\gamma(z,\vec{y},w;\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\gamma}(z,\vec{y},w,\vec{x};\vec{a}),\\
\delta(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\delta}(z,\vec{y},\vec{x};\vec{a}).
\end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under bounded recursion, define $\tilde{\alpha}$ by

$$\begin{aligned}\tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}) &= \tilde{\beta}(\vec{y},\vec{x};\vec{a}),\\ \tilde{\alpha}(z+1,\vec{y},\vec{x};\vec{a}) &= \tilde{\gamma}(z,\vec{y},\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}),\vec{x};\vec{a}).\end{aligned}$$

Now, we show $\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})$ by induction on z.
In the case $z = 0$,

$$\begin{aligned}\alpha(0,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \beta(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \tilde{\beta}(\vec{y},\vec{x};\vec{a}) = \tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}).\end{aligned}$$

In the case z, suppose that $\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})$.
In the case $z+1$,

$$\begin{aligned}&\alpha(z+1,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \gamma(z,\vec{y},\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}));\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \gamma(z,\vec{y},\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \tilde{\gamma}(z,\vec{y},\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}),\vec{x};\vec{a})\\ &= \tilde{\alpha}(z+1,\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Therefore, we have $\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})$ for all z.

Furthermore,

$$\begin{aligned}\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}) &= \alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &\le \delta(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \tilde{\delta}(z,\vec{y},\vec{x};\vec{a}) \quad \text{for all } z,\vec{y},\vec{x},\vec{a}.\end{aligned}$$

Since $\tilde{\alpha}$ is bounded from above by $\tilde{\delta} \in \tilde{\mathcal{C}}$, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$.
Case 1-BCVR:

Suppose that $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(z,\vec{y};\vec{b}) \in \mathbb{N}^{M+1} \times \mathbb{W}^N \to \mathbb{N}$ is defined by 1-bounded course-of-values recursion, then

$$\begin{aligned}\alpha(0,\vec{y};\vec{b}) &= \beta(\vec{y};\vec{b}),\\ \alpha(z+1,\vec{y};\vec{b}) &= \gamma(z,\vec{y},\langle\alpha(0,\vec{y};\vec{b}),\dots,\alpha(z,\vec{y};\vec{b})\rangle;\vec{b}),\\ \alpha(z,\vec{y};\vec{b}) &\le 1 \quad \text{for any } z,\vec{y},\vec{b},\end{aligned}$$

where $\beta,\gamma \in \tilde{\mathcal{C}}$.
By induction hypothesis, there exist $\tilde{\beta},\tilde{\gamma} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}\beta(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\beta}(\vec{y},\vec{x};\vec{a}),\\ \gamma(z,\vec{y},w;\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\gamma}(z,\vec{y},w,\vec{x};\vec{a}).\end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under 1-bounded course-of-values recursion, define $\tilde{\alpha}$ by

$$\tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}) = \tilde{\beta}(\vec{y},\vec{x};\vec{a}),$$
$$\tilde{\alpha}(z+1,\vec{y},\vec{x};\vec{a}) = \tilde{\gamma}(z,\vec{y},\langle\tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}),\dots,\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})\rangle,\vec{x};\vec{a}).$$

Now, we show $\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})$ by course-of-values induction on z.
In the case $z = 0$,

$$\begin{aligned}\alpha(0,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \beta(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \tilde{\beta}(\vec{y},\vec{x};\vec{a}) = \tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}).\end{aligned}$$

In the case $v(\le z)$, suppose that $\alpha(v,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(v,\vec{y},\vec{x};\vec{a})$.
In the case $z+1$,

$$\begin{aligned}&\alpha(z+1,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \gamma(z,\vec{y},\langle\alpha(0,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),\dots, \\ &\qquad \alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\rangle;\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \gamma(z,\vec{y},\langle\tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}),\dots,\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})\rangle;\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \tilde{\gamma}(z,\vec{y},\langle\tilde{\alpha}(0,\vec{y},\vec{x};\vec{a}),\dots,\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})\rangle,\vec{x};\vec{a}) \\ &= \tilde{\alpha}(z+1,\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Therefore, we have $\alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(z,\vec{y},\vec{x};\vec{a})$ for all z.
Furthermore,

$$\tilde{\alpha}(z,\vec{y},\vec{x};\vec{a}) = \alpha(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \le 1 \quad \text{for all } z,\vec{y},\vec{x},\vec{a}.$$

Since $\tilde{\alpha}$ is bounded from above by 1, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$. □

At the beginning of section 3.2, we assume that there are two functions ρ and π in $\mathcal{C}_{\mathbb{N}}$. Similarly, in the following lemma, we assume that these two functions ρ and π also belong to $\tilde{\mathcal{C}}$. That is, if y is a coded natural number $y = \langle y_0,\dots,y_{\ell-1}\rangle$, then $\rho(y;) = \ell$ and $\pi(i,y;) = y_i$ for $0 \le i < \ell$. The example of ρ and π mentioned for $\mathcal{C}_{\mathbb{N}}$ at the beginning of section 3.2 also holds for $\tilde{\mathcal{C}}$.

Lemma 3.25. *Let b be in $\mathbb{W}$, and let y be in $\mathbb{N}$ such that $y = \langle \mathrm{BIT}(0,b), \mathrm{BIT}(1,b),\dots,\mathrm{BIT}(|b|-1,b)\rangle$. Then for any $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(\vec{x};\vec{a},b) \in \mathbb{N}^m \times \mathbb{W}^{n+1} \to \mathbb{N}$, there exists $\tilde{\alpha} \in \tilde{\mathcal{C}}$ with $\tilde{\alpha}(\vec{x},y;\vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ such that*

$$\alpha(\vec{x};\vec{a},b) = \tilde{\alpha}(\vec{x},y;\vec{a}).$$

Proof. By induction on the structure of $\alpha \in \tilde{\mathcal{C}}$.
Basis.
Case $\alpha = \mathtt{p_{\mathbb{N}}}_i^{m,n+1}$:
Note that $0 \le i < m$. Let $\tilde{\alpha} = \mathtt{p_{\mathbb{N}}}_i^{m+1,n}$. Then

$$\alpha(\vec{x};\vec{a},b) = \mathtt{p_{\mathbb{N}}}_i^{m,n+1}(\vec{x};\vec{a},b) = x_i = \mathtt{p_{\mathbb{N}}}_i^{m+1,n}(\vec{x},y;\vec{a}) = \tilde{\alpha}(\vec{x},y;\vec{a}).$$

Case $\alpha = \mathtt{0}$:
Note that $m = n + 1 = 0$. Let $\tilde{\alpha} = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $\alpha = \mathtt{S}$:
Note that $m = 1, n + 1 = 0$. Let $\tilde{\alpha} = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1;)$.
Case $\alpha = +$:
Note that $m = 2, n + 1 = 0$. Let $\tilde{\alpha} = +$, then $x_1 + x_2 = x_1 + x_2$.
Case $\alpha = \times$:
Note that $m = 2, n + 1 = 0$. Let $\tilde{\alpha} = \times$, then $x_1 \times x_2 = x_1 \times x_2$.
Case $\alpha = |\cdot|_j^{m,n+1}$:
Note that $0 \le j \le n$.
In the case $0 \le j \le n - 1$, let $\tilde{\alpha}(\cdot) = |\cdot|_j^{m+1,n}$. Then

$$\alpha(\vec{x};\vec{a},b) = |(\vec{x};\vec{a},b)|_j^{m,n+1} = |a_j| = |(\vec{x},y;\vec{a})|_j^{m+1,n} = \tilde{\alpha}(\vec{x},y;\vec{a}).$$

In the case $j = n$, let $\tilde{\alpha}(\cdot) = \rho(\mathtt{p_{\mathbb{N}}}_m^{m+1,n}(\cdot);)$. Then

$$\begin{aligned}\alpha(\vec{x};\vec{a},b) &= |(\vec{x};\vec{a},b)|_n^{m,n+1} = |b| \\ &= \rho(y;) = \rho(\mathtt{p_{\mathbb{N}}}_m^{m+1,n}(\vec{x},y;\vec{a});) = \tilde{\alpha}(\vec{x},y;\vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{BIT}_j^{m+1,n+1}$:
Note that $0 \le j \le n$.
In the case $0 \le j \le n - 1$, let $\tilde{\alpha}(\cdot) = \mathtt{BIT}_j^{m+2,n}(\cdot)$. Then

$$\begin{aligned}\alpha(z,\vec{x};\vec{a},b) &= \mathtt{BIT}_j^{m+1,n+1}(z,\vec{x};\vec{a},b) = \mathrm{BIT}(z,a_j) \\ &= \mathtt{BIT}_j^{m+2,n}(z,\vec{x},y;\vec{a}) = \tilde{\alpha}(z,\vec{x},y;\vec{a}).\end{aligned}$$

In the case $j = n$, let $\tilde{\alpha}(\cdot) = \pi(\mathtt{p_{\mathbb{N}}}_0^{m+2,n}(\cdot), \mathtt{p_{\mathbb{N}}}_{m+1}^{m+2,n}(\cdot);)$. Then

$$\begin{aligned}\alpha(z,\vec{x};\vec{a},b) &= \mathtt{BIT}_n^{m+1,n+1}(z,\vec{x};\vec{a},b) = \mathrm{BIT}(z,b) = \pi(z,y;) \\ &= \pi(\mathtt{p_{\mathbb{N}}}_0^{m+2,n}(z,\vec{x},y;\vec{a}), \mathtt{p_{\mathbb{N}}}_{m+1}^{m+2,n}(z,\vec{x},y;\vec{a});) \\ &= \tilde{\alpha}(z,\vec{x},y;\vec{a}).\end{aligned}$$

Induction step.
Case COMP:

Suppose that

$$\alpha(\vec{x};\vec{a},b) = \gamma(\beta_0(\vec{x};\vec{a},b),\ldots,\beta_{L-1}(\vec{x};\vec{a},b);\vec{a},b),$$

whrere $\gamma,\beta_0,\ldots,\beta_{L-1} \in \tilde{\mathcal{C}}$. Then, by the induction hypothesis, there exist $\tilde{\gamma},\tilde{\beta}_0,\ldots,\tilde{\beta}_{L-1} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}\gamma(\vec{z};\vec{a},b) &= \tilde{\gamma}(\vec{z},y;\vec{a}),\\ \beta_j(\vec{x};\vec{a},b) &= \tilde{\beta}_j(\vec{x},y;\vec{a})\end{aligned}$$

for $0 \le j < L$.
Since $\tilde{\mathcal{C}}$ is closed under composition, define $\tilde{\alpha} \in \tilde{\mathcal{C}}$ by

$$\tilde{\alpha}(\vec{x},y;\vec{a}) = \tilde{\gamma}(\tilde{\beta}_0(\vec{x},y;\vec{a}),\ldots,\tilde{\beta}_{L-1}(\vec{x},y;\vec{a}),y;\vec{a}).$$

Then we have

$$\begin{aligned}\alpha(\vec{x};\vec{a},b) &= \gamma(\beta_0(\vec{x};\vec{a},b),\ldots,\beta_{L-1}(\vec{x};\vec{a},b);\vec{a},b)\\ &= \gamma(\tilde{\beta}_0(\vec{x},y;\vec{a}),\ldots,\tilde{\beta}_{L-1}(\vec{x},y;\vec{a});\vec{a},b)\\ &= \tilde{\gamma}(\tilde{\beta}_0(\vec{x},y;\vec{a}),\ldots,\tilde{\beta}_{L-1}(\vec{x},y;\vec{a}),y;\vec{a})\\ &= \tilde{\alpha}(\vec{x},y;\vec{a}).\end{aligned}$$

Case BR:

Suppose that $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(z,\vec{x};\vec{a},b) \in \mathbb{N}^{n+1} \times \mathbb{W}^{m+1} \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned}\alpha(0,\vec{x};\vec{a},b) &= \beta(\vec{x};\vec{a},b),\\ \alpha(z+1,\vec{x};\vec{a},b) &= \gamma(z,\vec{x},\alpha(z,\vec{x};\vec{a},b);\vec{a},b),\\ \alpha(z,\vec{x};\vec{a},b) &\le \delta(z,\vec{x};\vec{a},b) \quad \text{for any } z,\vec{x},\vec{a},b,\end{aligned}$$

where $\beta,\gamma,\delta \in \tilde{\mathcal{C}}$.
By induction hypothesis, there exist $\tilde{\beta},\tilde{\gamma},\tilde{\delta} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}\beta(\vec{x};\vec{a},b) &= \tilde{\beta}(\vec{x},y;\vec{a}),\\ \gamma(z,\vec{x},w;\vec{a},b) &= \tilde{\gamma}(z,\vec{x},w,y;\vec{a}),\\ \delta(z,\vec{x};\vec{a},b) &= \tilde{\delta}(z,\vec{x},y;\vec{a}).\end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under bounded recursion, define $\tilde{\alpha}$ by

$$\begin{aligned}\tilde{\alpha}(0,\vec{x},y;\vec{a}) &= \tilde{\beta}(\vec{x},y;\vec{a}),\\ \tilde{\alpha}(z+1,\vec{x},y;\vec{a}) &= \tilde{\gamma}(z,\vec{x},\tilde{\alpha}(z,\vec{x},y;\vec{a}),y;\vec{a}).\end{aligned}$$

Now, we show $\alpha(z,\vec{x};\vec{a},b) = \tilde{\alpha}(z,\vec{x},y;\vec{a})$ by induction on z.
In the case $z = 0$,

$$\alpha(0,\vec{x};\vec{a},b) = \beta(\vec{x};\vec{a},b) = \tilde{\beta}(\vec{x},y;\vec{a}) = \tilde{\alpha}(0,\vec{x},y;\vec{a}).$$

In the case z, suppose that $\alpha(z,\vec{x};\vec{a},b) = \tilde{\alpha}(z,\vec{x},y;\vec{a})$.
In the case $z+1$,

$$\begin{aligned}\alpha(z+1,\vec{x};\vec{a},b) &= \gamma(z,\vec{x},\alpha(z,\vec{x};\vec{a},b);\vec{a},b)\\ &= \gamma(z,\vec{x},\tilde{\alpha}(z,\vec{x},y;\vec{a});\vec{a},b)\\ &= \tilde{\gamma}(z,\vec{x},\tilde{\alpha}(z,\vec{x},y;\vec{a}),y;\vec{a})\\ &= \tilde{\alpha}(z+1,\vec{x},y;\vec{a}).\end{aligned}$$

Therefore, we have $\alpha(z,\vec{x};\vec{a},b) = \tilde{\alpha}(z,\vec{x},y;\vec{a})$ for all z.
Furthermore,

$$\begin{aligned}\tilde{\alpha}(z,\vec{x},y;\vec{a}) &= \alpha(z,\vec{x};\vec{a},b)\\ &\leq \delta(z,\vec{x};\vec{a},b) = \tilde{\delta}(z,\vec{x},y;\vec{a}) \quad \text{for all } z,\vec{x},y,\vec{a}.\end{aligned}$$

Since $\tilde{\alpha}$ is bounded from above by $\tilde{\delta} \in \tilde{\mathcal{C}}$, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$.
Case 1-BCVR:
Suppose that $\alpha \in \tilde{\mathcal{C}}$ with $\alpha(z,\vec{x};\vec{a},b) \in \mathbb{N}^{n+1} \times \mathbb{W}^{m+1} \to \mathbb{N}$ is defined by 1-bounded course-of-values recursion, then

$$\begin{aligned}\alpha(0,\vec{x};\vec{a},b) &= \beta(\vec{x};\vec{a},b),\\ \alpha(z+1,\vec{x};\vec{a},b) &= \gamma(z,\vec{x},\langle\alpha(0,\vec{x};\vec{a},b),\ldots,\alpha(z,\vec{x};\vec{a},b)\rangle;\vec{a},b),\\ \alpha(z,\vec{x};\vec{a},b) &\leq 1 \quad \text{for any } z,\vec{x},\vec{a},b,\end{aligned}$$

where $\beta,\gamma \in \tilde{\mathcal{C}}$.
By induction hypothesis, there exist $\tilde{\beta},\tilde{\gamma} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}\beta(\vec{x};\vec{a},b) &= \tilde{\beta}(\vec{x},y;\vec{a}),\\ \gamma(z,\vec{x},w;\vec{a},b) &= \tilde{\gamma}(z,\vec{x},w,y;\vec{a}).\end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under 1-bounded course-of-values recursion, define $\tilde{\alpha}$ by

$$\begin{aligned}\tilde{\alpha}(0,\vec{x},y;\vec{a}) &= \tilde{\beta}(\vec{x},y;\vec{a}),\\ \tilde{\alpha}(z+1,\vec{x},y;\vec{a}) &= \tilde{\gamma}(z,\vec{x},\langle\tilde{\alpha}(0,\vec{x},y;\vec{a}),\ldots,\tilde{\alpha}(z,\vec{x},y;\vec{a})\rangle,y;\vec{a}).\end{aligned}$$

Now, we show $\alpha(z,\vec{x};\vec{a},b) = \tilde{\alpha}(z,\vec{x},y;\vec{a})$ by course-of-values induction on z.
In the case $z = 0$,

$$\alpha(0,\vec{x};\vec{a},b) = \beta(\vec{x};\vec{a},b) = \tilde{\beta}(\vec{x},y;\vec{a}) = \tilde{\alpha}(0,\vec{x},y;\vec{a}).$$

In the case $v(\leq z)$, suppose that $\alpha(v,\vec{x};\vec{a},b) = \tilde{\alpha}(v,\vec{x},y;\vec{a})$.
In the case $z+1$,

$$\begin{aligned}\alpha(z+1,\vec{x};\vec{a},b) &= \gamma(z,\vec{x},\langle\alpha(0,\vec{x};\vec{a},b),\dots,\alpha(z,\vec{x};\vec{a},b)\rangle;\vec{a},b)\\ &= \gamma(z,\vec{x},\langle\tilde{\alpha}(0,\vec{x},y;\vec{a}),\dots,\tilde{\alpha}(z,\vec{x},y;\vec{a})\rangle;\vec{a},b)\\ &= \tilde{\gamma}(z,\vec{x},\langle\tilde{\alpha}(0,\vec{x},y;\vec{a}),\dots,\tilde{\alpha}(z,\vec{x},y;\vec{a})\rangle,y;\vec{a})\\ &= \tilde{\alpha}(z+1,\vec{x},y;\vec{a}).\end{aligned}$$

Therefore, we have $\alpha(z,\vec{x};\vec{a},b) = \tilde{\alpha}(z,\vec{x},y;\vec{a})$ for all z.

Furthermore,

$$\tilde{\alpha}(z,\vec{x},y;\vec{a}) = \alpha(z,\vec{x};\vec{a},b) \leq 1 \quad \text{for all } z,\vec{x},y,\vec{a}.$$

Since $\tilde{\alpha}$ is bounded from above by 1, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$. □

Proposition 3.26. *For each $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, there exist $\tilde{\alpha},\tilde{\varphi},\hat{\varphi} \in \tilde{\mathcal{C}}$ such that*

$$\begin{aligned}\alpha(\vec{x};\vec{a}) &= \tilde{\alpha}(\vec{x};\vec{a}),\\ \mathtt{BIT}(z;\varphi(\vec{x};\vec{a})) &= \tilde{\varphi}(z,\vec{x};\vec{a}),\\ |;\varphi(\vec{x};\vec{a})| &= \hat{\varphi}(\vec{x};\vec{a})\end{aligned}$$

for each $\vec{x},\vec{a}$ and z.

Proof. By simultaneous induction on the structures of $\alpha \in \mathcal{C}_{\mathbb{N}}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}_i}^{m,n}$:

Note that $0 \leq i < m$. Let $\tilde{\alpha} = \mathtt{p}_{\mathbb{N}_i}^{m,n}$. Then

$$\alpha(\vec{x};\vec{a}) = \mathtt{p}_{\mathbb{N}_i}^{m,n}(\vec{x};\vec{a}) = x_i = \tilde{\alpha}(\vec{x};\vec{a}).$$

Case $\varphi = \mathtt{p}_{\mathbb{W}_j}^{m,n}$:

Note that $0 \leq j < n$. Let $\tilde{\varphi} = \mathtt{BIT}_j^{m+1,n}$ and $\hat{\varphi} = |\cdot|_j^{m,n}$. Then

$$\begin{aligned}\mathtt{BIT}(z;\varphi(\vec{x};\vec{a})) &= \mathtt{BIT}(z;\mathtt{p}_{\mathbb{W}_j}^{m,n}(\vec{x};\vec{a})) = \mathtt{BIT}(z;a_j)\\ &= \mathtt{BIT}_j^{m+1,n}(z,\vec{x};\vec{a}) = \tilde{\varphi}(z,\vec{x};\vec{a}),\\ |;\varphi(\vec{x};\vec{a})| &= |;\mathtt{p}_{\mathbb{W}_j}^{m,n}(\vec{x};\vec{a})| = |;a_j|\\ &= |(\vec{x};\vec{a})|_j^{m,n} = \hat{\varphi}(\vec{x};\vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{0}$:

Note that $m = n = 0$. Let $\tilde{\alpha} = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.

Case $\alpha = \mathtt{S}$:

Note that $m = 1, n = 0$. Let $\tilde{\alpha} = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1;)$.

Case $\alpha = +$:

Note that $m = 2, n = 0$. Let $\tilde{\alpha} = +$, then $x_1 + x_2 = x_1 + x_2$.

Case $\alpha = \times$:

Note that $m = 2, n = 0$. Let $\tilde{\alpha} = \times$, then $x_1 \times x_2 = x_1 \times x_2$.

Case $\alpha = |\cdot|$:

Note that $m = 0, n = 1$. Let $\tilde{\alpha} = |\cdot|_0^{0,1}$. Then

$$\alpha(;a_1) = |;a_1| = |(;a_1)|_0^{0,1} = \tilde{\alpha}(;a_1).$$

Case $\alpha = \mathtt{BIT}$:

Note that $m = 1, n = 1$. Let $\tilde{\alpha} = \mathtt{BIT}_0^{1,1}$. Then

$$\alpha(z;a_1) = \mathtt{BIT}(z;a_1) = \mathtt{BIT}_0^{1,1}(z;a_1) = \tilde{\alpha}(z;a_1).$$

Induction step.

Case COMP($\in \mathcal{C}_{\mathbb{N}}$):

Suppose that

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x};\vec{a}), \dots, \beta_{M-1}(\vec{x};\vec{a}); \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\gamma, \beta_0, \dots, \beta_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\chi_0, \dots, \chi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\tilde{\gamma}, \tilde{\beta}_0, \dots, \tilde{\beta}_{M-1}, \tilde{\chi}_0, \dots, \tilde{\chi}_{N-1}, \hat{\chi}_0, \dots, \hat{\chi}_{N-1} \in \tilde{\mathcal{C}}$ such that

$$\gamma(\vec{y};\vec{b}) = \tilde{\gamma}(\vec{y};\vec{b}), \qquad \beta_i(\vec{x};\vec{a}) = \tilde{\beta}_i(\vec{x};\vec{a}),$$
$$\mathtt{BIT}(z;\chi_j(\vec{x};\vec{a})) = \tilde{\chi}_j(z,\vec{x};\vec{a}), \qquad |;\chi_j(\vec{x};\vec{a})| = \hat{\chi}_j(\vec{x};\vec{a})$$

for $0 \le i < M$ and $0 \le j < N$. Since the bit contents and lengths of $\chi_j(\vec{x};\vec{a})$ for $0 \le j < N$ are known, by Lemma 3.24, there exist $\tilde{\gamma}' \in \tilde{\mathcal{C}}$ such that

$$\tilde{\gamma}(\vec{y};\chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) = \tilde{\gamma}'(\vec{y},\vec{x};\vec{a}).$$

Since $\tilde{\mathcal{C}}$ is closed under composition, define $\tilde{\alpha} \in \tilde{\mathcal{C}}$ by the following formula using $\tilde{\gamma}'$ and $\tilde{\beta}_i$ for $0 \le i < M$:

$$\tilde{\alpha}(\vec{x};\vec{a}) = \tilde{\gamma}'(\tilde{\beta}_0(\vec{x};\vec{a}), \dots, \tilde{\beta}_{M-1}(\vec{x};\vec{a}), \vec{x};\vec{a}).$$

Then we have

$$\begin{aligned}
\alpha(\vec{x};\vec{a}) &= \gamma(\beta_0(\vec{x};\vec{a}), \dots, \beta_{M-1}(\vec{x};\vec{a}); \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\
&= \gamma(\tilde{\beta}_0(\vec{x};\vec{a}), \dots, \tilde{\beta}_{M-1}(\vec{x};\vec{a}); \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\
&= \tilde{\gamma}(\tilde{\beta}_0(\vec{x};\vec{a}), \dots, \tilde{\beta}_{M-1}(\vec{x};\vec{a}); \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\
&= \tilde{\gamma}'(\tilde{\beta}_0(\vec{x};\vec{a}), \dots, \tilde{\beta}_{M-1}(\vec{x};\vec{a}), \vec{x};\vec{a}) \\
&= \tilde{\alpha}(\vec{x};\vec{a}).
\end{aligned}$$

Case COMP($\in \mathcal{C}_{\mathbb{W}}$):

Suppose that

$$\varphi(\vec{x};\vec{a}) = \psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\beta_0,\dots,\beta_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\psi,\chi_0,\dots,\chi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\tilde{\beta}_0,\dots,\tilde{\beta}_{M-1},\tilde{\psi},\tilde{\chi}_0,\dots,\tilde{\chi}_{N-1},\hat{\psi},\hat{\chi}_0,\dots,\hat{\chi}_{N-1} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned} \beta_i(\vec{x};\vec{a}) &= \tilde{\beta}_i(\vec{x};\vec{a}), & & \\ \mathtt{BIT}(z;\psi(\vec{y};\vec{b})) &= \tilde{\psi}(z,\vec{y};\vec{b}), & |;\psi(\vec{y};\vec{b})| &= \hat{\psi}(\vec{y};\vec{b}), \\ \mathtt{BIT}(z;\chi_j(\vec{x};\vec{a})) &= \tilde{\chi}_j(z,\vec{x};\vec{a}), & |;\chi_j(\vec{x};\vec{a})| &= \hat{\chi}_j(\vec{x};\vec{a}) \end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since the bit contents and lengths of $\chi_j(\vec{x};\vec{a})$ for $0 \le j < N$ are known, by Lemma 3.24, there exist $\tilde{\psi}',\hat{\psi}' \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned} \tilde{\psi}(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\psi}'(z,\vec{y},\vec{x};\vec{a}), \\ \hat{\psi}(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \hat{\psi}'(\vec{y},\vec{x};\vec{a}). \end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under composition, define $\tilde{\varphi},\hat{\varphi} \in \tilde{\mathcal{C}}$ by the following formulas using $\tilde{\psi}',\hat{\psi}'$ and $\tilde{\beta}_i$ for $0 \le i < M$:

$$\begin{aligned} \tilde{\varphi}(z,\vec{x};\vec{a}) &= \tilde{\psi}'(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}), \\ \hat{\varphi}(\vec{x};\vec{a}) &= \hat{\psi}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}). \end{aligned}$$

Then we have

$$\begin{aligned} \mathtt{BIT}(z;\varphi(\vec{x};\vec{a})) &= \mathtt{BIT}(z;\psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))) \\ &= \tilde{\psi}(z,\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \tilde{\psi}(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \tilde{\psi}'(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}) \\ &= \tilde{\varphi}(z,\vec{x};\vec{a}), \\ |;\varphi(\vec{x};\vec{a})| &= |;\psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))| \\ &= \hat{\psi}(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \hat{\psi}(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) \\ &= \hat{\psi}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}) \\ &= \hat{\varphi}(\vec{x};\vec{a}). \end{aligned}$$

Case BR:

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(z, \vec{x}; \vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned} \alpha(0, \vec{x}; \vec{a}) &= \beta(\vec{x}; \vec{a}), \\ \alpha(z+1, \vec{x}; \vec{a}) &= \gamma(z, \vec{x}, \alpha(z, \vec{x}; \vec{a}); \vec{a}), \\ \alpha(z, \vec{x}; \vec{a}) &\leq \delta(z, \vec{x}; \vec{a}) \quad \text{for any } z, \vec{x}, \vec{a}, \end{aligned}$$

where $\beta, \gamma, \delta \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $\tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned} \beta(\vec{x}; \vec{a}) &= \tilde{\beta}(\vec{x}; \vec{a}), \\ \gamma(z, \vec{x}, w; \vec{a}) &= \tilde{\gamma}(z, \vec{x}, w; \vec{a}), \\ \delta(z, \vec{x}; \vec{a}) &= \tilde{\delta}(z, \vec{x}; \vec{a}). \end{aligned}$$

Since $\tilde{\mathcal{C}}$ is closed under bounded recursion, define $\tilde{\alpha}$ by

$$\begin{aligned} \tilde{\alpha}(0, \vec{x}; \vec{a}) &= \tilde{\beta}(\vec{x}; \vec{a}), \\ \tilde{\alpha}(z+1, \vec{x}; \vec{a}) &= \tilde{\gamma}(z, \vec{x}, \tilde{\alpha}(z, \vec{x}; \vec{a}); \vec{a}). \end{aligned}$$

Now, we show $\alpha(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{x}; \vec{a})$ by induction on z.
In the case $z = 0$,

$$\alpha(0, \vec{x}; \vec{a}) = \beta(\vec{x}; \vec{a}) = \tilde{\beta}(\vec{x}; \vec{a}) = \tilde{\alpha}(0, \vec{x}; \vec{a}).$$

In the case z, suppose that $\alpha(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{x}; \vec{a})$.
In the case $z + 1$,

$$\begin{aligned} \alpha(z+1, \vec{x}; \vec{a}) &= \gamma(z, \vec{x}, \alpha(z, \vec{x}; \vec{a}); \vec{a}) \\ &= \tilde{\gamma}(z, \vec{x}, \tilde{\alpha}(z, \vec{x}; \vec{a}); \vec{a}) \\ &= \tilde{\alpha}(z+1, \vec{x}; \vec{a}). \end{aligned}$$

Therefore, we have $\alpha(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{x}; \vec{a})$ for all z.

Furthermore,

$$\begin{aligned} \tilde{\alpha}(z, \vec{x}; \vec{a}) &= \alpha(z, \vec{x}; \vec{a}) \\ &\leq \delta(z, \vec{x}; \vec{a}) = \tilde{\delta}(z, \vec{x}; \vec{a}) \quad \text{for all } z, \vec{x}, \vec{a}. \end{aligned}$$

Since α' is bounded from above by $\tilde{\delta} \in \tilde{\mathcal{C}}$, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$.
Case BCVR:

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(z, \vec{x}; \vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by boolean course-of-values recursion, then

$$\begin{aligned}
\alpha(0, \vec{x}; \vec{a}) &= \beta(\vec{x}; \vec{a}), \\
\alpha(z+1, \vec{x}; \vec{a}) &= \gamma(z, \vec{x}; b, \vec{a}), \\
&\text{where } |;b| = z+1 \text{ and } \mathtt{BIT}(i; b) = \alpha(i, \vec{x}; \vec{a}) \text{ for } 0 \leq i \leq z, \\
\alpha(z, \vec{x}; \vec{a}) &\leq 1 \quad \text{for any } z, \vec{x}, \vec{a},
\end{aligned}$$

where $\beta, \gamma \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $\tilde{\beta}, \tilde{\gamma} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}
\beta(\vec{x}; \vec{a}) &= \tilde{\beta}(\vec{x}; \vec{a}), \\
\gamma(z, \vec{x}; b, \vec{a}) &= \tilde{\gamma}(z, \vec{x}; b, \vec{a}).
\end{aligned}$$

Here, for any $b \in \mathbb{W}$, let y be in $\mathbb{N}$ such that $y = \langle \mathrm{BIT}(0, b), \ldots, \mathrm{BIT}(|b| - 1, b) \rangle$.
Then, by Lemma 3.25, there exists $\tilde{\gamma}' \in \tilde{\mathcal{C}}$ such that

$$\tilde{\gamma}(z, \vec{x}; b, \vec{a}) = \tilde{\gamma}'(z, \vec{x}, y; \vec{a}).$$

Since $\tilde{\mathcal{C}}$ is closed under 1-bounded course-of-values recursion, define $\tilde{\alpha}$ by the following formula using $\tilde{\beta}$ and $\tilde{\gamma}'$:

$$\begin{aligned}
\tilde{\alpha}(0, \vec{x}; \vec{a}) &= \tilde{\beta}(\vec{x}; \vec{a}), \\
\tilde{\alpha}(z+1, \vec{x}; \vec{a}) &= \tilde{\gamma}'(z, \vec{x}, \langle \tilde{\alpha}(0, \vec{x}; \vec{a}), \ldots, \tilde{\alpha}(z, \vec{x}; \vec{a}) \rangle; \vec{a}).
\end{aligned}$$

Now, we show $\alpha(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{x}; \vec{a})$ by course-of-values induction on z.
In the case $z = 0$,

$$\alpha(0, \vec{x}; \vec{a}) = \beta(\vec{x}; \vec{a}) = \tilde{\beta}(\vec{x}; \vec{a}) = \tilde{\alpha}(0, \vec{x}; \vec{a}).$$

In the cases $w (\leq z)$, suppose that $\alpha(w, \vec{x}; \vec{a}) = \tilde{\alpha}(w, \vec{x}; \vec{a})$.
In the case $z + 1$,

$$\begin{aligned}
\alpha(z+1, \vec{x}; \vec{a}) &= \gamma(z, \vec{x}; b, \vec{a}), \\
&\qquad \text{where } |;b| = z+1 \text{ and } \mathtt{BIT}(i; b) = \alpha(i, \vec{x}; \vec{a}) \text{ for } 0 \leq i \leq z, \\
&= \tilde{\gamma}(z, \vec{x}; b, \vec{a}) \\
&= \tilde{\gamma}'(z, \vec{x}, \langle \alpha(0, \vec{x}; \vec{a}), \ldots, \alpha(z, \vec{x}; \vec{a}) \rangle; \vec{a}) \\
&= \tilde{\gamma}'(z, \vec{x}, \langle \tilde{\alpha}(0, \vec{x}; \vec{a}), \ldots, \tilde{\alpha}(z, \vec{x}; \vec{a}) \rangle; \vec{a}) \\
&= \tilde{\alpha}(z+1, \vec{x}; \vec{a}).
\end{aligned}$$

Therefore, we have $\alpha(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{x}; \vec{a})$ for all z.

Furthermore,

$$\tilde{\alpha}(z,\vec{x};\vec{a}) = \alpha(z,\vec{x};\vec{a}) \le 1 \quad \text{for all } z,\vec{x},\vec{a}.$$

Since $\tilde{\alpha}$ is bounded from above by 1, we conclude $\tilde{\alpha} \in \tilde{\mathcal{C}}$.
Case BC:

Suppose that $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi(z,\vec{x};\vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{W}$ is defined by bounded comprehension, then

$$\begin{aligned} &|;\varphi(z,\vec{x};\vec{a})| = z,\\ &\forall j < z\ [\mathtt{BIT}(j;\varphi(z,\vec{x};\vec{a})) = \mathtt{0} \leftrightarrow \alpha(j,\vec{x};\vec{a}) = \mathtt{0}], \end{aligned}$$

where $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha(j,\vec{x};\vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$.
By induction hypothesis, there exist $\tilde{\alpha} \in \tilde{\mathcal{C}}$ such that

$$\alpha(j,\vec{x};\vec{a}) = \tilde{\alpha}(j,\vec{x};\vec{a}).$$

With respect to $|;\varphi(z,\vec{x};\vec{a})|$, define $\tilde{\varphi} \in \tilde{\mathcal{C}}$ by $\tilde{\varphi}(z,\vec{x};\vec{a}) = z$. Then we have

$$|;\varphi(z,\vec{x};\vec{a})| = z = \tilde{\varphi}(z,\vec{x};\vec{a}).$$

With respect to $\mathtt{BIT}(j;\varphi(z,\vec{x};\vec{a}))$, since

$$\mathtt{BIT}(j;\varphi(z,\vec{x};\vec{a})) = \begin{cases} \mathtt{sg}(\alpha(j,\vec{x};\vec{a});) & \text{if } 0 \le j < z,\\ \mathtt{0} & \text{if } j \ge z, \end{cases}$$

define $\hat{\varphi} \in \tilde{\mathcal{C}}$ by[5]

$$\hat{\varphi}(j,z,\vec{x};\vec{a}) = \mathtt{cond}(j < z, \mathtt{0}, \mathtt{sg}(\tilde{\alpha}(j,\vec{x};\vec{a});););.$$

Then we have

$$\mathtt{BIT}(j;\varphi(z,\vec{x};\vec{a})) = \hat{\varphi}(j,z,\vec{x};\vec{a}).$$

□

Proposition 3.27. *For each $\tilde{\alpha} \in \tilde{\mathcal{C}}$ with $\tilde{\alpha} \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, there exists $f \in \mathcal{E}^{2+}$ such that*

$$\tilde{\alpha}(\vec{x};\mathtt{bin}(\vec{k})) = f(\vec{x},\vec{k})$$

for each $\vec{x}$ and $\vec{k}$.

[5] Notice that Lemma 3.4 and Lemma 3.5 also hold for $\tilde{\mathcal{C}}$.

Proof. By induction on the structure of $\tilde{\alpha} \in \tilde{\mathcal{C}}$.
Basis.
Case $\tilde{\alpha} = {\mathtt{p}_{\mathbb{N}}}_i^{m,n}$:
Note that $0 \le i < m$. Let $f = \mathtt{I}_i^{m,n}$. Then

$$\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = {\mathtt{p}_{\mathbb{N}}}_i^{m,n}(\vec{x}; \mathtt{bin}(\vec{k})) = x_i = \mathtt{I}_i^{m,n}(\vec{x}, \vec{k}) = f(\vec{x}, \vec{k}).$$

Case $\tilde{\alpha} = \mathtt{0}$:
Note that $m = n = 0$. Let $f = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $\tilde{\alpha} = \mathtt{S}$:
Note that $m = 1, n = 0$. Let $f = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1)$.
Case $\tilde{\alpha} = +$:
Note that $m = 2, n = 0$. Let $f = +$, then $x_1 + x_2 = x_1 + x_2$.
Case $\tilde{\alpha} = \times$:
Note that $m = 2, n = 0$. Let $f = \times$, then $x_1 \times x_2 = x_1 \times x_2$.
Case $\tilde{\alpha} = |\cdot|_j^{m,n}$:
Note that $0 \le j < n$. Since $|\cdot|$ belongs to $\mathcal{E}^{2+}$ by Proposition 2.24, let $f(\vec{x}, \vec{k}) = |\mathtt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})|$. Then

$$\begin{aligned}\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) &= |(\vec{x}; \mathtt{bin}(\vec{k}))|_j^{m,n} = |\mathtt{bin}(k_j)| \\ &= |k_j| = |\mathtt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})| = f(\vec{x}, \vec{k}).\end{aligned}$$

Case $\tilde{\alpha} = \mathtt{BIT}_j^{m+1,n}$:
Note that $0 \le j < n$. Since $\mathtt{BIT}$ belongs to $\mathcal{E}^{2+}$ by Proposition 2.24, let $f(z, \vec{x}, \vec{k}) = \mathtt{BIT}(\mathtt{I}_0^{m+1+n}(z, \vec{x}, \vec{k}), \mathtt{I}_{m+1+j}^{m+1+n}(z, \vec{x}, \vec{k}))$. Then

$$\begin{aligned}\tilde{\alpha}(z, \vec{x}; \mathtt{bin}(\vec{k})) &= \mathtt{BIT}_j^{m+1,n}(z, \vec{x}; \mathtt{bin}(\vec{k})) = \mathrm{BIT}(z, \mathtt{bin}(k_j)) \\ &= \mathtt{BIT}(z, k_j) = \mathtt{BIT}(\mathtt{I}_0^{m+1+n}(z, \vec{x}, \vec{k}), \mathtt{I}_{m+1+j}^{m+1+n}(z, \vec{x}, \vec{k})) = f(z, \vec{x}, \vec{k}).\end{aligned}$$

Induction step.
Case COMP:
Suppose that

$$\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = \tilde{\gamma}(\tilde{\beta}_0(\vec{x}; \mathtt{bin}(\vec{k})), \ldots, \tilde{\beta}_{L-1}(\vec{x}; \mathtt{bin}(\vec{k})); \mathtt{bin}(\vec{k})),$$

whrere $\tilde{\gamma}, \tilde{\beta}_0, \ldots, \tilde{\beta}_{L-1} \in \tilde{\mathcal{C}}$. Then, by the induction hypothesis, there exist $h, g_0, \ldots, g_{L-1} \in \mathcal{E}^{2+}$ such that

$$\begin{aligned}\tilde{\gamma}(\vec{y}; \mathtt{bin}(\vec{k})) &= h(\vec{y}, \vec{k}), \\ \tilde{\beta}_j(\vec{x}; \mathtt{bin}(\vec{k})) &= g_j(\vec{x}, \vec{k})\end{aligned}$$

for $0 \le j < L$.
Since $\mathcal{E}^{2+}$ is closed under composition, define $f \in \mathcal{E}^{2+}$ by

$$f(\vec{x},\vec{k}) = h(g_0(\vec{x},\vec{k}),\ldots,g_{L-1}(\vec{x},\vec{k}),\vec{k}).$$

Then we have

$$\begin{aligned}\tilde{\alpha}(\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\gamma}(\tilde{\beta}_0(\vec{x};\texttt{bin}(\vec{k})),\ldots,\tilde{\beta}_{L-1}(\vec{x};\texttt{bin}(\vec{k}));\texttt{bin}(\vec{k}))\\ &= \tilde{\gamma}(g_0(\vec{x},\vec{k}),\ldots,g_{L-1}(\vec{x},\vec{k});\texttt{bin}(\vec{k}))\\ &= h(g_0(\vec{x},\vec{k}),\ldots,g_{L-1}(\vec{x},\vec{k}),\vec{k})\\ &= f(\vec{x},\vec{k}).\end{aligned}$$

Case BR:

Suppose that $\tilde{\alpha} \in \tilde{\mathcal{C}}$ with $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) \in \mathbb{N}^{n+1} \times \mathbb{W}^m \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned}\tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})),\\ \tilde{\alpha}(z+1,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\gamma}(z,\vec{x},\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k}));\texttt{bin}(\vec{k})),\\ \tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) &\le \tilde{\delta}(z,\vec{x};\texttt{bin}(\vec{k})) \quad \text{for any } z,\vec{x},\vec{k},\end{aligned}$$

where $\tilde{\beta},\tilde{\gamma},\tilde{\delta} \in \tilde{\mathcal{C}}$.
By induction hypothesis, there exist $g,h,e \in \mathcal{E}^{2+}$ such that

$$\begin{aligned}\tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})) &= g(\vec{x},\vec{k}),\\ \tilde{\gamma}(z,\vec{x},y;\texttt{bin}(\vec{k})) &= h(z,\vec{x},y,\vec{k}),\\ \tilde{\delta}(z,\vec{x};\texttt{bin}(\vec{k})) &= e(z,\vec{x},\vec{k}).\end{aligned}$$

Since $\mathcal{E}^{2+}$ is closed under bounded recursion, define f by

$$\begin{aligned}f(0,\vec{x},\vec{k}) &= g(\vec{x},\vec{k}),\\ f(z+1,\vec{x},\vec{k}) &= h(z,\vec{x},f(z,\vec{x},\vec{k}),\vec{k}).\end{aligned}$$

Now, we show $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) = f(z,\vec{x},\vec{k})$ by induction on z.
In the case $z = 0$,

$$\tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})) = \tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})) = g(\vec{x},\vec{k}) = f(0,\vec{x},\vec{k}).$$

In the case z, suppose that $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) = f(z,\vec{x},\vec{k})$.
In the case $z+1$,

$$\begin{aligned}\tilde{\alpha}(z+1,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\gamma}(z,\vec{x},\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k}));\texttt{bin}(\vec{k}))\\ &= \tilde{\gamma}(z,\vec{x},f(z,\vec{x},\vec{k});\texttt{bin}(\vec{k}))\\ &= h(z,\vec{x},f(z,\vec{x},\vec{k}),\vec{k})\\ &= f(z+1,\vec{x},\vec{k}).\end{aligned}$$

Therefore, we have $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) = f(z,\vec{x},\vec{k})$ for all z.

Furthermore,

$$\begin{aligned} f(z,\vec{x},\vec{k}) &= \tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) \\ &\leq \tilde{\delta}(z,\vec{x};\texttt{bin}(\vec{k})) = e(z,\vec{x},\vec{k}) \quad \text{for all } z,\vec{x},\vec{k}. \end{aligned}$$

Since f is bounded from above by $e \in \mathcal{E}^{2+}$, we conclude $f \in \mathcal{E}^{2+}$.

Case 1-BCVR:

Suppose that $\tilde{\alpha} \in \tilde{\mathcal{C}}$ with $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) \in \mathbb{N}^{n+1} \times \mathbb{W}^m \to \mathbb{N}$ is defined by 1-bounded course-of-values recursion, then

$$\begin{aligned} \tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})), \\ \tilde{\alpha}(z+1,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\gamma}(z,\vec{x},\langle\tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})),\ldots,\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k}))\rangle;\texttt{bin}(\vec{k})), \\ \tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) &\leq 1 \quad \text{for any } z,\vec{x},\vec{k}, \end{aligned}$$

where $\tilde{\beta},\tilde{\gamma} \in \tilde{\mathcal{C}}$.

By induction hypothesis, there exist $g,h \in \mathcal{E}^{2+}$ such that

$$\begin{aligned} \tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})) &= g(\vec{x},\vec{k}), \\ \tilde{\gamma}(z,\vec{x},y;\texttt{bin}(\vec{k})) &= h(z,\vec{x},y,\vec{k}). \end{aligned}$$

Since $\mathcal{E}^{2+}$ is closed under 1-bounded course-of-values recursion, define f by

$$\begin{aligned} f(0,\vec{x},\vec{k}) &= g(\vec{x},\vec{k}), \\ f(z+1,\vec{x},\vec{k}) &= h(z,\vec{x},\langle f(0,\vec{x},\vec{k}),\ldots,f(z,\vec{x},\vec{k})\rangle,\vec{k}). \end{aligned}$$

Now, we show $\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k})) = f(z,\vec{x},\vec{k})$ by course-of-values induction on z.

In the case $z = 0$,

$$\tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})) = \tilde{\beta}(\vec{x};\texttt{bin}(\vec{k})) = g(\vec{x},\vec{k}) = f(0,\vec{x},\vec{k}).$$

In the case $w(\leq z)$, suppose that $\tilde{\alpha}(w,\vec{x};\texttt{bin}(\vec{k})) = f(w,\vec{x},\vec{k})$.

In the case $z+1$,

$$\begin{aligned} \tilde{\alpha}(z+1,\vec{x};\texttt{bin}(\vec{k})) &= \tilde{\gamma}(z,\vec{x},\langle\tilde{\alpha}(0,\vec{x};\texttt{bin}(\vec{k})),\ldots,\tilde{\alpha}(z,\vec{x};\texttt{bin}(\vec{k}))\rangle;\texttt{bin}(\vec{k})) \\ &= \tilde{\gamma}(z,\vec{x},\langle f(0,\vec{x},\vec{k}),\ldots,f(z,\vec{x},\vec{k})\rangle;\texttt{bin}(\vec{k})) \\ &= h(z,\vec{x},\langle f(0,\vec{x},\vec{k}),\ldots,f(z,\vec{x},\vec{k})\rangle,\vec{k}) \\ &= f(z+1,\vec{x},\vec{k}). \end{aligned}$$

Therefore, we have $\tilde{\alpha}(z, \vec{x}; \mathtt{bin}(\vec{k})) = f(z, \vec{x}, \vec{k})$ for all z.

Furthermore,

$$f(z, \vec{x}, \vec{k}) = \tilde{\alpha}(z, \vec{x}; \mathtt{bin}(\vec{k})) \leq 1 \quad \text{for all } z, \vec{x}, \vec{k}.$$

Since f is bounded from above by 1, we conclude $f \in \mathcal{E}^{2+}$. □

Combining Proposition 3.26 with Proposition 3.27, we can derive the following corollary:

Corollary 3.28. *For each $\alpha \in \mathcal{C}_{\mathbb{N}}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$, there exist $f, \tilde{g}, \hat{g} \in \mathcal{E}^{2+}$ such that*

$$\begin{aligned} \alpha(\vec{x}; \mathtt{bin}(\vec{k})) &= f(\vec{x}, \vec{k}), \\ \mathtt{BIT}(z; \varphi(\vec{x}; \mathtt{bin}(\vec{k})) &= \tilde{g}(z, \vec{x}, \vec{k}), \\ |; \varphi(\vec{x}; \mathtt{bin}(\vec{k}))| &= \hat{g}(\vec{x}, \vec{k}) \end{aligned}$$

for each $\vec{x}, \vec{k}$ and z.

3.4 Representation of $\mathcal{F}$PTIME functions by $\mathcal{C}_{\mathbb{N}}$ functions

In this section, we show that for any function in $\mathcal{F}$PTIME, the bit contents and length of its binary representation can be represented by some functions in $\mathcal{C}_{\mathbb{N}}$, that is,

$$\forall r \in \mathcal{F}\text{PTIME}\ \exists \alpha \in \mathcal{C}_{\mathbb{N}}\ [\text{BIT}(i, \text{bin}(r(\vec{x}))) = \alpha(i; \texttt{bin}(\vec{x}))],$$
$$\forall r \in \mathcal{F}\text{PTIME}\ \exists \beta \in \mathcal{C}_{\mathbb{N}}\ [|\text{bin}(r(\vec{x}))| = \beta(; \texttt{bin}(\vec{x}))]$$

(Proposition 3.29). In this Proposition, we take the following function algebra as $\mathcal{F}$PTIME:

$$\mathcal{F}\text{PTIME} = [\texttt{0}, \texttt{I}, \texttt{s}_0, \texttt{s}_1, \texttt{MOD2}, \texttt{msp}, \#; \text{COMP}, \text{FCRN}]$$

(Theorem 2.20). And, in this proposition and in the following discussion, the function bin$(\cdot)$ applied to a natural number in $\mathcal{F}$PTIME is an identity function which regards a natural number as its binary representation.

Note that in the above relations, if we apply bounded comprehension to $\alpha(i; \texttt{bin}(\vec{x})) \in \mathcal{C}_{\mathbb{N}}$, we obtain $\varphi(z; \texttt{bin}(\vec{x})) \in \mathcal{C}_{\mathbb{W}}$ satisfying

$$|; \varphi(z; \texttt{bin}(\vec{x}))| = z,$$
$$\forall i < z\ [\texttt{BIT}(i; \varphi(z; \texttt{bin}(\vec{x}))) = \texttt{0} \leftrightarrow \alpha(i; \texttt{bin}(\vec{x})) = \texttt{0}].$$

Then we have

$$\text{bin}(r(\vec{x})) = \varphi(\beta(; \texttt{bin}(\vec{x})); \texttt{bin}(\vec{x})),$$

that is, the above relations are the same thing as making $r \in \mathcal{F}$PTIME correspond to $\varphi \in \mathcal{C}_{\mathbb{W}}$.

Proposition 3.29. *For any $r \in \mathcal{F}$PTIME with $r \in \mathbb{N}^m \to \mathbb{N}$, there exist $\alpha, \beta \in \mathcal{C}_{\mathbb{N}}$ such that*

$$\text{(i)}\qquad \text{BIT}(i, \text{bin}(r(\vec{x}))) = \alpha(i; \texttt{bin}(\vec{x})),$$
$$\text{(ii)}\qquad |\text{bin}(r(\vec{x}))| = \beta(; \texttt{bin}(\vec{x}))$$

for each x and i.

Proof. By simultaneous induction of (i) and (ii) on the structures of $r \in \mathcal{F}$PTIME.

Basis.

Case $r = \texttt{0}$:

Note that $m = 0$ and bin$(\texttt{0}) = 0$, hence, BIT$(i, \text{bin}(\texttt{0})) = 0$ and $|\text{bin}(\texttt{0})| = 0$.

(i) Let $\alpha(i; \texttt{bin}(\vec{x})) = \texttt{0}$. Then BIT$(i, \text{bin}(\texttt{0})) = 0 = \alpha(i; \texttt{bin}(\vec{x}))$.

(ii) Let $\beta(; \texttt{bin}(\vec{x})) = \texttt{0}$. Then $|\text{bin}(\texttt{0})| = 0 = \beta(; \texttt{bin}(\vec{x}))$.

Case $r = \mathtt{I}^m_j$:

Note that $0 \le j < m$ and $\mathtt{I}^m_j(\vec{x}) = x_j$.

(i) Let $\alpha(i; \mathtt{bin}(\vec{x})) = \mathtt{BIT}(\mathrm{p}_{\mathbb{N}_0}{}^{1,m}(i; \mathtt{bin}(\vec{x})); \mathrm{p}_{\mathbb{W}_j}{}^{1,m}(i; \mathtt{bin}(\vec{x}))) = \mathtt{BIT}(i; \mathtt{bin}(x_j))$. Then

$$\mathrm{BIT}(i, \mathrm{bin}(\mathtt{I}^m_j(\vec{x}))) = \mathrm{BIT}(i, \mathrm{bin}(x_j)) = \mathtt{BIT}(i; \mathtt{bin}(x_j)) = \alpha(i; \mathtt{bin}(\vec{x})).$$

(ii) Let $\beta(; \mathtt{bin}(\vec{x})) = |; \mathrm{p}_{\mathbb{W}_j}{}^{0,m}(; \mathtt{bin}(\vec{x}))| = |; \mathtt{bin}(x_j)|$. Then

$$|\mathrm{bin}(\mathtt{I}^m_j(\vec{x}))| = |\mathrm{bin}(x_j)| = |; \mathtt{bin}(x_j)| = \beta(; \mathtt{bin}(\vec{x})).$$

Case $r = \mathtt{s}_0$:

Note that $m = 1$ and $\mathtt{s}_0(x) = 2 \cdot x$.

(i) Let $\alpha(i; \mathtt{bin}(x)) = \mathtt{cond}(i, 0, \mathtt{bit}(i \dot{-} 1; \mathtt{bin}(x));)$. Then

$$\mathrm{BIT}(i, \mathrm{bin}(\mathtt{s}_0(x))) = \mathrm{BIT}(i, \mathrm{bin}(2 \cdot x)) = \alpha(i; \mathtt{bin}(x)).$$

(ii) $|\mathrm{bin}(\mathtt{s}_0(x))|$ is $|\mathrm{bin}(x)| + 1$ if $x \neq 0$, and is $|\mathrm{bin}(\mathtt{0})| = 0$ if $x = 0$. Hence, let $\beta(; \mathtt{bin}(x)) = \mathtt{cond}(|; \mathtt{bin}(x)|, 0, |; \mathtt{bin}(x)| + 1;)$. Then

$$|\mathrm{bin}(\mathtt{s}_0(x))| = |\mathrm{bin}(2 \cdot x)| = \beta(; \mathtt{bin}(\vec{x})).$$

Case $r = \mathtt{s}_1$:

Note that $m = 1$ and $\mathtt{s}_1(x) = 2 \cdot x + 1$.

(i) Let $\alpha(i; \mathtt{bin}(x)) = \mathtt{cond}(i, 1, \mathtt{bit}(i \dot{-} 1; \mathtt{bin}(x));)$. Then

$$\mathrm{BIT}(i, \mathrm{bin}(\mathtt{s}_1(x))) = \mathrm{BIT}(i, \mathrm{bin}(2 \cdot x + 1)) = \alpha(i; \mathtt{bin}(x)).$$

(ii) Let $\beta(; \mathtt{bin}(x)) = |; \mathtt{bin}(x)| + 1$. Then

$$|\mathrm{bin}(\mathtt{s}_1(x))| = |\mathrm{bin}(2 \cdot x + 1)| = \beta(; \mathtt{bin}(\vec{x})).$$

Case $r = \mathtt{MOD2}$:

Note that $m = 1$ and $\mathtt{MOD2}(x) = x - \left\lfloor \frac{x}{2} \right\rfloor \cdot 2$, the residue of x divided by 2.

(i) Let $\alpha(i; \mathtt{bin}(x)) = \mathtt{cond}(i, \mathtt{bit}(i; \mathtt{bin}(x)), \mathtt{0};)$. Then

$$\mathrm{BIT}(i, \mathrm{bin}(\mathtt{MOD2}(x))) = \alpha(i; \mathtt{bin}(x)).$$

(ii) $|\mathrm{bin}(\mathtt{MOD2}(x))|$ is 0 if $\mathtt{MOD2}(x) = 0$, and is 1 if $\mathtt{MOD2}(x) = 1$. Hence, let $\beta(; \mathtt{bin}(x)) = \mathtt{bit}(\mathtt{0}; \mathtt{bin}(x))$. Then

$$|\mathrm{bin}(\mathtt{MOD2}(x))| = \beta(; \mathtt{bin}(\vec{x})).$$

Case $r = \mathtt{msp}$:

Note that $m = 2$ and $\texttt{msp}(x,y) = \left\lfloor \frac{x}{2^{|y|}} \right\rfloor$, that is, the number obtained by cutting the lower $|y|$ bits of x in the binary representation of x.
(i) Let $\alpha(i;\texttt{bin}(x),\texttt{bin}(y)) = \texttt{bit}(i + |;\texttt{bin}(y)|;\texttt{bin}(x))$. Then

$$\mathrm{BIT}(i,\mathrm{bin}(\texttt{msp}(x,y))) = \alpha(i;\texttt{bin}(x),\texttt{bin}(y)).$$

(ii) Let $\beta(;\texttt{bin}(x),\texttt{bin}(y)) = |;\texttt{bin}(x)| \dot{-} |;\texttt{bin}(y)|$. Then

$$|\mathrm{bin}(\texttt{msp}(x,y))| = \beta(;\texttt{bin}(x),\texttt{bin}(y)).$$

Case $r = \#$:
Note that $m = 2$ and $x\#y = 2^{|x|\cdot|y|}$. $2^{|x|\cdot|y|}$ is the number in which the $|x|\cdot|y|$-th bit is 1 and the other bits are 0 in its binary representation.
(i) Let $\alpha(i;\texttt{bin}(x),\texttt{bin}(y)) = \texttt{cond}(\chi_=(|;\texttt{bin}(x)| \times |;\texttt{bin}(y)|, i;), \texttt{0}, \texttt{S}(\texttt{0};);)$. Then

$$\mathrm{BIT}(i,\mathrm{bin}(x\#y)) = \alpha(i;\texttt{bin}(x),\texttt{bin}(y)).$$

(ii) Since $|x\#y| = |2^{|x|\cdot|y|}| = |x|\cdot|y| + 1$, let $\beta(;\texttt{bin}(x),\texttt{bin}(y)) = |;\texttt{bin}(x)| \times |;\texttt{bin}(y)| + 1$. Then

$$|\mathrm{bin}(x\#y)| = \beta(;\texttt{bin}(x),\texttt{bin}(y)).$$

Induction step.
Case COMP:
Suppose that

$$r(\vec{x}) = u(t_0(\vec{x}),\dots,t_{L-1}(\vec{x})),$$

whrere $u, t_0, \dots, t_{L-1} \in \mathcal{F}\mathrm{PTIME}$. Then, by the induction hypothesis, there exist $\gamma, \gamma^\ell, \beta_0, \dots, \beta_{L-1}, \beta_0^\ell, \dots, \beta_{L-1}^\ell \in \mathcal{C}_\mathbb{N}$ such that

$$\begin{aligned}
&\mathrm{BIT}(i,\mathrm{bin}(u(\vec{y}))) = \gamma(i;\texttt{bin}(\vec{y})),\\
&|\mathrm{bin}(u(\vec{y}))| = \gamma^\ell(;\texttt{bin}(\vec{y})),\\
&\mathrm{BIT}(i,\mathrm{bin}(t_j(\vec{x}))) = \beta_j(i;\texttt{bin}(\vec{x})),\\
&|\mathrm{bin}(t_j(\vec{x}))| = \beta_j^\ell(;\texttt{bin}(\vec{x}))
\end{aligned}$$

for $0 \le j < L$.
Applying bounded comprehension to $\beta_j(i;\texttt{bin}(\vec{x})) \in \mathcal{C}_\mathbb{N}$ for $0 \le j < L$, we obtain $\varphi_j(z;\texttt{bin}(\vec{x})) \in \mathcal{C}_\mathbb{W}$ satisfying

$$\begin{aligned}
&|;\varphi_j(z;\texttt{bin}(\vec{x}))| = z,\\
&\forall i < z\ [\texttt{BIT}(i;\varphi_j(z;\texttt{bin}(\vec{x}))) = \texttt{0} \leftrightarrow \beta_j(i;\texttt{bin}(\vec{x})) = \texttt{0}].
\end{aligned}$$

Then, each $\mathrm{bin}(t_j(\vec{x}))$ for $0 \le j < L$ is represented by some composite function taking only $\mathtt{bin}(\vec{x})$ as its arguments:

$$\mathrm{bin}(t_j(\vec{x})) = \varphi_j(\beta_j^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x})).$$

(i) Define $\alpha \in \mathcal{C}_{\mathbb{N}}$ by

$$\begin{aligned} &\alpha(i;\mathtt{bin}(\vec{x})) \\ &= \gamma(i;\varphi_0(\beta_0^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x})),\dots,\varphi_{L-1}(\beta_{L-1}^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x}))). \end{aligned}$$

Then we have

$$\begin{aligned} &\mathrm{BIT}(i,\mathrm{bin}(r(\vec{x}))) \\ &= \mathrm{BIT}(i,\mathrm{bin}(u(t_0(\vec{x}),\dots,t_{L-1}(\vec{x})))) \\ &= \gamma(i;\mathtt{bin}(t_0(\vec{x})),\dots,\mathtt{bin}(t_{L-1}(\vec{x}))) \\ &= \gamma(i;\varphi_0(\beta_0^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x})),\dots,\varphi_{L-1}(\beta_{L-1}^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x}))) \\ &= \alpha(i;\mathtt{bin}(\vec{x})). \end{aligned}$$

(ii) Define $\alpha^{\ell} \in \mathcal{C}_{\mathbb{N}}$ by

$$\begin{aligned} &\alpha^{\ell}(;\mathtt{bin}(\vec{x})) \\ &= \gamma^{\ell}(;\varphi_0(\beta_0^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x})),\dots,\varphi_{L-1}(\beta_{L-1}^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x}))). \end{aligned}$$

Then we have

$$\begin{aligned} &|\mathrm{bin}(r(\vec{x}))| \\ &= |\mathrm{bin}(u(t_0(\vec{x}),\dots,t_{L-1}(\vec{x})))| \\ &= \gamma^{\ell}(;\mathtt{bin}(t_0(\vec{x})),\dots,\mathtt{bin}(t_{L-1}(\vec{x}))) \\ &= \gamma^{\ell}(;\varphi_0(\beta_0^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x})),\dots,\varphi_{L-1}(\beta_{L-1}^{\ell}(;\mathtt{bin}(\vec{x}));\mathtt{bin}(\vec{x}))) \\ &= \alpha^{\ell}(;\mathtt{bin}(\vec{x})). \end{aligned}$$

Case FCRN:

First, we prepare some auxiliary functions in $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$.

1. The function $\mathtt{check}(i;a) \in \mathbb{N} \times \mathbb{W} \to \mathbb{N}$ computes the i-th bit counting from the most significant bit[6] of a for $0 \le i < |a|$. We have

$$\mathtt{check}(i;a) = \mathtt{BIT}(|;a| \dot{-} 1 \dot{-} i;a).$$

[6] The most significant bit of a is the $|a| - 1$-th bit of a.

2. The function $\texttt{rev}(;a) \in \mathbb{W} \to \mathbb{W}$ computes the reverse of a. Applying bounded comprehension to $\texttt{check}(i;a)$, we obtain $\varphi \in \mathcal{C}_{\mathbb{W}}$ such that

$$|;\varphi(z;a)| = z,$$
$$\forall i < z\ [\texttt{BIT}(i;\varphi(z;a) = \texttt{0} \leftrightarrow \texttt{check}(i;a) = 0].$$

Then we have

$$\texttt{rev}(;a) = \varphi(|;a|;a).$$

3. The function $\texttt{left}(i;a) \in \mathbb{N} \times \mathbb{W} \to \mathbb{W}$ computes a string of i-many bits counting from the most significant bit of a for $0 \leq i \leq |a|$. Using φ in 2. above, we have

$$\texttt{left}(i;a) = \texttt{rev}(\varphi(i;a)).$$

Suppose that $r \in \mathcal{F}\mathrm{PTIME}$ with $r \in \mathbb{N}^{m+1} \to \mathbb{N}$ is defined by full concatenation recursion on notation, then

$$r(0,\vec{y}) = t(\vec{y}),$$
$$r(\texttt{s}_0(x),\vec{y}) = \texttt{s}_{u_0(x,\vec{y},r(x,\vec{y}))}(r(x,\vec{y})) \quad (\text{if } x \neq 0),$$
$$r(\texttt{s}_1(x),\vec{y}) = \texttt{s}_{u_1(x,\vec{y},r(x,\vec{y}))}(r(x,\vec{y})),$$

where $t, u_0(\leq 1), u_1(\leq 1) \in \mathcal{F}\mathrm{PTIME}$.
By induction hypothesis, there exist $\beta, \beta^{\ell}, \gamma_0, \gamma_0^{\ell}, \gamma_1, \gamma_1^{\ell} \in \mathcal{C}_{\mathbb{N}}$ such that

$$\mathrm{BIT}(i, \mathrm{bin}(t(\vec{y}))) = \beta(i;\texttt{bin}(\vec{y})),$$
$$|\mathrm{bin}(t(\vec{y}))| = \beta^{\ell}(;\texttt{bin}(\vec{y})),$$
$$\mathrm{BIT}(i, \mathrm{bin}(u_0(x,\vec{y},z))) = \gamma_0(i;\texttt{bin}(x),\texttt{bin}(\vec{y}),\texttt{bin}(z)),$$
$$|\mathrm{bin}(u_0(x,\vec{y},z))| = \gamma_0^{\ell}(;\texttt{bin}(x),\texttt{bin}(\vec{y}),\texttt{bin}(z)),$$
$$\mathrm{BIT}(i, \mathrm{bin}(u_1(x,\vec{y},z))) = \gamma_1(i;\texttt{bin}(x),\texttt{bin}(\vec{y}),\texttt{bin}(z)),$$
$$|\mathrm{bin}(u_1(x,\vec{y},z))| = \gamma_1^{\ell}(;\texttt{bin}(x),\texttt{bin}(\vec{y}),\texttt{bin}(z)).$$

We want to show that there exist $\alpha, \alpha^{\ell} \in \mathcal{C}_{\mathbb{N}}$ such that

$$\mathrm{BIT}(i, \mathrm{bin}(r(x,\vec{y}))) = \alpha(i;\texttt{bin}(x),\texttt{bin}(\vec{y})),$$
$$|\mathrm{bin}(r(x,\vec{y}))| = \alpha^{\ell}(;\texttt{bin}(x),\texttt{bin}(\vec{y})).$$

(ii) Firstly, since in $\mathcal{F}\mathrm{PTIME}$, $|r(x,\vec{y})|$ is $|t(\vec{y})|$ plus $|x|$, we take

$$\alpha^{\ell}(;\texttt{bin}(x),\texttt{bin}(\vec{y})) = \beta^{\ell}(;\texttt{bin}(\vec{y})) + |;\texttt{bin}(x)|.$$

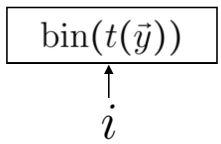
Then we have

$$|\mathrm{bin}(r(x,\vec{y}))| = \alpha^{\ell}(;\texttt{bin}(x),\texttt{bin}(\vec{y})).$$

(i) Secondly, $\mathcal{C}_{\mathbb{N}}$ is closed under boolean course-of-values recursion, we define $\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y})) \in \mathcal{C}_{\mathbb{N}}$ which computes the bit corresponding to the i-th bit counting from the most significant bit of $\mathrm{bin}(r(x,\vec{y}))$ for $0 \le i < |\mathrm{bin}(r(x,\vec{y}))|$ by BCVR.

Note that by the definition of FCRN, $\mathrm{bin}(r(x,\vec{y}))$ is obtained by attaching a bit sequence in which each bit of $\mathrm{bin}(x)$ is replaced by the value of u_0 or u_1 to the right end of $\mathrm{bin}(t(\vec{y}))$.

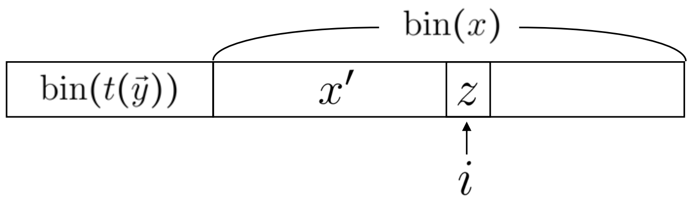
In the case $i < \beta^{\ell}(; \mathtt{bin}(\vec{y}))$,



$\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y}))$ corresponds to the i-th bit counting from the most significant bit of $\mathrm{bin}(t(\vec{y}))$, hence we have

$$\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y})) = \beta(\beta^{\ell}(; \mathtt{bin}(\vec{y})) \dot{-} 1 \dot{-} i; \mathtt{bin}(\vec{y})).$$

In the case $i \ge \beta^{\ell}(; \mathtt{bin}(\vec{y}))$,



$\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y}))$ corresponds to the i-th bit counting from the most significant bit of $\mathrm{bin}((r(x,\vec{y}))$, and when the value of the corresponding bit of $\mathrm{bin}(x)$ is z and the value of the left-side part of the corresponding bit of $\mathrm{bin}(x)$ is x', $\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y}))$ is $u_z(x', \vec{y}, r(x', \vec{y}))$. Since

$$\begin{aligned} u_z(x', \vec{y}, r(x', \vec{y})) &= \mathrm{BIT}(0, \mathrm{bin}(u_z(x', \vec{y}, r(x', \vec{y})))) \\ &= \gamma_z(0; \mathtt{bin}(x'), \mathtt{bin}(\vec{y}), \mathtt{bin}(r(x', \vec{y}))), \end{aligned}$$

we have

$$\begin{aligned} &\delta(i; \mathtt{bin}(x), \mathtt{bin}(\vec{y})) \\ = &\ \mathtt{cond}(\mathtt{check}(i \dot{-} \beta^{\ell}(; \mathtt{bin}(\vec{y})); \mathtt{bin}(x)), \\ &\qquad \gamma_0(0; \mathtt{left}(i \dot{-} \beta^{\ell}(; \mathtt{bin}(\vec{y})); \mathtt{bin}(x)), \mathtt{bin}(\vec{y}), \mathtt{bin}(r(x', \vec{y}))), \\ &\qquad \gamma_1(0; \mathtt{left}(i \dot{-} \beta^{\ell}(; \mathtt{bin}(\vec{y})); \mathtt{bin}(x)), \mathtt{bin}(\vec{y}), \mathtt{bin}(r(x', \vec{y})));) \end{aligned}$$

$$
\begin{aligned}
&= \texttt{cond}(\texttt{check}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\\
&\qquad \gamma_0(0;\texttt{left}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\texttt{bin}(\vec{y}),\texttt{rev}(;b)),\\
&\qquad \gamma_1(0;\texttt{left}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\texttt{bin}(\vec{y}),\texttt{rev}(;b));)\\
&\text{where } |;b| = i \text{ and } \texttt{BIT}(j;b) = \delta(j;\texttt{bin}(x),\texttt{bin}(\vec{y})) \text{ for } 0 \le j \le i-1.
\end{aligned}
$$

Therefore, unifying the above cases into the cases $i = 0$ and $i > 0$ in the BCVR form,we have

$$
\begin{aligned}
&\delta(0;\texttt{bin}(x),\texttt{bin}(\vec{y})) = \beta(\beta^{\ell}(;\texttt{bin}(\vec{y})) \dot{-} 1;\texttt{bin}(\vec{y})),\\
&\delta(i;\texttt{bin}(x),\texttt{bin}(\vec{y})) \qquad\qquad (i > 0)\\
&= \texttt{cond}(i \ge \beta^{\ell}(;\texttt{bin}(\vec{y})),\\
&\qquad \beta(\beta^{\ell}(;\texttt{bin}(\vec{y})) \dot{-} 1 \dot{-} i;\texttt{bin}(\vec{y})),\\
&\qquad \texttt{cond}(\texttt{check}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\\
&\qquad\qquad \gamma_0(0;\texttt{left}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\texttt{bin}(\vec{y}),\texttt{rev}(;b)),\\
&\qquad\qquad \gamma_1(0;\texttt{left}(i \dot{-} \beta^{\ell}(;\texttt{bin}(\vec{y}));\texttt{bin}(x)),\texttt{bin}(\vec{y}),\texttt{rev}(;b)););),\\
&\text{where } |;b| = i \text{ and } \texttt{BIT}(j;b) = \delta(j;\texttt{bin}(x),\texttt{bin}(\vec{y})) \text{ for } 0 \le j \le i-1.
\end{aligned}
$$

And clearly, $\delta(i;\texttt{bin}(x),\texttt{bin}(\vec{y})) \le 1$ for any $i, x, \vec{y}$.

Then, let

$$
\alpha(i;\texttt{bin}(x),\texttt{bin}(\vec{y})) = \delta(\beta^{\ell}(;\texttt{bin}(\vec{y})) + |;\texttt{bin}(x)| \dot{-} 1 \dot{-} i;\texttt{bin}(x),\texttt{bin}(\vec{y})),
$$

we have

$$
\mathrm{BIT}(i,\mathrm{bin}(r(x,\vec{y}))) = \alpha(i;\texttt{bin}(x),\texttt{bin}(\vec{y})).
$$

□

3.5 Representation of $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ functions by $\mathcal{F}$PTIME functions

In this section, we show that any function in $\mathcal{C}_{\mathbb{N}}$ is represented by some function in $\mathcal{F}$PTIME and that any function in $\mathcal{C}_{\mathbb{W}}$ is represented by some function is $\mathcal{F}$PTIME, that is,

$$\forall \alpha \in \mathcal{C}_{\mathbb{N}}\ \exists r \in \mathcal{F}\text{PTIME}\ [\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = |r(\vec{x}, \vec{k})|],$$
$$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists t \in \mathcal{F}\text{PTIME}\ [\varphi(|\vec{x}|; \mathtt{bin}(\vec{k})) = \text{bin}(t(\vec{x}, \vec{k}))]$$

(Proposition 3.30). Note that we make lengths of natural numbers in arguments of functions in $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ correspond to natural numbers in arguments of the corresponding functions in $\mathcal{F}$PTIME, and we make values of functions in $\mathcal{C}_{\mathbb{N}}$ correspond to lengths of values of the corresponding functions in $\mathcal{F}$PTIME. In this Proposition, we take the following function algebra as $\mathcal{F}$PTIME:

$$\mathcal{F}\text{PTIME} = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \#; \text{COMP}, \text{BRN}]$$

(Theorem 2.19).

To begin with, since the length function $|\cdot|$ is used in the above expression, we define $|\cdot|$ in $\mathcal{F}$PTIME.

The function $\mathtt{S}(x) = x + 1$ is defined in $\mathcal{F}$PTIME using bounded recursion on notation (BRN) as follows:

$$\begin{aligned} \mathtt{S}(\mathtt{0}) &= \mathtt{s}_1(\mathtt{0}), \\ \mathtt{S}(\mathtt{s}_0(x)) &= \mathtt{s}_1(x) \qquad (x \neq \mathtt{0}), \\ \mathtt{S}(\mathtt{s}_1(x)) &= \mathtt{s}_0(\mathtt{S}(x)), \\ \mathtt{S}(x) &\leq \mathtt{s}_1(x) \quad \text{for any } x. \end{aligned}$$

Hence, $\mathtt{S}(x) \in \mathcal{F}$PTIME. Notice that any constant belongs to $\mathcal{F}$PTIME by repeatedly applying the successor function $\mathtt{S}(x)$ to the constant $\mathtt{0}$.
The function $|x|$ which computes the length of x in binary is defined in $\mathcal{F}$PTIME using BRN as follows:

$$\begin{aligned} |\mathtt{0}| &= \mathtt{0}, \\ |\mathtt{s}_0(x)| &= \mathtt{S}(|x|) \qquad (x \neq \mathtt{0}), \\ |\mathtt{s}_1(x)| &= \mathtt{S}(|x|), \\ |x| &\leq x \quad \text{for any } x. \end{aligned}$$

Hence, $|x| \in \mathcal{F}$PTIME.

Proposition 3.30. *For each $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, there exist $r, t \in \mathcal{F}$PTIME, respectively, such that*

$$\alpha(|\vec{x}|; \texttt{bin}(\vec{k})) = |r(\vec{x}, \vec{k})|,$$
$$\varphi(|\vec{x}|; \texttt{bin}(\vec{k})) = \mathrm{bin}(t(\vec{x}, \vec{k}))$$

for each $\vec{x}, \vec{k}$.

Proof. By simultaneous induction on the structures of $\alpha \in \mathcal{C}_{\mathbb{N}}$ and $\varphi \in \mathcal{C}_{\mathbb{W}}$.
Basis.
Case $\alpha = \texttt{p}_{\mathbb{N}_i}^{m,n}$:
Note that $0 \le i < m$. Let $r = \texttt{I}_i^{m+n}$. Then

$$\begin{aligned}\alpha(|\vec{x}|; \texttt{bin}(\vec{k})) &= \texttt{p}_{\mathbb{N}_i}^{m,n}(|\vec{x}|; \texttt{bin}(\vec{k})) = |x_i| \\ &= |\texttt{I}_i^{m+n}(\vec{x}, \vec{k})| = |r(\vec{x}, \vec{k})|.\end{aligned}$$

Case $\varphi = \texttt{p}_{\mathbb{W}_j}^{m,n}$:
Note that $0 \le j < n$. Let $r = \texttt{I}_{m+j}^{m+n}$. Then

$$\begin{aligned}\alpha(|\vec{x}|; \texttt{bin}(\vec{k})) &= \texttt{p}_{\mathbb{W}_j}^{m,n}(|\vec{x}|; \texttt{bin}(\vec{k})) = \texttt{bin}(k_j) \\ &= \mathrm{bin}(k_j) = \mathrm{bin}(\texttt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})) = \mathrm{bin}(r(\vec{x}, \vec{k})).\end{aligned}$$

Case $\alpha = \texttt{0}$:
Note that $m = n = 0$. Let $r = \texttt{0}$, then $\texttt{0} = |\texttt{0}|$.
Case $\alpha = \texttt{S}$:
Note that $m = 1, n = 0$. Let $r = \texttt{s}_1$, Then

$$\alpha(|x|;) = \texttt{S}(|x|;) = |x| + 1 = |\texttt{s}_1(x)| = |r(x)|.$$

Case $\alpha = +$:
Note that $m = 2, n = 0$. Define the function $x * y$ which computes the concatenation of x and y in binary using BRN as follows:

$$\begin{aligned} x * \texttt{0} &= x, \\ x * \texttt{s}_0(y) &= \texttt{s}_0(x * y) \qquad (y \ne \texttt{0}), \\ x * \texttt{s}_1(y) &= \texttt{s}_1(x * y), \\ x * y &\le \texttt{s}_1(x) \# \texttt{s}_1(y) \quad \text{for any } x, y. \end{aligned}$$

In the last inequality above, notice that for any x, y we have

$$\begin{aligned} x * y &= 2^{|y|} \cdot x + y \le 2^{|y|} \cdot x + 2^{|y|} = 2^{|y|} \cdot (x+1) \le 2^{|y|} \cdot 2^{|x|} \\ &\le 2^{|x|+|y|} \le 2^{(|x|+1)\cdot(|y|+1)} = 2^{|\texttt{s}_1(x)| \cdot |\texttt{s}_1(y)|} = \texttt{s}_1(x) \# \texttt{s}_1(y). \end{aligned}$$

Hence, $x * y \in \mathcal{F}$PTIME. Let $r(x,y) = x * y$, then

$$\alpha(|x|,|y|;) = |x| + |y| = |x * y| = |r(x,y)|.$$

Case $\alpha = \times$:

Note that $m = 2, n = 0$. Define the function $\lfloor x/2 \rfloor$ which computes the quotient of x divided by 2 using BRN as follows:

$$\begin{aligned}
\lfloor 0/2 \rfloor &= 0, \\
\lfloor \mathtt{s}_0(x)/2 \rfloor &= x \qquad (x \neq 0), \\
\lfloor \mathtt{s}_1(x)/2 \rfloor &= x, \\
\lfloor x/2 \rfloor &\leq x \quad \text{for any } x.
\end{aligned}$$

Hence, $\lfloor x/2 \rfloor \in \mathcal{F}$PTIME.
Let $r(x,y) = \lfloor (x\#y)/2 \rfloor (= 2^{|x|\cdot|y|-1})$. Since $|\lfloor (x\#y)/2 \rfloor| = |x| \cdot |y|$, we have

$$\alpha(|x|,|y|;) = |x| \times |y| = \left| \left\lfloor \frac{x\#y}{2} \right\rfloor \right| = |r(x,y)|.$$

Case $\alpha = |\cdot|$:

Note that $m = 0, n = 1$. Let $r(k) = \mathtt{I}_0^1(k)$. Then

$$\alpha(;\mathtt{bin}(k)) = |;\mathtt{bin}(k)| = |k| = |\mathtt{I}_0^1(k)| = |r(k)|.$$

Case $\alpha = \mathtt{BIT}$:

In the following discussion, according to the proof of Lemma 2.3 in Ishihara [5], we construct the function $\mathtt{BIT}(y,x)$ in $\mathcal{F}$PTIME which computes the y-th bit in the binary representation of x.

1. The function $\mathtt{msp}(x,y) = \lfloor x/2^{|y|} \rfloor$ is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
\mathtt{msp}(x,0) &= x, \\
\mathtt{msp}(x,\mathtt{s}_0(y)) &= \lfloor \mathtt{msp}(x,y)/2 \rfloor \qquad (y \neq 0), \\
\mathtt{msp}(x,\mathtt{s}_1(y)) &= \lfloor \mathtt{msp}(x,y)/2 \rfloor \\
\mathtt{msp}(x,y) &\leq x \quad \text{for any } x.
\end{aligned}$$

2. The function $\mathtt{MOD2}(x) = x \bmod 2$ is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
\mathtt{MOD2}(0) &= 0, \\
\mathtt{MOD2}(\mathtt{s}_0(x)) &= 0 \qquad (x \neq 0), \\
\mathtt{MOD2}(\mathtt{s}_1(x)) &= 1, \\
\mathtt{MOD2}(x) &\leq x \quad \text{for any } x.
\end{aligned}$$

3. The function $\mathtt{prd}(x) = x \dot{-} 1$ is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
\mathtt{prd}(0) &= 0, \\
\mathtt{prd}(\mathtt{s}_0(x)) &= \mathtt{s}_1(\mathtt{prd}(x)) \qquad (x \neq 0), \\
\mathtt{prd}(\mathtt{s}_1(x)) &= \mathtt{s}_0(x), \\
\mathtt{prd}(x) &\leq x \quad \text{for any } x.
\end{aligned}$$

4. The function $y \dot{-} |x|$ is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
y \dot{-} |0| &= y, \\
y \dot{-} |\mathtt{s}_0(x)| &= \mathtt{prd}(y \dot{-} |x|) \qquad (x \neq 0), \\
y \dot{-} |\mathtt{s}_1(x)| &= \mathtt{prd}(y \dot{-} |x|), \\
y \dot{-} |x| &\leq y \quad \text{for any } x, y.
\end{aligned}$$

5. The function $\mathtt{cond}(x, y, z)$, whose value is y if $x = 0$ and z if $x > 0$, is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
\mathtt{cond}(0, y, z) &= y, \\
\mathtt{cond}(\mathtt{s}_0(x), y, z) &= z \qquad (x \neq 0), \\
\mathtt{cond}(\mathtt{s}_1(x), y, z) &= z, \\
\mathtt{cond}(x, y, z) &\leq y * z \quad \text{for any } x, y, z.
\end{aligned}$$

6. The function $2^{\mathtt{min}(|x|,y)}$ is in $\mathcal{F}$PTIME. Using BRN, we have

$$\begin{aligned}
2^{\mathtt{min}(|0|,y)} &= \mathtt{S}(0), \\
2^{\mathtt{min}(|\mathtt{s}_0(x)|,y)} &= \mathtt{cond}(y \dot{-} |x|, 2^{\mathtt{min}(|x|,y)}, \mathtt{s}_0(2^{\mathtt{min}(|x|,y)})) \qquad (x \neq 0), \\
2^{\mathtt{min}(|\mathtt{s}_1(x)|,y)} &= \mathtt{cond}(y \dot{-} |x|, 2^{\mathtt{min}(|x|,y)}, \mathtt{s}_0(2^{\mathtt{min}(|x|,y)})), \\
2^{\mathtt{min}(|x|,y)} &\leq 2^{|x|} = 1\#x \quad \text{for any } x, y.
\end{aligned}$$

7. The function $\mathtt{BIT}(y, x) = \mathrm{BIT}(y, x)$ is in $\mathcal{F}$PTIME. Using $\mathtt{msp}(x, y), \mathtt{MOD2}(x)$ and $2^{\mathtt{min}(|x|,y)}$, We have

$$\mathtt{BIT}(y, x) = \mathtt{MOD2}\left(\left\lfloor \frac{x}{2^{\mathtt{min}(|x|,y)}} \right\rfloor\right) = \mathtt{MOD2}\left(\mathtt{msp}\left(x, \left\lfloor \frac{2^{\mathtt{min}(|x|,y)}}{2} \right\rfloor\right)\right).$$

If $\alpha = \mathtt{BIT}$, note that $m = 1, n = 1$. Let $r(x, k) = \mathtt{BIT}(|x|, k)$. Then

$$\begin{aligned}
\alpha(|x|; \mathtt{bin}(k)) &= \mathtt{BIT}(|x|; \mathtt{bin}(k)) \\
&= \mathrm{BIT}(|x|, k) = |\mathtt{BIT}(|x|, k)| = |r(x, k)|.
\end{aligned}$$

Induction step.
Case COMP($\in \mathcal{C}_{\mathbb{N}}$):
Suppose that

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\gamma,\beta_0,\dots,\beta_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\chi_0,\dots,\chi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $t,u_0,\dots,u_{M-1},v_0,\dots,v_{N-1} \in \mathcal{F}$PTIME such that

$$\begin{aligned}
\gamma(|\vec{y}|;\mathtt{bin}(\vec{\ell})) &= |t(\vec{y},\vec{\ell})|,\\
\beta_i(|\vec{x}|;\mathtt{bin}(\vec{k})) &= |u_i(\vec{x},\vec{k})|,\\
\chi_j(|\vec{x}|;\mathtt{bin}(\vec{k})) &= \mathrm{bin}(v_j(\vec{x},\vec{k}))
\end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since $\mathcal{F}$PTIME is closed under composition, define $r \in \mathcal{F}$PTIME by

$$r(\vec{x},\vec{k}) = t(u_0(\vec{x},\vec{k}),\dots,t_{M-1}(\vec{x},\vec{k}),v_0(\vec{x},\vec{k}),\dots,v_{N-1}(\vec{x},\vec{k})).$$

Then we have[7]

$$\begin{aligned}
&\alpha(|\vec{x}|;\mathtt{bin}(\vec{k}))\\
&= \gamma(\beta_0(|\vec{x}|;\mathtt{bin}(\vec{k})),\dots,\beta_{M-1}(|\vec{x}|;\mathtt{bin}(\vec{k}));\\
&\qquad \chi_0(|\vec{x}|;\mathtt{bin}(\vec{k})),\dots,\chi_{N-1}(|\vec{x}|;\mathtt{bin}(\vec{k})))\\
&= \gamma(|u_0(\vec{x},\vec{k})|,\dots,|u_{M-1}(\vec{x},\vec{k})|;\mathrm{bin}(v_0(\vec{x},\vec{k})),\dots,\mathrm{bin}(v_{N-1}(\vec{x},\vec{k})))\\
&= |t(u_0(\vec{x},\vec{k}),\dots,u_{M-1}(\vec{x},\vec{k}),v_0(\vec{x},\vec{k}),\dots,v_{N-1}(\vec{x},\vec{k}))|\\
&= |r(\vec{x},\vec{k})|.
\end{aligned}$$

Case COMP($\in \mathcal{C}_{\mathbb{W}}$):
Suppose that

$$\varphi(\vec{x};\vec{a}) = \psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\beta_0,\dots,\beta_{M-1} \in \mathcal{C}_{\mathbb{N}}$ and $\psi,\chi_0,\dots,\chi_{N-1} \in \mathcal{C}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $u_0,\dots,u_{M-1},r,v_0,\dots,v_{N-1} \in \mathcal{F}$PTIME such that

$$\begin{aligned}
\psi(|\vec{y}|;\mathtt{bin}(\vec{\ell})) &= \mathrm{bin}(r(\vec{y},\vec{\ell})),\\
\beta_i(|\vec{x}|;\mathtt{bin}(\vec{k})) &= |u_i(\vec{x},\vec{k})|,\\
\chi_j(|\vec{x}|;\mathtt{bin}(\vec{k})) &= \mathrm{bin}(v_j(\vec{x},\vec{k}))
\end{aligned}$$

[7] To make the expression easier to see, we write the expression as this, but, more precisely, apply the 2nd equation of I.H. firstly, and then apply the 1st equation and 3rd equation of I.H. simultaneously.

for $0 \le i < M$ and $0 \le j < N$. Since $\mathcal{F}$PTIME is closed under composition, define $r \in \mathcal{F}$PTIME by

$$t(\vec{x},\vec{k}) = r(u_0(\vec{x},\vec{k}),\dots,t_{M-1}(\vec{x},\vec{k}),v_0(\vec{x},\vec{k}),\dots,v_{N-1}(\vec{x},\vec{k})).$$

Then we have[8]

$$\begin{aligned}
&\varphi(|\vec{x}|;\texttt{bin}(\vec{k}))\\
&= \psi(\beta_0(|\vec{x}|;\texttt{bin}(\vec{k})),\dots,\beta_{M-1}(|\vec{x}|;\texttt{bin}(\vec{k}));\\
&\qquad \chi_0(|\vec{x}|;\texttt{bin}(\vec{k})),\dots,\chi_{N-1}(|\vec{x}|;\texttt{bin}(\vec{k})))\\
&= \psi(|u_0(\vec{x},\vec{k})|,\dots,|u_{M-1}(\vec{x},\vec{k})|;\mathrm{bin}(v_0(\vec{x},\vec{k})),\dots,\mathrm{bin}(v_{N-1}(\vec{x},\vec{k})))\\
&= \mathrm{bin}(r(u_0(\vec{x},\vec{k}),\dots,u_{M-1}(\vec{x},\vec{k}),v_0(\vec{x},\vec{k}),\dots,v_{N-1}(\vec{x},\vec{k})))\\
&= \mathrm{bin}(t(\vec{x},\vec{k})).
\end{aligned}$$

Case BR:

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by bounded recursion, then

$$\begin{aligned}
\alpha(0,\vec{x};\vec{a}) &= \beta(\vec{x};\vec{a}),\\
\alpha(z+1,\vec{x};\vec{a}) &= \gamma(z,\vec{x},\alpha(z,\vec{x};\vec{a});\vec{a}),\\
\alpha(z,\vec{x};\vec{a}) &\le \delta(z,\vec{x};\vec{a}) \quad \text{for any } z,\vec{x},\vec{a},
\end{aligned}$$

where $\beta,\gamma,\delta \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $t,u,v \in \mathcal{F}$PTIME such that

$$\begin{aligned}
\beta(|\vec{x}|;\texttt{bin}(\vec{k})) &= |t(\vec{x},\vec{k})|,\\
\gamma(|z|,|\vec{x}|,|y|;\texttt{bin}(\vec{k})) &= |u(z,\vec{x},y,\vec{k})|,\\
\delta(|z|,|\vec{x}|;\texttt{bin}(\vec{k})) &= |v(z,\vec{x},\vec{k})|.
\end{aligned}$$

Since $\mathcal{F}$PTIME is closed under bounded recursion on notation, define r by

$$\begin{aligned}
r(0,\vec{x},\vec{k}) &= t(\vec{x},\vec{k}),\\
r(\texttt{s}_0(z),\vec{x},\vec{k}) &= u(z,\vec{x},r(z,\vec{x},\vec{k}),\vec{k}) \qquad (z \ne 0),\\
r(\texttt{s}_1(z),\vec{x},\vec{k}) &= u(z,\vec{x},r(z,\vec{x},\vec{k}),\vec{k}).
\end{aligned}$$

Now, we show $\alpha(|z|,|\vec{x}|;\texttt{bin}(\vec{k})) = |r(z,\vec{x},\vec{k})|$ by induction on $|z|$.
In the case $|0|$,

$$\alpha(|0|,|\vec{x}|;\texttt{bin}(\vec{k})) = \beta(|\vec{x}|;\texttt{bin}(\vec{k})) = |t(\vec{x},\vec{k})| = |r(0,\vec{x},\vec{k})|.$$

[8] Note the same as in the previous footnote.

In the case $|z|$, suppose that $\alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |r(z, \vec{x}, \vec{k})|$.
In the case $|\mathtt{s}_i(z)|$,

$$\begin{aligned}\alpha(|\mathtt{s}_i(z)|, |\vec{x}|; \mathtt{bin}(\vec{k})) &= \alpha(|z| + 1, |\vec{x}|; \mathtt{bin}(\vec{k})) \\ &= \gamma(|z|, |\vec{x}|, \alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})); \mathtt{bin}(\vec{k})) \\ &= \gamma(|z|, |\vec{x}|, |r(z, \vec{x}, \vec{k})|; \mathtt{bin}(\vec{k})) \\ &= |u(z, \vec{x}, r(z, \vec{x}, \vec{k}), \vec{k})| \\ &= |r(\mathtt{s}_i(z), \vec{x}, \vec{k})|.\end{aligned}$$

Therefore, we have $\alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |r(z, \vec{x}, \vec{k})|$ for all z.[9]

Furthermore,

$$|r(z, \vec{x}, \vec{k})| = \alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) \le \delta(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |v(z, \vec{x}, \vec{k})|.$$

Hence, we have

$$r(z, \vec{x}, \vec{k}) \le 2^{|v(z,\vec{x},\vec{k})|} = 1 \# v(z, \vec{x}, \vec{k}) \quad \text{for all } z, \vec{x}, \vec{k}.$$

Since r is bounded from above by some function in $\mathcal{F}$PTIME, we conclude $r \in \mathcal{F}$PTIME.

Case BCVR:

We prepare the function $\mathtt{rev}(x)$ which computes the number whose binary representaion is the reverse of the binary representaion of x. Firstly, we prepare an auxiliary function $\mathtt{rev0}(x, y) \in \mathcal{F}$PTIME using BRN as follows:

$$\begin{aligned}\mathtt{rev0}(x, 0) &= 0, \\ \mathtt{rev0}(x, \mathtt{s}_0(y)) &= \mathtt{s}_{\mathtt{BIT}(|y|,x)}(\mathtt{rev0}(x, y)) \qquad (y \ne 0), \\ \mathtt{rev0}(x, \mathtt{s}_1(y)) &= \mathtt{s}_{\mathtt{BIT}(|y|,x)}(\mathtt{rev0}(x, y)), \\ \mathtt{rev0}(x, y) &\le 2^{|y|} = 1 \# y \quad \text{for any } x, y.\end{aligned}$$

Then, we define $\mathtt{rev}(x) \in \mathcal{F}$PTIME by $\mathtt{rev}(x) = \mathtt{rev0}(x, x)$.[10]

Suppose that $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by boolean course-of-values recursion, then

$$\begin{aligned}&\alpha(0, \vec{x}; \vec{a}) = \beta(\vec{x}; \vec{a}), \\ &\alpha(z + 1, \vec{x}; \vec{a}) = \gamma(z, \vec{x}; b, \vec{a}), \\ &\quad \text{where } |; b| = z + 1 \text{ and } \mathtt{BIT}(i; b) = \alpha(i, \vec{x}; \vec{a}) \text{ for } 0 \le i \le z, \\ &\alpha(z, \vec{x}; \vec{a}) \le 1 \quad \text{for any } z, \vec{x}, \vec{a},\end{aligned}$$

[9] This proof by induction divides $\mathbb{N}$ into length classes $|z|$, and prove the equation for each $|z|$. Hence, the equation holds for all x included in $|z|$, i.e., $2^{|z|-1} \le x < 2^{|z|} - 1$.

[10] These constructions are based on Clote [2].

where $\beta, \gamma \in \mathcal{C}_{\mathbb{N}}$.
By induction hypothesis, there exist $t, u \in \mathcal{F}$PTIME such that

$$\beta(|\vec{x}|; \texttt{bin}(\vec{k})) = |t(\vec{x}, \vec{k})|,$$
$$\gamma(|z|, |\vec{x}|; \texttt{bin}(\ell), \texttt{bin}(\vec{k}) = |u(z, \vec{x}, \ell, \vec{k})|.$$

We want to show that there exists $r \in \mathcal{F}$PTIME such that

$$\alpha(|z|, |\vec{x}|; \texttt{bin}(\vec{k})) = |r(z, \vec{x}, \vec{k})|.$$

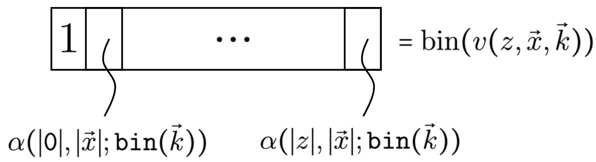
Since $\mathcal{F}$PTIME is closed under bounded recursion on notation, define v by[11]

$$\begin{aligned}
v(0, \vec{x}, \vec{k}) &= \texttt{s}_{|t(\vec{x},\vec{k})|}(1), \\
v(\texttt{s}_0(z), \vec{x}, \vec{k}) &= \texttt{s}_{|u(z,\vec{x},v'(z,\vec{x},\vec{k}),\vec{k})|}(v(z, \vec{x}, \vec{k})) \qquad (z \neq 0), \\
v(\texttt{s}_1(z), \vec{x}, \vec{k}) &= \texttt{s}_{|u(z,\vec{x},v'(z,\vec{x},\vec{k}),\vec{k})|}(v(z, \vec{x}, \vec{k})), \\
v'(z, \vec{x}, \vec{k}) &= \texttt{msp}(\texttt{rev}(v(z, \vec{x}, \vec{k})), 1), \\
v(z, \vec{x}, \vec{k}) &= \texttt{s}_0(\texttt{s}_0(1 \# z)) = 2^{|z|+2} \quad \text{for all } z, \vec{x}, \vec{k}.
\end{aligned}$$

Then, define $r \in \mathcal{F}$PTIME by

$$r(z, \vec{x}, \vec{k}) = \texttt{MOD2}(v(z, \vec{x}, \vec{k})).$$

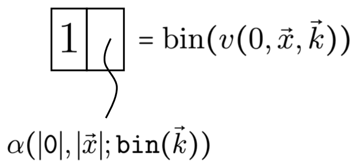
Now, we show that the string constructed by appending $\alpha(|0|, |\vec{x}|; \texttt{bin}(\vec{k}))$, $\ldots, \alpha(|z|, |\vec{x}|; \texttt{bin}(\vec{k}))$ to 1 is equal to $\text{bin}(v(z, \vec{x}, \vec{k}))$[12] by induction on $|z|$. We illustrate this in the following diagram.



In the case $|0|$,

$$v(0, \vec{x}, \vec{k}) = \texttt{s}_{|t(\vec{x},\vec{k})|}(1) = \texttt{s}_{\beta(|\vec{x}|;\texttt{bin}(\vec{k}))}(1) = \texttt{s}_{\alpha(|0|,|\vec{x}|;\texttt{bin}(\vec{k}))}(1),$$

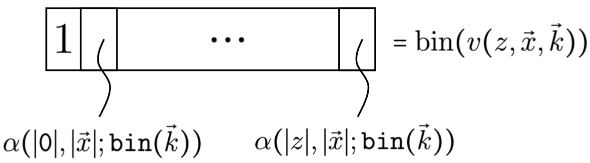
which means the following is true.



[11]Notice that $\beta, \gamma \leq 1$, hence $|t|, |u| \leq 1$.

[12]Since $\text{bin}(\cdot)$ is a identity function which regards a natural number as its binary representaion, we may or may not apply $\text{bin}(\cdot)$ to $v(z, \vec{x}, \vec{k})$.

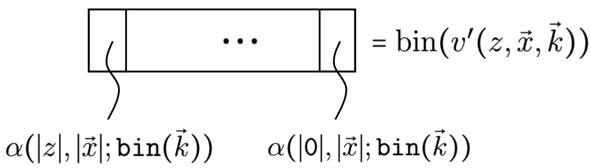
In the case $|z|$, suppose that the above statement holds.



In the case $|\mathtt{s}_i(z)|$,

$$v(\mathtt{s}_i(z), \vec{x}, \vec{k}) = \mathtt{s}_{|u(z,\vec{x},v'(z,\vec{x},\vec{k}),\vec{k})|}(v(z, \vec{x}, \vec{k})),$$
$$v'(z, \vec{x}, \vec{k}) = \mathtt{msp}(\mathtt{rev}(v(z, \vec{x}, \vec{k})), 1).$$

By induction hypothesis, $\mathrm{bin}(v'(z, \vec{x}, \vec{k}))$ is the number whose j-th bit is $\alpha(j, |\vec{x}|, \mathtt{bin}(\vec{k}))$ for $|0| \leq j \leq |z|$ shown as follows.



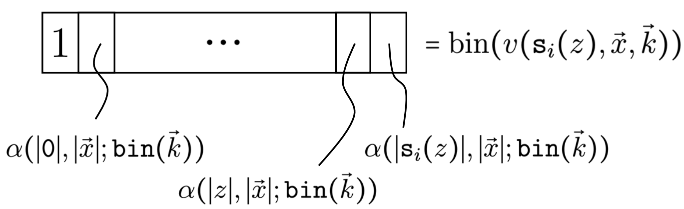
Hence,

$$\begin{aligned} |u(z, \vec{x}, v'(z, \vec{x}, \vec{k}), \vec{k})| &= \gamma(|z|, |\vec{x}|; b, \mathtt{bin}(\vec{k})) \\ &\quad \text{where } |; b| = |\mathtt{s}_i(z)| \text{ and } \mathtt{BIT}(j; b) = \alpha(j, |\vec{x}|; \mathrm{bin}(\vec{k})) \text{ for } |0| \leq j \leq |z|, \\ &= \alpha(|\mathtt{s}_i(z)|, |\vec{x}|; \mathrm{bin}(\vec{k})) \end{aligned}$$

Hence, we have

$$v(\mathtt{s}_i(z), \vec{x}, \vec{k}) = \mathtt{s}_{\alpha(|\mathtt{s}_i(z)|, |\vec{x}|; \mathrm{bin}(\vec{k}))}(v(z, \vec{x}, \vec{k})),$$

which shows by induction hypothesis the following is true.



Therefore, the above statement holds for all z.

Thus, we have

$$\begin{aligned} \alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) &= \mathtt{MOD2}(v(z, \vec{x}, \vec{k})) \\ &= r(z, \vec{x}, \vec{k}) = |r(z, \vec{x}, \vec{k})|. \end{aligned}$$

Case BC:

Firstly, we prepare an auxiliary function $\mathtt{ones}(x) \in \mathcal{F}$PTIME using BRN as follows:

$$\begin{aligned}
\mathtt{ones}(\mathtt{0}) &= \mathtt{0}, \\
\mathtt{ones}(\mathtt{s_0}(x)) &= \mathtt{s_1}(\mathtt{ones}(x)) \qquad (x \neq \mathtt{0}), \\
\mathtt{ones}(\mathtt{s_1}(x)) &= \mathtt{s_1}(\mathtt{ones}(x)), \\
\mathtt{ones}(x) &\leq 2^{|x|} = 1\#x \quad \text{for any } x.
\end{aligned}$$

Then, we define the signum function $\mathtt{sg}(x) \in \mathcal{F}$PTIME by $\mathtt{sg}(x) = \mathtt{MOD2}(\mathtt{ones}(x))$.

Suppose that $\varphi \in \mathcal{C}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{W}$ is defined by bounded comprehension, then

$$\begin{aligned}
&|; \varphi(z, \vec{x}; \vec{a})| = z, \\
&\forall j < z \; [\mathtt{BIT}(j; \varphi(z, \vec{x}; \vec{a})) = \mathtt{0} \leftrightarrow \alpha(j, \vec{x}; \vec{a}) = \mathtt{0}],
\end{aligned}$$

where $\alpha \in \mathcal{C}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$.
By induction hypothesis, there exists $r \in \mathcal{F}$PTIME such that

$$\alpha(|j|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |r(j, \vec{x}; \vec{k})|.$$

We want to show that there exists $t \in \mathcal{F}$PTIME such that

$$\varphi(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = \mathrm{bin}(t(z, \vec{x}, \vec{k})).$$

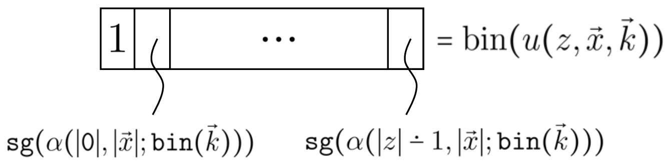
Since $\mathcal{F}$PTIME is closed under bounded recursion on notation, define u by

$$\begin{aligned}
u(\mathtt{0}, \vec{x}, \vec{k}) &= 1, \\
u(\mathtt{s_0}(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|r(z,\vec{x},\vec{k})|)}(u(z, \vec{x}, \vec{k})) \qquad (z \neq \mathtt{0}), \\
u(\mathtt{s_1}(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|r(z,\vec{x},\vec{k})|)}(u(z, \vec{x}, \vec{k})), \\
u(z, \vec{x}, \vec{k}) &= \mathtt{s_0}(1\#z) = 2^{|z|+1} \quad \text{for all } z, \vec{x}, \vec{k}.
\end{aligned}$$

Then, define $t \in \mathcal{F}$PTIME by

$$t(z, \vec{x}, \vec{k}) = \mathtt{msp}(\mathtt{rev}(u(z, \vec{x}, \vec{k})), 1).$$

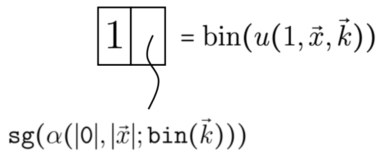
Now, we show that the string constructed by appending $\mathtt{sg}(\alpha(|0|, |\vec{x}|; \mathtt{bin}(\vec{k})))$, $\ldots, \mathtt{sg}(\alpha(|z| \dot{-} 1, |\vec{x}|; \mathtt{bin}(\vec{k})))$ to 1 is equal to $\mathrm{bin}(u(z, \vec{x}, \vec{k}))$ by induction on $|z|$. We illustrate this in the following diagram.



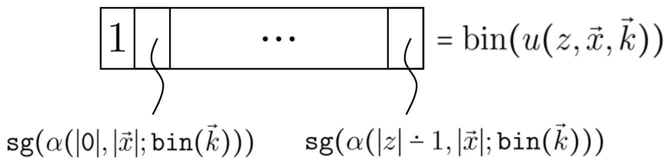
In the case $|1|$,

$$u(1,\vec{x},\vec{k}) = u(\mathtt{s}_1(0),\vec{x},\vec{k})$$
$$= \mathtt{s}_{\mathtt{sg}(|r(0,\vec{x},\vec{k})|)}(u(0,\vec{x},\vec{k})) = \mathtt{s}_{\mathtt{sg}(\alpha(|0|,|\vec{x}|;\mathtt{bin}(\vec{k})))}(1),$$

which means the following is true.



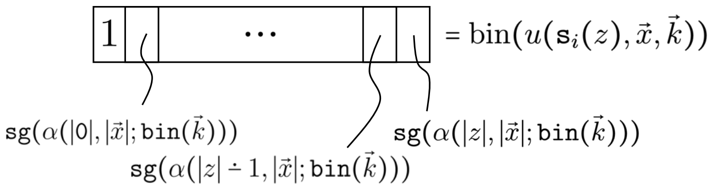
In the case $|z|$, suppose that the above statement holds.



In the case $|\mathtt{s}_i(z)|$,

$$u(\mathtt{s}_i(z),\vec{x},\vec{k})$$
$$= \mathtt{s}_{\mathtt{sg}(|r(z,\vec{x},\vec{k})|)}(u(z,\vec{x},\vec{k})) = \mathtt{s}_{\mathtt{sg}(\alpha(|z|,|\vec{x}|;\mathtt{bin}(\vec{k})))}(u(z,\vec{x},\vec{k})).$$

By induction hypothesis, the above expression means the following is true.



Therefore, the above statement holds for all $z(\geq 1)$.

Since by the definition of φ, $\varphi(|z|,|\vec{x}|;\mathtt{bin}(\vec{k}))$ is the string constructed by appeding from $\mathtt{sg}(\alpha(|z|\dot{-}1,|\vec{x}|;\mathtt{bin}(\vec{k})))$ to $\mathtt{sg}(\alpha(|0|,|\vec{x}|;\mathtt{bin}(\vec{k})))$, we have for $z(\geq 1)$

$$\varphi(|z|,|\vec{x}|;\mathtt{bin}(\vec{k})) = \mathrm{bin}(\mathtt{msp}(\mathtt{rev}(u(z,\vec{x},\vec{k})),1))$$
$$= \mathrm{bin}(t(z,\vec{x},\vec{k})).$$

Note that since $\varphi(|0|,|\vec{x}|;\mathtt{bin}(\vec{k})) = \varepsilon = \mathtt{bin}(0)$ and $\mathrm{bin}(t(\mathtt{0},\vec{x},\vec{k})) = \mathrm{bin}(\mathtt{0})$, the above equation holds for $z = 0$. □

3.6 Inclusion of PTIME by $\mathcal{E}_*^{2+}$

In section 3.2 and section 3.3, we have associated functions in $\mathcal{E}^{2+}$ with functions in $\mathcal{C}_{\mathbb{N}}$, and in section 3.4 and section 3.5, we have associated functions in $\mathcal{F}$PTIME with functions in $\mathcal{C}_{\mathbb{W}}$ (via $\mathcal{C}_{\mathbb{N}}$). Using these correspondences, we obtain the following theorem.

Theorem 3.31.

$$\text{PTIME} \subseteqq \mathcal{E}_*^{2+}.$$

Proof. Let s be any set in PTIME, then the charateristic function χ_s of s is in $\mathcal{F}$PTIME.
By Proposition 3.29 (i), there exists $\alpha \in \mathcal{C}_{\mathbb{N}}$ such that

$$\text{BIT}(i, \text{bin}(\chi_s(\vec{x}))) = \alpha(i; \texttt{bin}(\vec{x})).$$

By Corollary 3.28, there exists $f \in \mathcal{E}^{2+}$ such that

$$\alpha(i; \texttt{bin}(\vec{x})) = f(i, \vec{x}).$$

Hence, we have

$$\text{BIT}(i, \text{bin}(\chi_s(\vec{x}))) = f(i, \vec{x}).$$

Since χ_s is the characteristic function, we have

$$\chi_s(\vec{x}) = \text{BIT}(0, \text{bin}(\chi_s(\vec{x}))) = f(0, \vec{x}).$$

Hence,

$$\chi_s \in \mathcal{E}^{2+}.$$

Therefore,

$$s \in \mathcal{E}_*^{2+}.$$

□

4 Relationship between $\mathcal{M}^2$ and $\mathcal{F}$LH

In this capter, we study a relationship between $\mathcal{M}^2$ and $\mathcal{F}$LH.

Let $\mathcal{M}^2$ be the the second class in the hierarchy of bounded minimisation, that is,

$$\mathcal{M}^2 = [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \mathrm{COMP}, \mathrm{BMIN}].$$

As are defined in the next section, let $\mathcal{D}_\mathbb{N}$ and $\mathcal{D}_\mathbb{W}$ be classes of functions defined simultaneously and recursively over both the set of natural numbers $\mathbb{N}$ and the set of binary strings $\mathbb{W}$ such that functions in $\mathcal{D}_\mathbb{N}$ maps them to $\mathbb{N}$ and functions in $\mathcal{D}_\mathbb{W}$ maps them to $\mathbb{W}$.

Then, we associate functions in $\mathcal{M}^2$ with functions in $\mathcal{D}_\mathbb{N}$, and also associate functions in $\mathcal{F}$LH with functions in $\mathcal{D}_\mathbb{W}$.

Using these correspondences, with respect to their set classes $\mathcal{M}^2_*$ and LH, we show that

$$\mathrm{LH} \subsetneqq \mathcal{M}^2_*$$

(Theorem 4.16)[13].

4.1 Definitions of $\mathcal{D}_\mathbb{N}$ and $\mathcal{D}_\mathbb{W}$, and some basic functions

To begin with, we define function classes $\mathcal{D}_\mathbb{N}$ and $\mathcal{D}_\mathbb{W}$.

Definition 4.1. Classes $\mathcal{D}_\mathbb{N}$ and $\mathcal{D}_\mathbb{N}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, respectively, are generated simultaneously by the following clauses.

1. The *projection* functions $\mathtt{p}_{\mathbb{N}_i}^{m,n}$ and $\mathtt{p}_{\mathbb{W}_j}^{m,n}$ belong to $\mathcal{D}_\mathbb{N}$ and $\mathcal{D}_\mathbb{W}$, respectively:

$$\mathtt{p}_{\mathbb{N}_i}^{m,n}(x_0, \ldots, x_{m-1}; \vec{a}) = x_i \quad (0 \le i < m),$$
$$\mathtt{p}_{\mathbb{W}_j}^{m,n}(\vec{x}; a_0, \ldots, a_{n-1}) = a_j \quad (0 \le j < n);$$

2. the constant *zero* $\mathtt{0}$ belongs to $\mathcal{D}_\mathbb{N}$: $\mathtt{0} = 0$;

3. the *successor* function $\mathtt{S}$ belongs to $\mathcal{D}_\mathbb{N}$: $\mathtt{S}(x;) = \mathrm{S}x$;

4. the *cut-off subtraction* $\dot{-}$ belongs to $\mathcal{D}_\mathbb{N}$: $\dot{-}(x, y;) = x \dot{-} y$;

[13]It is known LTH = $\mathcal{M}^2_*$ (Theorem 2.22). And, it holds that LH $\subsetneqq$ LTH. Hence, it can be derived from existing knowledge that LH $\subsetneqq \mathcal{M}^2_*$. Accordingly, the contents of this chapter are another proof of this inclusion.

5. the *multiplication* $\times$ belongs to $\mathcal{D}_{\mathbb{N}}$: $\times(x, y;) = x \cdot y$;

6. the *length* function $|\cdot| \in \mathbb{W} \to \mathbb{N}$ belongs to $\mathcal{D}_{\mathbb{N}}$: $|; a| = |a|$;

7. the *bit* function $\mathtt{BIT} \in \mathbb{N} \times \mathbb{W} \to \mathbb{N}$ belongs to $\mathcal{D}_{\mathbb{N}}$: $\mathtt{BIT}(z; a) = \mathrm{BIT}(z, a)$;

8. $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ are closed under *composition* (COMP):
   if $\gamma, \beta_0, \ldots, \beta_{M-1} \in \mathcal{D}_{\mathbb{N}}$ and $\psi, \chi_0, \ldots, \chi_{N-1} \in \mathcal{D}_{\mathbb{W}}$ with $\gamma \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{N}$, $\beta_i \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, $\psi \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{W}$ and $\chi_j \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$ for $0 \le i < M$ and $0 \le j < N$, then there exist $\alpha \in \mathcal{D}_{\mathbb{N}}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}\alpha(\vec{x}; \vec{a}) &= \gamma(\beta_0(\vec{x}, \vec{a}), \ldots, \beta_{M-1}(\vec{x}; \vec{a}); \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a})),\\ \varphi(\vec{x}; \vec{a}) &= \psi(\beta_0(\vec{x}, \vec{a}), \ldots, \beta_{M-1}(\vec{x}; \vec{a}); \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a}));\end{aligned}$$

9. $\mathcal{D}_{\mathbb{N}}$ is closed under *bounded minimisation* (BMIN):
   if $\beta \in \mathcal{D}_{\mathbb{N}}$ with $\beta \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\alpha \in \mathcal{D}_{\mathbb{N}}$ satisfying

$$\alpha(z, \vec{x}; \vec{a}) = \begin{cases} \text{the least } y \le z \text{ such that } \beta(y, \vec{x}; \vec{a}) \ne \mathtt{0} & \text{if it exists,} \\ z + 1 & \text{otherwise;} \end{cases}$$

10. $\mathcal{D}_{\mathbb{W}}$ is closed under *bounded comprehension* (BC):
    if $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\varphi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}&|; \varphi(z, \vec{x}; \vec{a})| = z \quad \text{and}\\ &\forall i < z \, [\mathtt{BIT}(i; \varphi(z, \vec{x}; \vec{a})) = \mathtt{0} \leftrightarrow \alpha(i, \vec{x}; \vec{a}) = \mathtt{0}].\end{aligned}$$

Notation. We will use the same notations in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ as ones used in $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$, unless otherwise noted. In using bounded minimisation, we write $\min_{y \le z}\{\beta(y, \vec{x}; \vec{a}) \ne \mathtt{0}\}$ for $\alpha(z, \vec{x}; \vec{a})$.

In function algebras, the classes $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ are represented as follows:

$$\begin{aligned}\mathcal{D}_{\mathbb{N}} &= [\mathtt{p}_{\mathbb{N}}{}_i^{m,n}, \mathtt{0}, \mathtt{S}, \dot{-}, \times, |\cdot|, \mathtt{BIT}; \mathrm{COMP}, \mathrm{BMIN}],\\ \mathcal{D}_{\mathbb{W}} &= [\mathtt{p}_{\mathbb{W}}{}_j^{m,n}; \mathrm{COMP}, \mathrm{BC}].\end{aligned}$$

Note that any constant belongs to $\mathcal{D}_{\mathbb{N}}$ by repeatedly applying the successor function $\mathtt{S}$ to the constant $\mathtt{0}$.

Now, we introduce some useful functions to $\mathcal{D}_{\mathbb{N}}$ which will be used in subsequent lemmas and propositions.

Lemma 4.2. *The following functions belong to* $\mathcal{D}_{\mathbb{N}}$:

1. *the predecessor fucntion* $\mathtt{prd} : \mathtt{prd}(x;) = \mathrm{prd}(x)$;
2. *the addition* $+ : +(x,y;) = x + y$.

Proof. The predecessor function is defined by $\mathtt{prd}(x;) = x \dot{-} \mathtt{S}(\mathtt{0};)$.

The addition is defined by

$$+(x,y;) = \mathtt{S}(x;) \times \mathtt{S}(y;) \dot{-} x \times y \dot{-} 1.$$

□

Lemma 4.3. *The following functions belong to* $\mathcal{D}_{\mathbb{N}}$:

1. *the signum function* $\mathtt{sg} : \mathtt{sg}(x;) = \mathrm{sg}(x)$;
2. *the inverse signum fucntion* $\overline{\mathtt{sg}} : \overline{\mathtt{sg}}(x;) = \overline{\mathrm{sg}}(x)$;
3. *the maximum function* $\mathtt{max} : \mathtt{max}(x,y;) = \max(x,y)$;
4. *the minimum function* $\mathtt{min} : \mathtt{min}(x,y;) = \min(x,y)$;
5. *the conditional function* $\mathtt{cond} : \mathtt{cond}(x,y,z;) = \mathrm{cond}(x,y,z)$;
6. *the characteristic function* $\chi_=$ *of* $=$: $\chi_=(x,y;) = \chi_=(x,y)$;
7. *the characteristic function* $\chi_\leq$ *of* $\leq$: $\chi_\leq(x,y;) = \chi_\leq(x,y)$;
8. *the characteristic function* $\chi_<$ *of* $<$: $\chi_<(x,y;) = \chi_<(x,y)$.

Proof. See the proof of Lemma 3.5. □

Lemma 4.4. *The following logical functions belong to* $\mathcal{D}_{\mathbb{N}}$:

1. *the characteristic function* $\chi_\neg$ *of* $\neg$: $\chi_\neg(x;) = \chi_\neg(x)$;
2. *the characteristic function* $\chi_\wedge$ *of* $\wedge$: $\chi_\wedge(x,y;) = \chi_\wedge(x,y)$;
3. *the characteristic function* $\chi_\vee$ *of* $\vee$: $\chi_\vee(x,y;) = \chi_\vee(x,y)$;
4. *the characteristic function* $\chi_\rightarrow$ *of* $\rightarrow$: $\chi_\rightarrow(x,y;) = \chi_\rightarrow(x,y)$.

Proof. See the proof of Lemma 3.6. □

At the beginning of section 4.3. we will construct $\mathtt{bin}(n;) \in \mathcal{D}_{\mathbb{W}}$ which computes the binary string of the binary representation of n.

4.2 Representation of $\mathcal{M}^2$ functions by $\mathcal{D}_\mathbb{N}$ functions

In this section, we show that any function in $\mathcal{M}^2$ is represented by some function in $\mathcal{D}_\mathbb{N}$, that is,

$$\forall f \in \mathcal{M}^2 \ \exists \alpha \in \mathcal{D}_\mathbb{N} \ [f(\vec{x}) = \alpha(\vec{x};)].$$

Proposition 4.5. *For each $f \in \mathcal{M}^2$ with $f \in \mathbb{N}^m \to \mathbb{N}$, there exists $\alpha \in \mathcal{D}_\mathbb{N}$ such that*

$$f(\vec{x}) = \alpha(\vec{x};)$$

for each $\vec{x}$.

Proof. By induction on the structure of $f \in \mathcal{M}^2$.
Basis.
Case $f = \mathtt{0}$:
Note that $m = 0$. Let $\alpha = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $f = \mathtt{I}_i^m$:
Note that $0 \leq i < m$. Let $\alpha = \mathtt{p_\mathbb{N}}_i^{m,0}$, then

$$f(\vec{x}) = \mathtt{I}_i^m(\vec{x}) = x_i = \mathtt{p_\mathbb{N}}_i^{m,0}(\vec{x};) = \alpha(\vec{x};).$$

Case $f = \mathtt{S}$:
Note that $m = 1$. Let $\alpha = \mathtt{S}$, then $\mathtt{S}(x_1) = \mathtt{S}(x_1;)$.
Case $f = +$:
Note that $m = 2$. Let $\alpha = +$, then $x_1 + x_2 = x_1 + x_2$.
Case $f = \times$:
Note that $m = 2$. Let $\alpha = \times$, then $x_1 \times x_2 = x_1 \times x_2$.

Induction step.
Case COMP:
Suppose that

$$f(\vec{x}) = h(g_0(\vec{x}), \dots, g_{L-1}(\vec{x})),$$

whrere $h, g_0, \dots, g_{L-1} \in \mathcal{M}^2$. Then, by the induction hypothesis, there exist $\gamma, \beta_0, \dots, \beta_{L-1} \in \mathcal{D}_\mathbb{N}$ such that

$$h(\vec{y}) = \gamma(\vec{y};),$$
$$g_i(\vec{x}) = \beta_i(\vec{x};)$$

for $0 \leq i < L$.
Since $\mathcal{D}_\mathbb{N}$ is closed under composition, define $\alpha \in \mathcal{D}_\mathbb{N}$ by

$$\alpha(\vec{x};) = \gamma(\beta_0(\vec{x};), \dots, \beta_{L-1}(\vec{x};);).$$

Then we have

$$\begin{aligned} f(\vec{x}) &= h(g_0(\vec{x}), \dots, g_{L-1}(\vec{x})) \\ &= \gamma(\beta_0(\vec{x};), \dots, \beta_{L-1}(\vec{x};);) \\ &= \alpha(\vec{x};). \end{aligned}$$

Case BMIN:

Suppose that $f \in \mathcal{M}^2$ is defined by bounded minimisation, then

$$f(z, \vec{x}) = \begin{cases} \text{the least } y \leq z \text{ such that } g(y, \vec{x}) = 0 & \text{if it exists,} \\ 0 & \text{otherwise,} \end{cases}$$

where $g \in \mathcal{M}^2$.
By induction hypothesis, there exists $\beta \in \mathcal{D}_{\mathbb{N}}$ such that

$$g(y, \vec{x}) = \beta(y, \vec{x};).$$

Since $\mathcal{D}_{\mathbb{N}}$ is closed under bounded minimisation, define α by

$$\begin{aligned} \alpha(z, \vec{x};) = \mathtt{cond}\Big(&\chi_{=}\Big(\min_{y \leq z}\{\overline{\mathtt{sg}}(\beta(y, \vec{x};)) \neq \mathtt{0}\}, z + 1\Big), \\ &\min_{y \leq z}\{\overline{\mathtt{sg}}(\beta(y, \vec{x};)) \neq \mathtt{0}\}, \\ &\mathtt{0}\Big). \end{aligned}$$

Then we have

$$f(z, \vec{x}) = \alpha(z, \vec{x};).$$

□

4.3 Representation of $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ functions by $\mathcal{M}^2$ functions

Firstly, the function $\mathtt{bin}(\vec{k})$ will be used in the arguments of functions in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$, we explain the construction of $\mathtt{bin}(k;)$ in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$.

In section 3.1, we construct the $\mathtt{bin}(k;)$ function in $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$, but bounded recursion is used in the definition of the function $\mathtt{None} \in \mathbb{N} \to \mathbb{N}$, which is needed in the definitions of the functions $|\cdot| \in \mathbb{N} \to \mathbb{N}$, $\mathtt{exp} \in \mathbb{N}^2 \to \mathbb{N}$, $\mathtt{BIT} \in \mathbb{N}^2 \to \mathbb{N}$ and $\mathtt{bin} \in \mathbb{N} \to \mathbb{W}$. Hence, this construction can not be used in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$.

However, by Proposition 4.5 in the previous section, we have shown that

$$\forall f \in \mathcal{M}^2 \; \exists \alpha \in \mathcal{D}_{\mathbb{N}} \; [f(\vec{x}) = \alpha(\vec{x};)].$$

By Proposition 2.25, we know that the functions $|\cdot| \in \mathbb{N} \to \mathbb{N}$ and $\mathtt{BIT} \in \mathbb{N}^2 \to \mathbb{N}$ belong to $\mathcal{M}^2$, hence, these functions also belong to $\mathcal{D}_{\mathbb{N}}$. Therefore, we obtain the following lemma:

Lemma 4.6. *The function* $\mathtt{bin}(k;) \in \mathbb{N} \to \mathbb{W}$ *belongs to* $\mathcal{D}_{\mathbb{W}}$:[14]

$$\mathtt{bin}(k;) = \textit{the binary string of the binary representation of } k.$$

Proof. In the definition of bounded comprehension, let $\alpha(i, k;) = \mathtt{BIT}(i, k;)$, then there is $\varphi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$|; \varphi(z, k;)| = z,$$
$$\forall i < z \; [\mathtt{BIT}(i; \varphi(z, k;)) = 0 \leftrightarrow \alpha(i, k;) = 0].$$

Then we have

$$\mathtt{bin}(k;) = \varphi(|k;|, k;).$$

Notice that $\mathtt{bin}(0;) = \varepsilon$ $(\in \mathbb{W})$. □

In this section, we would like to show that any function in $\mathcal{D}_{\mathbb{N}}$ is represented by some function in $\mathcal{M}^2$ and that any function in $\mathcal{D}_{\mathbb{W}}$ is represented by some function in $\mathcal{M}^2$, that is,

$$\forall \alpha \in \mathcal{D}_{\mathbb{N}} \; \exists f \in \mathcal{M}^2 \; [\alpha(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})],$$
$$\forall \varphi \in \mathcal{D}_{\mathbb{W}} \; \exists g \in \mathcal{M}^2 \; [\varphi(\vec{x}; \mathtt{bin}(\vec{k})) = \mathrm{bin}(g(\vec{x}, \vec{k}))].$$

Just as there was a problem in $\mathcal{C}_{\mathbb{W}}$ and $\mathcal{C}_{\mathbb{W}}$, there is the same problem. If a function $\varphi \in \mathcal{D}_{\mathbb{W}}$ is defined by bounded comprehension, we have

[14] Again, we will omit a semicolon in $\mathtt{bin}(k;)$.

$|; \varphi(z, \vec{x}; \mathtt{bin}(\vec{k}))| = z$. Hence we must construct a fucntion $g \in \mathcal{M}^2$ such that $g = \Theta(2^z)$. However, we cannot construct such a function in $\mathcal{M}^2$ because $\mathcal{M}^2$ does not contain exponential function. Therefore, the latter formula above does not hold.

For this reason, we consider an intermediate class $\tilde{\mathcal{D}}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$, and for each function in $\mathcal{D}_{\mathbb{N}}$ we construct a function in $\tilde{\mathcal{D}}$ of the same values, and for each function in $\mathcal{D}_{\mathbb{W}}$ we construct two functions in $\tilde{\mathcal{D}}$, one giving its bit contents and the other giving its length, that is,

$$\begin{aligned}
&\forall \alpha \in \mathcal{D}_{\mathbb{N}}\ \exists \tilde{\alpha} \in \tilde{\mathcal{D}}\ [\alpha(\vec{x}; \vec{a}) = \tilde{\alpha}(\vec{x}, \vec{a})],\\
&\forall \varphi \in \mathcal{D}_{\mathbb{W}}\ \exists \tilde{\varphi} \in \tilde{\mathcal{D}}\ [\mathtt{BIT}(z; \varphi(\vec{x}; \vec{a})) = \tilde{\varphi}(z, \vec{x}; \vec{a})] \text{ and}\\
&\qquad\qquad \exists \hat{\varphi} \in \tilde{\mathcal{D}}\ [|; \varphi(\vec{x}; \vec{a})| = \hat{\varphi}(\vec{x}; \vec{a})]
\end{aligned}$$

(Proposition 4.9). And then, for each function in $\tilde{\mathcal{D}}$ we construct a function in $\mathcal{M}^2$ of the same values, that is,

$$\forall \tilde{\alpha} \in \tilde{\mathcal{D}}\ \exists f \in \mathcal{M}^2\ [\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})]$$

(Proposition 4.10).

The contents of thie section are based on Ishihara [6].

To begin with, we define an intermediate class $\tilde{\mathcal{D}}$.

Definition 4.7. A class $\tilde{\mathcal{D}}$ of functions of types $\mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ is generated by the following clauses.

1. The *projection* functions ${\mathtt{p}_{\mathbb{N}}}_i^{m,n}$ belong to $\tilde{\mathcal{D}}$:

$${\mathtt{p}_{\mathbb{N}}}_i^{m,n}(x_0, \dots, x_{m-1}; \vec{a}) = x_i \quad (0 \le i < m);$$

2. the constant *zero* $\mathtt{0}$ belongs to $\tilde{\mathcal{D}}$: $\mathtt{0} = 0$;

3. the *successor* function $\mathtt{S}$ belongs to $\tilde{\mathcal{D}}$: $\mathtt{S}(x;) = \mathrm{S}x$;

4. the *cut-off subtraction* $\dot{-}$ belongs to $\tilde{\mathcal{D}}$: $-(x, y;) = x \dot{-} y$;

5. the *multiplication* $\times$ belongs to $\tilde{\mathcal{D}}$: $\times(x, y;) = x \cdot y$;

6. the *projective length* function $|\cdot|_j^{m,n}$ belongs to $\tilde{\mathcal{D}}$:

$$|(\vec{x}; a_0, \dots, a_{n-1})|_j^{m,n} = |a_j| \quad (0 \le j < n);$$

7. the *projective bit* function $\mathtt{BIT}_j^{m+1,n}$ belongs to $\tilde{\mathcal{D}}$:

$$\mathtt{BIT}_j^{m+1,n}(z, \vec{x}; a_0, \dots, a_{n-1}) = \mathrm{BIT}(z, a_j) \quad (0 \le j < n);$$

8. $\tilde{\mathcal{D}}$ is closed under *composition* (COMP):
   if $\beta_0, \dots, \beta_{L-1}, \gamma \in \tilde{\mathcal{D}}$ with $\beta_i \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\gamma \in \mathbb{N}^L \times \mathbb{W}^n \to \mathbb{N}$ for $0 \le i < L$, then there exist $\alpha \in \tilde{\mathcal{D}}$ satisfying

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x},\vec{a}), \dots, \beta_{L-1}(\vec{x};\vec{a});\vec{a});$$

9. $\tilde{\mathcal{D}}$ is closed under *bounded minimisation* (BMIN):
   if $\beta \in \tilde{\mathcal{D}}$ with $\beta \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$, then there is $\alpha \in \tilde{\mathcal{D}}$ satisfying

$$\alpha(z,\vec{x};\vec{a}) = \begin{cases} \text{the least } y \le z \text{ such that } \beta(y,\vec{x};\vec{a}) \ne \mathtt{0} & \text{if it exists,} \\ z+1 & \text{otherwise.} \end{cases}$$

In function algebra, the class $\tilde{\mathcal{D}}$ is represented as follows:

$$\tilde{\mathcal{D}} = [\mathtt{p}_{\mathbb{N}}{}_i^{m,n}, \mathtt{0}, \mathtt{S}, \dot{-}, \times, |\cdot|_j^{m,n}, \mathtt{BIT}_j^{m+1,n}; \text{COMP}, \text{BMIN}].$$

We show a lemma used in the proposition 4.9.

Lemma 4.8. *Let* $\chi_0, \dots, \chi_{N-1}$ *be in* $\mathcal{D}_{\mathbb{W}}$ *with* $\chi_j \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$ *for* $0 \le j < N$, *and suppose that there exist* $\chi_0^{|\cdot|}, \dots, \chi_{N-1}^{|\cdot|}$ *and* $\chi_0^{\mathtt{BIT}}, \dots, \chi_{N-1}^{\mathtt{BIT}}$ *in* $\tilde{\mathcal{D}}$ *such that*

$$|\chi_j(\vec{x};\vec{a})| = \chi_j^{|\cdot|}(\vec{x};\vec{a}), \qquad \text{BIT}(z, \chi_j(\vec{x};\vec{a})) = \chi_j^{\mathtt{BIT}}(z,\vec{x};\vec{a})$$

for each $0 \le j < N$. *Then for any* $\alpha \in \tilde{\mathcal{D}}$ *with* $\alpha \in \mathbb{N}^M \times \mathbb{W}^N \to \mathbb{N}$, *there exists* $\tilde{\alpha} \in \tilde{\mathcal{D}}$ *with* $\tilde{\alpha} \in \mathbb{N}^{M+m} \times \mathbb{W}^n \to \mathbb{N}$ *such that*

$$\alpha(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) = \tilde{\alpha}(\vec{y},\vec{x};\vec{a}).$$

Proof. By induction on the structure of $\alpha \in \tilde{\mathcal{D}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}}{}_i^{M,N}$:
Note that $0 \le i < M$. Let $\tilde{\alpha} = \mathtt{p}_{\mathbb{N}}{}_i^{M+m,n}$. Then

$$\begin{aligned} \alpha(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= \mathtt{p}_{\mathbb{N}}{}_i^{M,N}(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= y_i = \mathtt{p}_{\mathbb{N}}{}_i^{M+m,n}(\vec{y},\vec{x};\vec{a}) = \tilde{\alpha}(\vec{y},\vec{x};\vec{a}). \end{aligned}$$

Case $\alpha = \mathtt{0}$:
Note that $M = N = 0$. Let $\tilde{\alpha} = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.
Case $\alpha = \mathtt{S}$:
Note that $M = 1, N = 0$. Let $\tilde{\alpha} = \mathtt{S}$, then $\mathtt{S}(y_1;) = \mathtt{S}(y_1;)$.
Case $\alpha = \dot{-}$:
Note that $M = 2, N = 0$. Let $\tilde{\alpha} = \dot{-}$, then $y_1 \dot{-} y_2 = y_1 \dot{-} y_2$.

Case $\alpha = \times$:

Note that $M = 2, N = 0$. Let $\tilde{\alpha} = \times$, then $y_1 \times y_2 = y_1 \times y_2$.

Case $\alpha = |\cdot|_j^{M,N}$:

Note that $0 \le j < N$. Let $\tilde{\alpha}(\vec{y}, \vec{x}; \vec{a}) = \chi_j^{|\cdot|}(\vec{x}; \vec{a})$. Then

$$\begin{aligned}\alpha(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= |(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a}))|_j^{M,N} \\ &= |\chi_j(\vec{x};\vec{a})| = \chi_j^{|\cdot|}(\vec{x};\vec{a}) = \tilde{\alpha}(\vec{y}, \vec{x}; \vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{BIT}_j^{M+1,N}$:

Note that $0 \le j < N$. Let $\tilde{\alpha}(z, \vec{y}, \vec{x}; \vec{a}) = \chi_j^{\mathtt{BIT}}(z, \vec{x}; \vec{a})$. Then

$$\begin{aligned}&\alpha(z, \vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \mathtt{BIT}_j^{M+1,N}(z, \vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \mathrm{BIT}(z, \chi_j(\vec{x};\vec{a})) = \chi_j^{\mathtt{BIT}}(z, \vec{x}; \vec{a}) = \tilde{\alpha}(z, \vec{y}, \vec{x}; \vec{a}).\end{aligned}$$

Induction step.

Case COMP:

Suppose that

$$\alpha(\vec{y}; \vec{b}) = \gamma(\beta_0(\vec{y};\vec{b}), \dots, \beta_{L-1}(\vec{y};\vec{b}); \vec{b}),$$

whrere $\gamma, \beta_0, \dots, \beta_{L-1} \in \tilde{\mathcal{D}}$. Then, by the induction hypothesis, there exist $\tilde{\gamma}, \tilde{\beta}_0, \dots, \tilde{\beta}_{L-1} \in \tilde{\mathcal{C}}$ such that

$$\begin{aligned}\gamma(\vec{z}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\gamma}(\vec{z}, \vec{x}; \vec{a}), \\ \beta_j(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\beta}_j(\vec{y}, \vec{x}; \vec{a})\end{aligned}$$

for $0 \le j < L$.

Since $\tilde{\mathcal{D}}$ is closed under composition, define $\tilde{\alpha} \in \tilde{\mathcal{D}}$ by

$$\tilde{\alpha}(\vec{y}, \vec{x}; \vec{a}) = \tilde{\gamma}(\tilde{\beta}_0(\vec{y}, \vec{x}; \vec{a}), \dots, \tilde{\beta}_{L-1}(\vec{y}, \vec{x}; \vec{a}), \vec{x}; \vec{a}).$$

Then we have

$$\begin{aligned}&\alpha(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \gamma(\beta_0(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})), \dots, \beta_{L-1}(\vec{y}; \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})); \\ &\qquad \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \gamma(\tilde{\beta}_0(\vec{y}, \vec{x}; \vec{a}), \dots, \tilde{\beta}_{L-1}(\vec{y}, \vec{x}; \vec{a}); \chi_0(\vec{x};\vec{a}), \dots, \chi_{N-1}(\vec{x};\vec{a})) \\ &= \tilde{\gamma}(\tilde{\beta}_0(\vec{y}, \vec{x}; \vec{a}), \dots, \tilde{\beta}_{L-1}(\vec{y}, \vec{x}; \vec{a}), \vec{x}; \vec{a}) \\ &= \tilde{\alpha}(\vec{y}, \vec{x}; \vec{a}).\end{aligned}$$

Case BMIN:

Suppose that $\alpha \in \tilde{\mathcal{D}}$ is defined by bounded minimisation, then

$$\alpha(z, \vec{y}; \vec{b}) = \begin{cases} \text{the least } w \leq z \text{ such that } \beta(w, \vec{y}; \vec{b}) \neq 0 & \text{if it exists,} \\ z + 1 & \text{otherwise,} \end{cases}$$

where $\beta \in \tilde{\mathcal{D}}$.
By induction hypothesis, there exists $\beta \in \tilde{\mathcal{D}}$ such that

$$\beta(w, \vec{y}; \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a})) = \tilde{\beta}(w, \vec{y}, \vec{x}; \vec{a}).$$

Since $\tilde{\mathcal{D}}$ is closed under bounded minimisation, define $\tilde{\alpha} \in \tilde{\mathcal{D}}$ by

$$\tilde{\alpha}(z, \vec{y}, \vec{x}; \vec{a}) = \begin{cases} \text{the least } w \leq z \text{ such that } \tilde{\beta}(w, \vec{y}, \vec{x}; \vec{a}) \neq 0 & \text{if it exists,} \\ z + 1 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} & \alpha(z, \vec{y}; \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a})) \\ = & \begin{cases} \text{the least } w \leq z \text{ such that } \beta(w, \vec{y}; \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a})) \neq 0 & \text{(if it exists)} \\ z + 1 & \text{(otherwise)} \end{cases} \\ = & \begin{cases} \text{the least } w \leq z \text{ such that } \tilde{\beta}(w, \vec{y}, \vec{x}; \vec{a}) \neq 0 & \text{(if it exists)} \\ z + 1 & \text{(otherwise)} \end{cases} \\ = & \tilde{\alpha}(z, \vec{y}, \vec{x}; \vec{a}). \end{aligned}$$

□

Proposition 4.9. *For each $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, there exist $\tilde{\alpha}, \tilde{\varphi}, \hat{\varphi} \in \tilde{\mathcal{D}}$ such that*

$$\begin{aligned} \alpha(\vec{x}; \vec{a}) &= \tilde{\alpha}(\vec{x}; \vec{a}), \\ \mathtt{BIT}(z; \varphi(\vec{x}; \vec{a})) &= \tilde{\varphi}(z, \vec{x}; \vec{a}), \\ |; \varphi(\vec{x}; \vec{a})| &= \hat{\varphi}(\vec{x}; \vec{a}) \end{aligned}$$

for each $\vec{x}, \vec{a}$ and z.

Proof. By simultaneous induction on the structures of $\alpha \in \mathcal{D}_{\mathbb{N}}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}_i}^{m,n}$:

Note that $0 \leq i < m$. Let $\tilde{\alpha} = \mathtt{p}_{\mathbb{N}_i}^{m,n}$. Then

$$\alpha(\vec{x}; \vec{a}) = \mathtt{p}_{\mathbb{N}_i}^{m,n}(\vec{x}; \vec{a}) = x_i = \tilde{\alpha}(\vec{x}; \vec{a}).$$

Case $\varphi = \mathtt{p_{\mathbb{W}}}{}_j^{m,n}$:

Note that $0 \le j < n$. Let $\tilde{\varphi} = \mathtt{BIT}_j^{m+1,n}$ and $\hat{\varphi} = |\cdot|_j^{m,n}$. Then

$$\begin{aligned}\mathtt{BIT}(z;\varphi(\vec{x};\vec{a})) &= \mathtt{BIT}(z;\mathtt{p_{\mathbb{W}}}{}_j^{m,n}(\vec{x};\vec{a})) = \mathtt{BIT}(z;a_j)\\ &= \mathtt{BIT}_j^{m+1,n}(z,\vec{x};\vec{a}) = \tilde{\varphi}(z,\vec{x};\vec{a}),\\ |;\varphi(\vec{x};\vec{a})| &= |;\mathtt{p_{\mathbb{W}}}{}_j^{m,n}(\vec{x};\vec{a})| = |;a_j|\\ &= |(\vec{x};\vec{a})|_j^{m,n} = \hat{\varphi}(\vec{x};\vec{a}).\end{aligned}$$

Case $\alpha = \mathtt{0}$:

Note that $m = n = 0$. Let $\tilde{\alpha} = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.

Case $\alpha = \mathtt{S}$:

Note that $m = 1, n = 0$. Let $\tilde{\alpha} = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1;)$.

Case $\alpha = \dot{-}$:

Note that $m = 2, n = 0$. Let $\tilde{\alpha} = \dot{-}$, then $x_1 \dot{-} x_2 = x_1 \dot{-} x_2$.

Case $\alpha = \times$:

Note that $m = 2, n = 0$. Let $\tilde{\alpha} = \times$, then $x_1 \times x_2 = x_1 \times x_2$.

Case $\alpha = |\cdot|$:

Note that $m = 0, n = 1$. Let $\tilde{\alpha} = |\cdot|_0^{0,1}$. Then

$$\alpha(;a_1) = |;a_1| = |(;a_1)|_0^{0,1} = \tilde{\alpha}(;a_1).$$

Case $\alpha = \mathtt{BIT}$:

Note that $m = 1, n = 1$. Let $\tilde{\alpha} = \mathtt{BIT}_0^{1,1}$. Then

$$\alpha(z;a_1) = \mathtt{BIT}(z;a_1) = \mathtt{BIT}_0^{1,1}(z;a_1) = \tilde{\alpha}(z;a_1).$$

Induction step.

Case COMP($\in \mathcal{D}_{\mathbb{N}}$):

Suppose that

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\gamma,\beta_0,\dots,\beta_{M-1} \in \mathcal{D}_{\mathbb{N}}$ and $\chi_0,\dots,\chi_{N-1} \in \mathcal{D}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\tilde{\gamma},\tilde{\beta}_0,\dots,\tilde{\beta}_{M-1},\tilde{\chi}_0,\dots,\tilde{\chi}_{N-1},\hat{\chi}_0,\dots,\hat{\chi}_{N-1} \in \tilde{\mathcal{D}}$ such that

$$\begin{aligned}\gamma(\vec{y};\vec{b}) &= \tilde{\gamma}(\vec{y};\vec{b}), & \beta_i(\vec{x};\vec{a}) &= \tilde{\beta}_i(\vec{x};\vec{a}),\\ \mathtt{BIT}(z;\chi_j(\vec{x};\vec{a})) &= \tilde{\chi}_j(z,\vec{x};\vec{a}), & |;\chi_j(\vec{x};\vec{a})| &= \hat{\chi}_j(\vec{x};\vec{a})\end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since the bit contents and lengths of $\chi_j(\vec{x};\vec{a})$ for $0 \le j < N$ are known, by Lemma 4.8, there exist $\tilde{\gamma}' \in \tilde{\mathcal{D}}$ such that

$$\tilde{\gamma}(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) = \tilde{\gamma}'(\vec{y},\vec{x};\vec{a}).$$

Since $\tilde{\mathcal{D}}$ is closed under composition, define $\tilde{\alpha} \in \tilde{\mathcal{D}}$ by the following formula using $\tilde{\gamma}'$ and $\tilde{\beta}_i$ for $0 \le i < M$:

$$\tilde{\alpha}(\vec{x};\vec{a}) = \tilde{\gamma}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}).$$

Then we have

$$\begin{aligned}\alpha(\vec{x};\vec{a}) &= \gamma(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \gamma(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \tilde{\gamma}(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\ &= \tilde{\gamma}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a})\\ &= \tilde{\alpha}(\vec{x};\vec{a}).\end{aligned}$$

Case COMP($\in \mathcal{D}_{\mathbb{W}}$):

Suppose that

$$\varphi(\vec{x};\vec{a}) = \psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\beta_0,\dots,\beta_{M-1} \in \mathcal{D}_{\mathbb{N}}$ and $\psi,\chi_0,\dots,\chi_{N-1} \in \mathcal{D}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $\tilde{\beta}_0,\dots,\tilde{\beta}_{M-1},\tilde{\psi},\tilde{\chi}_0,\dots,\tilde{\chi}_{N-1},\hat{\psi},\hat{\chi}_0,\dots,\hat{\chi}_{N-1} \in \tilde{\mathcal{D}}$ such that

$$\begin{aligned}\beta_i(\vec{x};\vec{a}) &= \tilde{\beta}_i(\vec{x};\vec{a}),\\ \mathtt{BIT}(z;\psi(\vec{y};\vec{b})) &= \tilde{\psi}(z,\vec{y};\vec{b}), & |;\psi(\vec{y};\vec{b})| &= \hat{\psi}(\vec{y};\vec{b}),\\ \mathtt{BIT}(z;\chi_j(\vec{x};\vec{a})) &= \tilde{\chi}_j(z,\vec{x};\vec{a}), & |;\chi_j(\vec{x};\vec{a})| &= \hat{\chi}_j(\vec{x};\vec{a})\end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since the bit contents and lengths of $\chi_j(\vec{x};\vec{a})$ for $0 \le j < N$ are known, by Lemma 4.8, there exist $\tilde{\psi}',\hat{\psi}' \in \tilde{\mathcal{D}}$ such that

$$\begin{aligned}\tilde{\psi}(z,\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \tilde{\psi}'(z,\vec{y},\vec{x};\vec{a}),\\ \hat{\psi}(\vec{y};\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})) &= \hat{\psi}'(\vec{y},\vec{x};\vec{a}).\end{aligned}$$

Since $\tilde{\mathcal{D}}$ is closed under composition, define $\tilde{\varphi},\hat{\varphi} \in \tilde{\mathcal{D}}$ by the following formulas using $\tilde{\psi}',\hat{\psi}'$ and $\tilde{\beta}_i$ for $0 \le i < M$:

$$\begin{aligned}\tilde{\varphi}(z,\vec{x};\vec{a}) &= \tilde{\psi}'(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}),\\ \hat{\varphi}(\vec{x};\vec{a}) &= \hat{\psi}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a}).\end{aligned}$$

Then we have

$$
\begin{aligned}
\mathtt{BIT}(z;\varphi(\vec{x};\vec{a})) &= \mathtt{BIT}(z;\psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a})))\\
&= \tilde{\psi}(z,\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \tilde{\psi}(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \tilde{\psi}'(z,\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a})\\
&= \tilde{\varphi}(z,\vec{x};\vec{a}),\\
|;\varphi(\vec{x};\vec{a})| &= |;\psi(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))|\\
&= \hat{\psi}(\beta_0(\vec{x};\vec{a}),\dots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \hat{\psi}(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\dots,\chi_{N-1}(\vec{x};\vec{a}))\\
&= \hat{\psi}'(\tilde{\beta}_0(\vec{x};\vec{a}),\dots,\tilde{\beta}_{M-1}(\vec{x};\vec{a}),\vec{x};\vec{a})\\
&= \hat{\varphi}(\vec{x};\vec{a}).
\end{aligned}
$$

Case BMIN:

Suppose that $\alpha \in \mathcal{D}_{\mathbb{N}}$ is defined by bounded minimisation, then

$$
\alpha(z,\vec{x};\vec{a}) = \begin{cases} \text{the least } y \le z \text{ such that } \beta(y,\vec{x};\vec{a}) \neq 0 & \text{if it exists,}\\ z+1 & \text{otherwise,} \end{cases}
$$

where $\beta \in \mathcal{D}_{\mathbb{N}}$.
By induction hypothesis, there exists $\tilde{\beta} \in \tilde{\mathcal{D}}$ such that

$$
\beta(y,\vec{x};\vec{a}) = \tilde{\beta}(y,\vec{x};\vec{a}).
$$

Since $\tilde{\mathcal{D}}$ is closed under bounded minimisation, define $\tilde{\alpha} \in \tilde{\mathcal{D}}$ by

$$
\tilde{\alpha}(z,\vec{x};\vec{a}) = \begin{cases} \text{the least } y \le z \text{ such that } \tilde{\beta}(y,\vec{x};\vec{a}) \neq 0 & \text{if it exists,}\\ z+1 & \text{otherwise.} \end{cases}
$$

Then we have

$$
\alpha(z,\vec{x};\vec{a}) = \tilde{\alpha}(z,\vec{x};\vec{a}).
$$

Case BC:

Suppose that $\varphi \in \mathcal{D}_{\mathbb{W}}$ with $\varphi(z,\vec{x};\vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{W}$ is defined by bounded comprehension, then

$$
\begin{aligned}
&|;\varphi(z,\vec{x};\vec{a})| = z,\\
&\forall j < z\ [\mathtt{BIT}(j;\varphi(z,\vec{x};\vec{a})) = 0 \leftrightarrow \alpha(j,\vec{x};\vec{a}) = 0],
\end{aligned}
$$

where $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha(j, \vec{x}; \vec{a}) \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$.
By induction hypothesis, there exist $\tilde{\alpha} \in \tilde{\mathcal{D}}$ such that

$$\alpha(j, \vec{x}; \vec{a}) = \tilde{\alpha}(j, \vec{x}; \vec{a}).$$

With respect to $|; \varphi(z, \vec{x}; \vec{a})|$, define $\tilde{\varphi} \in \tilde{\mathcal{D}}$ by $\tilde{\varphi}(z, \vec{x}; \vec{a}) = z$.
Then we have

$$|; \varphi(z, \vec{x}; \vec{a})| = z = \tilde{\varphi}(z, \vec{x}; \vec{a}).$$

With respect to $\mathtt{BIT}(j; \varphi(z, \vec{x}; \vec{a}))$, since

$$\mathtt{BIT}(j; \varphi(z, \vec{x}; \vec{a})) = \begin{cases} \mathtt{sg}(\alpha(j, \vec{x}; \vec{a});) & \text{if } 0 \le j < z, \\ \mathtt{0} & \text{if } j \ge z, \end{cases}$$

define $\hat{\varphi} \in \tilde{\mathcal{D}}$ by[15]

$$\hat{\varphi}(j, z, \vec{x}; \vec{a}) = \mathtt{cond}(j < z, \mathtt{0}, \mathtt{sg}(\tilde{\alpha}(j, \vec{x}; \vec{a});););.$$

Then we have

$$\mathtt{BIT}(j; \varphi(z, \vec{x}; \vec{a})) = \hat{\varphi}(j, z, \vec{x}; \vec{a}).$$

□

Proposition 4.10. *For each $\tilde{\alpha} \in \tilde{\mathcal{D}}$ with $\tilde{\alpha} \in \mathbb{N}^m \times \mathbb{W}^m \to \mathbb{N}$, there exists $f \in \mathcal{M}^2$ such that*

$$\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})$$

for each $\vec{x}$ and $\vec{k}$.

Remark. In the discussion on the function algebra

$$A_0 = [\mathtt{0}, \mathtt{I}, \mathtt{s_0}, \mathtt{s_1}, \mathtt{BIT}, |\cdot|, \#; \mathrm{COMP}, \mathrm{CRN}]$$

in Clote [2], it is shown that cut-off subtraction $\dot{-}$ belongs to A_0 without smash function #. By Proposition 2.25, $[\mathtt{0}, \mathtt{I}, \mathtt{s_0}, \mathtt{s_1}, |\cdot|, \mathtt{BIT}; \mathrm{COMP}, \mathrm{CRN}]$ is contained in $\mathcal{M}^2$. Hence, cut-off subtraction $\dot{-}$ also belongs to $\mathcal{M}^2$. In addition, inverse signum function $\overline{\mathtt{sg}}(x) = 1 \dot{-} x$, signum function $\mathtt{sg}(x) = \overline{\mathtt{sg}}(\overline{\mathtt{sg}}(x))$ and conditional function $\mathtt{cond}(x, y, z) = \overline{\mathtt{sg}}(x) \times y + \mathtt{sg}(x) \times z$ belong to $\mathcal{M}^2$.

Proof. By induction on the structure of $\tilde{\alpha} \in \tilde{\mathcal{D}}$.
Basis.
Case $\tilde{\alpha} = \mathtt{p}_{\mathbb{N}}{}_i^{m,n}$:

[15] Notice that Lemma 4.2 and Lemma 4.3 also hold for $\tilde{\mathcal{D}}$.

Note that $0 \le i < m$. Let $f = \mathtt{I}_i^{m,n}$. Then

$$\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = \mathtt{p}_{\mathbb{N}_i}^{m,n}(\vec{x}; \mathtt{bin}(\vec{k})) = x_i = \mathtt{I}_i^{m,n}(\vec{x}, \vec{k}) = f(\vec{x}, \vec{k}).$$

Case $\tilde{\alpha} = \mathtt{0}$:

Note that $m = n = 0$. Let $f = \mathtt{0}$, then $\mathtt{0} = \mathtt{0}$.

Case $\tilde{\alpha} = \mathtt{S}$:

Note that $m = 1, n = 0$. Let $f = \mathtt{S}$, then $\mathtt{S}(x_1;) = \mathtt{S}(x_1)$.

Case $\tilde{\alpha} = \dot{-}$:

Note that $m = 2, n = 0$. Let $f = \dot{-}$, then $x_1 \dot{-} x_2 = x_1 \dot{-} x_2$.

Case $\tilde{\alpha} = \times$:

Note that $m = 2, n = 0$. Let $f = \times$, then $x_1 \times x_2 = x_1 \times x_2$.

Case $\tilde{\alpha} = |\cdot|_j^{m,n}$:

Note that $0 \le j < n$. Since $|\cdot|$ belongs to $\mathcal{M}^2$ by Proposition 2.25, let $f(\vec{x}, \vec{k}) = |\mathtt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})|$. Then

$$\begin{aligned}\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) &= |(\vec{x}; \mathtt{bin}(\vec{k}))|_j^{m,n} = |\mathtt{bin}(k_j)| \\ &= |k_j| = |\mathtt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})| = f(\vec{x}, \vec{k}).\end{aligned}$$

Case $\tilde{\alpha} = \mathtt{BIT}_j^{m+1,n}$:

Note that $0 \le j < n$. Since $\mathtt{BIT}$ belongs to $\mathcal{M}^2$ by Proposition 2.25, let $f(z, \vec{x}, \vec{k}) = \mathtt{BIT}(\mathtt{I}_0^{m+1+n}(z, \vec{x}, \vec{k}), \mathtt{I}_{m+1+j}^{m+1+n}(z, \vec{x}, \vec{k}))$. Then

$$\begin{aligned}\tilde{\alpha}(z, \vec{x}; \mathtt{bin}(\vec{k})) &= \mathtt{BIT}_j^{m+1,n}(z, \vec{x}; \mathtt{bin}(\vec{k})) = \mathrm{BIT}(z, \mathtt{bin}(k_j)) \\ &= \mathtt{BIT}(z, k_j) = \mathtt{BIT}(\mathtt{I}_0^{m+1+n}(z, \vec{x}, \vec{k}), \mathtt{I}_{m+1+j}^{m+1+n}(z, \vec{x}, \vec{k})) = f(z, \vec{x}, \vec{k}).\end{aligned}$$

Induction step.

Case COMP:

Suppose that

$$\tilde{\alpha}(\vec{x}; \mathtt{bin}(\vec{k})) = \tilde{\gamma}(\tilde{\beta}_0(\vec{x}; \mathtt{bin}(\vec{k})), \dots, \tilde{\beta}_{L-1}(\vec{x}; \mathtt{bin}(\vec{k})); \mathtt{bin}(\vec{k})),$$

whrere $\tilde{\gamma}, \tilde{\beta}_0, \dots, \tilde{\beta}_{L-1} \in \tilde{\mathcal{D}}$. Then, by the induction hypothesis, there exist $h, g_0, \dots, g_{L-1} \in \mathcal{M}^2$ such that

$$\begin{aligned}\tilde{\gamma}(\vec{y}; \mathtt{bin}(\vec{k})) &= h(\vec{y}, \vec{k}), \\ \tilde{\beta}_j(\vec{x}; \mathtt{bin}(\vec{k})) &= g_j(\vec{x}, \vec{k})\end{aligned}$$

for $0 \le j < L$.

Since $\mathcal{M}^2$ is closed under composition, define $f \in \mathcal{M}^2$ by

$$f(\vec{x}, \vec{k}) = h(g_0(\vec{x}, \vec{k}), \dots, g_{L-1}(\vec{x}, \vec{k}), \vec{k}).$$

Then we have

$$\begin{aligned}\tilde{\alpha}(\vec{x};\mathtt{bin}(\vec{k})) &= \tilde{\gamma}(\tilde{\beta}_0(\vec{x};\mathtt{bin}(\vec{k})),\dots,\tilde{\beta}_{L-1}(\vec{x};\mathtt{bin}(\vec{k}));\mathtt{bin}(\vec{k}))\\ &= \tilde{\gamma}(g_0(\vec{x},\vec{k}),\dots,g_{L-1}(\vec{x},\vec{k});\mathtt{bin}(\vec{k}))\\ &= h(g_0(\vec{x},\vec{k}),\dots,g_{L-1}(\vec{x},\vec{k}),\vec{k})\\ &= f(\vec{x},\vec{k}).\end{aligned}$$

Case BMIN:

Suppose that $\tilde{\alpha} \in \tilde{\mathcal{D}}$ is defined by bounded minimisation, then

$$\tilde{\alpha}(z,\vec{x};\mathtt{bin}(\vec{k})) = \begin{cases}\text{the least } y \le z \text{ such that } \tilde{\beta}(y,\vec{x};\mathtt{bin}(\vec{k})) \ne \mathtt{0} & \text{if it exists,}\\ z+1 & \text{otherwise,}\end{cases}$$

where $\tilde{\beta} \in \tilde{\mathcal{D}}$.
By induction hypothesis, there exists $g \in \mathcal{M}^2$ such that

$$\tilde{\beta}(y,\vec{x};\mathtt{bin}(\vec{k})) = g(y,\vec{x},\vec{k}).$$

Since $\mathcal{M}^2$ is closed under bounded minimisation, define $f \in \mathcal{M}^2$ by[16]

$$\begin{aligned}f(z,\vec{x};\vec{k}) = \mathtt{cond}(&\mu y \le z[\overline{\mathtt{sg}}(g(y,\vec{x},\vec{k})) = 0],\\ &\mathtt{cond}(\overline{\mathtt{sg}}(g(0,\vec{x},\vec{k})),0,z+1),\\ &\mu y \le z[\overline{\mathtt{sg}}(g(y,\vec{x},\vec{k})) = 0]).\end{aligned}$$

Then we have

$$\tilde{\alpha}(z,\vec{x};\mathtt{bin}(\vec{k})) = f(z,\vec{x};\vec{k}).$$

□

Combining Proposition 4.9 with Proposition 4.10, we can derive the following corollary:

Corollary 4.11. *For each $\alpha \in \mathcal{D}_{\mathbb{N}}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$, there exist $f, \tilde{g}, \hat{g} \in \mathcal{M}^2$ such that*

$$\begin{aligned}\alpha(\vec{x};\mathtt{bin}(\vec{k})) &= f(\vec{x},\vec{k}),\\ \mathtt{BIT}(z;\varphi(\vec{x};\mathtt{bin}(\vec{k})) &= \tilde{g}(z,\vec{x},\vec{k}),\\ |;\varphi(\vec{x};\mathtt{bin}(\vec{k}))| &= \hat{g}(\vec{x},\vec{k})\end{aligned}$$

for each $\vec{x}, \vec{k}$ and z.

[16] We write BMIN in $\mathcal{M}^2$ using function $h(z,\vec{x})$ as $\mu y \le z[h(y,\vec{x}) = 0]$.

4.4 Representation of $\mathcal{F}$LH functions by $\mathcal{D}_{\mathbb{W}}$ functions

We know that

$$\mathcal{F}\text{LH} = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{BIT}, |\cdot|, \#; \text{COMP}, \text{CRN}].$$

(Theorem 2.21). In this function class, the function $\mathtt{BIT}(x, y)$ equals MOD2 $(\lfloor x/2^y \rfloor)$. However, it is convenient for us to use the function $\mathtt{bit}(x, y) = \text{MOD2}(\lfloor x/2^{|y|} \rfloor)$ instead of BIT. Hence, firstly, we define a function algebra in which BIT in $\mathcal{F}$LH is replaced with bit, and prove their equivalence.

Definition 4.12.

$$A_0' = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{bit}, |\cdot|, \#; \text{COMP}, \text{CRN}],$$

where 0 is constant 0, I is projection function, $\mathtt{s}_0$ and $\mathtt{s}_1$ are binary successor functions, bit is (lowercase) bit, $|\cdot|$ is length function, # is smash function, COMP is composition and CRN is concatenation recursion on notation.

Remark. In the discussion on the function algebra $A_0 = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{BIT}, |\cdot|, \#; \text{COMP}, \text{CRN}]$ in Clote [2], it is shown that the reverse function $\mathtt{rev}(x)$, the signum function $\mathtt{sg}(x)$, the inverse signum function $\overline{\mathtt{sg}}(x)$, the conditional function $\mathtt{cond}(x, y, z)$ and the most significant part function $\mathtt{msp}(x, y)$ belong to A_0. In these definitions, when BIT is used, we can replace BIT with bit, hence these functions also belong to A_0'. We use this fact in the proof of the next proposition.

Proposition 4.13 (Ishihara)**.**

$$A_0' = \mathcal{F}\text{LH}.$$

Proof. ($A_0' \subseteqq \mathcal{F}$LH) It is trivial by the euqation $\mathtt{bit}(x, y) = \mathtt{BIT}(|x|, y) \in \mathcal{F}\text{LH}$. ($\mathcal{F}$LH $\subseteqq A_0'$) Since $\chi_\neg(x) = \overline{\mathtt{sg}}(x)$, $\chi_\wedge(x, y) = \mathtt{cond}(x, 0, \mathtt{sg}(y))$ and $\chi_\vee(x, y) = \mathtt{cond}(x, \mathtt{sg}(y), 1)$, we have $\chi_\oplus(x, y) = (\neg x \wedge y) \vee (x \vee \neg y)$ belonging to A_0'.

Using CRN, we define $\mathtt{dif}(x, y) \in A_0'$ which computes the number whose bit is 1 if the bit of its position in x and the bit of its position in y are different, and whose bit is 0 if the bit of its position in x and the bit of its position in y are the same, provided that $|x| = |y|$, as follows:

$$\begin{aligned} v(\mathtt{0}, y) &= \mathtt{0}, \\ v(\mathtt{s}_0(x), y) &= \mathtt{s}_{\chi_\oplus(\mathtt{0}, \mathtt{bit}(x,y))}(v(x, y)) \qquad (\text{if } x \neq \mathtt{0}), \\ v(\mathtt{s}_1(x), y) &= \mathtt{s}_{\chi_\oplus(\mathtt{1}, \mathtt{bit}(x,y))}(v(x, y)), \\ \mathtt{dif}(x, y) &= v(x, \mathtt{rev}(\mathtt{s}_1(y))). \end{aligned}$$

Note that if $x = y$ then $\mathtt{dif}(x, y) = \mathtt{0}$, and if $x \neq y$ then $|\mathtt{dif}(x, y)|$ is the maximum bit number of different bits between x and y, where $1 \leq |\mathtt{dif}(x, y)| \leq |x|$.

And, we have $|y| \dot{-} |x| = |\mathtt{msp}(y, x)| \in A'_0$.

The predicate $x < y$ is true if $|x| < |y|$, or true if $|x| = |y|$ and the y's bit of the maximum bit number of different bits between x and y is 1. Hence, we can define $\chi_<(x, y) \in A'_0$ as follows:

$$
\begin{aligned}
\chi_<(x, y) = \mathtt{cond}(&|y| \dot{-} |x|,\\
&\mathtt{cond}(|x| \dot{-} |y|,\\
&\qquad \mathtt{cond}(\mathtt{dif}(x, y), \mathtt{0}, \mathtt{bit}(\mathtt{msp}(\mathtt{dif}(x, y), 1), y)),\\
&\qquad 0),\\
&1).
\end{aligned}
$$

Next, using CRN, we define $r(x, i) = \min_{z \leq x}\{|z| = i\}$ in A'_0 as follows:

$$
\begin{aligned}
t(\mathtt{0}, i) &= \mathtt{0},\\
t(\mathtt{s}_0(x), i) &= \mathtt{s}_{u(x,i)}(t(x, i)) \qquad (\text{if } x \neq 0),\\
t(\mathtt{s}_1(x), i) &= \mathtt{s}_{u(x,i)}(t(x, i)),\\
u(x, i) &= \mathtt{cond}(\chi_<(|x|, i), 0, 1),\\
r(x, i) &= \mathtt{rev}(t(x, i)).
\end{aligned}
$$

Then, we can define $\mathtt{BIT}(i, x) \in A'_0$ as follows:

$$\mathtt{BIT}(i, x) = \mathtt{bit}(r(x, i), x).$$

□

In the subsequent discussion of this section and in the next section, we adopt A'_0 as $\mathcal{F}$LH:

$$\mathcal{F}\mathrm{LH} = [\mathtt{0}, \mathtt{I}, \mathtt{s}_0, \mathtt{s}_1, \mathtt{bit}, |\cdot|, \#; \mathrm{COMP}, \mathrm{CRN}].$$

Next, we show that any function in $\mathcal{F}$LH is represented by some function in $\mathcal{D}_{\mathbb{W}}$, that is,

$$\forall r \in \mathcal{F}\mathrm{LH}\ \exists \varphi \in \mathcal{C}_{\mathbb{W}}\ [\mathrm{bin}(r(\vec{x})) = \varphi(; \mathtt{bin}(\vec{x}))],$$

(Proposition 4.14). In this proposition and in the following discussion, the function bin(·) applied to a natural number in $\mathcal{F}$LH is an identity function which regards a natural number as its binary representation.

Proposition 4.14. *For any $r \in \mathcal{F}\mathrm{LH}$ with $r \in \mathbb{N}^m \to \mathbb{N}$, there exists $\varphi \in \mathcal{C}_{\mathbb{W}}$ such that*

$$\mathrm{bin}(r(\vec{x})) = \varphi(;\mathtt{bin}(\vec{x}))$$

for each x.

Proof. By induction on the structures of $r \in \mathcal{F}\mathrm{LH}$.
Basis.
Case $r = \mathtt{0}$:
Note that $m = 0$. Let $\alpha(i;) = \mathtt{0}$. By bounded comprehension, there exists $\psi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}&|;\psi(z;)| = z,\\ &\forall i < z\ [\mathtt{BIT}(i;\psi(z;)) = \mathtt{0} \leftrightarrow \alpha(i;) = \mathtt{0}].\end{aligned}$$

Let $\varphi(;) = \mathtt{zero}(;) = \psi(\mathtt{S}(\mathtt{0};);)$, then we have

$$\mathrm{bin}(\mathtt{0}) = 0 = \mathtt{zero}(;) = \varphi(;).$$

Case $r = \mathtt{I}^m_j$:
Note that $0 \le j < m$ and $\mathtt{I}^m_j(\vec{x}) = x_j$.
Let $\varphi = \mathtt{p_{\mathbb{W}}}{}^{0,m}_j$. Then

$$\mathrm{bin}(\mathtt{I}^m_j(\vec{x})) = \mathrm{bin}(x_j) = \mathtt{bin}(x_j) = \mathtt{p_{\mathbb{W}}}{}^{0,m}_j(;\mathtt{bin}(\vec{x})) = \varphi(;\mathtt{bin}(\vec{x})).$$

Case $r = \mathtt{s_0}$:
Note that $m = 1$ and $\mathtt{s_0}(x) = 2 \cdot x$.
Let $\alpha(i;a) = \mathtt{cond}(i, 0, \mathtt{BIT}(i \dot{-} 1; a);)$. By bounded comprehension, there exists $\psi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}&|;\psi(z;a)| = z,\\ &\forall i < z\ [\mathtt{BIT}(i;\psi(z;a)) = \mathtt{0} \leftrightarrow \alpha(i;a) = \mathtt{0}].\end{aligned}$$

Let $\varphi(;a) = \psi(\mathtt{S}(|;a|;);a)$, then we have

$$\mathrm{bin}(\mathtt{s_0}(x)) = \mathtt{bin}(x)\mathtt{0} = \varphi(;\mathtt{bin}(x)).$$

Case $r = \mathtt{s_1}$:
Note that $m = 1$ and $\mathtt{s_1}(x) = 2 \cdot x + 1$.
Let $\alpha(i;a) = \mathtt{cond}(i, \mathtt{S}(\mathtt{0};), \mathtt{BIT}(i \dot{-} 1; a);)$. By bounded comprehension, there exists $\psi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}&|;\psi(z;a)| = z,\\ &\forall i < z\ [\mathtt{BIT}(i;\psi(z;a)) = \mathtt{0} \leftrightarrow \alpha(i;a) = \mathtt{0}].\end{aligned}$$

Let $\varphi(;a) = \psi(\mathtt{S}(|;a|;);a)$, then we have

$$\mathrm{bin}(\mathtt{s}_1(x)) = \mathtt{bin}(x)1 = \varphi(;\mathtt{bin}(x)).$$

Case $r = \mathtt{bit}$:

Note that $m = 2$ and $\mathtt{bit}(i,x) = \mathtt{MOD2}(x/2^{|i|})$.
Let $\varphi(;a,b) = \mathtt{BIT}(|;a|;b)$. Then

$$\mathrm{bin}(\mathtt{bit}(i,x)) = \varphi(;\mathtt{bin}(i),\mathtt{bin}(x)).$$

Case $r = |\cdot|$:

Note that $m = 1$ and $|x| = \lceil \log_2(x+1) \rceil$.
Let $\varphi(;a) = \mathtt{cond}(|;a|,\mathtt{zero}(;),\mathtt{bin}(|;a|);)$. Then

$$\mathrm{bin}(|x|) = \varphi(;\mathtt{bin}(x)).$$

Case $r = \#$:

Note that $m = 2$ and $x\#y = 2^{|x|\cdot|y|}$. $2^{|x|\cdot|y|}$ is the number in which the $|x|\cdot|y|$-th bit is 1 and the other bits are 0 in its binary representation.
Let $\alpha(i;a,b) = \mathtt{cond}(\chi_=(|;a|\times|;b|,i;),\mathtt{0},\mathtt{S}(\mathtt{0};);)$. By bounded comprehension, there exists $\psi \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$|;\psi(z;a,b)| = z,$$
$$\forall i < z\ [\mathtt{BIT}(i;\psi(z;a,b)) = \mathtt{0} \leftrightarrow \alpha(i;a,b) = \mathtt{0}].$$

Let $\varphi(;a,b) = \psi(\mathtt{S}(|;a|\times|;b|;);a,b)$. Then

$$\mathrm{bin}(x\#y) = \varphi(;\mathtt{bin}(x),\mathtt{bin}(y)).$$

Induction step.
Case COMP:

Suppose that
$$r(\vec{x}) = u(t_0(\vec{x}),\ldots,t_{L-1}(\vec{x})).$$

whrere $u,t_0,\ldots,t_{L-1} \in \mathcal{F}\mathrm{LH}$. Then, by the induction hypothesis, there exist $\psi,\chi_0,\ldots,\chi_{L-1} \in \mathcal{D}_{\mathbb{W}}$ such that

$$\mathrm{bin}(u(\vec{y})) = \psi(;\mathtt{bin}(\vec{y})),$$
$$\mathrm{bin}(t_j(\vec{x})) = \chi_j(;\mathtt{bin}(\vec{x}))$$

for $0 \le j < L$.
Since $\mathcal{D}_{\mathbb{W}}$ is closed under composition, define $\varphi \in \mathcal{D}_{\mathbb{W}}$ by

$$\varphi(;\mathtt{bin}(\vec{x})) = \psi(;\chi_0(;\mathtt{bin}(\vec{x})),\ldots,\chi_{L-1}(;\mathtt{bin}(\vec{x}))).$$

Then we have[17]

$$\begin{aligned}\mathrm{bin}(r(\vec{x})) &= \mathrm{bin}(u(t_0(\vec{x}),\dots,t_{L-1}(\vec{x})))\\ &= \psi(;\mathtt{bin}(t_0(\vec{x})),\dots,\mathtt{bin}(t_{L-1}(\vec{x})))\\ &= \psi(;\chi_0(;\mathtt{bin}(\vec{x})),\dots,\chi_{L-1}(;\mathtt{bin}(\vec{x})))\\ &= \varphi(;\mathtt{bin}(\vec{x})).\end{aligned}$$

Case CRN:

First, we prepare some auxiliary functions in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$.

1. The function $\mathtt{check}(i;a) \in \mathbb{N}\times\mathbb{W} \to \mathbb{N}$ computes the i-th bit counting from the most significant bit of a for $0 \le i < |a|$. We have
$$\mathtt{check}(i;a) = \mathtt{BIT}(|;a| \dot{-} 1 \dot{-} i; a).$$

2. The function $\mathtt{rev}(;a) \in \mathbb{W} \to \mathbb{W}$ computes the reverse of a. Applying bounded comprehension to $\mathtt{check}(i;a)$, we obtain $\varphi \in \mathcal{D}_{\mathbb{W}}$ such that
$$\begin{aligned}&|;\varphi(z;a)| = z,\\ &\forall i < z\ [\mathtt{BIT}(i;\varphi(z;a) = \mathtt{0} \leftrightarrow \mathtt{check}(i;a) = \mathtt{0}].\end{aligned}$$
Then we have
$$\mathtt{rev}(;a) = \varphi(|;a|;a).$$

3. The function $\mathtt{left}(i;a) \in \mathbb{N}\times\mathbb{W} \to \mathbb{W}$ computes a string of i-many bits counting from the most significant bit of a for $0 \le i \le |a|$. Using φ in 2. above, we have
$$\mathtt{left}(i;a) = \mathtt{rev}(\varphi(i;a)).$$

Suppose that $r \in \mathcal{F}\mathrm{LH}$ with $r \in \mathbb{N}^{m+1} \to \mathbb{N}$ is defined by concatenation recursion on notation, then

$$\begin{aligned}r(\mathtt{0},\vec{y}) &= t(\vec{y}),\\ r(\mathtt{s}_0(x),\vec{y}) &= \mathtt{s}_{u_0(x,\vec{y})}(r(x,\vec{y})) \qquad (\text{if } x \neq 0),\\ r(\mathtt{s}_1(x),\vec{y}) &= \mathtt{s}_{u_1(x,\vec{y})}(r(x,\vec{y})),\end{aligned}$$

where $t, u_0(\le 1), u_1(\le 1) \in \mathcal{F}\mathrm{LH}$.
By induction hypothesis, there exist $\psi, \chi_0, \chi_1 \in \mathcal{D}_{\mathbb{W}}$ such that

$$\begin{aligned}\mathrm{bin}(t(\vec{y})) &= \psi(;\mathtt{bin}(\vec{y})),\\ \mathrm{bin}(u_0(x,\vec{y})) &= \chi_0(;\mathtt{bin}(x),\mathtt{bin}(\vec{y})),\\ \mathrm{bin}(u_1(x,\vec{y})) &= \chi_1(;\mathtt{bin}(x),\mathtt{bin}(\vec{y})).\end{aligned}$$

[17] To make the expression easier to see, we write the expression as this, but, more precisely, apply the 1st equation and 2rd equation of I.H. simultaneously.

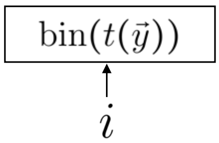
We want to show that there exists $\varphi \in \mathcal{D}_{\mathbb{W}}$ such that

$$\mathrm{bin}(r(x,\vec{y})) = \varphi(;\mathtt{bin}(x),\mathtt{bin}(\vec{y})).$$

We define $\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y})) \in \mathcal{D}_{\mathbb{N}}$ which computes the bit corresponding to the i-th bit counting from the most significant bit of $\mathrm{bin}(r(x,\vec{y}))$ for $0 \le i < |\mathrm{bin}(r(x,\vec{y}))|$.

Note that by the definition of CRN, $\mathrm{bin}(r(x,\vec{y}))$ is obtained by attaching a bit sequence in which each bit of $\mathrm{bin}(x)$ is replaced by the value of u_0 or u_1 to the right end of $\mathrm{bin}(t(\vec{y}))$.

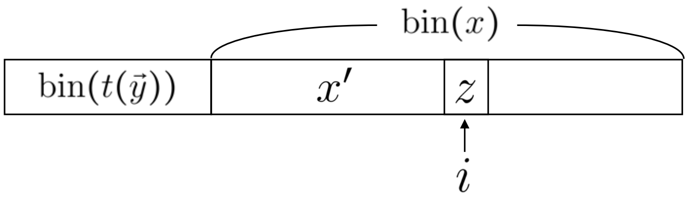
In the case $i < |\psi(;\mathtt{bin}(\vec{y}))|$,



$\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))$ corresponds to the i-th bit counting from the most significant bit of $\mathrm{bin}(t(\vec{y}))$, hence we have

$$\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y})) = \mathtt{BIT}(|\psi(;\mathtt{bin}(\vec{y}))| \dot{-} 1 \dot{-} i;\psi(;\mathtt{bin}(\vec{y}))).$$

In the case $i \ge |\psi(;\mathtt{bin}(\vec{y}))|$,



$\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))$ corresponds to the i-th bit counting from the most significant bit of $\mathrm{bin}((r(x,\vec{y}))$, and when the value of the corresponding bit of $\mathrm{bin}(x)$ is z and the value of the left-side part of the corresponding bit of $\mathrm{bin}(x)$ is x', $\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))$ is $u_z(x',\vec{y}) = \mathtt{BIT}(0;\chi_z(;\mathtt{bin}(x'),\mathtt{bin}(\vec{y})))$. Hence we have

$$\begin{aligned}
&\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))\\
= {}&\mathtt{cond}(\mathtt{check}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\\
&\qquad \mathtt{BIT}(0;\chi_0(;\mathtt{left}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\mathtt{bin}(\vec{y}))),\\
&\qquad \mathtt{BIT}(0;\chi_1(;\mathtt{left}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\mathtt{bin}(\vec{y})));).
\end{aligned}$$

Therefore, unifying the above cases, we have

$$\begin{aligned}
&\delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))\\
=\ &\mathtt{cond}(i \geq |\psi(;\mathtt{bin}(\vec{y}))|,\\
&\qquad \mathtt{BIT}(|\psi(;\mathtt{bin}(\vec{y}))| \dot{-} 1 \dot{-} i;\psi(;\mathtt{bin}(\vec{y}))),\\
&\qquad \mathtt{cond}(\mathtt{check}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\\
&\qquad\qquad \mathtt{BIT}(0;\chi_0(;\mathtt{left}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\mathtt{bin}(\vec{y}))),\\
&\qquad\qquad \mathtt{BIT}(0;\chi_1(;\mathtt{left}(i \dot{-} |\psi(;\mathtt{bin}(\vec{y}))|;\mathtt{bin}(x)),\mathtt{bin}(\vec{y})));\,);\,).
\end{aligned}$$

Applying bounded comprehension to δ, there exists $\varphi' \in \mathcal{D}_{\mathbb{W}}$ satisfying

$$\begin{aligned}
&|;\varphi'(z;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))| = z,\\
&\forall i < z\ [\mathtt{BIT}(i;\varphi'(z;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))) = \mathtt{0} \leftrightarrow \delta(i;\mathtt{bin}(x),\mathtt{bin}(\vec{y})) = \mathtt{0}].
\end{aligned}$$

Then, let

$$\varphi(;\mathtt{bin}(x),\mathtt{bin}(\vec{y})) = \mathtt{rev}(\varphi'(|\psi(;\mathtt{bin}(\vec{y}))| + |;\mathtt{bin}(x)|;\mathtt{bin}(x),\mathtt{bin}(\vec{y}))),$$

we have

$$\mathrm{bin}(r(x,\vec{y})) = \varphi(;\mathtt{bin}(x),\mathtt{bin}(\vec{y})).$$

□

4.5 Representation of $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ functions by $\mathcal{F}$LH functions

In this section, we show that any function in $\mathcal{D}_{\mathbb{N}}$ is represented by some function in $\mathcal{F}$LH and that any function in $\mathcal{D}_{\mathbb{W}}$ is represented by some function is $\mathcal{F}$LH, that is,

$$\forall \alpha \in \mathcal{D}_{\mathbb{N}}\ \exists r \in \mathcal{F}\mathrm{LH}\ [\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = |r(\vec{x}, \vec{k})|],$$
$$\forall \varphi \in \mathcal{D}_{\mathbb{W}}\ \exists t \in \mathcal{F}\mathrm{LH}\ [\varphi(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathrm{bin}(t(\vec{x}, \vec{k}))]$$

(Proposition 4.15). Note that we make lengths of natural numbers in arguments of functions in $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ correspond to natural numbers in arguments of the corresponding functions in $\mathcal{F}$LH, and we make values of functions in $\mathcal{D}_{\mathbb{N}}$ correspond to lengths of values of the corresponding functions in $\mathcal{F}$LH.

Proposition 4.15. *For each $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{N}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^m \times \mathbb{W}^n \to \mathbb{W}$, there exist $r, t \in \mathcal{F}$LH, respectively, such that*

$$\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = |r(\vec{x}, \vec{k})|,$$
$$\varphi(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathrm{bin}(t(\vec{x}, \vec{k}))$$

for each $\vec{x}, \vec{k}$.

Proof. By simultaneous induction on the structures of $\alpha \in \mathcal{D}_{\mathbb{N}}$ and $\varphi \in \mathcal{D}_{\mathbb{W}}$.
Basis.
Case $\alpha = \mathtt{p}_{\mathbb{N}}{}_i^{m,n}$:
Note that $0 \le i < m$. Let $r = \mathtt{I}_i^{m+n}$. Then

$$\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathtt{p}_{\mathbb{N}}{}_i^{m,n}(|\vec{x}|; \mathtt{bin}(\vec{k})) = |x_i| = |\mathtt{I}_i^{m+n}(\vec{x}, \vec{k})| = |r(\vec{x}, \vec{k})|.$$

Case $\varphi = \mathtt{p}_{\mathbb{W}}{}_j^{m,n}$:
Note that $0 \le j < n$. Let $r = \mathtt{I}_{m+j}^{m+n}$. Then

$$\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathtt{p}_{\mathbb{W}}{}_j^{m,n}(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathtt{bin}(k_j) = \mathrm{bin}(k_j) = \mathrm{bin}(\mathtt{I}_{m+j}^{m+n}(\vec{x}, \vec{k})) = \mathrm{bin}(r(\vec{x}, \vec{k})).$$

Case $\alpha = \mathtt{0}$:
Note that $m = n = 0$. Let $r = \mathtt{0}$, then $\mathtt{0} = |\mathtt{0}|$.
Case $\alpha = \mathtt{S}$:
Note that $m = 1, n = 0$. Let $r = \mathtt{s}_1$, Then

$$\alpha(|x|;) = \mathtt{S}(|x|;) = |x| + 1 = |\mathtt{s}_1(x)| = |r(x)|.$$

Case $\alpha = \dot{-}$:

Note that $m = 2, n = 0$. Define the most significant part function $\mathtt{msp}(x,y) = \lfloor x/2^{|y|} \rfloor$ using CRN as follows:

$$\begin{aligned}
\mathtt{msp}(0,y) &= 0, \\
\mathtt{msp}(\mathtt{s}_0(x),y) &= \mathtt{s}_{\mathtt{bit}(y,\mathtt{s}_0(x))}(\mathtt{msp}(x,y)) \qquad (x \neq \mathtt{0}), \\
\mathtt{msp}(\mathtt{s}_1(x),y) &= \mathtt{s}_{\mathtt{bit}(y,\mathtt{s}_1(x))}(\mathtt{msp}(x,y)).
\end{aligned}$$

Let $r(x,y) = \mathtt{msp}(x,y)$. Since $|\mathtt{msp}(x,y)| = |x| \dot{-} |y|$, we have

$$\alpha(|x|,|y|;) = |x| \dot{-} |y| = |\mathtt{msp}(x,y)| = |r(x,y)|.$$

Case $\alpha = \times$:

Note that $m = 2, n = 0$. The function $\lfloor x/2 \rfloor = \mathtt{msp}(x, \mathtt{s}_1(\mathtt{0}))$ belongs to $\mathcal{F}$LH. Let $r(x,y) = \lfloor (x\#y)/2 \rfloor (= 2^{|x|\cdot|y|-1})$. Since $|\lfloor (x\#y)/2 \rfloor| = |x|\cdot|y|$, we have

$$\alpha(|x|,|y|;) = |x| \times |y| = \left| \left\lfloor \frac{x\#y}{2} \right\rfloor \right| = |r(x,y)|.$$

Case $\alpha = |\cdot|$:

Note that $m = 0, n = 1$. Let $r(k) = \mathtt{I}^1_0(k)$. Then

$$\alpha(;\mathtt{bin}(k)) = |;\mathtt{bin}(k)| = |k| = |\mathtt{I}^1_0(k)| = |r(k)|.$$

Case $\alpha = \mathtt{BIT}$:

Note that $m = 1, n = 1$. Let $r(x,k) = \mathtt{bit}(x,k)$. Then

$$\begin{aligned}
\alpha(|x|;\mathtt{bin}(k)) &= \mathtt{BIT}(|x|;\mathtt{bin}(k)) \\
&= \mathrm{BIT}(|x|,k) = |\mathtt{bit}(x,k)| = |r(x,k)|.
\end{aligned}$$

Induction step.

Case COMP($\in \mathcal{D}_{\mathbb{N}}$):

Suppose that

$$\alpha(\vec{x};\vec{a}) = \gamma(\beta_0(\vec{x};\vec{a}),\ldots,\beta_{M-1}(\vec{x};\vec{a});\chi_0(\vec{x};\vec{a}),\ldots,\chi_{N-1}(\vec{x};\vec{a})),$$

whrere $\gamma, \beta_0, \ldots, \beta_{M-1} \in \mathcal{D}_{\mathbb{N}}$ and $\chi_0, \ldots, \chi_{N-1} \in \mathcal{D}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $t, u_0, \ldots, u_{M-1}, v_0, \ldots, v_{N-1} \in \mathcal{F}$LH such that

$$\begin{aligned}
\gamma(|\vec{y}|;\mathtt{bin}(\vec{\ell})) &= |t(\vec{y},\vec{\ell})|, \\
\beta_i(|\vec{x}|;\mathtt{bin}(\vec{k})) &= |u_i(\vec{x},\vec{k})|, \\
\chi_j(|\vec{x}|;\mathtt{bin}(\vec{k})) &= \mathrm{bin}(v_j(\vec{x},\vec{k}))
\end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since $\mathcal{F}\mathrm{LH}$ is closed under composition, define $r \in \mathcal{F}\mathrm{LH}$ by

$$r(\vec{x}, \vec{k}) = t(u_0(\vec{x}, \vec{k}), \ldots, t_{M-1}(\vec{x}, \vec{k}), v_0(\vec{x}, \vec{k}), \ldots, v_{N-1}(\vec{x}, \vec{k})).$$

Then we have[18]

$$\begin{aligned}
&\alpha(|\vec{x}|; \texttt{bin}(\vec{k})) \\
&= \gamma(\beta_0(|\vec{x}|; \texttt{bin}(\vec{k})), \ldots, \beta_{M-1}(|\vec{x}|; \texttt{bin}(\vec{k})); \\
&\qquad \chi_0(|\vec{x}|; \texttt{bin}(\vec{k})), \ldots, \chi_{N-1}(|\vec{x}|; \texttt{bin}(\vec{k}))) \\
&= \gamma(|u_0(\vec{x}, \vec{k})|, \ldots, |u_{M-1}(\vec{x}, \vec{k})|; \mathrm{bin}(v_0(\vec{x}, \vec{k})), \ldots, \mathrm{bin}(v_{N-1}(\vec{x}, \vec{k}))) \\
&= |t(u_0(\vec{x}, \vec{k}), \ldots, u_{M-1}(\vec{x}, \vec{k}), v_0(\vec{x}, \vec{k}), \ldots, v_{N-1}(\vec{x}, \vec{k}))| \\
&= |r(\vec{x}, \vec{k})|.
\end{aligned}$$

Case COMP($\in \mathcal{D}_{\mathbb{W}}$):

Suppose that

$$\varphi(\vec{x}; \vec{a}) = \psi(\beta_0(\vec{x}; \vec{a}), \ldots, \beta_{M-1}(\vec{x}; \vec{a}); \chi_0(\vec{x}; \vec{a}), \ldots, \chi_{N-1}(\vec{x}; \vec{a})),$$

whrere $\beta_0, \ldots, \beta_{M-1} \in \mathcal{D}_{\mathbb{N}}$ and $\psi, \chi_0, \ldots, \chi_{N-1} \in \mathcal{D}_{\mathbb{W}}$. Then, by the induction hypothesis, there exist $u_0, \ldots, u_{M-1}, r, v_0, \ldots, v_{N-1} \in \mathcal{F}\mathrm{LH}$ such that

$$\begin{aligned}
\psi(|\vec{y}|; \texttt{bin}(\vec{\ell})) &= \mathrm{bin}(r(\vec{y}, \vec{\ell})), \\
\beta_i(|\vec{x}|; \texttt{bin}(\vec{k})) &= |u_i(\vec{x}, \vec{k})|, \\
\chi_j(|\vec{x}|; \texttt{bin}(\vec{k})) &= \mathrm{bin}(v_j(\vec{x}, \vec{k}))
\end{aligned}$$

for $0 \le i < M$ and $0 \le j < N$. Since $\mathcal{F}\mathrm{LH}$ is closed under composition, define $r \in \mathcal{F}\mathrm{LH}$ by

$$t(\vec{x}, \vec{k}) = r(u_0(\vec{x}, \vec{k}), \ldots, t_{M-1}(\vec{x}, \vec{k}), v_0(\vec{x}, \vec{k}), \ldots, v_{N-1}(\vec{x}, \vec{k})).$$

Then we have[19]

$$\begin{aligned}
&\varphi(|\vec{x}|; \texttt{bin}(\vec{k})) \\
&= \psi(\beta_0(|\vec{x}|; \texttt{bin}(\vec{k})), \ldots, \beta_{M-1}(|\vec{x}|; \texttt{bin}(\vec{k})); \\
&\qquad \chi_0(|\vec{x}|; \texttt{bin}(\vec{k})), \ldots, \chi_{N-1}(|\vec{x}|; \texttt{bin}(\vec{k}))) \\
&= \psi(|u_0(\vec{x}, \vec{k})|, \ldots, |u_{M-1}(\vec{x}, \vec{k})|; \mathrm{bin}(v_0(\vec{x}, \vec{k})), \ldots, \mathrm{bin}(v_{N-1}(\vec{x}, \vec{k}))) \\
&= \mathrm{bin}(r(u_0(\vec{x}, \vec{k}), \ldots, u_{M-1}(\vec{x}, \vec{k}), v_0(\vec{x}, \vec{k}), \ldots, v_{N-1}(\vec{x}, \vec{k}))) \\
&= \mathrm{bin}(t(\vec{x}, \vec{k})).
\end{aligned}$$

[18] To make the expression easier to see, we write the expression as this, but, more precisely, apply the 2nd equation of I.H. firstly, and then apply the 1st equation and 3rd equation of I.H. simultaneously.

[19] Note the same as in the previous footnote.

Case BMIN:

Suppose that $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$ is defined by bounded minimisation, then

$$\alpha(z, \vec{x}; \vec{a}) = \begin{cases} \text{the least } y \le z \text{ such that } \beta(y, \vec{x}; \vec{a}) \ne 0 & \text{if it exists,} \\ z + 1 & \text{otherwise,} \end{cases}$$

where $\beta \in \mathcal{D}_{\mathbb{N}}$.
By induction hypothesis, there exists $t \in \mathcal{F}\mathrm{LH}$ such that

$$\beta(|y|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |t(y, \vec{x}; \vec{k})|.$$

Since $\mathcal{F}\mathrm{LH}$ is closed under concatenation recursion on notation, define u by

$$\begin{aligned} u(0, \vec{x}, \vec{k}) &= \mathtt{sg}(|t(0, \vec{x}, \vec{k})|), \\ u(\mathtt{s}_0(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|t(\mathtt{s}_0(z), \vec{x}, \vec{k})|)}(u(z, \vec{x}, \vec{k})) \qquad (z \ne 0), \\ u(\mathtt{s}_1(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|t(\mathtt{s}_1(z), \vec{x}, \vec{k})|)}(u(z, \vec{x}, \vec{k})). \end{aligned}$$

And, define $r \in \mathcal{F}\mathrm{LH}$ by

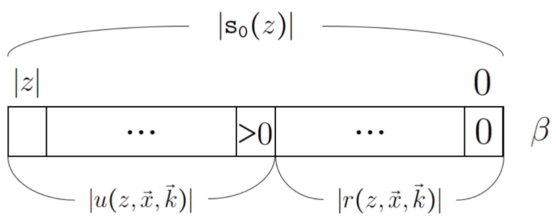
$$r(z, \vec{x}, \vec{k}) = \mathtt{msp}(\mathtt{s}_0(z), u(z, \vec{x}, \vec{k})).$$

Then, since

$$|r(z, \vec{x}, \vec{k})| = |\mathtt{s}_0(z)| \dot{-} |u(z, \vec{x}, \vec{k})|,$$

we have

$$\alpha(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |r(z, \vec{x}, \vec{k})|.$$



Case BC:

Suppose that $\varphi \in \mathcal{D}_{\mathbb{W}}$ with $\varphi \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{W}$ is defined by bounded comprehension, then

$$\begin{aligned} &|; \varphi(z, \vec{x}; \vec{a})| = z, \\ &\forall j < z \; [\mathtt{BIT}(j; \varphi(z, \vec{x}; \vec{a})) = 0 \leftrightarrow \alpha(j, \vec{x}; \vec{a}) = 0], \end{aligned}$$

where $\alpha \in \mathcal{D}_{\mathbb{N}}$ with $\alpha \in \mathbb{N}^{m+1} \times \mathbb{W}^n \to \mathbb{N}$.
By induction hypothesis, there exists $r \in \mathcal{F}$LH such that

$$\alpha(|j|, |\vec{x}|; \mathtt{bin}(\vec{k})) = |r(j, \vec{x}; \vec{k})|.$$

Since $\mathcal{F}$LH is closed under concatenation recursion on notation, define u by

$$\begin{aligned} u(\mathtt{0}, \vec{x}, \vec{k}) &= 1, \\ u(\mathtt{s_0}(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|r(z,\vec{x},\vec{k})|)}(u(z, \vec{x}, \vec{k})) \qquad (z \neq \mathtt{0}), \\ u(\mathtt{s_1}(z), \vec{x}, \vec{k}) &= \mathtt{s}_{\mathtt{sg}(|r(z,\vec{x},\vec{k})|)}(u(z, \vec{x}, \vec{k})). \end{aligned}$$

And, define $t \in \mathcal{F}$LH by

$$t(z, \vec{x}, \vec{k}) = \mathtt{msp}(\mathtt{rev}(u(z, \vec{x}, \vec{k})), 1).$$

Then, we have[20]

$$\varphi(|z|, |\vec{x}|; \mathtt{bin}(\vec{k})) = \mathrm{bin}(t(z, \vec{x}, \vec{k})).$$

□

[20]For correctness of this equation, see the proof of *Case* BC in Proposition 3.30.

4.6 Inclusion of LH by $\mathcal{M}^2_*$

In section 4.2 and section 4.3, we have associated functions in $\mathcal{M}^2$ with functions in $\mathcal{C}_{\mathbb{N}}$, and in section 4.4 and section 4.5, we have associated functions in $\mathcal{F}$LH with functions in $\mathcal{D}_{\mathbb{W}}$. Using these correspondences, we obtain the following theorem.

Theorem 4.16.

$$\mathrm{LH} \subseteqq \mathcal{M}^2_*.$$

Proof. Let s be any set in LH, then the charateristic function χ_s of s is in $\mathcal{F}$LH.
By Proposition 4.14, there exists $\varphi \in \mathcal{C}_{\mathbb{W}}$ such that

$$\mathrm{bin}(\chi_s(\vec{x})) = \varphi(;\mathtt{bin}(\vec{x})).$$

By Corollary 4.11, there exists $\tilde{g} \in \mathcal{M}^2$ such that

$$\mathtt{BIT}(j;\varphi(;\mathtt{bin}(\vec{x}))) = \tilde{g}(j,\vec{x}).$$

Hence, we have

$$\mathtt{BIT}(j;\mathrm{bin}(\chi_s(\vec{x}))) = \tilde{g}(j,\vec{x}).$$

Since χ_s is the characteristic function, we have

$$\chi_s(\vec{x}) = \mathtt{BIT}(0;\mathrm{bin}(\chi_s(\vec{x}))) = \tilde{g}(0,\vec{x}).$$

Hence,

$$\chi_s \in \mathcal{M}^2.$$

Therefore,

$$s \in \mathcal{M}^2_*.$$

□

5 Concluding remarks

5.1 Conclusions

We tried to take the correspondence between $\mathcal{E}^2$ and $\mathcal{F}$PTIME and the correspondence between $\mathcal{M}^2$ and $\mathcal{F}$LH, and using these correspondences and the relation $\mathcal{F}\mathrm{LH} \subsetneq \mathcal{F}\mathrm{PTIME}$, we tried to show that $\mathcal{M}^2$ is properly included in $\mathcal{E}^2$. This plan did not do well, however, we were able to prove the following matters:

In chpter 3, we introduced 1-bounded course-of-values recursion (1-BCVR), and defined the class $\mathcal{E}^{2+}$ as

$$\mathcal{E}^{2+} = [\mathtt{0}, \mathtt{I}, \mathtt{S}, +, \times; \mathrm{COMP}, \mathrm{BR}, \text{1-BCVR}].$$

We also defined the two intermediate classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$ as follows:

$$\mathcal{C}_{\mathbb{N}} = [\mathtt{p}_{\mathbb{N}}{}_i^{m,n}, \mathtt{0}, \mathtt{S}, +, \times, |\cdot|, \mathtt{BIT}; \mathrm{COMP}, \mathrm{BR}, \mathrm{BCVR}],$$
$$\mathcal{C}_{\mathbb{W}} = [\mathtt{p}_{\mathbb{W}}{}_j^{m,n}; \mathrm{COMP}, \mathrm{BC}].$$

Then, we have shown the following relations between $\mathcal{E}^{2+}$ and $\mathcal{F}$PTIME by means of the two intermediate classes $\mathcal{C}_{\mathbb{N}}$ and $\mathcal{C}_{\mathbb{W}}$:

(3a) $\forall f \in \mathcal{E}^{2+}\ \exists \alpha \in \mathcal{C}_{\mathbb{N}}\ [f(\vec{x}) = \alpha(\vec{x};)]$,

(3b) $\forall \alpha \in \mathcal{C}_{\mathbb{N}}\ \exists f \in \mathcal{E}^{2+}\ [\alpha(\vec{x}; \mathtt{bin}(\vec{k})) = f(\vec{x}, \vec{k})]$,
$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists \tilde{g} \in \mathcal{E}^{2+}\ [\mathtt{BIT}(z; \varphi(\vec{x}; \mathtt{bin}(\vec{k}))) = \tilde{g}(z, \vec{x}, \vec{k})]$,
$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists \hat{g} \in \mathcal{E}^{2+}\ [|; \varphi(\vec{x}; \mathtt{bin}(\vec{k}))| = \hat{g}(\vec{x}, \vec{k})]$,

(3c) $\forall r \in \mathcal{F}\mathrm{PTIME}\ \exists \alpha \in \mathcal{C}_{\mathbb{N}}\ [\mathrm{BIT}(i, \mathrm{bin}(r(\vec{x}))) = \alpha(i; \mathtt{bin}(\vec{x}))]$,
$\forall r \in \mathcal{F}\mathrm{PTIME}\ \exists \beta \in \mathcal{C}_{\mathbb{N}}\ [|\mathrm{bin}(r(\vec{x}))| = \beta(; \mathtt{bin}(\vec{x}))]$,

(3d) $\forall \alpha \in \mathcal{C}_{\mathbb{N}}\ \exists r \in \mathcal{F}\mathrm{PTIME}\ [\alpha(|\vec{x}|; \mathtt{bin}(\vec{k})) = |r(\vec{x}, \vec{k})|]$,
$\forall \varphi \in \mathcal{C}_{\mathbb{W}}\ \exists t \in \mathcal{F}\mathrm{PTIME}\ [\varphi(|\vec{x}|; \mathtt{bin}(\vec{k})) = \mathrm{bin}(t(\vec{x}, \vec{k}))]$.

Using these relations, with respect to their set classes $\mathcal{E}^{2+}_*$ and PTIME, we have shown that

$$\mathrm{PTIME} \subseteq \mathcal{E}^{2+}_*.$$

Similarly, in chapter 4, We defined the two intermediate classes $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$ as follows:

$$\mathcal{D}_{\mathbb{N}} = [\mathtt{p}_{\mathbb{N}}{}_i^{m,n}, \mathtt{0}, \mathtt{S}, \dot{-}, \times, |\cdot|, \mathtt{BIT}; \mathrm{COMP}, \mathrm{BMIN}],$$
$$\mathcal{D}_{\mathbb{W}} = [\mathtt{p}_{\mathbb{W}}{}_j^{m,n}; \mathrm{COMP}, \mathrm{BC}].$$

Then, we have shown the following relations between $\mathcal{M}^2$ and $\mathcal{F}$LH by means of the two intermediate classes $\mathcal{D}_{\mathbb{N}}$ and $\mathcal{D}_{\mathbb{W}}$:

(4a) $\forall f \in \mathcal{M}^2\ \exists \alpha \in \mathcal{D}_{\mathbb{N}}\ [f(\vec{x}) = \alpha(\vec{x};)]$,

(4b) $\forall \alpha \in \mathcal{D}_{\mathbb{N}}\ \exists f \in \mathcal{M}^2\ [\alpha(\vec{x};\mathtt{bin}(\vec{k})) = f(\vec{x},\vec{k})]$,

$\forall \varphi \in \mathcal{D}_{\mathbb{W}}\ \exists \tilde{g} \in \mathcal{M}^2\ [\mathtt{BIT}(z;\varphi(\vec{x};\mathtt{bin}(\vec{k}))) = \tilde{g}(z,\vec{x},\vec{k})]$,

$\forall \varphi \in \mathcal{D}_{\mathbb{W}}\ \exists \hat{g} \in \mathcal{M}^2\ [|;\varphi(\vec{x};\mathtt{bin}(\vec{k}))| = \hat{g}(\vec{x},\vec{k})]$,

(4c) $\forall r \in \mathcal{F}\mathrm{LH}\ \exists \varphi \in \mathcal{D}_{\mathbb{W}}\ [\mathrm{bin}(r(\vec{x})) = \varphi(;\mathtt{bin}(\vec{x}))]$,

(4d) $\forall \alpha \in \mathcal{D}_{\mathbb{N}}\ \exists r \in \mathcal{F}\mathrm{LH}\ [\alpha(|\vec{x}|;\mathtt{bin}(\vec{k})) = |r(\vec{x},\vec{k})|]$,

$\forall \varphi \in \mathcal{D}_{\mathbb{W}}\ \exists t \in \mathcal{F}\mathrm{LH}\ [\varphi(|\vec{x}|;\mathtt{bin}(\vec{k})) = \mathrm{bin}(t(\vec{x},\vec{k}))]$.

Using these relations, with respect to their set classes $\mathcal{M}^2_*$ and LH, we have shown that

$$\mathrm{LH} \subseteqq \mathcal{M}^2_*.$$

5.2 Directions for further research

In the following, we show some directions for further research.

1. $\mathcal{E}^{2+}_* \subseteqq$ PTIME

   To prove that PTIME $\subseteqq \mathcal{E}^{2+}_*$, we only use the relations (3b) and (3c). It seems difficult to show that $\mathcal{E}^{2+}_* \subseteqq$ PTIME with the current formulas in the relations (3a) and (3d), but it may be possible to show that by modifying these relations. If we can show the opposite inclusion, it will hold that PTIME = $\mathcal{E}^{2+}_*$, which means that the relationship between $\mathcal{E}^2_*$ and PTIME has been elucidated in a sense.

2. $\mathcal{M}^2_* \subseteqq$ LH

   Similarly, to prove that LH $\subseteqq \mathcal{M}^2_*$, we only use the relations (4b) and (4c). It seems difficult to show that $\mathcal{M}^2_* \subseteqq$ LH with the current formulas in the relations (4a) and (4d), but it may be possible to show that by modifying these relations. If we can show the opposite inclusion, it will hold that LH = $\mathcal{M}^2_*$. On the other hand, it is know that LTH = $\mathcal{M}^2_*$ (Theorem 2.22), hence it will also hold that LH = LTH.

3. $\mathcal{M}^2 \subsetneqq \mathcal{E}^{2+}$

   It is know that $\mathcal{F}$LH $\subsetneqq \mathcal{F}$PTIME. It may be possible to prove that $\mathcal{M}^2 \subsetneqq \mathcal{E}^{2+}$ by using this fact and the above relations (3a)~(3d), (4a)~(4d) or their modified relations.

4. $\mathcal{M}^2 \subsetneqq \mathcal{E}^2$

Though we have changed $\mathcal{E}^2$ to $\mathcal{E}^{2+}$ in order to make $\mathcal{E}^2$ correspond with $\mathcal{F}$PTIME, we return to $\mathcal{E}^2$. The difference between $\mathcal{M}^2$ and $\mathcal{E}^2$ is whether the recursive operator is bounded minimisation or bounded recursion, and the difference of computational powers between them seems to be large. Hence, it seems to be true that $\mathcal{M}^2 \subsetneqq \mathcal{E}^2$.

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- Don't focus on studying, focus on thinking about the problems. (However, if you do not study, you will not grow.)

- Do trial and error.

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