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Commutative residuated lattices and intuitionistic linear logic FLe

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1 Introduction

Substractural logics are logics obtained from classical logic or intuitionistic logic by deleting some or all of structural rules. They include e.g. relevant logics which have no weakening rule in general, linear logic which has neither weakening rule nor contraction rule, and FLew which has no contraction rule. In proof-theoretic studies of substructural logics, many of important results like the disjunction property and the decidability come out as consequences of the cut elimination theorem. But when we try to study general properties of substructural logics as a whole, we need to use semantical methods. Recently algebraic study of substructural logics has been developed much.

Main topics of the present thesis is a study of commutative residuated lattices as algebraic structures for logics over intuitionistic linear logic FLe. Here, by FLe we mean the logic obtained from intuitionistic logic by deleting both weakening and contraction rule. In particular, relevant logics forms an important class of logics over FLe.

By using recent results of residuated lattices, we show relations between algebraic and logical properties.

Our main result says that there exist uncontably many minimal varieties of commutative residuated lattices. This is equivalent to say that there exist uncontably many maximal consistent logics over FLe.

2 Logics over FLe

FLe is a logic obtained from intuitionistic logic by deleting both weakening and contraction rule.

In order to treat logics in an abstract and general way, we define a logic as a set of formulas which is closed under substitution and modus ponens. Clearly set of all formulas Φ is a logic, which is the single inconsistent logic, and the set of all formulas which are provable in FLe, denoted by **FLe**, is also a logic. Now we define $\langle \mathcal{L}, \cap, \vee, \Phi, \mathbf{FLe} \rangle$ as follows.

1. $\mathcal{L} := \{L \mid L \textit{ is logic and } \mathbf{FLe} \subseteq L\}$
2. *for* $L_1, L_2 \in \mathcal{L}$,

   $L_1 \vee L_2 :=$ *the smallest logic which includes* $L_1 \cup L_2$.

Then $\langle \mathcal{L}, \cap, \vee, \Phi, \mathbf{FLe} \rangle$ forms a bounded lattice whose lattice order is the set inclusion, which has Φ and **FLe** as the greatest and the smallest element, respectively.

3 Commutative residuated lattices

An algebra $\mathbf{M} = \langle M, \cap, \cup, \cdot, \rightarrow, \top, \bot, 0, 1 \rangle$ is a commutative residuated lattice if

(R1) $\langle M, \cap, \cup, 0, 1 \rangle$ is a bounded lattice with the greatest element $\top$ and the smallest element $\bot$,

(R2) $\langle M, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,

(R3) *for* $x, y, z \in M \quad x \cdot y \leq z \iff x \leq y \rightarrow z$

4 Varieties

Definition 1 *Let $\mathcal{K}$ be a class of algebras. Define operations S, H, P as follows.*

- $\mathbf{A} \in S(\mathcal{K}) \iff \mathbf{A}$ *is a subalgebra of some member of $\mathcal{K}$.*
- $\mathbf{A} \in H(\mathcal{K}) \iff \mathbf{A}$ *is a homomorphic image of some member of $\mathcal{K}$.*
- $\mathbf{A} \in P(\mathcal{K}) \iff \mathbf{A}$ *is a direct product of a nonempty family of algebras in $\mathcal{K}$.*

A nonempty class $\mathcal{K}$ is called a variety if it is closed under subalgebras, homomorphic images, and direct products. Let $V(\mathcal{K})$ denote the smallest variety containing $\mathcal{K}$, and is called the variety generated by $\mathcal{K}$.

Proposition 1 *For any class $\mathcal{K}$,*

$$V(\mathcal{K}) = HSP(\mathcal{K}).$$

Proposition 2 *If $\mathcal{K}$ is a variety, then every member of $\mathcal{K}$ is isomorphic to a subdirect product of subdirectly irreducible members of $\mathcal{K}$.*

So, every variety is generated by subdirectly irreducible members of it.

5 Minimal varieties

A variety V is minimal if V does not have a nontrivial subvariety, where a class K is trivial variety if any member of K is trivial algebra. So if V is minimal variety then its subvarieties are either a trivial variety or V. Our main theorem is the following.

Theorem 1 *There exist continuum minimal varieties of commutative residuated lattices.*

6 Conclusion

Let **CRL** be the variety of all residuated lattices and TV be trivial variety. Now we define $\langle \mathcal{V}, \cap, \vee, TV, \mathbf{CRL} \rangle$ as follows.

1. $\mathcal{V} := \{V \mid V$ is a variety of commutative residuated lattices $\}$

2. for $V_1, V_2 \in \mathcal{V}$,

   $V_1 \vee V_2$ is the smallest variety which includes $V_1 \cup V_2$.

Then $\langle \mathcal{V}, \cap, \vee, TV, \mathbf{CRL} \rangle$ forms a bounded lattice whose lattice order is the set inclusion, which has TV and **CRL** as the greatest and the smallest element, respectively . So $\langle \mathcal{V}, \cap, \vee, TV, \mathbf{CRL} \rangle$ is dual isomorphic to $\langle \mathcal{L}, \cap, \vee, \Phi, \mathbf{FLe} \rangle$.

By theorem 1, there exist continuum maximal logics which include **FLe**.